GRADIENT ESTIMATES AND A FAILURE OF THE MEAN VALUE PRINCIPLE FOR p-HARMONIC FUNCTIONS

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List of Notation

We use the following notation that is assumed to be familiar to the reader.

$\mathbb{R}^n$	Euclidean n-space
$\partial u/\partial x_i$	ith partial derivative of $u : \mathbb{R}^n \to \mathbb{R}$
$\partial u/\partial \nu$	outer normal derivative of u
∇u	gradient of u
$\|E\|$	Lebesgue measure of a set $E \subset \mathbb{R}^n$
∂E	topological boundary of E
$\overline{E}$	closure of E
$\mathcal{H}^k$	k-dimensional Hausdorff measure
$B_r(x)$	$\{y \in \mathbb{R}^n : \|x-y\| < r\}$
$U \subset\subset \Omega$	$\overline{U}$ is compact and $\overline{U} \subset \Omega$
$\operatorname{osc}_A u$	diameter of the set $\{u(x) : x \in A\}$
$\operatorname{spt} u$	support of u, $\operatorname{spt} u = \overline{\{x : u(x) \neq 0\}}$
$C(\Omega)$	continuous functions on Ω
$C^k(\Omega)$	k times continuously differentiable functions on Ω
$C_0^\infty(\Omega)$	functions $u \in C^\infty(\Omega)$ such that $\operatorname{spt} u \subset\subset \Omega$
$C^{0,\alpha}(\Omega)$	locally Hölder continuous functions on Ω
$C^{k,\alpha}(\Omega)$	functions in $C^k(\Omega)$ whose k-th order derivatives are locally Hölder continuous on Ω
$L^p(\Omega)$	p-th power Lebesgue integrable functions on Ω
$L^p_{\text{loc}}(\Omega)$	functions in $L^p(K)$ for each $K \subset\subset \Omega$
$\|\|u\|\|_{p;\Omega}$ or $\|\|u\|\|_p$	L^p norm of u in Ω
$W^{1,p}(\Omega)$	Sobolev $1,p$ class, i.e. functions in $L^p(\Omega)$ whose distributional first-order derivatives are in $L^p(\Omega)$
$W^{1,p}_{\text{loc}}(\Omega)$	functions in $W^{1,p}(K)$ for each $K \subset\subset \Omega$
$\|\|u\|\|_{1,p;\Omega}$ or $\|\|u\|\|_{1,p}$	Sobolev $1,p$ norm of u; $\|\|u\|\|_{1,p;\Omega} = \|\|u\|\|_p + \|\|\nabla u\|\|_p$
$W^{1,p}_0(\Omega)$	closure of $C_0^\infty(\Omega)$ wrt. the Sobolev $1,p$ norm
$W^{1,p}(\Omega)^*$	dual of $W^{1,p}(\Omega)$, i.e. the space of bounded linear functionals $W^{1,p}(\Omega) \to \mathbb{R}$
u_E	mean value of u over E, $u_E := ⨍_E u(x)dx := \|E\|^{-1} \int_E u(x)dx$
$u_{x,r}$ or u_r	$u_{B_r(x)}$
$\operatorname{dist}(x, E)$	distance from the point x to the set E.

If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are vectors in $\mathbb{R}^n$, their tensor product $a \otimes b$ is an $n \times n$ matrix with

$$(a \otimes b)_{ij} = a_i b_j.$$

Finally, the ubiquitous constant whose values may change will generally be called C.

1. Introduction

In this text we consider weak solutions to the elliptic p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1}$$

in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Here $p > 1$ is a fixed real parameter, and the weak solutions to (1) are called p-harmonic functions. By a weak solution to (1) we mean a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

The p-Laplace equation (1) is the Euler-Lagrange equation for the p-Dirichlet integral

$$\int_\Omega |\nabla u|^p dx.$$

With $p = 2$, we recover the classical Dirichlet integral and harmonic functions, but for $p \neq 2$ the equation (1) is nonlinear and the ellipticity breaks down at critical points, i.e. points where $\nabla u = 0$. The equation (1) is called singular for $1 < p < 2$ and degenerate for $p > 2$. The former case is considered more difficult and is less studied than the latter.

The operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

is called the p-Laplace operator, or the p-Laplacian. It serves as a prototype operator for more general classes of nonlinear operators with p-growth. The elliptic theory for a wide class of similar operators is developed in Heinonen, Kilpeläinen and Martio [15], and the parabolic theory for a corresponding class in DiBenedetto [7].

In the first part of this text, we are concerned with the local $C^{1,\alpha}$ regularity of p-harmonic functions. That p-harmonic functions are in the class $C^{1,\alpha}$ for some $\alpha = \alpha(n,p) \in (0,1)$ was originally proven by Uhlenbeck [38] and Ural'tseva [39] for $p > 2$, and independently by DiBenedetto [6], Lewis [23] and Tolksdorf [37] for $1 < p < \infty$. Moreover, for any $p > 2$ and any $n \geq 2$, there exist p-harmonic functions without continuous second partials, so any stronger regularity than $C^{1,\alpha}$ is not available for $p > 2$. Lewis [22] was perhaps the first to point this out (he attributed the idea to Krol' [19]); Bojarski and Iwaniec [4] constructed other examples by showing that the complex gradient of a p-harmonic function in the plane is a quasiregular map.

The optimal regularity in the planar case was settled by Iwaniec and Manfredi [16], who showed that p-harmonic functions ($1 < p < \infty$, $p \neq 2$) in the plane are optimally in the class $C^{k+\alpha}$, where

$$k + \alpha = \frac{1}{6}\left(7 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\right).$$

Here $4/3 < k + \alpha < 2$ for $p > 2$ and $k + \alpha > 2$ for $1 < p < 2$, with $k + \alpha \to \infty$ as $p \to 1$ and $k + \alpha \to 4/3$ as $p \to \infty$. Thus the question is settled for $n = 2$; note the difference between the cases $p > 2$ and $1 < p < 2$. In higher dimensions the question about optimal regularity seems to be wide open.

Our first result is the following local $C^{1,\alpha}$ estimate for p-harmonic functions:

1.1. **Theorem.** *Let $1 < p < \infty$ be fixed. Let u be p-harmonic in Ω, and let $q \geq \min\{2, p\}$. Then*

$$\begin{aligned}
&\left(\fint_{B_r(x_0)} |\nabla u(x) - (\nabla u)_{x_0,r}|^q dx\right)^{\frac{1}{q}} \\
&\leq C\left(\frac{r}{R}\right)^{\sigma}\left(\fint_{B_R(x_0)} |\nabla u(x) - (\nabla u)_{x_0,R}|^q dx\right)^{\frac{1}{q}}
\end{aligned} \tag{2}$$

for each $B_R(x_0) \subset\subset \Omega$ and each $0 < r \leq R$. Here $C = C(n, p, q, \sigma) \geq 1$ and $\sigma = \sigma(n, p) \in (0, 1)$. Moreover,

$$\operatorname*{osc}_{B_r(x_0)} |\nabla u| \leq C_{n,p,\sigma}\left(\frac{r}{R}\right)^{\sigma} \operatorname*{osc}_{B_R(x_0)} |\nabla u|$$

for each $B_R(x_0) \subset\subset \Omega$ and each $0 < r < R/2$.

The estimate (2) with $q = p$ was proved by Lieberman [24] for solutions, and with $q = 2$ (independently) by DiBenedetto and Manfredi [8] for systems. We are not aware of a previously published version of the estimate for other values of q. The supremum of the numbers σ for which the estimate (2) holds is henceforth denoted by $\sigma_0 = \sigma_0(n, p)$.

Another result also improves a Theorem by Lieberman [25], who proved that the $C^{1,\alpha}$ regularity of the solutions for the homogeneous equation (1) is preserved also in a measure data equation, provided that the measure obeys certain growth conditions. Our result states the following:

1.2. **Theorem.** *Let $1 < p < \infty$, and let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution to*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu, \tag{3}$$

i.e. let

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx = \int_\Omega \varphi\, d\mu \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

where μ is a signed Radon measure satisfying the growth condition

$$|\mu|(B_R) \le CR^{n-1+\alpha} \tag{4}$$

for some $\alpha \in (0,1)$ and for all $B_R \subset \Omega$. Moreover, let

$$\delta = \begin{cases} \alpha, & \text{if } p = 2 \\ \min\left\{\dfrac{\alpha}{p-1}, \sigma_0^-\right\}, & \text{if } p > 2 \\ \min\{\alpha, \sigma_0^-\}, & \text{if } 1 < p < 2, \end{cases} \tag{5}$$

where σ_0^- is any number smaller than σ_0. Then $u \in C^{1,\delta}(\Omega)$.

Lieberman [25] proved that, for a nonnegative measure μ, the condition (4) implies that the solution u is in $C^{1,\beta}(\Omega)$ for some $0 < \beta < \sigma_0$. We allow for signed measures, and the new feature here is the explicit α-dependence in the Hölder modulus of the gradient.

