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POROSITY AND DIMENSION OF SETS AND MEASURES

TAPIO RAJALA

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Jyväskylä, July 2009

Tapio Rajala

List of included articles

This dissertation consists of an introductory part and the following four publications:

[A] M. Csörnyei, A. Käenmäki, T. Rajala and V. Suomala, *Upper conical density results for general measures on* $\mathbb{R}^n$, Proc. Edinb. Math. Soc. (2), to appear.

[B] D. Beliaev, E. Järvenpää, M. Järvenpää, A. Käenmäki, T. Rajala, S. Smirnov and V. Suomala, *Packing dimension of mean porous measures*, J. Lond. Math. Soc. (2), to appear.

[C] E. Järvenpää, M. Järvenpää, A. Käenmäki, T. Rajala, S. Rogovin and V. Suomala, *Packing dimension and Ahlfors regularity of porous sets in metric spaces*, Math. Z., to appear.

[D] T. Rajala, *Large porosity and dimension of sets in metric spaces*, Ann. Acad. Sci. Fenn. Math. **34** (2009), no. 2, 565–581.

The author of this dissertation has actively taken part in research of the joint papers [A], [B] and [C].

Introduction

This thesis is in the field of geometric measure theory. From one perspective, it is concerned with conical densities. From another perspective the thesis deals with porosities of sets and measures. These two concepts are closely related. For example, some of the dimension estimates for porous sets have been obtained using conical density results. See [24], [19] and Section 2 of this introduction for more information. Conical densities are also closely related to rectifiability and approximate tangent planes. For more information on rectifiability and approximate tangents see Subsection 1.1 of this introduction, [25] and [28].

1 Conical densities

The study of conical densities goes back to Besicovitch. See [2] and [3]. He studied the conical density properties of sets in the plane. Since then many authors have worked in the field. These include Morse and Randolph [27], Marstrand [22], Federer [11], Salli [30] and Mattila [24]. These works dealt mainly with distributions of sets with respect to Hausdorff measures. For more general Hausdorff (and packing) type measures conical densities were studied by Käenmäki and Suomala in [18]. See also [33]. The article [A] of this thesis continues the study of conical densities in the direction shown by [18].

Let $n, m \in \mathbb{N}$ and $0 \leq m < n$. By S^{n-1} we denote the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ and by $G(n, n-m)$ the space of $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$. For a point $x \in \mathbb{R}^n$, a direction $\theta \in S^{n-1}$ and an 'angle' $0 \leq \alpha \leq 1$, define

$$H(x, \theta, \alpha) = \{y \in \mathbb{R}^n : (y - x) \cdot \theta > \alpha |y - x|\}.$$

The one-sided cone $H(x, \theta, \alpha)$ with a small α is nearly a halfspace. A cone around a subspace $V \in G(n, n-m)$ is defined as

$$X(x, V, \alpha) = \{y \in \mathbb{R}^n : \operatorname{dist}(y - x, V) < \alpha |x - y|\}.$$

Notice that when $n - m = 1$, a cone $X(x, V, \alpha)$ is two-sided. By $\operatorname{dist}(z, A)$ we denote the distance from a point $z \in \mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$. Note that here small α means small cone. Look at Figure 1 to get an idea what the cones look like.

A typical upper density theorem states that for $\mathcal{H}^s$-almost every point $x \in \mathbb{R}^n$ an s-dimensional set $A \subset \mathbb{R}^n$ is spread around every $n-m$-dimensional subspace going through x when $s > m$. More precisely, let $V \in G(n, n-m)$, $0 < \alpha < 1$,

