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REGULARITY OF SOBOLEV–LORENTZ MAPPINGS ON NULL SETS

THOMAS ZÜRCHER

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Jyväskylä, November 2009

Thomas Zürcher

List of Included Articles

This dissertation consists of an introductory part and the following publications:

[BRZ04] Zoltán M. Balogh, Kevin Rogovin, and Thomas Zürcher. The Stepanov differentiability theorem in metric measure spaces. *The Journal of Geometric Analysis*, 14:405–422, 2004.

[WZ08] Kevin Wildrick and Thomas Zürcher. Peano cubes with derivatives in a Lorentz space. `http://www.math.jyu.fi/research/pspdf/371.pdf`, 2008. Preprint 371, Department of Mathematics and Statistics, University of Jyväskylä; To be published in Illinois J. Math.

[WZ09a] Kevin Wildrick and Thomas Zürcher. Mappings with an upper gradient in a Lorentz space. `http://www.math.jyu.fi/research/pspdf/382.pdf`, 2009. Preprint 382, Department of Mathematics and Statistics, University of Jyväskylä.

[WZ09b] Kevin Wildrick and Thomas Zürcher. Space filling with metric measure spaces. `http://www.math.jyu.fi/research/pspdf/381.pdf`, 2009. Preprint 381, Department of Mathematics and Statistics, University of Jyväskylä.

The author of this dissertation has actively taken part in research of the joint papers [WZ08], [WZ09a], [WZ09b], and [BRZ04].

Contents of The Introductory Part

1. Preface

We start with the definition of Lusin's condition N in Section 2. We discuss some of its consequences and then investigate what kind of mappings satisfy this condition. To show the sharpness of our results, we compare condition N with space filling curves and their generalizations in Section 4. Both sections consider mappings with domains in $\mathbb{R}^n$. In Section 5, we look at more general domains.

2. Lusin's Condition N in Euclidean Space

The main topic of this thesis is the study of *Lusin's condition N*. We say that a mapping $u\colon \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, satisfies[1] *Lusin's condition N* if whenever $\mathcal{L}^n(E) = 0$, $E \subset \Omega$, then $\mathcal{L}^n(u(E)) = 0$. Let us give some examples to underline the importance of this condition.

Elasticity theory: Let us assume that $\Omega \subset \mathbb{R}^3$ is a body that is subjected to a homeomorphic deformation $u\colon \Omega \to u(\Omega)$. Any model of deformations of elastic bodies should take into account that deformations do not create matter. In mathematical terms, this requires that the function u satisfies Lusin's condition N.

Change of variable formulas: J. Malý compiled in Theorem 1.1 in [Mal03] area and coarea formulas. They are due to H. Federer, [Fed69]. We refer the reader as well to P. Hajłasz's treatment of the subject in [Haj00].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $E \subset \Omega$ be a measurable set and $f\colon \Omega \to \mathbb{R}^d$ be a Lipschitz function. Let $u\colon \Omega \to \mathbb{R}$ be a measurable function. Suppose that $m \in \{1, \dots, n\}$ and that one of the following situations occurs:*

(a) $m = n$
(b) $m = d$
(c) $m \in \{1, \dots, d\}$ *and $f(\Omega)$ is $\mathcal{H}^m$-rectifiable, i.e. $f(\Omega)$ can be written — up to a set of $\mathcal{H}^m$-measure zero — as countable union of images of subsets of $\mathbb{R}^m$ under Lipschitz maps.*

Then

$$\int_E u(x)|J_m f(x)|\, dx = \int_{\mathbb{R}^d} \left(\int_{E\cap f^{-1}(y)} u(x)\, d\mathcal{H}^{n-m}(x) \right) d\mathcal{H}^m(y), \tag{1}$$

provided the integral on the left makes sense. The Jacobian J_m is defined as

$$J_m f(x) = \sqrt{\sum_{a(x)\in\mathcal{M}(x)} a(x)^2},$$

where $\mathcal{M}(x)$ is the set of all $m \times m$-minors[2] *of $Df(x)$.*

[1] The reader might want to consult Definition 3.2 in [WZ09a] for a definition that is applicable to metric measure spaces.

[2] If $m > d$, then the set $\mathcal{M}(x)$ is empty and hence $J_m f(x) = 0$.

