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DEPARTMENT OF MATHEMATICS
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NONLINEAR POTENTIAL THEORY ON METRIC SPACES

TERO MÄKÄLÄINEN

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Editor: Pekka Koskela
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Jyväskylä, June 2008
Tero Mäkäläinen

Included papers

This dissertation consists of the introductory part and the following publications:

[A] T. Mäkäläinen, *Adams inequality on metric measure spaces*, to appear in Rev. Mat. Iberoamericana.

[B] A. Björn, J. Björn, T. Mäkäläinen, M. Parviainen, *Nonlinear balayage on metric spaces.*

[C] T. Mäkäläinen, *Removable sets for Hölder continuous p-harmonic functions on metric measure spaces*, Ann. Acad. Sci. Fenn. Math. 33 (2008), 605-624.

1 Introduction

This dissertation is about analysis on metric spaces. More precisely, we study embedding inequalities, nonlinear potential theory and p-harmonic functions in the setting of metric measure spaces. In this section, we give a short overview of the analysis on metric spaces supporting a Poincaré inequality with a doubling measure. In subsequent sections we give a short overview of the included papers.

1.1 Doubling measure and Poincaré inequality

We make two assumptions on the metric measure space. The first one is on the measure and the second on the geometry of the space. Precisely, we assume that

1. the measure μ is doubling;
2. X admits a weak Poincaré inequality.

A measure μ is *doubling* if balls have positive and finite measure and there exists a constant $C_d \geq 1$ such that for all balls $B(x,r)$ in X,

$$\mu(B(x,2r)) \leq C_d \mu(B(x,r)).$$

Note that the doubling measure μ has a density lower bound, see [He]: There exist constants c, $s > 0$ that depend only on the doubling constant of μ, such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq c\left(\frac{r}{R}\right)^s, \tag{1}$$

whenever $r < R$, $x \in X$ and $y \in B(x,R)$. Usually we consider s to be the natural dimension of the space X, and we assume that $s > 1$. However, s is, in general, not equal to the topological dimension of the space.

Sometimes one assumes that the metric space is doubling, that is, there exists a finite constant N such that every ball of radius r can be covered with N balls of radii $r/2$. If a metric space supports a doubling measure, then it is doubling. Converse is true in the following sense: we can construct a doubling measure to every complete doubling space, see [LS]. However, there are non-complete doubling metric spaces which do not support doubling measures, see [Sa].

When the measure is doubling, the space has many useful properties. For instance, if the space is complete, as we assume in this work, then it is proper, that is, all closed and bounded subsets are compact. Also many tools such as Vitali-type covering theorem, Lebesgue theorem and Hardy–Littlewood maximal theorem are available.

Before defining a Poincaré inequality, we need a substitute for the Sobolev gradient in metric spaces.

A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all compact rectifiable paths $\gamma : [0, l_\gamma] \to X$, we have

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\, ds \tag{2}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2) holds for p-almost every path, then g is a *p-weak upper gradient* of f, see Definition 2.1 in [Sh1].

Notice that in Euclidean spaces, modulus of Sobolev gradient is an upper gradient of a smooth function. From inequality (2), we immediately see that upper gradient is not unique and $g \equiv \infty$ is an upper gradient of any function. Therefore it is natural to define "the smallest" upper gradient. If f has the upper gradient in $L^p(X)$, then it has the *minimal p-weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of f, $g_f \le g$ μ-a.e., see Corollary 3.7 in [Sh2].

The geometric assumption on the space is the following Poincaré inequality. Let $1 \le p < \infty$. A metric measure space (X, d, μ) is said to admit a *weak $(1, p)$-Poincaré inequality* if there are constants $C_p > 0$ and $\tau \ge 1$ such that

$$⨍_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le C_p r \left(⨍_{B(x,\tau r)} g^p\, d\mu \right)^{1/p} \tag{3}$$

for all balls $B(x, r) \subset X$, for all integrable functions u in $B(x, r)$ and for all upper gradients g of u. Here we use the notation $u_{B(x,r)} = ⨍_{B(x,r)} u\, d\mu = \mu(B(x, r))^{-1} \int_{B(x,r)} u\, d\mu$. In the definition the word *weak* refers to the possibility that $\tau > 1$.