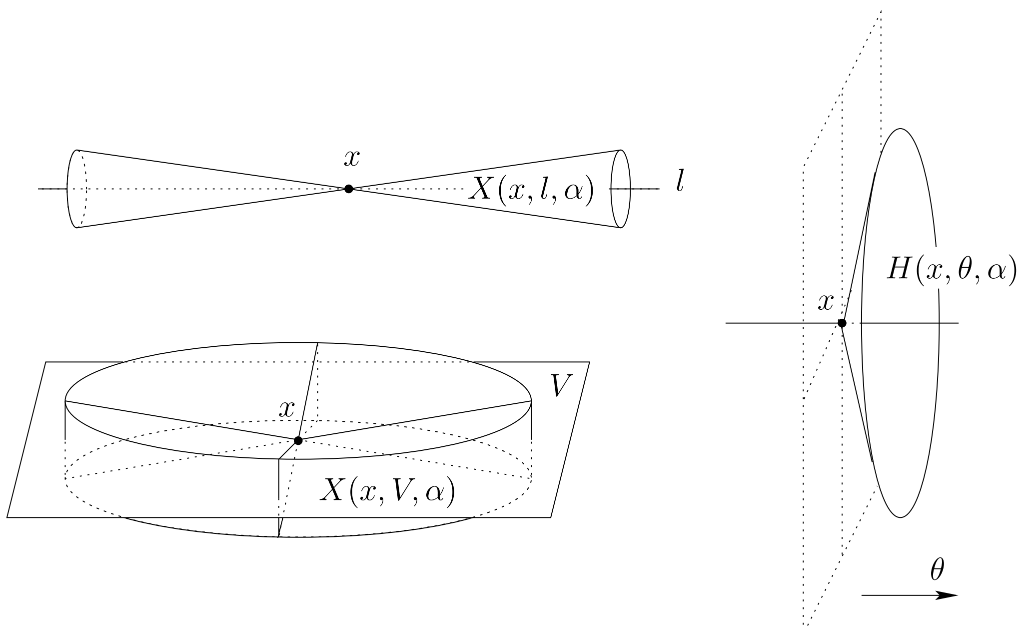

Figure 1: Here are illustrated the cones $X(x,l,\alpha)$, $X(x,V,\alpha)$ and $H(x,\theta,\alpha)$ with a small angle α, a hyperplane $V \in G(3,2)$, a line $l \in G(3,1)$ and a direction $\theta \in S^2$.

and $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$ where $s > m$. Then there exists a positive constant c depending only on n, m, s and α so that

$$\limsup_{r\to 0} \inf_{V\in G(n,n-m)} \frac{\mathcal{H}^s(A \cap X(x,V,\alpha) \cap B(x,r))}{(2r)^s} \geq c.$$

Here $B(x,r)$ is the open ball $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. See [24, Theorem 3.3] for the proof of a more general version of this result.

For a general measure μ taking the limit with a denominator $(2r)^s$ does not usually make any sense. Therefore it is replaced by $\mu(B(x,r))$. The article [A] consists of examples and the following two theorems. We will come back to the examples in the next subsection.

Theorem 1.1. *[A, Theorem 4.1] Suppose μ is a non-atomic measure on $\mathbb{R}^n$ and $0 < \alpha \leq 1$. If $\dim_{\mathrm{H}}(\mu) \geq s > m \in \{0,1,\ldots,n-1\}$, then for μ-almost every $x \in \mathbb{R}^n$*

$$\limsup_{r\to 0} \inf_{\substack{\theta\in S^{n-1}\\ V\in G(n,n-m)}} \frac{\mu(B(x,r) \cap X(x,V,\alpha) \setminus H(x,\theta,\alpha))}{\mu(B(x,r))} \geq c$$

with the constant $c > 0$ depending only on α, s, n and m.

By $\dim_{\mathrm{H}}(\mu)$ we denote the Hausdorff dimension of μ which can be defined using the Hausdorff dimension of sets as

$$\dim_{\mathrm{H}}(\mu) = \inf\{\dim_{\mathrm{H}}(A) : A \text{ is a Borel set with } \mu(A) > 0\}.$$

Theorem 1.1 means that measures with sufficiently large Hausdorff dimension are spread near every subspace in many directions. Leaving out the subspace V we see by the next theorem that any measure (that is not atomic at a point) is locally spread in many directions.