Before continuing, let us make the following definition:

Definition 2.2 (Lipschitz off small sets)**.** Assume that $E_1 \supset E_2 \supset \cdots$ is a sequence of subsets in Ω, $\Omega \subset \mathbb{R}^n$, such that $\mathcal{L}^n(E_k) \leq 1/k$. Suppose that $f\colon \Omega \to \mathbb{R}^m$ is such that each restriction of f to any of the sets $\Omega \setminus E_k$ is Lipschitz. Then we say that f is *Lipschitz off small sets.*

Does (1) still hold for a mapping that is Lipschitz off small sets? Note that in opposition to (c), we decompose the domain and not the image of the mapping. We see that the validity of (1) needs only to be checked for the set $\cap_{j=1}^{\infty} E_j$, which has measure zero. Note that we can rewrite (1) as

$$\int_E u(x)|J_m f(x)|\, dx = \int_{f(E)} \left(\int_{E \cap f^{-1}(y)} u(x)\, d\mathcal{H}^{n-m}(x) \right) d\mathcal{H}^m(y),$$

and hence (1) holds in the case $m = n$ if f satisfies Lusin's condition N.

What kind of mappings satisfy Lusin's condition N? To start the discussion, let us first consider real-valued functions of a real variable. It is not hard to see that mappings that are Lipschitz send sets of measure zero to sets of measure zero. However, Lipschitz continuity is not needed to achieve Lusin's condition N. The function $g\colon (0,1) \to \mathbb{R}$, where $g(x) = 1/x$, is only locally Lipschitz, and by writing E as a countable union of sets where g is Lipschitz, we see that g satisfies Lusin's condition N. We can even give an example of a function that is not locally Lipschitz but satisfies Lusin's condition N. For this, let $h\colon \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = 0$ if $x < 0$ and $\sqrt{x}$ otherwise. If we can write the domain of a function as union of measurable sets where it is Lipschitz and a countable set, then Lusin's condition N still holds.

We note that by the classical Rademacher theorem, see for example [Rad19, Satz I] or [EG92, p. 81], locally Lipschitz functions are differentiable almost everywhere. The functions we have seen so far have been differentiable almost everywhere as well. However, differentiability almost everywhere is not enough to conclude Lusin's condition N. The Cantor function, see for example [Sag94, Section 5.3], maps the middle third Cantor set, which has 1-dimensional Lebesgue measure zero, onto the interval $[0,1]$. The issue with the Cantor function is that its derivative, which exists only almost everywhere, does not reflect enough of the behavior of the Cantor function. That is, the fundamental theorem of calculus fails to hold for the Cantor function. In fact, on a closed interval $[a,b] \subset \mathbb{R}$, the fundamental theorem of calculus holds if and only if the function is absolutely continuous. *Absolute continuity* means that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\sum_{i=1}^{N} (b_i - a_i) < \delta$$

implies

$$\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \varepsilon$$

for any collection $\{(a_i, b_i)\}$ of pairwise disjoint intervals. Absolute continuity, and hence the fundamental theorem of calculus, clearly imply Lusin's condition N.

The converse, that continuous functions that satisfy Lusin's condition N are locally absolutely continuous, is not exactly true. However, by Theorem 7.1.38 in [KK96], we have equivalence for functions in $\mathrm{BV}_{\mathrm{loc}}$.

In dimension 1, there is yet another characterization of absolutely continuous functions in terms of being continuous and enjoying the membership in the Sobolev space $W^{1,1}(\Omega)$, see for example [Hei07, Section 3]. What can we say about the case $n \geq 2$? M. Marcus and V. J. Mizel showed in [MM73] that a continuous mapping in $W^{1,p}(\Omega; \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$ a bounded domain, sends sets of measure zero to sets of measure zero provided $p > n$. On the other hand, as seen by the constructions of O. Martio and J. Malý, for each $n \geq 2$ there exists a continuous mapping $f\colon \Omega \to \mathbb{R}^n$ in $W^{1,n}(\Omega; \mathbb{R}^n)$ such that the Jacobian determinant J_f of f satisfies $J_f = 0$ a.e. and f maps a set of n-dimensional Lebesgue measure zero onto an n-dimensional cube, see [MM95, Section 5]. In the case $p < n$, S. P. Ponomarev demonstrated in [Pon87] that for all $n > 1$, there exists even a homeomorphism $f\colon [0,1]^n \to [0,1]^n$ such that f is the identity on the boundary, it is contained in $W^{1,p}([0,1]^n;[0,1]^n)$ for all $p<n$, the inverse f^{-1} is in $W^{1,p}([0,1]^n;[0,1]^n)$ for all $p>0$, and yet f fails to satisfy Lusin's condition N. See also [Pon71]. Yu. G. Reshetnyak showed in [Res87] that under a certain stability condition, a continuous mapping in $W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ satisfies Lusin's condition N. Further information can be found in his book [Res89], especially Section II.6.2. We also mention the article [MZ92] by O. Martio and W. Ziemer where it is shown that $J_f>0$ almost everywhere is a sufficient condition for a continuous mapping f in $W^{1,n}(\Omega;\mathbb{R}^n)$ to satisfy Lusin's condition N. If $f^{-1}(y)$ is totally disconnected for each $y\in\mathbb{R}^n$, then it is possible to replace $J_f>0$ by $J_f\geq 0$ in above statement. Finally, in the aforementioned article [MM95], the authors list further requirements we can impose on a continuous mapping in $W^{1,n}(\Omega;\mathbb{R}^n)$ to enforce Lusin's condition N. For example each of these three additional requirements is enough: pseudomonotonicity, openness, and Hölder continuity. However, these concepts are not appropriate to obtain a complete space with mappings satisfying Lusin's condition N.