A consequence of Theorem 1.2 is the following non-removability result for p-harmonic functions:

1.3. **Corollary.** *Let $1 < p < \infty$, let $0 < \alpha < 1$, and let δ be as in (5). If $E \subset \Omega$ is a closed set with Hausdorff content $\mathcal{H}^{n-1+\alpha}(E) > 0$, then there exists a function $u \in C^{1,\delta}(\Omega)$ which is p-harmonic in $\Omega \setminus E$, but which does not have a p-harmonic extension to Ω.*

Proof. Let $K \subset E$ be a compact set with $0 < \mathcal{H}^{n-1+\alpha}(K) < \infty$. By Frostman's Lemma (see e.g. Adams and Hedberg [1, Theorem 5.1.1]), there exists a nonnegative Radon measure μ such that $\operatorname{spt}\mu \subset K$, $\mu(K) > 0$ and $\mu(B_R) \le CR^{n-1+\alpha}$ whenever $B_R \subset \Omega$.

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be any weak solution to the equation $-\Delta_p u = \mu$ in Ω; such solutions exist because the growth condition obeyed by the measure μ implies that μ belongs to the dual of $W^{1,p}_0(\Omega)$ (see Theorem 3.2 on page 22), and thereafter the existence and uniqueness of solutions (with prescribed boundary values) follows from the theory of monotone operators, see e.g. Kinderlehrer and Stampacchia [18], Corollary III.1.8, page 87.

The solution u is p-harmonic in $\Omega \setminus E$ (because $\operatorname{spt}\mu \subset E$), and $u \in C^{1,\delta}(\Omega)$ by Theorem 1.2. Now the interior of K is empty, so there is only one continuous extension of u, namely u itself. But u is not p-harmonic, since $\mu(K) > 0$. □

The optimal removability and non-removability results for $C^{1,\alpha}$-smooth p-harmonic functions was proved by Pokrovskii [33], but Theorem 1.2 is not covered in his treatment. See also Kilpeläinen and Zhong [17] for the optimal result at the $C^{0,\alpha}$ level.

At this point, let us mention one simple result as an aside.

1.4. **Theorem.** *Let $1 < p < \infty$, $p \neq 2$, and let H denote the set of p-harmonic functions in an open set $\Omega \in \mathbb{R}^n$, $n \geq 2$. Further, let*

$$F = \{f \in H : f + h \in H \text{ for all } h \in H\}.$$

Then F is the set of constants.

Proof. Obviously constants belong to F. Assume that there exists a nonconstant $f \in F$. Then there exists a point $x \in \Omega$ such that $\nabla f \neq 0$ at x. Lewis [21] proved that p-harmonic functions are real analytic outside critical points (and also [22] that real analytic nonconstant p-harmonic functions do not have any critical points), so f is real analytic near x. Take a function $h \in H$ that is not real analytic near x. Then h has a critical point at x, so $\nabla(f + h)(x) = \nabla f(x) \neq 0$. Thus $f + h$ is real analytic in a neighborhood of x, but this implies that h is real analytic in a neighborhood of x, a contradiction. □

The second part of this work is devoted to the boundary behavior of p-harmonic functions. We explain and partly extend a result in a manuscript of the late Thomas H. Wolff [40]. In the paper, Wolff (1954-2000) proves the following.

1.5. **Theorem** (Wolff 1984). *For $p > 2$, there exists a bounded solution of $\Delta_p u = 0$ in the upper half-plane $\mathbb{R}^2_+$ such that the set*

$$\{x \in \mathbb{R} : \text{ the limit } \lim_{y \to 0} u(x, y) \text{ exists}\}$$

has one-dimensional Lebesgue measure zero, and there exists a bounded positive solution of $\Delta_p v = 0$ in $\mathbb{R}^2_+$ such that the set

$$\{x \in \mathbb{R} : \limsup_{y \to 0} v(x, y) > 0\}$$

has one-dimensional Lebesgue measure zero.

For $p = 2$, a Fatou Theorem states that a bounded harmonic function has radial limits almost everywhere on the boundary of its domain, and that these limits are positive if the function is positive. Thus Wolff's Theorem is an anti-Fatou Theorem for p-harmonic functions. We are motivated by the following quote from Wolff's paper:

> Theorem 1.5 generalizes to $\mathbb{R}^{n+1}_+$, $n \geq 1$, by adding dummy variables. It must also generalize to other domains but we have not carried this out; the argument is easiest in a half space since Δ_p behaves nicely under the Euclidean operations.

For example, one can take $p = 3$ in Theorem 1.5, add one dummy variable, and map the half-space $\mathbb{R}^3_+$ to a ball $B \in \mathbb{R}^3$ using a conformal map. By using the fact that the n-Laplacian is conformally invariant (see e.g. Reshetnyak [34], Bojarski and Iwaniec [3], Heinonen, Kilpeläinen, and Martio [15, Chapter 14], and Rickman [35]), one obtains the anti-Fatou theorem for 3-harmonic functions in a ball $B \subset \mathbb{R}^3$ with minimal effort. This procedure does not work when $n = 2$, since the conformal invariance of the p-Laplacian is lost for $p \neq 2$.

The core of Wolff's proof is the construction of a function $\Phi \in \mathrm{Lip}(\overline{\mathbb{R}^2_+})$ that is p-harmonic in $\mathbb{R}^2_+$, has period 1 in the x variable,

$$\int_0^1 \Phi(x,0)\,dx = \eta \neq 0,$$

and $\lim_{y\to\infty} \Phi(x,y) = 0$.

(This is a failure of the mean value principle, and cannot be done for $p = 2$.) After this, let $T_j \geq 1$ be an increasing sequence of integers, and consider the function $\Phi(T_j x, T_j y)$ for a fixed j. It is p-harmonic, it has period T_j^{-1} in the x variable, it still approaches zero when $y \to \infty$, and

$$\int_0^1 \Phi(T_j x, 0)\,dx = \eta$$

independent of j. As we will see, all these features are heavily used in the rest of Wolff's construction; this is the nice behavior that Wolff refers to in the quote above.

We have tried to prove Wolff's Theorem in the unit disc $\mathbb{D}$, but the best we can currently do is the following:

1.6. **Lemma.** *If $p > 2$, there exists a sequence of functions $\Phi_j \in \mathrm{Lip}(\overline{\mathbb{D}})$ such that $||\Phi_j||_{L^\infty(\mathbb{D})} \leq C < \infty$ for all j, $\Delta_p \Phi_j = 0$ in $\mathbb{D}$, Φ_j has period $\lambda_j > 0$ (dividing 2π) in the θ variable, $\lambda_j \to 0$ as $j \to \infty$,*

$$\lim_{r\to 0} \Phi_j(r,\theta) = 0,$$

and

$$\int_0^{2\pi} \Phi_j(1,\theta) d\theta = \eta_j > 0.$$

In order to prove the anti-Fatou Theorem in the unit disc using the method of Wolff, we would like to bound the sequence η_j from below by a sequence like $(\log j)^{-1}$, but we currently do not know whether this is possible or not.

Let us mention some related results. Lewis [20] extended Wolff's Theorem to the case $1 < p < 2$. Manfredi and Weitsman [30] proved that if $\Omega \subset \mathbb{R}^n$ is a smooth domain, if $1 < p < 3 + \frac{2}{n-2}$ and if $\Delta_p u = 0$ in Ω, then the set

$$E = \{x \in \partial\Omega : \ u \text{ has a radial limit at } x\}$$

has $\dim_{\mathcal{H}}(E) \geq \beta$ for some $\beta = \beta(n, p) > 0$. Manfredi and Weitsman were able to extend their result to all $p \in (1, \infty)$, but that result was never published. At the same time Fabes, Garofalo, Marín-Malave and Salsa [9] proved the analogous result that covers Lipschitz domains as well (with β depending on the Lipschitz character of the domain). Finally, Wolff's proof has recently been used by Llorente, Manfredi, and Wu [28] to construct a p-harmonic measure ω relative to $\mathbb{R}^2_+$ that is not subadditive at the zero level[1]: there exist sets $A, B \subset \mathbb{R}$ such that $\omega(A) = \omega(B) = 0$, but $\omega(A \cup B) > 0$. Also this construction remains to be done in the disc.

[1]In the literature, there are two notions that carry the name p-harmonic measure, one of them is a measure and the other one is not (if $p \neq 2$). This one is the non-measure.

Part I: Interior Estimates

2. Hölder Continuity

Let $\alpha \in (0,1)$, and let u be a real-valued function on Ω. We say that u is uniformly α-Hölder continuous on Ω if

$$[u]_{\alpha,\Omega} = \sup \left\{ \frac{|u(x) - u(y)|}{|x-y|^\alpha} : x, y \in \Omega, x \neq y \right\} < \infty,$$

and that u is locally α-Hölder continuous on Ω if $[u]_{\alpha,\Omega'} < \infty$ for each $\Omega' \subset\subset \Omega$. We denote this latter space by $C^\alpha(\Omega)$.