Theorem 1.2. *[A, Theorem 3.1] Suppose μ is a measure on $\mathbb{R}^n$ and $0 < \alpha \leq 1$. Then for μ-almost every $x \in \mathbb{R}^n$*

$$\limsup_{r \to 0} \inf_{\theta \in S^{n-1}} \frac{\mu(B(x,r) \setminus H(x,\theta,\alpha))}{\mu(B(x,r))} \geq c$$

with the constant $c > 0$ depending only on n and α.

In the proof of Theorem 1.2 we look at the distribution of the measure on a sequence of doubling scales. On each of the scales we discretize the possible directions of cones and conclude that the measure cannot be concentrated on any of those cones.

The proof of Theorem 1.1 is a bit more technical. Again we do our reasoning on doubling scales. First we need enough balls with large mass. To find them we use the concept of average homogeneity and the results obtained by E. Järvenpää and M. Järvenpää in [14]. From the collection of balls we find some of them in suitable relative positions using the geometric results of Erdős and Füredi [10]. This forces the conclusion of the theorem to be valid.

1.1 Approximate tangent planes and rectifiability

A set $A \subset \mathbb{R}^n$ is m-rectifiable if there exist Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^n$, $i \in \mathbb{N}$ such that

$$\mathcal{H}^m(A \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^m)) = 0.$$

A set $B \subset \mathbb{R}^n$ is called purely m-unrectifiable if $\mathcal{H}^m(B \cap A) = 0$ for every m-rectifiable set A.

Let $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $V \in G(n,m)$. We say that V is an approximate tangent m-plane for A at x if

$$\limsup_{r \to 0} \frac{\mathcal{H}^m(A \cap B(x,r))}{(2r)^m} > 0$$

and for all $0 < \alpha < 1$

$$\lim_{r\to 0} \frac{\mathcal{H}^m(A \cap B(x,r) \setminus X(x,V,\alpha))}{r^m} = 0.$$

Rectifiability and approximate tangent planes are related to each other in the following manner:

Theorem 1.3. *[25, Corollary 15.20] Let $A \subset \mathbb{R}^n$ be an $\mathcal{H}^m$-measurable set of $\mathbb{R}^n$ with $\mathcal{H}^m(A) < \infty$. Then A is purely m-unrectifiable if and only if for $\mathcal{H}^m$ almost all $x \in A$ there are no approximative tangent m-planes for A at x.*

Moreover, it is known that if we have a purely m-unrectifiable set $A \subset \mathbb{R}^n$ and if we fix $V \in G(n, n-m)$ and $0 < \alpha < 1$ there exists a positive constant c depending only on α and n so that

$$\limsup_{r\to 0} \frac{\mathcal{H}^m(A \cap B(x,r) \cap X(x,V,\alpha))}{(2r)^m} > c \tag{1}$$

for $\mathcal{H}^m$ almost all $x \in A$. See [25, Corollary 15.16]. In particular,

$$\inf_{V\in G(n,n-m)} \limsup_{r\to 0} \frac{\mathcal{H}^m(A \cap B(x,r) \cap X(x,V,\alpha))}{(2r)^m} > c.$$

One could ask if this could be improved to

$$\limsup_{r\to 0} \inf_{V\in G(n,n-m)} \frac{\mathcal{H}^m(A \cap B(x,r) \cap X(x,V,\alpha))}{(2r)^m} \geq c.$$

This is not possible as can be seen by the Example 5.4 in [A]: There exists a purely 1-unrectifiable compact set $A \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(A) < \infty$ so that for every $0 < \alpha \leq 1$

$$\lim_{r\to 0} \inf_{l\in G(2,1)} \frac{\mathcal{H}^1(A \cap B(x,r) \cap X(x,l,\alpha))}{2r} = 0$$

for every $x \in A$.

A measure μ on $\mathbb{R}^n$ is called purely m-unrectifiable if $\mu(A) = 0$ for every m-rectifiable set $A \subset \mathbb{R}^n$. The article [A] gives an example showing that an estimate like (1) does not hold for measures: There is $l \in G(2,1)$ and a measure μ in $\mathbb{R}^2$ so that μ is purely 1-unrectifiable and for every $0 < \alpha < 1$

$$\lim_{r\downarrow 0} \frac{\mu(X(x,l,\alpha) \cap B(x,r))}{\mu(B(x,r))} = 0$$

for μ-almost all $x \in \mathbb{R}^2$.