We mentioned above that $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ is small enough to guarantee Lusin's condition N for its continuous members if $n<p$. On the other hand, $W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ is too large; it contains continuous mappings violating Lusin's condition N. To continue our inquiry, we consider spaces larger than $\cup_{p>n}W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ but smaller than $W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. Especially, we need to have a finer categorization of the regularity of the derivative at hand. The Lorentz spaces $L^{n,q}_{\mathrm{loc}}(\Omega)$ lie between the Lebesgue spaces. We have for $1\leq p\leq n\leq q\leq\infty$:

$$L^{n,1}_{\mathrm{loc}}(\Omega)\subset L^{n,p}_{\mathrm{loc}}(\Omega)\subset L^{n,n}_{\mathrm{loc}}(\Omega)=L^{n}_{\mathrm{loc}}(\Omega)\subset L^{n,q}_{\mathrm{loc}}(\Omega)\subset L^{n,\infty}_{\mathrm{loc}}(\Omega).$$

J. Kauhanen, P. Koskela, and J. Malý treated Lusin's condition N in the framework of Lorentz spaces in [KKM99]. As an important tool, they use a characterization of the membership in a Lorentz space by an Orlicz space-like condition. This gives the possibility to estimate the norm of a function without taking recourse to the nonincreasing rearrangement. Their Theorem C is as follows:

Theorem 2.3. *Assume that $\Omega\subset\mathbb{R}^n$ is a domain and suppose that $u\in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ is a continuous mapping whose weak partial derivatives belong to $L^{n,1}(\Omega)$. Then u satisfies Lusin's condition N, i.e. $\mathcal{H}^n(u(E))=0$ for every set $E\subset\Omega$ with $\mathcal{L}^n(E)=0$.*

Theorem B in the same article sheds more light on the connection between Lusin's condition N and differentiability, see also [Ste81]:

Theorem 2.4. *Suppose that $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a function whose weak partial derivative belongs to $L^{n,1}(\Omega)$. Then there is a representative of u that is continuous and differentiable a.e. in Ω.*

The key component in the proof of Theorem 2.3 made its first appearance in [RR55]. A function $f\colon\Omega\to\mathbb{R}^n$ satisfies the *Rado-Reichelderfer condition (RR)* if there exists $g\in L^1(\Omega)$ such that

$$\left(\operatorname{osc}_{B(x,r)} f\right)^n \leq \int_{B(x,r)} g\,dy \tag{2}$$

for every ball $B(x,r)\subset\Omega$. In this case, we call g a *weight*. In the 1-dimensional case, we easily deduce (RR) from the fundamental theorem of calculus:

$$\operatorname{osc}_{B(x_0,r)} f=\sup_{x,y\in B(x_0,r)}|f(y)-f(x)|\leq\sup_{x,y\in B(x_0,r)}\left|\int_x^y f'(z)\,dz\right|\leq\int_{B(x_0,r)}|f'(z)|\,dz.$$

The Rado-Reichelderfer condition implies not only Lusin's condition N but also n-absolute continuity (a generalization of absolute continuity) and $\operatorname{Lip} f(x)<\infty$ almost everywhere. The finiteness of $\operatorname{Lip} f$ together with Stepanov's theorem, see [Mal99a], implies differentiability almost everywhere. We take up this theme again in Section 5.3.

3. Lusin's Condition N with Metric Space Targets

One goal of this thesis is to consider Lusin's condition N for mappings between spaces other than $\mathbb{R}^n$ equipped with the Euclidean metric. One example that we investigate more carefully are the Heisenberg groups. They have underlying set $\mathbb{R}^n$ but possess a different metric, see for example [Sem03] or [CDPT07] for more information. However, our inquiry is not bounded to spaces whose underlying set is a subset of $\mathbb{R}^n$ nor do we require the metric spaces to be manifolds.

Before continuing with the discussion, let us say something about mappings with values in any separable metric space. The Lebesgue theory of integration extends to Banach-valued mappings. In this case, one calls the integral *Bochner-integral*, see for example [HKST01].