It is easy to see that $u \in C^\alpha(\Omega)$ if and only if

$$\operatorname*{osc}_{B_R(x_0)} u \leq CR^\alpha \tag{6}$$

for each ball $B_R(x_0) \subset\subset \Omega$. The constant C may depend on α, u, and $\operatorname{dist}(B_R(x_0), \partial\Omega)$, but not on x_0 or R.

By Campanato 1963 [5], the oscillation in (6) may be replaced by an L^p norm for any $1 \leq p < \infty$. That is, a function u (or a representative in L^p) belongs to $C^\alpha(\Omega)$ if and only if

$$\left(\fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^p \, dx \right)^{\frac{1}{p}} \leq CR^\alpha \tag{7}$$

for each ball $B_R(x_0) \subset\subset \Omega$.

We say that a function u is $C^\alpha(\Omega)$ *scalable* if

$$\operatorname*{osc}_{B_r(x_0)} u \leq C \left(\frac{r}{R} \right)^\alpha \operatorname*{osc}_{B_R(x_0)} u \tag{8}$$

for each $B_R(x_0) \subset\subset \Omega$ and for each $0 < r \leq R$. If $u \in L^\infty_{\mathrm{loc}}(\Omega)$, the condition (8) readily implies $u \in C^\alpha(\Omega)$. The converse is not true, i.e. there exist functions in $C^\alpha(\Omega)$ that are bounded in Ω, but are not $C^\alpha(\Omega)$ scalable. For unbounded domains this is easy to see, but for bounded domains some work is required:

2.1. **Remark.** *Let $\Omega = (0,1) \subset \mathbb{R}$. For given $0 < \alpha < 1$, there exists a bounded function $u \in C^\alpha(\Omega)$ that is not $C^\beta(\Omega)$ scalable for any $0 < \beta < 1$.*

Proof. For an interval $J \subset (0,1)$, let x_0 be the midpoint of J, and let u_J be the tent function in J with maximum value $|J|^\alpha$, i.e.

$$u_J(x) = \begin{cases} 2|J|^{\alpha-1}\left(-|x-x_0| + \frac{|J|}{2}\right), & \text{when } x \in J, \\ 0, & \text{otherwise.} \end{cases}$$

Take a sequence of disjoint intervals $I_i \subset (0,1)$ converging to a point, for example

$$I_i = \left(2^{-(2i+1)}, 2^{-2i}\right).$$

For each interval I_i, let $0 < \lambda_i < 1$, and take a subinterval $J_i \subset I_i$ such that $|J_i| = \lambda_i |I_i|$. Then (denote $u_i = u_{J_i}$)

$$\frac{\operatorname{osc}_{J_i} u_i}{|J_i|^\beta} = \frac{\operatorname{osc}_{I_i} u_i}{\lambda_i^\beta |I_i|^\beta},$$

i.e.

$$\operatorname*{osc}_{J_i} u_i = \frac{1}{\lambda_i^\beta}\left(\frac{|J_i|}{|I_i|}\right)^\beta \operatorname*{osc}_{I_i} u_i.$$

Note that also

$$\operatorname*{osc}_{J_i}(a_i u_i) = \frac{1}{\lambda_i^\beta}\left(\frac{|J_i|}{|I_i|}\right)^\beta \operatorname*{osc}_{I_i}(a_i u_i)$$

for any $a_i \in \mathbb{R}$. Choose the sequence a_i such that $\sum_{i=1}^\infty |a_i| < \infty$ (in order to stay in the class C^α), choose the sequence λ_i such that $\lambda_i \to 0$ as $i \to \infty$, and set

$$u = \sum_{i=1}^\infty a_i u_i.$$

Then u has the required properties. □

Let us also remark that the same method can be used to show that the converse of Morrey's theorem (see e.g. Theorem 7.19 in [12]) does not hold.

The following Theorem, originally by Manfredi and H. I. Choe [29], is a scalable version of Campanato's result.

2.2. **Theorem.** *Let $0 < \alpha < 1$ and $1 \le p < \infty$. Assume $u \in L^\infty_{loc}(\Omega)$ satisfies*

$$\left(\fint_{B_r(x_0)} |u(x) - u_{x_0,r}|^p dx\right)^{\frac{1}{p}} \le C_{n,p,\alpha}\left(\frac{r}{R}\right)^\alpha \left(\fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^p dx\right)^{\frac{1}{p}} \tag{9}$$

for all $B_R(x_0) \subset\subset \Omega$ *and for all* $0 < r \leq R$. *Then, for any* $q > p$,

$$\left(\fint_{B_r(x_0)} |u(x) - u_{x_0,r}|^q dx \right)^{\frac{1}{q}} \leq C_{n,p,q,\alpha} \left(\frac{r}{R}\right)^\alpha \left(\fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^q dx \right)^{\frac{1}{q}} \tag{10}$$

for all $B_R(x_0) \subset\subset \Omega$ *and for all* $0 < r \leq R$. *Moreover,*

$$\operatorname*{osc}_{B_r(x_0)} u \leq C_{n,p,\alpha} \left(\frac{r}{R}\right)^\alpha \operatorname*{osc}_{B_R(x_0)} u \tag{11}$$

for all $B_R(x_0) \subset\subset \Omega$ *and for all* $0 < r < R/2$.

We will need the following two Lemmas:

2.3. **Lemma.** *Let* $1 \leq p < \infty$, *and let* $f \in L^p_{loc}(\Omega; \mathbb{R}^N)$, $N \geq 1$. *Then there exists a constant* $C = C_p > 0$ *such that for each* $L \in R^N$ *and each* $B_r(x_0) \subset \Omega$,

$$\fint_{B_r(x_0)} |f(x) - f_{x_0,r}|^p \, dx \leq C_p \fint_{B_r(x_0)} |f(x) - L|^p \, dx.$$

Proof.

$$\fint_{B_r(x_0)} |f(x) - f_{x_0,r}|^p \, dx \leq C_p \left(\fint_{B_r(x_0)} |f(x) - L|^p \, dx + \fint_{B_r(x_0)} |L - f_{x_0,r}|^p \, dx \right),$$

and, by Hölder's inequality,

$$\fint_{B_r(x_0)} |L - f_{x_0,r}|^p \, dx = \left| \fint_{B_r(x_0)} \big(f(x) - L\big) \, dx \right|^p \leq \fint_{B_r(x_0)} |f(x) - L|^p \, dx.$$

□

2.4. **Lemma.** *Let* $1 \leq p < \infty$, *and let* $f \in L^p_{loc}(\Omega; \mathbb{R}^N)$, $N \geq 1$. *Assume that, for some* $0 < \rho < 1$,

$$\left(\fint_{B_r(x_0)} |f(x) - f_{x_0,r}|^p dx \right)^{\frac{1}{p}} \leq C_{n,p,\alpha} \left(\frac{r}{R}\right)^\alpha \left(\fint_{B_R(x_0)} |f(x) - f_{x_0,R}|^p dx \right)^{\frac{1}{p}}$$

for all $B_R(x_0) \subset \Omega$ *and all* $0 < r < \rho R$. *Then*

$$\left(\fint_{B_r(x_0)} |f(x) - f_{x_0,r}|^p dx \right)^{\frac{1}{p}}$$
$$\leq C_{n,p,\alpha,\rho} \left(\frac{r}{R}\right)^{\alpha} \left(\fint_{B_R(x_0)} |f(x) - f_{x_0,R}|^p dx \right)^{\frac{1}{p}}$$

for all $0 < r \leq R$.

Proof. Let $r \geq \rho R$. Then $R/r \leq \rho^{-1}$, and by Lemma 2.3,

$$\fint_{B_r(x_0)} |f(x) - f_{x_0,r}|^p dx \leq C_p \frac{|B_R|}{|B_r|} \fint_{B_R(x_0)} |f(x) - f_{x_0,R}|^p dx$$
$$\leq C_{n,p,\rho} \fint_{B_R(x_0)} |f(x) - f_{x_0,R}|^p dx.$$

Moreover, since $1 \leq \rho^{-p\alpha}(r/R)^{p\alpha}$, we obtain

$$\fint_{B_r(x_0)} |f(x) - f_{x_0,r}|^p dx \leq C_{n,p,s,\alpha} \left(\frac{r}{R}\right)^{p\alpha} \fint_{B_R(x_0)} |f(x) - f_{x_0,R}|^p dx,$$

as wanted. □

Proof of Theorem 2.2.