2 Porosity

The definition of porosity was used already by Denjoy in the 1920's [7]. He introduced a quantity which is nowadays called upper porosity. The term *porosity* in this context is due to Dolženko [8]. Porosities have been studied, for example, in connection with boundary interpolation sets [5], quasiconformal and quasisymmetric maps [20], [23], [32], [34], [35], harmonic measures [12], complex dynamics [29] and so on. In order to define porosity we first let $A \subset \mathbb{R}^n$, take a point $x \in A$ and a radius $r > 0$ and set

$$\operatorname{por}(A, x, r) = \sup\{\alpha \geq 0 : B(y, \alpha r) \subset B(x, r) \setminus A \text{ for some } y \in \mathbb{R}^n\}. \tag{2}$$

From this we define the lower-porosity of A at x as

$$\operatorname{por}(A, x) = \liminf_{r \to 0} \operatorname{por}(A, x, r). \tag{3}$$

Finally the lower-porosity of A is defined as

$$\operatorname{por}(A) = \inf_{x \in A} \operatorname{por}(A, x).$$

We say that A is lower-porous if $\operatorname{por}(A) > 0$. The set A is called ϱ-porous if $\operatorname{por}(A) \geq \varrho$. Taking the limit superior instead of limit inferior in (3) gives the notion of upper-porosity. We will not study upper-porosity here as it does not imply a drop in the dimension. Indeed, there are upper-porous sets on $\mathbb{R}^n$ with Hausdorff dimension n, see for example [13] or [25]. From now on we will refer to lower-porosity as porosity. For an extensive review on upper-porosity and σ-porosity see the survey papers of Zajíček [36], [37].

The first implications of large porosity to dimension were obtained by Mattila in [24] when he answered to the following question posed by Martio: *Let $E \subset \mathbb{R}^n$ with $\operatorname{por}(E, x) = \frac{1}{2}$ for every $x \in E$. Is the Hausdorff dimension of E at most $n-1$?* Mattila proved that this is the case and also gave a stronger result:

Theorem 2.1. *[24, Corollary 3.4] For $0 < \varrho < \frac{1}{2}$ there is $d_n(\varrho)$, $n-1 \leq d_n(\varrho) \leq n$, such that $\lim_{\varrho \to \frac{1}{2}} d_n(\varrho) = n-1$ and $\dim_{\mathrm{H}}(E) \leq d_n(\varrho)$ for every ϱ-porous set $E \subset \mathbb{R}^n$.*

This theorem was proved as a corollary to a conical density result. The transition from holes to cones goes as follows: Let $\operatorname{por}(E, x, r) > \varrho$. Then there is a point $z \in \mathbb{R}^n$ such that $B(z, \varrho r) \subset B(x, r) \setminus E$ and so

$$H(x, \theta, \alpha) \cap B(x, \frac{r}{2}) \subset B(x, 2(1-2\varrho)r) \cup B(z, \varrho r),$$

where θ is the direction from x to z and α is a suitable angle depending on ϱ. The conical density result implies then that the dimension of E cannot be too large. If this were not the case, then a cone like $H(x,\theta,\alpha) \cap B(x,\frac{r}{2})$ would typically have large measure compared to $B(x, 2(1-2\varrho)r)$.

It is natural to ask what the optimal function d_n in Theorem 2.1 is. By optimal function d_n we mean the function

$$d_n(\varrho) = \sup\{\dim_{\mathrm{H}}(E) : E \subset \mathbb{R}^n \text{ is } \varrho\text{-porous}\}.$$

Salli proved in [31] that for the optimal $d_n(\varrho)$ there are positive constants C_1 and C_2 so that

$$n - 1 + \frac{C_1}{\log(\frac{1}{1-2\varrho})} \leq d_n(\varrho) \leq n - 1 + \frac{C_2}{\log(\frac{1}{1-2\varrho})}. \tag{4}$$

Here the constant C_1 is absolute and the constant C_2 depends only on n. Moreover, he proved the upper bound for the packing dimension, which we will denote by $\dim_{\mathrm{p}}$. Recall that for a set $A \subset \mathbb{R}^n$ we always have $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{p}}(A)$.