Let us now recall the Lorentz spaces, [Lor50]. In order to do that, we need some more notation.

Definition 3.1. Given a metric measure space (X, d, μ) and a Banach space $(V, \|\cdot\|_V)$, we denote by $\mathcal{M}$ and $\mathcal{M}_0$ the following classes of functions, respectively:

$$\mathcal{M} := \{f : X \to V : \ f \ \mu\text{-measurable}\},$$
$$\mathcal{M}_0 := \{\|f\|_V \in \mathcal{M} : f \text{ finite } \mu\text{-almost everywhere}\}.$$

Definition 3.2 (distribution function, nonincreasing rearrangement)**.** Given a metric measure space (X, d, μ), we let $f \in \mathcal{M}_0$. Then we define the *distribution function* $\omega_f : [0, \infty) \to [0, \infty]$ and the *nonincreasing rearrangement* $f^* : [0, \infty) \to [0, \infty]$ as

$$\omega_f(\alpha) := \mu(\{x \in X : \|f(x)\|_V > \alpha\}),$$
$$f^*(t) := \inf\{\alpha \geq 0 : \omega_f(\alpha) \leq t\}.$$

Now, we are able to introduce the *Lorentz spaces*:

Definition 3.3 (Lorentz spaces)**.** Let (X, d, μ) be a metric measure space. Let $1 \leq Q \leq \infty$ and $0 < q \leq \infty$. The (Q, q)-*Lorentz class* consists of those functions $f \in \mathcal{M}_0(X)$ such that the quantity

$$\|f\|_{Q,q} := \begin{cases} (\int_0^\infty (t^{1/Q} f^*(t))^q \frac{dt}{t})^{1/q}, & 0 < q < \infty, \\ \sup_{0<t<\infty} \{t^{1/Q} f^*(t)\}, & Q < \infty \text{ and } q = \infty, \\ f^*(0), & q = Q = \infty \end{cases}$$

is finite. If $1 \leq q \leq Q$, then $\|\cdot\|_{Q,q}$ defines a semi-norm on the (Q, q)-Lorentz class, and the corresponding normed space $(L^{Q,q}(X), \|\cdot\|_{Q,q})$ is a Banach space. We refer to it as the (Q, q)-*Lorentz space.*

As in the Euclidean case, the following definition is built keeping the integration by parts formula at the back of our minds.

Definition 3.4 (Banach-valued Sobolev space)**.** We assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a domain equipped with the n-dimensional Hausdorff measure $\mathcal{H}^n$. We let further $(V, \|\cdot\|_V)$ be a Banach space and $1 \leq q \leq n$. Given $f \in L^1_{\text{loc}}(\Omega; V)$, we call a Bochner measurable function $v : \Omega \to V$ an i^{th} weak partial derivative of f if for every $\phi \in \mathcal{C}_0^\infty(\Omega)$,

$$\int_\Omega \frac{\partial \phi}{\partial x_i} f \, d\mathcal{H}^n = - \int_\Omega \phi v \, d\mathcal{H}^n.$$

If an i^{th} weak partial derivative exists, then it is unique, and we denote it by $\partial_i f$.
Finally, above f belongs to the *Sobolev space* $W^{1,n}(\Omega; V)$ if:

(i) $f \in L^n(\Omega)$,
(ii) f has weak partial derivatives $\partial_1 f, \ldots, \partial_n f$ in the space $L^n(\Omega; V)$.

Finally, we make the step from Banach-valued mappings to mappings with metric spaces as targets by isometrically embedding the metric spaces into Banach spaces. Analogously, we define the space $W^{1,n,q}(\Omega; V)$, where the weak partial derivatives are required to be in $L^{n,q}(\Omega)$.

In this thesis, we provide a version of Theorem 2.3 for metric valued mappings.

Theorem 3.5 (Theorem 1.3 in [WZ08])**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain, and let Y be a separable*[3] *metric space. If $f : \Omega \to Y$ is a locally integrable and continuous mapping with weak partial derivatives in $L^{n,1}(\Omega)$, then f satisfies Lusin's condition N.*

[3] A metric space is *separable* if it contains a countable dense subset.

The above statement has the following corollary.

Corollary 3.6 (Theorem 1.2 in [WZ08]). *Let Y be a length-compact metric space. For all $n \geq 1$, if Y is the image of a continuous surjection in the Sobolev-Lorentz space $W^{1,n,1}([0,1]^n; Y)$, then the Hausdorff dimension of Y is at most n.*

Above results render obvious that the higher dimensional analogue of $W^{1,1}$ is not $W^{1,n}$ but $W^{1,n,1}$.