The above definition is due to Heinonen and Koskela [HeK]. There are various formulation for a Poincaré inequality on a metric measure space. For example, we could require the inequality (3) for all Lipschitz functions and replace the upper gradient by the local Lipschitz constant as done in paper [A]. When the space is complete and is equipped with a doubling Borel regular measure, these definitions coincide, see e.g. [K1], [K2] and [KR].

Hölder inequality gives that any complete metric space that admits a $(1, p)$-Poincaré inequality, admits a $(1, \tilde{p})$-Poincaré inequality for every $\tilde{p} \ge p$. The converse is not true in general, but by a deep result in [KeZ], we have that a weak $(1, p)$-Poincaré inequality implies a weak $(1, t)$-Poincaré inequality for some $t < p$, which is needed in paper [A].

We can also change the exponent on the left hand side of the Poincaré inequality, and we obtain the Sobolev–Poincaré inequalities. It is shown in [HaK], that a weak $(1, p)$-Poincaré inequality also implies (q, p)-Poincaré inequality for some $q > p$, with possibly a different τ.

It is hard to check if a given space admits a Poincaré inequality. Some results about sufficient conditions are available, see [Se].

1.2 Analysis on metric measure spaces

Next we discuss about the analysis on metric measure space that admits the assumptions introduced in section 1.1. Indeed, from now on, we assume that the measure is doubling and the space admits a weak Poincaré inequality. Some examples of such spaces are for example Euclidean spaces with Lebesgue measure, weighted Euclidean spaces with Muckenhoupt weights, complete Riemannian manifolds with nonnegative Ricci curvature, many graphs and Carnot groups, see [He] and [HaK].

When these assumptions on the space and on the measure hold, the space has nice geometric properties and allows us to conduct analysis on such a space, and recently such analysis was done in many areas of studies. For instance, many results from Sobolev spaces, nonlinear potential theory, geometric measure theory and quasiconformal mappings in Euclidean setting can be obtained on such spaces, see [AT], [HaK], [He], [HeK] [KM1], [KoM] and [Sh2].

In the first order calculus in Euclidean spaces, Sobolev spaces play an important role. To define Sobolev spaces, one needs weak derivatives. Classical definitions do not work in general metric spaces, because the space has not a priori smooth structure. However, more general approaches have been found lately to define Sobolev type spaces on metric spaces that are, of course, equivalent to the classical Sobolev spaces in the Euclidean spaces.

Recently there has been progress in the theory of Sobolev spaces in general metric measure spaces, see for instance [Ch], [HaK], [Ha], [HeK], [KKM], [KoM], [Sh1] and references therein. There are various approaches to define Sobolev type spaces in the setting of metric spaces. In [Sh1], Shanmugalingam constructs a Sobolev type space on metric spaces, which yields the same space studied by Cheeger in [Ch], when $p > 1$. When the metric space is complete and admits a Poincaré inequality with a doubling measure, the Sobolev type spaces introduced by Hajłasz [Ha] also coincide with the spaces mentioned above, see [KeZ].

Also nonlinear potential theory can be generalized to metric spaces, see [KM1], [KM2], [KS] and [Sh2]. Cheeger's [Ch] definition of partial derivatives makes it possible to study partial differential equations on such spaces, see [BMS], [BBS1], [KS2] and [B3].

To study nonlinear potential theory and partial differential equations on metric measure spaces, the natural way is to generalize theory from the Euclidean setting. In Euclidean spaces, we may study the following p-Laplace equation:

$$\operatorname{div}(|Du|^{p-2}|Du|) = 0. \tag{4}$$

Or equivalently, we may consider the following nonlinear variational problem: Find a minimizer for the p-Dirichlet integral

$$\int_\Omega |Du|^p \, dx \tag{5}$$

among all functions $u : \Omega \to \mathbb{R}$ with prescribed boundary value. Here we need to assume that the functions u belong to a suitable Sobolev space.

In metric space setting, we may study both problems. However, there are two reasonable approaches that we are interested in. Those are upper gradient minimizers and (Cheeger) p-harmonic functions. We study minimizers in paper [B] and we give a short overview in section 3. Cheeger p-harmonic functions are studied in paper [C], on which we give a short survey in section 4.