We start by repeating the proof of Campanato's Theorem (modified from [11]). Fix a ball $B_R(x_0) \subset\subset \Omega$, and let $0 < \rho \leq r \leq R$. Then, for an arbitrary $x \in B_\rho(x_0)$,

$$|u_{x_0,r} - u_{x_0,\rho}|^p \leq C_p \big(|u(x) - u_{x_0,r}|^p + |u(x) - u_{x_0,\rho}|^p\big). \tag{12}$$

Integrating both sides of (12) over the ball $B_\rho(x_0)$, and using Lemma 2.3 and (9), yields

$$|u_{x_0,r} - u_{x_0,\rho}|^p |B_\rho(x_0)|$$
$$\leq C_p \left(\int_{B_\rho(x_0)} |u(x) - u_{x_0,r}|^p dx + \int_{B_\rho(x_0)} |u(x) - u_{x_0,\rho}|^p dx \right)$$
$$\leq C_p \int_{B_r(x_0)} |u(x) - u_{x_0,r}|^p dx$$
$$\leq C_{n,p,\alpha} r^{n+p\alpha} R^{-(n+p\alpha)} \int_{B_R(x_0)} |u(x) - u_{x_0,R}|^p dx,$$

i.e.

$$|u_{x_0,r} - u_{x_0,\rho}| \leq C \, \rho^{-n/p} r^{n/p+\alpha} \tag{13}$$

for each $0 < \rho \le r \le R$, where

$$C = C_{n,p,\alpha} R^{-\alpha} \left(\fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^p dx \right)^{\frac{1}{p}} < \infty, \tag{14}$$

since $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Keep r and R fixed, and let $r_i = 2^{-i} r$. Then (13) implies

$$|u_{x_0,r_i} - u_{x_0,r_{i+1}}| \le C\, r_{i+1}^{-n/p} r_i^{n/p+\alpha} = C r_i^\alpha, \tag{15}$$

which gives, for any integer $k \ge 1$,

$$|u_{x_0,r_i} - u_{x_0,r_{i+k}}| \le C\, r^\alpha \sum_{j=i}^\infty 2^{-\alpha j}.$$

This approaches zero as $i \to \infty$, so the sequence $\{u_{x_0,r_i}\}_{i=1}^\infty$ is a Cauchy sequence for all $x_0 \in \Omega$. Now we claim that the limit

$$\widetilde{u}(x_0) = \lim_{i\to\infty} u_{x_0,r_i}$$

does not depend on the choice of r (or R). Indeed, for $\rho < r$, let $\rho_i = 2^{-i}\rho$ and $r_i = 2^{-i} r$, and for each i choose $j \ge i$ such that $r_{j+1} < \rho_i \le r_j$. Then, using (15) and (13), we obtain

$$\begin{aligned} |u_{x_0,r_i} - u_{x_0,\rho_i}| &\le |u_{x_0,r_i} - u_{x_0,r_j}| + |u_{x_0,r_j} - u_{x_0,\rho_i}| \\ &\le C(r_i^\alpha + r_j^\alpha) \le C r_i^\alpha, \end{aligned}$$

as claimed. On the other hand $u_{x,\rho}$ converges, as $\rho \to 0$, in $L^1(\Omega)$ to the function u, so we have $u = \widetilde{u}$ a.e. Now, summing from zero to infinity in (15), and taking (14) into account, yields the important estimate

$$|u(x_0) - u_{x_0,r}| \le C_{n,p,\alpha} \left(\frac{r}{R}\right)^\alpha \left(\fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^p dx \right)^{\frac{1}{p}}. \tag{16}$$

for each $B_R(x_0) \subset\subset \Omega$ and for each $0 < r \le R$. This implies

$$\operatorname*{osc}_{B_r(x_0)} u \le C \left(\frac{r}{R}\right)^\alpha \left(\fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^p dx \right)^{\frac{1}{p}} \tag{17}$$

for each $B_R(x_0) \subset\subset \Omega$ and for each $0 < r < R/2$. Indeed, let $B_R(x_0) \subset\subset \Omega$, let $0 < r < R/2$, and let $x \in B_r(x_0)$. Then

$$|u(x) - u(x_0)| \le |u(x) - u_{x,r}| + |u_{x,r} - u_{x_0,r}| + |u(x_0) - u_{x_0,r}|. \tag{18}$$

The first and third terms on the right-hand side of (18) are estimated in (16). (For the first term, use the ball $B_{R/2}(x)$ as the larger ball in (16), and then use Lemma 2.4.) For the second term we have

$$|u_{x,r} - u_{x_0,r}| \le |u_{x,r} - u(y)| + |u(y) - u_{x_0,r}|,$$

and integrating with respect to y over $B_r(x_0) \cap B_r(x)$ yields

$$\begin{aligned} &\int_{B_r(x_0)\cap B_r(x)} |u_{x,r} - u_{x_0,r}|\, dy \\ &\leq \int_{B_r(x)} |u(y) - u_{x,r}|\, dy + \int_{B_r(x_0)} |u(y) - u_{x_0,r}|\, dy. \end{aligned} \tag{19}$$

Since $|B_r(x_0) \cap B_r(x)| = Cr^n$, using Hölder's inequality and dividing both sides by r^n leaves us with

$$\begin{aligned} &|u_{x,r} - u_{x_0,r}| \\ &\leq \left(\fint_{B_r(x)} |u(y) - u_{x,r}|^p\, dy\right)^{\frac{1}{p}} + \left(\fint_{B_r(x_0)} |u(y) - u_{x_0,r}|^p\, dy\right)^{\frac{1}{p}} \\ &\leq C\left(\frac{r}{R}\right)^{\alpha} \left(\fint_{B_R(x_0)} |u(y) - u_{x_0,R}|^p\, dy\right)^{\frac{1}{p}}. \end{aligned}$$

Thus (17) holds, and the estimate (11) is proved.

If $B_R(x_0) \subset\subset \Omega$, if $0 < r < R/2$, and if $p < q < \infty$, then by (17), by the assumption (9), and by Hölder's inequality,

$$\begin{aligned} &\fint_{B_r(x_0)} |u(x) - u_{x_0,r}|^q dx \leq \big(\operatorname*{osc}_{B_r(x_0)} u\big)^{q-p} \fint_{B_r(x_0)} |u(x) - u_{x_0,r}|^p dx \\ &\leq C\left(\frac{r}{R}\right)^{\alpha(q-p)} \left(\fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^q dx\right)^{\frac{q-p}{q}} \fint_{B_r(x_0)} |u(x) - u_{x,r}|^p dx \\ &\leq C\left(\frac{r}{R}\right)^{q\alpha} \left(\fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^q dx\right)^{\frac{q-p}{q}} \fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^p dx \\ &\leq C\left(\frac{r}{R}\right)^{q\alpha} \left(\fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^q dx\right)^{\frac{q-p}{q}} \left(\fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^q dx\right)^{\frac{p}{q}} \\ &= C\left(\frac{r}{R}\right)^{q\alpha} \fint_{B_R(x_0)} |u(x) - u_{x_0,R}|^q dx. \end{aligned}$$

By Lemma 2.4, this holds for all $0 < r \leq R$, and thus also (10) is proved. □

This also completes the proof of Theorem 1.1 (page 8), since the Theorem holds for $q = \min\{2, p\}$, see DiBenedetto and Manfredi [8] and Lieberman [24].

The following perturbation lemma is needed in the next section.

2.5. **Lemma.** *Let $B_R = B_R(x_0) \subset \mathbb{R}^n$ be a ball, let $u \in W^{1,p}(B_R)$ be an arbitrary Sobolev function, and let $h \in W^{1,p}_{loc}(B_R)$ be an arbitrary*

p-harmonic function in B_R. Then, for each $r \in (0, R]$,

$$
\begin{aligned}
&\int_{B_r(x_0)} |\nabla u(x) - (\nabla u)_{x_0,r}|^p dx \\
&\leq C \left\{ \left(\frac{r}{R}\right)^{n+p\sigma} \int_{B_R(x_0)} |\nabla u(x) - (\nabla u)_{x_0,R}|^p dx \right. \\
&\qquad \left. + \int_{B_R(x_0)} |\nabla u(x) - \nabla h(x)|^p dx \right\}
\end{aligned} \tag{20}
$$

with any $0 < \sigma < \sigma_0$, where $\sigma_0 = \sigma_0(n, p)$ is the number defined on page 8.

Proof. Let $r \in (0, R]$. By adding and subtracting $(\nabla u - \nabla h)_{x_0,r}$,

$$
\begin{aligned}
&\int_{B_r(x_0)} |\nabla u(x) - (\nabla u)_{x_0,r}|^p dx \\
&\leq C(p) \left\{ \int_{B_r(x_0)} |\nabla u(x) - (\nabla h)_{x_0,r}|^p dx \right. \\
&\qquad \left. + \int_{B_r(x_0)} |(\nabla u - \nabla h)_{x_0,r}|^p dx \right\}.
\end{aligned} \tag{21}
$$