We direct the reader to [Kar07] and [Dud07] for area and coarea formulas similar to Theorem 2.1 in this context.

4. Space-filling

To see the sharpness with respect to the Lorentz-scale of the above results, we need to construct a mapping with (weak) gradients in $L^{n,q}(\mathbb{R}^n)$ for $q > 1$ that send sets of measure zero to sets of positive measure. We even go a step further and investigate the existence of continuous surjections from one space onto an other. We will talk of *space-filling* in this context. The reader is encouraged to have a look at [Sag94]. Let us illustrate the contrast to Lusin's condition N by the following example. Suppose that the mapping $f\colon [0,1]^n \to \mathbb{R}^{n+1}$ is Lipschitz off small sets and maps the cube $[0,1]^n$ onto a set, which has positive $(n+\varepsilon)$-dimensional Hausdorff measure for some $\varepsilon > 0$. Then there exists a set in the cube with n-dimensional Hausdorff measure zero that is mapped to a set of positive n-dimensional Hausdorff measure, as can be seen for example by Lemma 6.7 in [WZ09b]. Having a Hajłasz upper gradient in L^1 is for example a sufficient condition for being Lipschitz off small sets.

What spaces can we write as image of — let us say — the unit interval $[0,1]$ under a continuous surjection? There is a classical result known as Hahn-Mazurkiewicz theorem, [Hah14],[Maz20], [Sag94, Theorem 6.8], which characterizes continuous images of $[0,1]$ as precisely those topological spaces that are compact, connected, locally connected and metrizable. The Hahn-Mazurkiewicz theorem says nothing about the regularity of the space-filling. The following result, which is Theorem 1.3 in [HT08], takes regularity into account. Here, length-compact means that Y is compact with respect to the induced path metric, see Definition 5.1:

Theorem 4.1 (Hajłasz-Tyson). *For all $n \geq 2$ and each length-compact metric space Y there is a continuous surjection $f\colon [0,1]^n \to Y$ in the Sobolev class $W^{1,n}([0,1]^n; Y)$, that is a.e. metrically differentiable.*

We see that Theorem 4.1 is almost what we need to guarantee the sharpness of Lusin's condition N. We generalized it as follows:

Theorem 4.2 (Theorem 1.2 in [WZ08]). *For all $n \geq 2$ and $1 < q \leq n$, each length-compact metric space Y is the image of a continuous surjection in the Sobolev-Lorentz space $W^{1,n,q}([0,1]^n; Y)$.*

We note here that the space-filling can be chosen such that it is metrically differentiable almost everywhere. This can be seen by applying a Stepanov-type theorem.

5. Metric Setting Results

In the previous sections, we have considered mappings with nice domains in $\mathbb{R}^n$. Now, we would like to allow more general domains not only in $\mathbb{R}^n$ but in general metric measure spaces. This is much more delicate than considering metric targets. Let us start with the discussion of some properties of metric spaces.

5.1. **Basic Properties of Metric Measure Spaces.** We start with a version of compactness.

Definition 5.1 (length-compact). A metric space (X,d) is length-compact if the space (X,d_l) is compact, where d_l is the path metric.

We note that a length-compact metric space is automatically separable. The *Hilbert cube* H is an example of a length-compact space that is not finite-dimensional. It can be defined by

$$H := \left\{ (a_n)_n \in l^2 : 0 \leq a_n \leq \frac{1}{n} \right\}$$

equipped with metric induced by the ℓ^2-metric. An application of Tychonoff's theorem shows that it is compact with respect to the product topology, which agrees with the topology induced by the metric. Since ℓ^2 is geodesic[4] and H is convex, its path metric is the same as the ℓ^2-metric.

Definition 5.2 (metric measure space)**.** A *metric measure space* is a triple (X, d, μ) where (X, d) is a metric space, and μ is a Borel regular outer measure on X that assigns positive and finite measure to each ball in X.

In a metric measure space as defined above, there is not much relation between the metric and the measure. It is often crucial to have the right interplay between them. The *doubling condition* and the *Q-regularity* take this into account:

Definition 5.3 (doubling)**.** Let (X, d, μ) be a metric measure space. We call the measure μ *doubling* if there is a constant $C \geq 1$ such that for all $x \in X$ and $r > 0$

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Doubling measures enforce the metric space (X, d) to be *doubling*, meaning that there exists a constant C such that every ball $B(x, r)$ in X can be covered by at most C balls $B(x_i, r/2)$. In a doubling space, some parts of the space may be of lower dimension than other parts. This contrasts with *Ahlfors regular spaces*:

Definition 5.4 (Ahlfors Q-regular)**.** A metric measure space (X, d, μ) is called *Ahlfors Q-regular*, $Q \geq 0$, if there exists a constant $C \geq 1$ such that for all points $x \in X$ and radii $0 < r \leq \operatorname{diam} X$, we have

$$\frac{r^Q}{C} \leq \mu(B(x, r)) \leq Cr^Q. \tag{3}$$

If only the latter inequality holds, then X is said to be *upper Ahlfors Q-regular*. We call the space X *Ahlfors Q-regular at scales below* $r_0 > 0$ if (3) holds for all radii r strictly smaller than r_0.