Certain theorems in papers [A] and [B] are essential to solve problems studied in paper [C].

2 Adams-type inequalities

In the Euclidean spaces we have the following Adams inequality, see e.g. [AH], [Ma], [Tu] or [Zi]:

Theorem 2.1. *Let ν be a Radon measure on $\mathbb{R}^n$ and let $1 \le p < q < \infty$ with $p < n$. Suppose that there is a constant M such that for all balls $B(x,r) \subset \mathbb{R}^n$,*

$$\nu(B(x,r)) \le Mr^{\alpha},$$

where $\alpha = q(n-p)/p$. Then

$$\left(\int_{\mathbb{R}^n} |u|^q d\nu\right)^{1/q} \le CM^{1/q} \left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{1/p}, \tag{6}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, where $C = C(p,q,n) > 0$.

In the Euclidean setting a necessary and sufficient condition for trace type theorems is obtained, see e.g. [AH, Chapter 7.2]. For Sobolev functions, inequality (6) is an extension of the Sobolev inequality, since if ν is n-dimensional Lebesgue measure, then $q = p^* = np/(n-p)$.

In paper [A], we extend the Adams inequality, Theorem 2.1, to the setting of metric measure spaces. The results are formulated for Lipschitz functions. In a metric space (X,d), a function $u : X \to \mathbb{R}$ is said to be *Lipschitz continuous*, denoted by $u \in \mathrm{Lip}(X)$, if for some constant $L > 0$

$$|u(x) - u(y)| \le Ld(x,y),$$

for every $x, y \in X$. We also use the notation $u \in \mathrm{Lip}_0(X)$ when the function u has compact support. For a Lipschitz function $u : X \to \mathbb{R}$, we define

$$\mathrm{Lip}\, u(x) := \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x,y)}.$$

Recall that we assume that the metric measure space (X, d, μ) is complete, μ is doubling and the space admits a weak $(1, p)$-Poincaré inequality. For Adams inequality, we have several cases depending on the value of p.

The case $p = 1$ needs a special treatment as usual. The following theorem is proven in paper [A].

Theorem 2.2. *[A, Theorem 1.3] Let (X, d, μ) be a complete metric measure space such that it admits a weak $(1, 1)$-Poincaré inequality and μ is a doubling Radon measure. Let ν be a Radon measure on X. Suppose that there are $M \geq 0$ and $q \geq 1$, such that for all balls $B(x, r) \subset X$ of radius $r < \operatorname{diam} X$, it holds*

$$\frac{\nu(B(x,r))}{\mu(B(x,r))^q} \leq M r^{-q}.$$

Then

$$\left(\int_X |u|^q d\nu \right)^{1/q} \leq C M^{1/q} \int_X \operatorname{Lip} u \; d\mu,$$

for all $u \in \operatorname{Lip}_0(X)$, where the constant $C > 0$ depends only on q, s, the doubling constant and the constants in the Poincaré inequality.

The proof is based on the recently developed theory of BV-functions in metric spaces, see [Am] and [Mi]. We need isoperimetric inequality and the co-area formula in this setting. Moreover, we need a covering theorem referred as boxing inequality, see Lemma 3.1 in paper [A].

Next we move into the case $1 < p < s$. We follow the outline of the proof in Euclidean spaces, where Riesz potentials play a role.

The *Riesz potential* of a nonnegative, measurable function f on a metric measure space (X, d, μ) is

$$I_{1,A}(f)(x) = \int_A \frac{f(y) d(x,y)}{\mu(B(x, d(x,y)))} d\mu(y),$$

for a measurable set $A \subset X$.

To prove Adams inequality in this case, we first need the Fractional Integration Theorem, which we call the Adams-type inequality for the Riesz potential.

Theorem 2.3. *[A, Corollary 4.2] Let (X, d, μ) be a metric measure space, where μ is a doubling Radon measure, and $1 < p < s$. Assume that ν is a Radon measure such that*

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \leq M r^{\frac{sq}{p} - s - q}$$

for all balls $B(x, r) \subset X$ of radius $r < \operatorname{diam} X$, where M is a positive constant and $1 < p < q < \infty$. If $f \in L^p(B_0, \mu)$ for some ball $B_0 = B(x_0, r_0) \subset X$, we have

$$\left(\int_{B_0} I_{1,B_0}(|f|)^q d\nu \right)^{1/q} \leq C \mu(B_0)^{1/q - 1/p} \, r_0^{\frac{s}{p} - \frac{s}{q}} M^{1/q} \left(\int_{B_0} |f|^p d\mu \right)^{1/p},$$

where $C = C(p, q, C_d, s) > 0$ is a constant.