Salli proved his result by investigating carefully the properties of unions of balls with equal radii. In particular, he proved that the boundary of the union can be locally covered with a δ-neighbourhood of a convex set which could in turn be covered with $C\delta^{n-1}$ balls of radius δ. Carrying out the calculations with $\delta = (1 - 2\,\mathrm{por}(A))$ gives the upper bound in estimate (4). The lower bound was shown by looking at Cantor sets and their cartesian products with unit intervals.

The results of Mattila and Salli can be generalized in many directions. One can replace the Euclidean spaces with more general spaces, pass on from porosities of sets to porosities of measures, look at sets having holes in different directions or consider sets that are porous only on some percentage of scales. We will look at these generalizations in the next subsections.

2.1 k-porosity

Käenmäki and Suomala introduced the notion of k-porosity in [19]. In k-porosity we look for the maximal k holes in orthogonal directions. With an integer $k \in \{1,\ldots,n\}$, define for a set $A \subset \mathbb{R}^n$, a point $x \in A$ and a radius $r > 0$

$$\begin{aligned}\mathrm{por}_k(A,x,r) = \sup\{\alpha \geq 0 \,:\, &\text{there are } y_1,\ldots y_k \in \mathbb{R}^n \text{ such that for every } i \\ &B(y_i,\alpha r) \subset B(x,r) \setminus A, \text{ and } (y_i - x)\cdot(y_j - x) = 0 \text{ if } j \neq i\}\end{aligned}$$

and as before

$$\mathrm{por}_k(A,x) = \liminf_{r\to 0} \mathrm{por}_k(A,x,r).$$

Finally let

$$\mathrm{por}_k(A) = \inf_{x\in A} \mathrm{por}_k(A,x).$$

Käenmäki and Suomala proved the following result which is a generalization of Theorem 2.1.

Theorem 2.2. *[19, Theorem 3.2] Suppose* $0 < k \leq n$. *Then*

$$\sup\{s > 0 : \mathrm{por}_k(A) > \varrho \text{ and } \dim_{\mathrm{H}}(A) > s \text{ for some } A \subset \mathbb{R}^n\} \to n - k$$

as $\varrho \to \frac{1}{2}$.

Also the proof of Theorem 2.2 uses the ideas of the proof of Theorem 2.1. It is based on a conical density result for sets in $\mathbb{R}^n$ which is similar to Theorem 1.1.

Theorem 2.2 was later made more precise by Järvenpää, Järvenpää, Käenmäki and Suomala in [17] where they proved a result similar to that of Salli's.

Theorem 2.3. *[17, Corollary 2.6] Let* $0 < \varrho < \frac{1}{2}$ *and suppose* $A \subset \mathbb{R}^n$ *with* $\mathrm{por}_k(A) > \varrho$. *Then*

$$\dim_{\mathrm{p}}(A) \leq n - k + \frac{c}{\log(\frac{1}{1-2\varrho})},$$

where c *depends only on* n *and* k.

In the proof of Theorem 2.3 one looks at a convex set. This time the convex set is estimated with planar subsets and when the set A is k-porous with $k \geq 2$ we have $k-1$-porosity inside these planar subsets. The process of taking unions of balls, estimating the boundary of the union with a convex set and then moving into planar subsets can be repeated $k-1$ times. At the end by calculating the estimates from 1-porosity up to k-porosity the final result is obtained.