If the measure is Ahlfors regular then it is doubling as well.

Let us delve a little bit more into the geometry of metric measure spaces. There is a close relationship between the geometry of the space and which functions are considered to be "differentiable". We need to generalize the notion of the derivative to the setting of metric measure spaces.

For Sobolev mappings with metric spaces as targets, we refer to [Res07] by Yu. G. Reshetnyak, which summarizes his work done in [Res97], [Res04], and [Res06]. M. Troyanov investigates in [Tro00] connections between a generalization of Hajłasz spaces and the spaces introduced by Reshetnyak, where the domain is a Riemannian manifold. We also mention here P. Hajłasz's article [Haj09] concerning Sobolev spaces of mappings between metric spaces, especially when the domain is a manifold. For yet an other approach to mappings with metric space as targets, we cite [KS93], see [HKST01] as well. These approaches mainly deal with the case where the domain is a Euclidean space or a manifold. In Euclidean spaces, we can talk of directional derivatives — in metric spaces we have to deal with "derivatives along curves" instead. The following concept is a generalization of the norm of the derivative:

Definition 5.5 (upper gradient)**.** Let $f \colon X \to Y$ be a mapping between metric spaces. An *upper gradient* of f is a Borel function $g \colon X \to [0, \infty]$ such that for each rectifiable path $\gamma \colon [0, 1] \to X$,

$$d_Y(f(\gamma(0)), f(\gamma(1))) \leq \int_\gamma g\, ds.$$

[4]Every pair of points can be joined by a curve whose length is the distance between the points.

It turns out that in order to construct space-fillings, we need auxiliary functions attaining the value 1 in a small ball and 0 outside a bigger ball such that there is an upper gradient with small norm. If a space has only few curves, then philosophically, functions possess more upper gradients and hence the infimum over all their norms tends to be small.

This is where the geometry of the spaces comes into the play. If a space contains a lot of short curves, then there is less chance of space-filling. There is a family of inequalities — the Poincaré inequalities — that mirror the amount of curves a space possesses:

Definition 5.6 (Poincaré inequality)**.** A metric measure space (X, d, μ) is said to admit a p-Poincaré inequality, for $p \geq 1$, with two positive constants C_P and σ if

$$\fint_B |u - u_B| \, d\mu \leq C_P (\operatorname{diam} B) \left(\fint_{\sigma B} g^p \, d\mu \right)^{\frac{1}{p}} \tag{4}$$

for all balls $B \subset X$ and every measurable function $u \colon X \to \mathbb{R}$ and upper gradient g.
We denote

$$u_B := \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

There are different ways of defining the Poincaré inequality. However, if the space is complete and the measure is doubling, then there is a bunch of other definitions equivalent to the given one, see Theorem 2 in [Kei03].

We refer the reader to the discussion of the connection between the Loewner condition — a property described with help of the amount of curves — and the Q-Poincaré inequality, where Q is the "dimension" of the space, in [HK98], especially Corollary 5.13. An other good source is [Hei01]. R. Korte has investigated geometric implications of the Poincaré inequalities in [Kor07]. We think that although the geometric meaning of the Loewner condition is more evident than the one of the Poincaré inequality, it is easier to handle the inequality. That is why we prefer to work with the inequality rather than with the Loewner condition.

Theorem 9.27 in [Hei01], tells us that the Heisenberg group fulfills the requirements we put on our domains, especially, by Hölder's inequality, it is Loewner:

Theorem 5.7. *The metric measure space $\left(\mathbb{H}^1, d, \mathcal{H}^3\right)$ is a proper, geodesic, and 4-regular space admitting a 1-Poincaré inequality.*

5.2. **Back to the Track.** The topic of Lusin's condition and Lorentz spaces in metric measure spaces has been studied by A. S. Romanov in [Rom08] and A. Ranjbar-Motlagh in [RM09]. A. S. Romanov shows a result about the absolute continuity of mappings with an upper gradient in $L^{Q,1}(\Omega)$. Similar results are given by A. Ranjbar-Motlagh in a very general setting. He deals mainly with slightly different mappings than the ones showing up in this dissertation. With some extra work, we may deduce the following result from the work of A. S. Romanov, see Section 6 in [WZ09b]. A full and detailed proof that does not rely on the literature is presented in [WZ09a]:

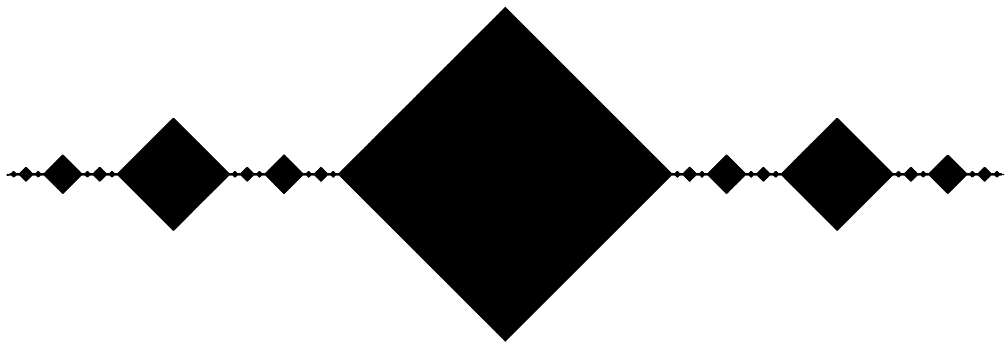
Theorem 5.8 (Theorem 1.1 in [WZ09a])**.** *Assume that (X, d, μ) is a complete and doubling metric measure space that supports a Q-Poincaré inequality, $Q > 1$, and is Ahlfors Q-regular at small scales. Let Y be a separable metric space, and suppose that $f \in L^1_{\mathrm{loc}}(X; Y)$ is continuous and has an upper gradient $g \in L^{Q,1}(X)$. Then f satisfies the Q-Rado-Reichelderfer condition with a weight that depends only on the constants associated to the assumptions and g. Consequently, the mapping f satisfies Lusin's condition N and satisfies* $\operatorname{Lip} f(x) < \infty$ *for almost every $x \in X$.*

In Corollary 5.13, we will note that if $Y = \mathbb{R}^m$, then f is differentiable almost everywhere.

The work of N. Marola and W. P. Ziemer also considers Lusin's condition N. The difference is that they assume higher regularity for the functions but less for the space, see Section 6 in [MZ08].

Again, as in the Euclidean case, there is a counter part provided by space-fillings:

Theorem 5.9 (Theorem 1.3 in [WZ09b])**.** *Let (X, d, μ) be a locally compact metric measure space, let Y be any length-compact metric space, and let $1 < q \leq Q$. Suppose that there is a non-empty set $P \subset X$ that has no isolated points and compact closure, and that X is upper Q-regular at each point of P. Then there is a continuous surjection $f \colon X \to Y$ that has an upper gradient in the Lorentz space $L^{Q,q}(X)$.*



The Cantor Diamond

We will look more carefully at the differentiability of above mappings in Corollary 5.13.

In Theorem 5.8, we saw that mappings with upper gradients in $L^{Q,1}(X)$ satisfy Lusin's condition N. However, we needed the presence of the Q-Poincaré inequality to conclude this. As the following result shows, if the Poincaré inequality is too weak, then it is sometimes possible to obtain space-filling and hence condition N cannot hold, see Theorem 1.5 in [WZ09b]:

Theorem 5.10. *For any $\varepsilon > 0$, there is a compact Ahlfors 2-regular metric space X that supports a $(2+\varepsilon)$-Poincaré inequality with the following property: for any $1 \leq p < 2+\varepsilon$, and any length-compact metric space Y, there is a continuous surjection $f\colon X \to Y$ that is constant off a set of finite measure and has an upper gradient in the space $L^p(X)$. In particular, there is a continuous and integrable surjection $f\colon X \to Y$ with an upper gradient in the space $L^{2,1}(X)$.*

The spaces we use in the previous result were introduced in [KM98], see the figure. The mappings that we construct build on some sort of Cantor functions. How can there exist upper gradients of functions acting like the Cantor functions with small norms? It is as Wilhelm Tell said, when he planned his ambush on Gessler [Sch04, Vierter Aufzug, Dritte Scene]: "Durch diese hohle Gasse muß er kommen, es führt kein andrer Weg nach Küßnacht[5]." Our curves have not much freedom in choosing their paths since if they want to go from one square to another there are plenty of points they *have* to pass. And as Tell for Gessler, we wait in these points to give high values to the upper gradients in a neighborhood of these points. As Gessler does not really recognize the danger, so the 2-dimensional measure does not take these points enough into account.