Analogously, by adding and subtracting $\nabla h(x)$,

$$
\begin{aligned}
&\int_{B_r(x_0)} |\nabla u(x) - (\nabla h)_{x_0,r}|^p dx \\
&\leq C(p) \left\{ \int_{B_r(x_0)} |\nabla h(x) - (\nabla h)_{x_0,r}|^p dx \right. \\
&\qquad \left. + \int_{B_r(x_0)} |\nabla u(x) - \nabla h(x)|^p dx \right\},
\end{aligned}
$$

and using Hölder's inequality yields

$$
\begin{aligned}
&\int_{B_r(x_0)} |(\nabla u - \nabla h)_{x_0,r}|^p dx \\
&= \left(\frac{1}{|B_r(x_0)|}\right)^{p-1} \left| \int_{B_r(x_0)} \big(\nabla u(x) - \nabla h(x)\big)\, dx \right|^p \\
&\leq \int_{B_r(x_0)} |\nabla u(x) - \nabla h(x)|^p dx,
\end{aligned}
$$

so all in all we obtain, combined with the estimate (10) for $q = p$,

$$
\begin{aligned}
&\int_{B_r(x_0)} |\nabla u(x) - (\nabla u)_{x_0,r}|^p dx \\
&\leq C(p) \left\{ \int_{B_r(x_0)} |\nabla h(x) - (\nabla h)_{x_0,r}|^p dx \right. \\
&\qquad \left. + \int_{B_r(x_0)} |\nabla u(x) - \nabla h(x)|^p dx \right\} \\
&\leq C(n,p,\sigma) \left\{ \left(\frac{r}{R} \right)^{n+\sigma p} \int_{B_R(x_0)} |\nabla h(x) - (\nabla h)_{x_0,R}|^p dx \right. \\
&\qquad \left. + \int_{B_R(x_0)} |\nabla u(x) - \nabla h(x)|^p dx \right\}.
\end{aligned}
\tag{22}
$$

Again, adding and subtracting as before yields

$$
\begin{aligned}
&\int_{B_R(x_0)} |\nabla h(x) - (\nabla h)_{x_0,R}|^p dx \\
&\leq C(p) \left\{ \int_{B_R(x_0)} |\nabla u(x) - (\nabla u)_{x_0,R}|^p dx \right. \\
&\qquad \left. + \int_{B_R(x_0)} |\nabla u(x) - \nabla h(x)|^p dx \right\},
\end{aligned}
\tag{23}
$$

and the result follows by inserting (23) into (22). □

3. Measure Data

In this section we prove Theorem 1.2 (page 9). Our main tool to handle measure data is the following Theorem by Maz'ya [32, p.54]. Let us note that in Maz'ya's Theorem it is assumed that the boundary of Ω is smooth, but the assumption is never employed in the proof. This observation was already made by Hajłasz and Koskela [13].

3.1. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let μ be a measure on Ω such that*

$$\sup\left\{\frac{\mu(B_R)}{R^{n-1}} : B_R \subset\subset \Omega\right\} =: M < \infty. \tag{24}$$

Then

$$\int_\Omega |\varphi| d\mu \leq C(n)M \int_\Omega |\nabla\varphi| dx \quad \text{for all } \varphi \in C_0^1(\Omega).$$

□

We also need to test our measure data equation (3) with a function in $W_0^{1,p}(\Omega)$, i.e. we want our measure to be in the dual of $W_0^{1,p}(\Omega)$. This is indeed the case by the following Theorem, originally proven by Hedberg and Wolff [14]:

3.2. **Theorem.** *A nonnegative Radon measure μ on Ω (open and bounded) belongs to the dual of $W_0^{1,p}(\Omega)$ if and only if*

$$\int_\Omega \int_0^1 \left(r^{p-n}\mu(B_r(x))\right)^{1/(p-1)} \frac{dr}{r} d\mu(x) < \infty.$$

□

Indeed, since both the positive part μ^+ and the negative part μ^- of our measure μ in Theorem 1.2 satisfy $\mu^\pm(B_r(x)) \leq Cr^{n-1+\alpha}$, we have

$$\begin{aligned}&\int_\Omega \int_0^1 \left(r^{p-n}\mu^\pm(B_r(x))\right)^{1/(p-1)} \frac{dr}{r} d\mu^\pm(x)\\ &\leq C \int_\Omega \int_0^1 r^{\alpha/(p-1)} dr\, d\mu^\pm(x) = C\mu^\pm(\Omega) < \infty.\end{aligned}$$

Thus both μ^+ and μ^-, and thereby also μ, belong to the dual of $W_0^{1,p}(\Omega)$.

In addition to Theorems 3.1 and 3.2, we will use the following results that are well known.

3.3. **Lemma.** *If $\Omega \subset \mathbb{R}^n$ is open and bounded, and if $u \in W^{1,p}(\Omega)$, there exists a unique p-harmonic $h \in W^{1,p}(\Omega)$ such that $u - h \in W_0^{1,p}(\Omega)$.* $\square$

3.4. **Lemma.** *If $h \in W^{1,p}(\Omega)$ is p-harmonic, then*

$$\int_\Omega |\nabla h|^p dx \leq \int_\Omega |\nabla v|^p dx$$

for each $v \in W^{1,p}(\Omega)$ with $h - v \in W_0^{1,p}(\Omega)$. $\square$

3.5. **Lemma.** *The following estimate holds for any $\xi, \zeta \in \mathbb{R}^n$, $p \geq 2$:*

$$|\xi - \zeta|^p \leq 2^{p-1}(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta) \cdot (\xi - \zeta).$$

$\square$

3.6. **Lemma.** *The following estimate holds for any $\xi, \zeta \in \mathbb{R}^n$, $1 < p < 2$ and $\varepsilon > 0$:*

$$(p-1)|\xi - \zeta|^2(\varepsilon + |\xi|^2 + |\zeta|^2)^{\frac{p-2}{2}} \leq 2^{p-1}(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta) \cdot (\xi - \zeta).$$

$\square$

The proofs of Lemmas 3.3 and 3.4 can be found e.g. in Heinonen, Kilpeläinen and Martio [15] (3,17 and 3.13, respectively), and the proofs of Lemmas 3.5 and 3.6 can be found e.g. in the lecture notes by Lindqvist [26] (section 10).

We now set out to prove Theorem 1.2. It suffices to prove that the gradient of u satisfies the growth condition in Campanato's theorem:

$$\int_{B_R} |\nabla u - (\nabla u)_R|^p dx \leq CR^{n+p\delta} \tag{25}$$

whenever $B_R \subset\subset \Omega$.

Going one step further back, (25) follows after we prove that our function u satisfies, for each $B_R(x_0) \subset\subset \Omega$ and for each $r \in (0, R]$,

$$\begin{aligned} &\int_{B_r(x_0)} |\nabla u - (\nabla u)_r|^p dx \\ &\leq C\left\{\left(\frac{r}{R}\right)^{n+p\sigma(n,p)} \int_{B_R(x_0)} |\nabla u - (\nabla u)_R|^p dx + R^{n+p\tau}\right\}, \end{aligned} \tag{26}$$

where $\tau = \alpha \min\{1, (p-1)^{-1}\}$. This gives (25) because we have the following result, also by Campanato (see e.g. [11, Lemma 2.1]):

3.7. **Lemma.** *Let Φ be a nonnegative and nondecreasing function on $[0, R_0] \subset \mathbb{R}$ such that for all $r, R \in (0, R_0]$, $r \leq R$,*

$$\Phi(r) \leq A\left(\frac{r}{R}\right)^{\gamma} \Phi(R) + BR^{\beta}, \tag{27}$$

where $A, B \geq 0$ and $0 < \beta < \gamma$. Then

$$\Phi(R) \leq CR^{\beta}, \tag{28}$$

where $C = C(A, B, \gamma, \beta, R_0, \Phi(R_0)) > 0$. □

The condition $\beta < \gamma$ in (27) yields our result only if $\tau < \sigma_0$. (This is exactly the condition on the number δ in Theorem 1.2. Note also that for $p = 2$, we have $\sigma(n,p) = 1$, see e.g. [11].)

Finally, (26) follows when Lemma 2.5 (page 19) is combined with the following result:

3.8. **Lemma.** *Let u, μ, and α be as in Theorem 1.2, and let $B_R \subset\subset \Omega$. Then there exists a p-harmonic function $h \in W^{1,p}(B_R)$ such that*

$$\int_{B_R} |\nabla u - \nabla h|^p dx \leq CR^{n+p\tau}, \tag{29}$$

where $\tau = \alpha \min\{1, (p-1)^{-1}\}$.

Proof. Let $B_R \subset\subset \Omega$, and let h be the p-harmonic function in B_R with $u - h \in W_0^{1,p}(B_R)$, given by Lemma 3.3. We start with the case $p \geq 2$. By using the inequality of Lemma 3.5, and by testing our equation with $\varphi = u - h$, we obtain

$$\begin{aligned}
&\int_{B_R} |\nabla u - \nabla h|^p dx \\
&\leq 2^{p-1} \int_{B_R} \left(|\nabla u|^{p-2}\nabla u - |\nabla h|^{p-2}\nabla h\right) \cdot (\nabla u - \nabla h) dx \\
&= 2^{p-1} \int_{B_R} |\nabla u|^{p-2}\nabla u \cdot (\nabla u - \nabla h) dx \\
&= 2^{p-1} \int_{B_R} (u - h) d\mu \leq 2^{p-1} \int_{B_R} |u - h| d|\mu|.
\end{aligned} \tag{30}$$

Next, use Theorem 3.1, the measure growth condition (4), and Hölder's inequality to obtain

$$\begin{aligned}&\int_{B_R} |u-h|d|\mu| \le CR^\alpha \int_{B_R} |\nabla u - \nabla h| dx\\ &= CR^{\alpha+\frac{p-1}{p}n}(\int_{B_R} |\nabla u-\nabla h|^p dx)^{1/p}.\end{aligned}$$

We have arrived at

$$\int_{B_R} |\nabla u-\nabla h|^p dx \le CR^{\alpha+\frac{p-1}{p}n}\left(\int_{B_R} |\nabla u-\nabla h|^p dx\right)^{\frac{1}{p}}.$$

Raising both sides to the power p, dividing by the integral term (may be assumed nonzero), and raising both sides to the power $1/(p-1)$, gives

$$\int_{B_R} |\nabla u-\nabla h|^p dx \le C(n,p)R^{n+\frac{p}{p-1}\alpha}$$

as wanted.