Second, we consider the pointwise inequality, which follows from the Poincaré inequality and a chain condition, see Section 3 in paper [A].

Theorem 2.4. *[A,Remark 3.3] Assume that (X, d, μ) admits a weak $(1, p)$-Poincaré inequality with a doubling Borel measure μ and let $B(y, r) \subset X$ such that $r < \operatorname{diam} X/10$. Let $u \in \mathrm{Lip}_0(B(y, r))$. Then for each $x \in B(y, r)$*

$$|u(x)|^p \leq C r^{p-1} I_{1,B(y,r)}((\operatorname{Lip} u)^p)(x).$$

By combining Theorem 2.3 and Theorem 2.4, we obtain our main theorem: the Adams inequality in the case $1 < p < s$.

Theorem 2.5. *[A, Theorem 1.4] Let (X, d, μ) be a complete metric measure space such that it admits a weak $(1, t)$-Poincaré inequality for some $1 \leq t < p$, and μ is a doubling Radon measure. Suppose that ν is a Radon measure on X, satisfying*

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \leq M r^{\alpha} \qquad \textit{with} \qquad \alpha = \frac{sq}{p} - s - \frac{q}{t},$$

for all balls $B(x, r) \subset X$ of radius $r < \operatorname{diam} X$, where $1 < p < q < \infty$, $p/t < s$ and M is a positive constant. Here s is from (1). If $u \in \mathrm{Lip}_0(B_0)$ for some ball $B_0 = B(x_0, r_0) \subset X$, for which $r_0 < \operatorname{diam} X/10$, we have

$$\left(\int_{B_0} |u|^q d\nu\right)^{1/q} \leq C\mu(B_0)^{1/q-1/p}\, r_0^{\frac{t-1}{t}+\frac{s}{p}-\frac{s}{q}} M^{1/q} \left(\int_{B_0} (\operatorname{Lip} u)^p d\mu\right)^{1/p},$$

where $C = C(p, q, s, t, C_d, C_p, \tau) > 0$.

The case in which $p = s$ and the space admits a weak $(1, 1)$-Poincaré inequality is not included in Theorem 2.5. We prove the following theorem in this borderline case.

Theorem 2.6. *[A, Theorem 6.2] Let (X, d, μ) be a complete metric measure space such that it supports weak $(1, 1)$-Poincaré inequality and μ is a doubling Radon measure. Let $B_0 = B(x_0, r_0) \subset X$ such that $r_0 < \operatorname{diam} X/10$ and suppose that ν is a Radon measure in B_0 with*

$$\nu(B(x, r)) \leq M \left(\log \frac{r_0}{r}\right)^{\frac{1-s}{s}q},$$

for all balls $B(x, r) \subset X$ such that $x \in 2B_0$ and $r < r_0/2$. Here $1 < s < q < \infty$ and M is a positive constant. Then

$$\left(\int_{B_0} |u|^q d\nu\right)^{1/q} \leq C r_0 \mu(B_0)^{-1/s} M^{1/q} \left(\int_{B_0} (\operatorname{Lip} u)^s d\mu\right)^{1/s},$$

for all $u \in \mathrm{Lip}_0(B_0)$, where $C = C(q, s, C_d, C_p, \tau) > 0$ is a constant.

The case $p > s$ follows from the Theorem 5.1 (3) in [HaK].

3 Nonlinear potential theory: balayage

Recently, nonlinear potential theory has been generalized to the setting of metric measure spaces. Main references that include basic results for minimizers, superminimizers and superharmonic functions, are [KM2], [KM1], [Sh2] and [B1]. The Dirichlet problem has been studied in [BBS1] and the Perron method in [BBS2]. Harnack's inequalities are found in [KS] and Moser's iteration in [BM]. Boundary regularity is studied in [BB1] and polar sets in [KS2]. Recent progress of this topic is presented in [BB3]. The list is not exhaustive by any means.