2.2 Mean porosity

In mean porosity we require holes to occur only at some percentage of scales. A set $A \subset \mathbb{R}^n$ is called mean (α, p)-porous at a point $x \in A$ if

$$\liminf_{i \to \infty} \frac{\#\{1 \leq j \leq i \ : \ \mathrm{por}(A, x, 2^{-j}) \geq \alpha\}}{i} \geq p.$$

The set A is called mean (α, p)-porous if it is mean (α, p)-porous at every point $x \in A$. Mean porosity was introduced by Koskela and Rohde in [20]. They used a porosity definition where the holes were searched from annuli: Let $0 < \epsilon, c \leq 1$ and denote by $A_m(x)$ an annulus

$$A_m(x) = \{y \in \mathbb{R}^n \ : \ (1+\epsilon)^{-m} < |x - y| < (1+\epsilon)^{-m+1}\}.$$

Then a set E is porous at level m at a point $x \in E$ if there exsists $y \in A_m(x)$ with $d(y, E) > c\epsilon|x - y|$.

Koskela and Rohde proved an asymptotic dimension estimate for small mean porosity. For large mean porosity the correct asymptotics were obtained by Beliaev and Smirnov [1]. The asymptotics are again similar to the ones Salli proved.

Theorem 2.4. *[1, Corollary 1] There is a positive constant C depending only on n so that for every mean (α, p)-porous set $A \subset \mathbb{R}^n$ one has*

$$\dim_{\mathrm{p}}(A) \leq n - p + \frac{C}{\log(\frac{1}{1-2\alpha})}.$$

The proof is based on the techniques used by Salli: the treatment of boundaries, convex sets and neighbourhoods. However, dealing with mean porosity requires some new tricks. First of all Beliaev and Smirnov moved from porosity defined with respect to balls to that defined with respect to cubes to ensure that there is no overlapping that could destroy the estimates. Secondly they collected the information on porosity using different weights depending on whether a cube is porous or not.

2.3 Porosity of measures on $\mathbb{R}^n$

For measures porosity was first considered by Eckmann, Järvenpää and Järvenpää [9]. Let μ be a Radon measure on $\mathbb{R}^n$. For a point $x \in \mathbb{R}^n$, a radius $r > 0$ and a real number $\epsilon > 0$ define

$$\begin{aligned} \mathrm{por}(\mu, x, r, \epsilon) = \sup\{\alpha \geq 0 \,:\, & \text{there is } y \in X \text{ such that} \\ & B(y, \alpha r) \subset B(x, r) \text{ and } \mu(B(y, \alpha r)) \leq \epsilon\mu(B(x, r))\}. \end{aligned}$$

We define the porosity of μ at a point $x \in \mathbb{R}^n$ as

$$\mathrm{por}(\mu, x) = \lim_{\epsilon \to 0} \liminf_{r \to 0} \mathrm{por}(\mu, x, r, \epsilon). \tag{5}$$

The porosity of the measure μ is finally defined as

$$\mathrm{por}(\mu) = \mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \mathrm{por}(\mu, x).$$

Note that if we change the order of taking limits in (5) we get the porosity of the support of μ. Eckmann, Järvenpää and Järvenpää proved in [9] that if a measure μ on $\mathbb{R}^n$ is doubling in the sense that at μ-almost every point $x \in \mathbb{R}^n$

$$\limsup_{r \to 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty,$$

then

$$\mathrm{por}(\mu) = \sup\{\mathrm{por}(A) \,:\, A \text{ is a Borel set with } \mu(A) > 0\}. \tag{6}$$

They also gave an example showing that this is not generally true for non-doubling measures. Equality (6) together with Salli's result gives the asymptotic behaviour of the optimal upper bound for the dimension of doubling porous measures.

Large porosity of measures on the real line was investigated by Järvenpää and Järvenpää in [16]. An estimate analogous to that of Salli's for the Hausdorff measure of general porous measures was proved by Järvenpää and Järvenpää in [15]. (See also [14].)