In the case $1 < p < 2$, we use a device taken from Manfredi [31]. Set $w = u - h$, fix $\varepsilon > 0$, and write

$$|\nabla w|^p = \left\{(\varepsilon + |\nabla u|^2 + |\nabla h|^2)^{\frac{p-2}{2}}|\nabla w|^2\right\}^{\frac{p}{2}} \left\{\varepsilon + |\nabla u|^2 + |\nabla h|^2\right\}^{(\frac{2-p}{2})\frac{p}{2}}.$$

Use Hölder's inequality with exponents $2/p$, $2/(2-p)$, and denote $A = \varepsilon + |\nabla u|^2 + |\nabla h|^2$, to obtain

$$\int_{B_R} |\nabla w|^p dx \le \left\{\int_{B_R} A^{\frac{p-2}{2}}|\nabla w|^2 dx\right\}^{\frac{p}{2}} \left\{\int_{B_R} A^{\frac{p}{2}} dx\right\}^{\frac{2-p}{2}}. \tag{31}$$

We now estimate the first factor on the right-hand side of (31) as in the case $p \ge 2$, this time using the inequality of Lemma 3.6:

$$\begin{aligned}&\int_{B_R} (\varepsilon + |\nabla u|^2 + |\nabla h|^2)^{\frac{p-2}{2}}|\nabla w|^2\, dx\\ &\le C(p)\int_{B_R} (|\nabla u|^{p-2}\nabla u - |\nabla h|^{p-2}\nabla h)\cdot \nabla w\, dx\\ &= C(p)\int_{B_R} w\, d\mu \le C(n,p)R^\alpha \int_{B_R} |\nabla w|\, dx.\end{aligned}$$

For the second factor on the right-hand side of (31), we establish

$$\begin{aligned}&\int_{B_R}(\varepsilon+|\nabla u|^2+|\nabla h|^2)^{\frac{p}{2}}dx\\ &\leq \int_{B_R}(\varepsilon+|\nabla u|^2)^{\frac{p}{2}}\,dx+\int_{B_R}|\nabla h|^p\,dx\\ &\leq 2\int_{B_R}(\varepsilon+|\nabla u|^2)^{\frac{p}{2}}dx,\end{aligned}$$

by Lemma 3.4, and since $p<2$. Rewriting (31) with these estimates yields

$$\int_{B_R}|\nabla w|^p dx \leq C(n,p)\left\{R^\alpha\int_{B_R}|\nabla w|dx\right\}^{\frac{p}{2}}\left\{\int_{B_R}(\varepsilon+|\nabla u|^2)^{\frac{p}{2}}dx\right\}^{\frac{2-p}{2}}.$$

Now Hölder's inequality yields

$$\begin{aligned}&\int_{B_R}|\nabla w|^p dx\\ &\leq C(n,p)R^{\frac{p}{2}(\alpha+\frac{p-1}{p}n)}\left\{\int_{B_R}|\nabla w|^p dx\right\}^{\frac{1}{2}}\left\{\int_{B_R}(\varepsilon+|\nabla u|^2)^{\frac{p}{2}}dx\right\}^{\frac{2-p}{2}},\end{aligned}$$

so that

$$\int_{B_R}|\nabla w|^p dx \leq C(n,p)R^{p\alpha+(p-1)n}\left\{\int_{B_R}(\varepsilon+|\nabla u|^2)^{\frac{p}{2}}dx\right\}^{2-p}.$$

Since $\nabla u\in L^\infty_{\mathrm{loc}}(\Omega)$ (see e.g. Manfredi [31]), we have

$$\int_{B_R}(\varepsilon+|\nabla u|^2)^{\frac{p}{2}}dx\leq (C+\varepsilon)R^n,$$

and by letting $\varepsilon\to 0$ we arrive at our final estimate:

$$\int_{B_R}|\nabla u-\nabla h|^p dx\leq C(n,p)R^{n+p\alpha}, \tag{32}$$

since $p\alpha+(p-1)n+(2-p)n=n+p\alpha$. □

Part II: Boundary Behavior

4. The Wolff Story

In this section we explain the big picture of Wolff's proof, concentrating on the first form of his anti-Fatou Theorem:

4.1. **Theorem.** *For $p > 2$, there exists a bounded solution of $\Delta_p u = 0$ in $\mathbb{R}^2_+$ such that the set*

$$\{x \in \mathbb{R} : \text{ the limit } \lim_{y \to 0} u(x, y) \text{ exists}\}$$

has measure zero.

The proof of the second form (see page 11) is similar, as commented in the end of this section (see page 32).

The reader is advised to have Wolff's paper [40] at hand throughout the rest of this text.

Let us start with a classical theorem, found in many probability theory books. Our proof is modified from Llorente [27]; we have stripped it of the probabilistic terminology.

4.2. **Theorem.** *Consider the formal series*

$$\sum_{j=1}^{\infty} \pm \frac{1}{j}, \tag{33}$$

where the independent signs are chosen by a coin toss at each stage. The series converges with probability one.

Proof. Let us first rephrase the problem. Let $x \in [0, 1)$, and consider the binary representation of x:

$$x = \sum_{j=1}^{\infty} \frac{\xi_j(x)}{2^j},$$

where $\xi_j(x) \in \{0, 1\}$ for each j. Rescale these numbers to obtain the sequence

$$R_j(x) = 1 - 2\xi_j(x),$$

where $R_j(x) \in \{-1, 1\}$ for each j. Now, saying that the series (33) converges with probability one is equivalent to saying that the series

$$\sum_{j=1}^{\infty} \frac{1}{j} R_j(x) \tag{34}$$

converges almost everywhere in $[0, 1)$. If we denote

$$s_k(x) = \sum_{j=1}^{k} \frac{1}{j} R_j(x),$$

then we want the sequence $\{s_k(x)\}_{k=1}^{\infty}$ to be Cauchy for almost every $x \in [0, 1)$. It is rather easy to see that it suffices to prove, for $\lambda > 0$ fixed, that

$$\left| \{x \in [0,1) : \sup_{m \le k \le n} |s_k(x) - s_m(x)| \ge \lambda\} \right| \to 0, \tag{35}$$

when first $n \to \infty$, and then $m \to \infty$. To this end, we prove the following maximal inequality[2]: for all $1 \le m \le n < \infty$,

$$\begin{aligned} &\left| \{x \in [0,1) : \sup_{m \le k \le n} |s_k(x) - s_m(x)| \ge \lambda\} \right| \\ &\le \frac{1}{\lambda^2} \int_0^1 (s_n(x) - s_m(x))^2 \, dx. \end{aligned}$$

Fix $1 \le m \le n < \infty$. Denote

$$T_k(x) = s_k(x) - s_m(x)$$

for $k = m, \ldots, n$, and define a *stopping time*

$$\tau : [0, 1) \to \{m, \ldots, n\} \cup \{\infty\}$$

as the least index $k \in \{m, \ldots, n\}$ where

$$|T_k(x)| > \lambda.$$

(If no such index exists, we define $\tau(x) = \infty$.) We now "stop" the sequence

$$\{T_k(x)\}_{k=m}^{n}$$

as soon as $|T_k(x)| > \lambda$, by defining the *stopped sequence*

$$T_k^{\tau}(x) = \begin{cases} T_k(x), & \text{if } k \le \tau(x) \\ T_{\tau(x)}(x), & \text{otherwise.} \end{cases}$$

[2]This is a special case of Kolmogorov's inequality.