In the Euclidean spaces a nonlinear balayage is studied in [HK1], [HK2] and [HKM]. In paper [B], we develop the basic theory of balayage on metric measure spaces. To give the definition of the balayage, we need the notion of superharmonic function.

A function u from the Newtonian space $N^{1,p}_{\mathrm{loc}}(\Omega)$ (see [B, Definition 2.2]) is a *minimizer* in a domain Ω if for all $\varphi \in N^{1,p}_0(\Omega)$ we have

$$\int_{\varphi\neq 0} g_u^p \, d\mu \leq \int_{\varphi\neq 0} g_{u+\varphi}^p \, d\mu. \tag{7}$$

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *superminimizer* in Ω if (7) holds for all nonnegative $\varphi \in N^{1,p}_0(\Omega)$.

Now superharmonic functions are defined as lower semicontinuous functions (not identically ∞ in any component of Ω), that admit a certain comparison principle with minimizers, see [B, Definition 3.3] and [B1].

To define the balayage, we also need the $\liminf$*-regularization* of a function $f : \Omega \to \overline{\mathbf{R}}$, which is

$$\hat{f}(x) = \lim_{r\to 0} \inf_{\Omega\cap B(x,r)} f, \quad x \in \Omega.$$

In paper [B], we give two definitions for balayage as follows. Let

$$\begin{aligned}
\Phi^\psi &= \Phi^\psi(\Omega) = \{u : u \text{ is superharmonic in } \Omega \text{ and } u \geq \psi \text{ in } \Omega\},\\
\Psi^\psi &= \Psi^\psi(\Omega) = \{u : u \text{ is superharmonic in } \Omega \text{ and } u \geq \psi \text{ q.e. in } \Omega\},\\
R^\psi &= R^\psi(\Omega) = \inf \Phi^\psi,\\
Q^\psi &= Q^\psi(\Omega) = \inf \Psi^\psi.
\end{aligned}$$

The $\liminf$-regularizations $\widehat{R}^\psi$ and $\widehat{Q}^\psi$ are called the R- and Q-*balayage* of ψ in Ω, respectively. If $\Phi^\psi = \emptyset$, we set $\widehat{R}^\psi = \infty$ and similarly for $\widehat{Q}^\psi$. From now on assume that $\Phi^\psi \neq \emptyset$.

In paper [B], we show that the balayage is superharmonic and study the properties of balayage when the obstacle function, or the domain, is varying, see Theorem 4.4, Propositions 4.11 and 4.12 in [B]. An interesting question is to find, whether R- and Q-balayages are equal, in other words, whether the sets of capacity zero can be neglected, see Section 11 in [B]. We give some conditions, that give us positive answer to this question.

Theorem 3.1. *[B, Proposition 4.6, Theorem 4.10, Corollary 5.4] Assume that Ω is bounded. If ψ is lower semicontinuous or $\widehat{Q}^\psi \in N^{1,p}(\Omega)$ or $\psi \in N^{1,p}(\Omega)$, then $\widehat{R}^\psi = \widehat{Q}^\psi$.*

When the obstacle function is continuous and bounded above, we obtain the following useful result needed in paper [C].

Theorem 3.2. *[B, Proposition 4.9, Corollary 6.9] If ψ is a continuous and bounded in Ω, then $\widehat{Q}^\psi = \widehat{R}^\psi$ is a continuous p-supersolution with $\widehat{R}^\psi \geq \psi$. Moreover, $\widehat{R}^\psi$ is p-harmonic in the open set $\{\hat{R}^\psi > \psi\}$.*

We study the connection between the balayage and the solution of the obstacle problem and prove the following theorem.

Theorem 3.3. *[B, Proposition 5.6] Assume that $V \subset \Omega$ is open and bounded and that $\widehat{Q}^\psi \in N^{1,p}(V)$. Then $\widehat{Q}^\psi$ is the solution of the obstacle problem in V, with the obstacle ψ and boundary values $\widehat{Q}^\psi$.*

Boundary regularity for minimizers was previously studied in [BB1], where several definitions for regular boundary points are given. In Theorems 7.5 and 7.8 in [B], we show that equivalent characterizations for regular boundary points in terms of balayage are available. We also give characterizations for polar sets in terms of balayage, see Theorem 8.2 in [B]. These characterizations are shown to coincide with the definitions of polar sets given in [KS2].