Mean porosity of measures can be defined similarly to that of sets. A measure μ on $\mathbb{R}^n$ is called mean (α, p)-porous at a point $x \in \mathbb{R}^n$ if

$$\lim_{\epsilon \to 0} \liminf_{i \to \infty} \frac{\#\{1 \leq j \leq i \;:\; \mathrm{por}(\mu, x, 2^{-j}, \epsilon) \geq \alpha\}}{i} \geq p.$$

Finally, μ is mean (α, p)-porous if there is a set $A \subset \mathbb{R}^n$ so that $\mu(A) > 0$ and μ is mean (α, p)-porous at every point $x \in A$. In the article [B] of this thesis we prove that for any mean (α, p)-porous measure μ we have

$$\dim_{\mathrm{p}} \mu \leq n - p + \frac{C}{\log(\frac{1}{1-2\alpha})}, \tag{7}$$

where C depends only on n.

Quite surprisingly, it is impossible to estimate mean porous measures by mean porous sets - at least on every scale at once. This is shown in the last section of [B] by constructing a $\frac{1}{8}$-porous measure with $\mu(A) = 0$ for any $0 < p \leq 1$ and $0 < \alpha \leq \frac{1}{2}$ and for any mean (α, p)-porous set A. Therefore the proof of the estimate (7) is more involved with the actual definition of porosity of measures. As in the proof of Theorem 2.4 we pass on from porosity defined with respect to balls to that defined with respect to cubes. Although this guarantees that we have no overlapping it makes the proof a bit more technical. This is because we deal with general measures and therefore there is no control on the relative measures of neighbouring cubes.

The problem of neighbouring cubes is overcome by looking separately at the cubes which are on a 'boundary region' of a larger cube and the cubes which are in an 'inner region'. This separation gives an error in the estimate which is eventually made smaller and smaller by induction. Mean porosity is dealt with similar weights as in the proof of Theorem 2.4.

2.4 Porosity in metric spaces

In metric spaces the above mentioned definition of porosity of sets (in particular (2)) is not the most natural one. One reason for this is the fact that with the previous definition the maximum porosity is just some number between 0 and 1 that depends on the metric space. We want the maximum porosity not to exceed $\frac{1}{2}$ so we instead use the definition introduced in [26]. Let (X, d) be a metric space. For a set $A \subset X$, a radius $r > 0$ and a point $x \in A$ let

$$\begin{aligned}\operatorname{por}^*(A, x, r) = \sup\{\varrho \geq 0 : &\text{there is } y \in X \text{ such that } B(y, \varrho r) \cap A = \emptyset \\ &\text{and } \varrho r + d(x, y) \leq r\}.\end{aligned}$$

and

$$\operatorname{por}^*(A, x) = \liminf_{r \to 0} \operatorname{por}^*(A, x, r).$$

We say that A is ϱ-porous, if $\operatorname{por}^*(A, x) \geq \varrho$ at every point $x \in A$.

One way to investigate the structure of a metric space is to look at the measures it supports. The first class of measures we use here is the class of regular measures and the second one is the class of doubling measures.

Let $s > 0$. A metric space X is called s-regular if it supports an s-regular measure μ. A measure μ is said to be s-regular if there are constants $0 < a_\mu \leq b_\mu$ and $r_\mu > 0$ so that

$$a_\mu r^s \leq \mu(B(x, r)) \leq b_\mu r^s$$

for all $x \in X$ and $0 < r < r_\mu$.

David and Semmes proved in [6] that for ϱ-porous sets A in s-regular spaces

$$\dim_{\mathrm{p}}(A) \leq s - d_s(\varrho), \tag{8}$$

where $d_s(\varrho) > 0$ depends on s and ϱ. Bonk, Heinonen and Rohde showed in [4] that inequality (8) is true also for the Assouad dimension of A. Recall that the Assouad dimension is always at least the packing dimension. In the article [C] we show that $d_s(\varrho) = c\varrho^s$ is the asymptotically sharp function in (8). The constant c depends on the constants a_μ, b_μ and s. The proof of (8) is based on careful estimation on the measure of the neighbourhoods of porous sets.

We call a metric space X doubling if it supports a doubling measure μ. Among complete metric spaces this definition of a doubling metric space agrees with the usual definition of a doubling metric space where we require that for any $r > 0$ all balls of radius $2r$ can be covered by a fixed number of balls of radius r. See [21]. A measure μ is doubling on X if there exists a constant $c_\mu \geq 1$, called the doubling constant, so that there exists some $r_0 > 0$ such that

$$0 < \mu(B(x, 2r)) \leq c_\mu \mu(B(x, r)) < \infty$$

for every $x \in X$ and $0 < r < r_0$.