Note that

$$\left\{x \in [0,1) : \sup_{m \le k \le n} |s_k(x) - s_m(x)| \ge \lambda\right\} = \{x \in [0,1) : \tau(x) \le n\}$$

and that in this set

$$|T_n^\tau(x)| \ge \lambda.$$

Thus we use Chebychev's inequality:

$$\begin{aligned} &\left|\{x \in [0,1) : \sup_{m \le k \le n} |s_k(x) - s_m(x)| \ge \lambda\}\right| \\ &= |\{x \in [0,1) : \tau(x) \le n\}| \le \frac{1}{\lambda^2}\int_0^1 T_n^\tau(x)^2 dx. \end{aligned}$$

Next, we compute the representation (note that $T_m^\tau(x) = 0$)

$$\begin{aligned} T_n^\tau(x) &= \sum_{k=m+1}^{n} \big(T_k^\tau(x) - T_{k-1}^\tau(x)\big) + T_m^\tau(x) \\ &= \sum_{k=m+1}^{n} \big(T_k(x) - T_{k-1}(x)\big)\chi_{\{\tau(x)\ge k\}}(x) \\ &= \sum_{k=m+1}^{n} \frac{1}{k} R_k(x)\chi_{\{\tau(x)\ge k\}}(x), \end{aligned}$$

so

$$\int_0^1 T_n^\tau(x)^2 dx = \int_0^1 \sum_{k=m+1}^{n} \left(\frac{1}{k} R_k(x)\chi_{\{\tau(x)\ge k\}}(x)\right)^2 dx,$$

where we also used the orthogonality of the sequence $R_j(x)\chi_{\{\tau(x)\ge j\}}$ (left as an excercise to the reader):

$$i \ne j \implies \int_0^1 R_i(x)\chi_{\{\tau(x)\ge i\}} R_j(x)\chi_{\{\tau(x)\ge j\}} = 0. \tag{36}$$

Moreover,

$$\begin{aligned} &\sum_{k=m+1}^{n} \left(\frac{1}{k} R_k(x)\chi_{\{\tau(x)\ge k\}}(x)\right)^2 \le \sum_{k=m+1}^{n} \left(\frac{1}{k} R_k(x)\right)^2 \\ &\le \left(s_n(x) - s_m(x)\right)^2. \end{aligned}$$

Summing up,

$$\begin{aligned} &\left|\{x \in [0,1) : \sup_{m \le k \le n} |s_k(x) - s_m(x)| \ge \lambda\}\right| \\ &\le \frac{1}{\lambda^2}\int_0^1 \left(s_n(x) - s_m(x)\right)^2 dx, \end{aligned} \tag{37}$$

as wanted. After this, proving (35) is simple. Note that by (36),

$$\int_0^1 s_n(x)^2 dx = \sum_{j=1}^{n} \int_0^1 \frac{1}{j^2} R_j(x)^2 dx,$$

and that this sequence increases to M, say, as $n \to \infty$. Using (37) and (36) yields

$$\begin{aligned}
&\left| \{x \in [0,1) : \sup_{m \le k \le n} |s_k(x) - s_m(x)| \ge \lambda\} \right| \\
&\le \frac{1}{\lambda^2} \int_0^1 (s_n(x) - s_m(x))^2 \, dx \\
&= \frac{1}{\lambda^2} \int_0^1 \left(s_n(x)^2 - s_m(x)^2 \right) dx \le \frac{1}{\lambda^2} \int_0^1 \left(M - s_m(x)^2 \right) dx.
\end{aligned}$$

Let $m \to \infty$ to finish the proof. □

We now know that the formal series

$$\sum_{j=1}^{\infty} \frac{1}{j} R_j(x) \tag{38}$$

converges to a finite limit a.e. $x \in [0,1)$. It immediately follows that iff $\eta \in \mathbb{R}$ is nonzero, the formal series

$$\sum_{j=1}^{\infty} \left(\eta + \frac{1}{j} R_j(x) \right)$$

diverges for almost every $x \in [0,1)$. Wolff proves this even more generally (Lemma 2.1 in Wolff):

4.3. **Lemma.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ satisfy $\phi(x+1) = \phi(x)$ and $\phi \in \mathrm{Lip}(\mathbb{R})$. If*

$$\int_0^1 \phi(x) dx = \eta \neq 0$$

and if T_j is any sequence of integers such that $T_{j+1}/T_j > q > 1$, then the sequence

$$\sum_{j=1}^{k} \frac{1}{j} \phi(T_j x), \tag{39}$$

diverges as $k \to \infty$ for almost every $x \in \mathbb{R}$. □

This result is well known; the philosophy is that $\phi(T_j x)$ behaves like $\eta + R_j(x)$ in our basic example.

With Lemma 4.3 as the starting point, Wolff's proof now consists of three other Lemmas. In the first one, the function ϕ in Lemma 4.3 is fixed as $\phi(x) = \Phi(x, 0)$, where $\Phi\colon \overline{\mathbb{R}^2_+} \to \mathbb{R}$ is as in the following Lemma (Lemma 1.1 in Wolff):

4.4. **Lemma.** *If $p > 2$, there exists a function $\Phi \in \mathrm{Lip}(\overline{\mathbb{R}^2_+})$ such that $\Delta_p \Phi = 0$ in $\mathbb{R}^2_+$, Φ has period 1 in the x variable,*

$$\lim_{y\to\infty} \Phi(x, y) = 0 \qquad \text{for all } x \in \mathbb{R},$$

and

$$\int_0^1 \Phi(x, 0)dx = \eta \neq 0.$$

□

This is the most important Lemma, we will prove it in detail in the upcoming sections.

The purpose of the next Lemma (Lemma 2.12 in Wolff) is to insert "stopping times" L_j into the sequence (39), so as to make it uniformly bounded. For this to work, one needs fast growth in the sequence T_j:

4.5. **Lemma.** *Let Φ be the function constructed in Lemma 4.4, and define the function $\phi\colon \mathbb{R} \to \mathbb{R}$ by $\phi(x) = \Phi(x, 0)$. If T_j is any sequence of integers such that $T_{j+1} > \rho(T_j)$ for some function $\rho\colon [0, \infty) \to [0, \infty)$ having faster than linear growth (Wolff has $\rho(x) = x(\log(2 + x))^3$, then there exists a sequence of 1-periodic Lipschitz functions L_j on $\mathbb{R}$ such that $0 \le L_j \le 1$, and such that the sequence*

$$\sigma_k(x) = \sum_{j=1}^{k} \frac{1}{j} L_j(x)\phi(T_j x) \tag{40}$$

is uniformly bounded, and diverges a.e. as $k \to \infty$. □

We will skip the proof.

For a continuous and bounded function $f\colon \mathbb{R} \to \mathbb{R}$, denote by $\widehat{f}$ the unique p-harmonic function in $\mathbb{R}^2_+$ with boundary values f on $\mathbb{R}$. It is rather simple to prove that there exists a uniform limit

$$\lim_{y\to\infty} \widehat{f}(x, y) = \mu \in \mathbb{R}$$

for all x; see Wolff's Lemma 1.3.

Now, return to the sequence (40) and for each k, consider the p-harmonic function $\widehat{\sigma}_k$. Eventually, after a clever choice of the sequence T_j, this (uniformly bounded) sequence of p-harmonic functions will converge, as $k \to \infty$, to a p-harmonic limit function that gives the anti-Fatou example. To make this work, a sophisticated comparison principle (Lemma 1.6 in Wolff) is needed. We don't even state the Lemma here due to its messy appearance. Let us just state that it enables us to fix the sequence T_j in a specific way, in order to obtain a decreasing sequence of positive numbers $\beta_k \to 0$, along with estimates of the following form:

$$|\widehat{\sigma}_{k+1}(x,y) - \widehat{\sigma}_k(x,y)| < \frac{1}{2^{k+1}} \qquad \text{when } y > \beta_k \tag{41}$$

$$|\widehat{\sigma}_{k+1}(x,y) - \sigma_k(x)| < \frac{1}{2^k} + \frac{1}{k} \qquad \text{when } y \leq \beta_k. \tag{42}$$

We skip this proof as well.

After these Lemmas, Theorem 4.1 follows: the estimate (41) yields that the sequence $\widehat{\sigma}_k$ converges locally uniformly to a limit function G, that this limit function is p-harmonic in $\mathbb{R}^2_+$, and that

$$|G(x,y) - \widehat{\sigma}_k(x,y)| < \frac{1}{2^k} \qquad \text{when } y > \beta_k.$$

Moreover, by (42),

$$\begin{aligned} |G(x,y) - \sigma_k(x)| &\leq |G(x,y) - \widehat{\sigma}_{k+1}(x,y)| + |\widehat{\sigma}_{k+1}(x,y) - \sigma_k(x)| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} + \frac{1}{k}, \end{aligned}$$

when $\beta_{k+1} < y \leq \beta_k$. Let $k \to \infty$ to obtain the Theorem. □

To prove the second form of the anti-Fatou Theorem, Lemma 4.5 is modified in such a way that the sequence

$$\sigma_k(x) = \sum_{j=1}^{k} \frac{1}{j} L_j(x)\phi(T_j x)$$

is positive for each k, is uniformly bounded, and such that

$$\lim_{k\to\infty} \sigma_k(x) = 0$$

for almost every x. (This is Lemma 2.13 in Wolff.) The rest of the proof remains unchanged.

5. Failure of the Mean Value Property in the Half-Plane

We now set out to prove Lemma 4.4. In this section we give the big picture in the half-plane case of Wolff, detailed proofs for the disc case (applicable also in the half-plane) are given in the next two sections. Throughout, we have $p > 2$.

Start by fixing a p-harmonic function in $\mathbb{R}^2_+$:

5.1. **Lemma.** *The function $f(x,y) = e^{-y}a(x)$ satisfies $\Delta_p f = 0$ in $\mathbb{R}^2_+$ if $a\colon \mathbb{R} \to \mathbb{R}$ satisfies the ordinary differential equation*

$$a_{xx} + V(a, a_x)a = 0, \tag{43}$$

where

$$V(a, a_x) = \frac{(2p-3)a_x^2 + (p-1)a^2}{(p-1)a_x^2 + a^2}.$$

□

Note that for $p = 2$ we have $V \equiv 1$.