As application of our results, we mention here a corollary concerning the Heisenberg group. The precise reader may object that the Heisenberg group is not length-compact. We overcome this issue, see [WZ09b, Corollary 1.6]:

Corollary 5.11. *For each $n \geq 1$, and each $1 < q \leq 4$, there is a continuous surjection $f\colon \mathbb{H}^1 \to \mathbb{H}^n$ that is constant off a set of finite measure, has finite local Lipschitz constant off a set of Hausdorff dimension 0, and has an upper gradient in the space $L^{4,q}(\mathbb{H}^1)$. On the other hand, if $f\colon \mathbb{H}^1 \to \mathbb{H}^n$ is a continuous mapping with an upper gradient in the space $L^{4,1}(\mathbb{H}^1)$, then the image of f has Hausdorff dimension at most 4.*

5.3. **Differentiability and Differentiable Structures.** An additional property that the space filling mappings we constructed fulfill is that they have finite local Lipschitz constant $\operatorname{Lip} f$ off a set of zero measure. In the Euclidean case, this puts them in a row with other classes of mappings like quasiconformal mappings — homeomorphisms that map infinitesimally circles to ellipses whose ratio of the axes is bounded — and some of their generalizations such as Q-homeomorphisms

[5][SPBW47] Trough this hollow way must he come; There leads no other way to Küssnacht.

for $Q \in L^1_{\mathrm{loc}}(\Omega)$. We refer the reader to [Väi71], [Car74], [MRSY09] and [Sal08] for the precise definitions. It turns out that quasiconformal mappings and Q-homeomorphisms are differentiable almost everywhere, and the key point used to prove this is the finiteness of the local Lipschitz constant, see for example Chapter 4 in [MRSY09]. Once this fact is known, one can apply Stepanov's theorem to conclude differentiability almost everywhere.

It is very delicate to generalize differentiability to spaces other than finite dimensional Banach spaces. There are even Lipschitz mappings from $(0,1)$ into $L^1((0,1))$ that are nowhere differentiable, see Example II on p. 169 in [Aro76] and [CK06]. In contrast to this example where the target is very general, J. Cheeger studied in [Che99] the situation where the domain is a metric space. He introduced *strong measurable differentiable structures* to provide a framework for doing calculus on more general spaces than manifolds. He showed that spaces with a doubling measure supporting a Poincaré inequality admit a differentiable structure. S. Keith found a weaker condition implying the existence of a strong measurable differentiable structure, see Theorem 2.3.1 in [Kei04].

Strong measurable differentiable structures have the almost everywhere differentiability of Lipschitz functions built in. In the classical situation, Stepanov's theorem follows from the fact that Lipschitz functions are differentiable almost everywhere, see [Ste23] or [Mal99b]. It is a natural question to ask if there is also a Stepanov like theorem in the setting of strong measurable differentiable structures. The following result is Theorem 3.1 in [BRZ04]. It provides an affirmative answer.

Theorem 5.12. *Let (X,d,μ) be a metric measure space. Let the measure μ be doubling. Assume that there is a strong measurable differentiable structure $\{(X_\alpha,\varphi_\alpha)\}$ for (X,d,μ) with respect to* $\mathrm{LIP}(X)$*, the space of all Lipschitz functions on X. Then a function $f\colon X\to\mathbb{R}$ is μ-a.e. differentiable in*

$$S(f):=\left\{x\in X:\ \mathrm{Lip}\, f(x)=\limsup_{r\to 0}\ \sup_{y\in B(x,r)}\frac{|f(x)-f(y)|}{r}<\infty\right\}$$

with respect to the structure $\{(X_\alpha,\varphi_\alpha)\}$.

We note that Stepanov's theorem now gives the almost everywhere differentiability of the mappings in Theorem 5.8 and in Theorem 5.9:

Corollary 5.13. *If $Y=\mathbb{R}^m$, then the mappings considered in Theorem 5.8 as well as the ones in Theorem 5.9 are differentiable almost everywhere.*

In contrast, E. Stein notes in [Ste81] that for $1<q\le n$ there exist mappings $f\in L^{n,q}(\mathbb{R}^n;\mathbb{R})$ that are not differentiable almost everywhere.

As further corollary of Stepanov's theorem, we obtain sort of weak differentiability for quasiconformal homeomorphisms:

Corollary 5.14 (Corollary 4.9 in [BRZ04])**.** *Let $f\colon (X,d_X,\mu)\to(Y,d_Y,\nu)$ be a quasiconformal homeomorphism between metric measure spaces, where both are Q-regular and X admits a strong measurable differentiable structure. Then for each Lipschitz function $\varphi\colon Y\to\mathbb{R}$ the function $\varphi\circ f$ is differentiable μ-almost everywhere in X.*

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