In the case of doubling metric spaces a similar calculation gives a weaker result. This is because there might in general be too many empty annuli in the space. Also in a doubling metric space there might be a 'bad part' which destroys any estimates. However, this part has always zero measure with any doubling measure on the space.

Theorem 2.5. *[C, Theorem 4.10] Suppose that μ is a doubling measure on X. Then there is a set $N \subset X$ with $\mu(N) = 0$ so that*

$$\dim_{\mathrm{p}}(A) \leq \dim_{\mathrm{p}}(X) - c(\mu)(\log \frac{1}{\varrho})^{-1} \varrho^{\log_2 c_\mu}$$

for any ϱ-porous set $A \subset X \setminus N$.

In article [C] the following connection between uniformly porous and regular subsets of complete regular metric spaces is verified: Let X be a complete s-regular metric space. A set $A \subset X$ is uniformly porous if and only if there is $0 < t < s$ and a t-regular set $F \subset X$ such that $A \subset F$. By uniform porosity of A we mean that there exist constants $r_p > 0$ and $\varrho > 0$ so that $\mathrm{por}^*(A, x, r) > \varrho$ for every $x \in A$ and $0 < r < r_p$.

From the definition of porosity one sees that its maximum value in any metric space is $\frac{1}{2}$. Therefore it is natural to ask in what spaces the estimate (4) of Salli's is valid. In the article [D] I give partial answers to this. First of all, the estimate in s-regular geodesic metric spaces can not generally be of the form $s - 1 + d_s(\varrho)$ with $d_s(\varrho) \to 0$ as $\varrho \to \frac{1}{2}$. This is essentially because 'a direction' does not have to contribute a whole $+1$ in the dimension. On the other hand, if we drop geodesity there is in general no hope for an $n-1$-type estimate even with bi-Lipschitz images of the Euclidean space $\mathbb{R}^n$. Reason for this is that without an assumption like geodesity the boundaries of holes do not have to come close to the porous set.

Making the two above mentioned heuristic properties into assumptions on local mappings from the metric space to a Euclidean space one arrives to an estimate like the one of Salli's. For example, normed vector spaces and step two Carnot groups satisfy these assumptions.

3 Open Questions

In the study of conical densities and porosity there are still several open questions. Let us list some of the immediate questions one considers while reading the theory.

Question 1. In Theorem 1.1 we had to require that $\dim_{\mathrm{H}}(\mu) > m$. Would the claim hold also with $\dim_{\mathrm{H}}$ replaced by $\dim_{\mathrm{p}}$?

Question 2. [A, Question 5.1] Given $\alpha > 0$ and $n \in \mathbb{N}$ does there exist a constant $c > 0$ depending only on α and n so that for every non-atomic measure μ on $\mathbb{R}^n$ one could pick $\theta = \theta(x) \in S^{n-1}$ for μ almost all $x \in \mathbb{R}^n$ so that

$$\limsup_{r\to 0} \frac{\min \mu(B(x,r) \cap H(x,\theta,\alpha)), \mu(B(x,r) \cap H(x,-\theta,\alpha))}{\mu(B(x,r))} > c?$$

Question 3. In [B] our proof for the dimension estimate of porous measures works only with 1-porosity. Is it true also that for measures μ in $\mathbb{R}^n$ with k-porosity at least ϱ we have

$$\dim_{\mathrm{p}}(\mu) \leq n - k + \frac{C}{\log(\frac{1}{1-2\varrho})}$$

with some constant C depending only on n?

Question 4. In [B] we considered the case where porosity is large. For small porosity one could ask if there exists a constant c depending only on n so that for any (α, p)-porous measure μ on $\mathbb{R}^n$ we have

$$\dim_{\mathrm{p}}(\mu) \leq n - cp\alpha^n.$$

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