4 Partial differential equations: removability

Another approach in the study of p-harmonic functions is based on derivatives due to Cheeger. In [Ch] Cheeger showed that under our general assumptions the metric space has a differentiable structure, under which Lipschitz functions have derivatives almost everywhere. This deep theorem allows us to consider the following equation for a function u in a domain Ω:

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = 0, \tag{8}$$

where $1 < p < \infty$ is a fixed number and D denotes the derivation operation, see [Ch]. A continuous function u is *(Cheeger) p-harmonic* in a domain Ω if $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and (8) holds for all Lipschitz testing functions φ with compact support in Ω. A function $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-supersolution* in Ω if for every nonnegative Lipschitz functions φ with compact support in Ω, the inequality "$\geq$" holds in (8).

Cheeger p-harmonic functions are studied, for example, in [BMS], [BBS1], [KS2] and [B3]. In the proof of Theorem 5.2 in [KS], it is shown that there exists

$0 < \kappa \le 1$ such that for every p-harmonic function h in Ω satisfies the local Hölder continuity estimate

$$\operatorname{osc}(h, B(x,r)) \le C\left(\frac{r}{R}\right)^{\kappa} \operatorname{osc}(h, B(x,R)), \tag{9}$$

where $0 < r < R$, $B(x, 2R) \subset\subset \Omega$, and C and κ are independent of r, R and h.

In paper [C], we study the removable sets for p-harmonic functions. We say that a compact set $E \subset \Omega$ is *removable* for Hölder continuous p-harmonic functions, if every function that is Hölder continuous in Ω and p-harmonic outside E, is actually p-harmonic in Ω.

In paper [C], we give the sharp result on removability defined above. We characterize the removable sets in terms of weighted Hausdorff measure. For the definition of weighted Hausdorff measure, see Definition 2.5 in paper [C].

Theorem 4.1. *[C, Theorem 1.1] Let X be a complete metric measure space with a doubling measure μ supporting a weak $(1,p)$-Poincaré inequality. Let $\Omega \subset X$ be open and bounded, and let $0 < \alpha < \kappa$, where κ is from (9). A closed set $E \subset \Omega$ is removable for α-Hölder continuous p-harmonic functions if and only if E is of weighted $(-p + \alpha(p-1))$-Hausdorff measure zero.*

For $\mathcal{A}$-harmonic functions in $\mathbb{R}^n$, where $\mathcal{A}$ is of p-Laplacian type, see [HKM, Chapter 3], the above theorem was proven in [KiZ].

To show that the given function is actually p-harmonic in the whole domain, the balayage is a key tool in our methods. In paper [B], we have shown that the balayage of a bounded continuous function is a supersolution. To prove the above removability result, we need the connection between the Riesz measure and the supersolution. This is given by the following equation proven in [BMS], see also [C, (13)]. There is a one to one correspondence between supersolutions $u \in N_0^{1,p}(\Omega)$ and Radon measures ν given by

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_\Omega \varphi \, d\nu,$$

whenever $\varphi \in N_0^{1,p}(\Omega)$. We say that ν is a *Riesz measure associated with u.*

The proof of Theorem 4.1 is based on the following result, which gives the optimal Hölder continuity of p-supersolutions in terms of the associated Riesz measure. It has interest of its own.

Theorem 4.2. *[C, Theorem 1.3] Let $\Omega \subset X$ be open and bounded, and $0 < \alpha < \kappa$, where κ is as in (9). Assume that u is a p-supersolution in Ω and $\nu \in N_0^{1,p}(\Omega)^*$ is a Riesz measure associated with u. Then $u \in C^{0,\alpha}_{loc}(\Omega)$ if and only if there is a constant $M > 0$ such that*

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \le M r^{-p+\alpha(p-1)}, \tag{10}$$

for all balls $B(x, 4r) \subset \Omega$.

In the proof of Theorem 4.2, the Adams inequality, Theorem 2.5, plays a key role.

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Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland.
E-mail address: **tjmakala@jyu.fi**

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