It turns out that the equation (43) has a unique solution in $\mathbb{R}$ for given initial data, that a and a_x cannot vanish simultaneously (unless $a \equiv 0$), and that $a \in C^\infty(\mathbb{R})$. Moreover, a turns out to be periodic with period $\lambda = \lambda(p) > 0$, and if we fix the solution with initial data $a(0) = 0$, $a'(0) = 1$, then

$$\int_0^\lambda a(x)dx = 0$$

with

$$\int_0^{\lambda/2} a(x)dx = -\int_{\lambda/2}^\lambda a(x)dx.$$

Moreover,

$$a(x + \lambda/2) = -a(\lambda/2 - x)$$

for all x, so that this fixed solution a behaves much like the sine function.

Fix a as above, and let $f(x,y) = e^{-y}a(x)$. Then f is p-harmonic, but

$$\int_0^\lambda f(x,y)dx = 0$$

for all y, so f as such does not fail the mean value principle. We perturbate f by considering solutions v to the following linear equation:

5.2. **Lemma.** *The formal equation*

$$-\frac{d}{d\varepsilon}|_{\varepsilon=0}\Delta_p(f+\varepsilon v)=0$$

reduces to

$$-\operatorname{div}(A\nabla v)=0, \tag{44}$$

where

$$\begin{aligned} A(x,y) &= |\nabla f|^{p-4}\begin{pmatrix} |\nabla f|^2+(p-2)f_x^2 & (p-2)f_xf_y \\ (p-2)f_xf_y & |\nabla f|^2+(p-2)f_y^2 \end{pmatrix} \\ &= e^{-(p-2)y}(a_x^2+a^2)^{\frac{p-4}{2}}\begin{pmatrix} (p-1)a_x^2+a^2 & (2-p)aa_x \\ (2-p)aa_x & a_x^2+(p-1)a^2 \end{pmatrix}. \end{aligned}$$

□

Note that the equation (44) is linear, degenerate elliptic (the ellipticity degenerates like $e^{-(p-2)y}$ at infinity), and that the partials f_x and f_y of f solve (44).

Now, the crucial step (which takes many pages in the proof and cannot be done if $p=2$) is to solve a Neumann problem for the equation (44), and thereby to obtain a solution $v\in C^\infty(\overline{\mathbb{R}^2_+})\cap W^{1,\infty}(\mathbb{R}^2_+)$ (with period λ in the x variable) such that

$$\frac{\partial v}{\partial \nu}(x,0)=h(x),$$

where

$$\int_0^\lambda h(x)dx=M>0.$$

In other words, by denoting

$$F(y)=\int_0^\lambda v(x,y)dx,$$

we have

$$F'(0)=M>0.$$

This implies the existence of small numbers $y_2>y_1>0$ such that

$$|F(y_2)-F(y_1)|>b>0,$$

i.e.

$$\left|\int_0^\lambda v(x,y_2)-v(x,y_1)dx\right|>b>0. \tag{45}$$

Finally, take a small $\varepsilon > 0$, and consider the function $f+\varepsilon v \in C^\infty(\overline{\mathbb{R}^2_+})$. Take its values on $\mathbb{R}$ (i.e. $y = 0$), and solve the Dirichlet problem to obtain the p-harmonic function $\widehat{f+\varepsilon v}$. Denote the uniform limit at infinity, $\lim_{y\to\infty}(\widehat{f+\varepsilon v})(x,y)$, by μ. A suitable translate to the y direction of the function

$$\widehat{f+\varepsilon v} - \mu$$

now gives what we want, for ε small enough. To see this, denote first

$$q_\varepsilon(x,y) = (\widehat{f+\varepsilon v})(x,y) - (f+\varepsilon v)(x,y).$$

Next, one proves the estimate (Lemma 3.19 in Wolff)

$$\int_{(0,\lambda)\times(0,1)} |\nabla q_\varepsilon|\, dxdy \le C\varepsilon^\tau, \tag{46}$$

where $\tau = \tau(p) > 1$ and $C = C(p, ||\nabla f||_\infty, ||\nabla v||_\infty) \ge 1$.

Now, using the fact that

$$\int_0^\lambda f(x,y)dx = 0$$

for each y, along with (45) and (46), we obtain

$$\begin{aligned}
&|\int_0^\lambda (\widehat{f+\varepsilon v})(x,y_2) - (\widehat{f+\varepsilon v})(x,y_1)dx| \\
&\ge \varepsilon|\int_0^\lambda v(x,y_2) - v(x,y_1)dx| - |\int_0^\lambda q_\varepsilon(x,y_2) - q_\varepsilon(x,y_1)dx| \\
&\ge \varepsilon b - C\varepsilon^\tau.
\end{aligned}$$

Since $\tau > 1$, this is greater than zero for ε small enough. For such an ε, we thus have a gap $\eta > 0$ between

$$\int_0^\lambda (\widehat{f+\varepsilon v})(x,y_2)dx$$

and

$$\int_0^\lambda (\widehat{f+\varepsilon v})(x,y_1)dx.$$

Hence one of the functions

$$\Phi(x,y) = (\widehat{f+\varepsilon v})(x,y+y_1) - \mu$$

or

$$\Phi(x,y) = (\widehat{f+\varepsilon v})(x,y+y_2) - \mu$$

must fail the mean value principle as desired.

Before we move on to the calculations in the disc case, let us repeat once again the problem of proving the actual anti-Fatou Theorem for the disc.

In the half-plane, Wolff needs to construct only one function Φ (of period 1 in the x variable) that fails the mean value principle. Thereafter, the functions $q_j(x) = \phi(T_j x) = \Phi(T_j x, 0)$ are of period $1/T_j$,

$$\int_0^1 q_j(x)dx = \eta \tag{47}$$

with a uniform $\eta \neq 0$, and the functions

$$\widehat{q}_j(x,y) = \widehat{\phi}(T_j x) = \Phi(T_j x, T_j y) \tag{48}$$

all have the property that

$$\lim_{y\to\infty} \widehat{q}_j(x,y) = 0. \tag{49}$$

The property (49) is essential in obtaining the estimates (41) and (42) (see page 32). On the other hand, a uniform $\eta \neq 0$ in (47) is essential for the argument to work[3], since the behavior of $q_j(x)$ has to mimic the behavior of $\eta + R_j(x)$ (see section 4) with $\eta \neq 0$.

In the unit disc $\mathbb{D}$, we currently lose either the uniformity of η in (47) or the uniform limit zero in (49). Indeed, we are able to construct a function Φ that has the properties of lemma 4.4:

5.3. **Lemma.** *If $p > 2$, there exists a function $\Phi \in \mathrm{Lip}(\overline{\mathbb{D}})$ such that $\Delta_p \Phi = 0$ in $\mathbb{D}$, Φ is periodic in the θ variable,*

$$\lim_{r\to 0} \Phi(r,\theta) = 0,$$

and

$$\int_0^{2\pi} \Phi(1,\theta)d\theta = \eta \neq 0.$$

□

But if we now set $q_j(\theta) = \Phi(1, T_j\theta)$, then we do not know of any reason why (49), i.e.

$$\lim_{r\to 0} \widehat{q}_j(r,\theta) = 0,$$

[3]Actually, a sequence like $\eta_j = (\log j)^{-1}$ would work as well.

should hold for all j. Moreover, the scaling in the θ variable does not preserve the p-harmonicity, and we do not know how to express $\widehat{q_j}(r,\theta)$ in terms of the original function Φ.

We are, however, able to prove Lemma 1.6, let us restate it here.

5.4. **Lemma.** *If $p > 2$, there exists a sequence of functions $\Phi_j \in \mathrm{Lip}(\overline{\mathbb{D}})$ such that $||\Phi_j||_{L^\infty(\mathbb{D})} \leq C < \infty$ for all j, $\Delta_p \Phi_j = 0$ in $\mathbb{D}$, Φ_j has period $\lambda_j > 0$ (dividing 2π) in the θ variable, $\lambda_j \to 0$ as $j \to \infty$,*

$$\lim_{r \to 0} \Phi_j(r,\theta) = 0,$$

and

$$\int_0^{2\pi} \Phi_j(1,\theta) d\theta = \eta_j > 0.$$

□

Then the behavior of $q_j(\theta) = \Phi_j(1,\theta)$ is like that of $\eta_j + R_j(\theta)$, but this time we do not know how to estimate the sequence η_j.

In the rest of the text, we will prove Lemma 5.4 in detail.

6. Failure of the Mean Value Property in the Disc

In the unit disc $\mathbb{D}$, the big picture is the same as in the half-plane. However, using polar coordinates yields messy terms in the matrix A of the linearized equation, and solving the Neumann problem seems to be difficult. Luckily, the calculations are much more elegant in the moving frame

$$e_r = \frac{\partial}{\partial r}, \qquad e_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}.$$

The intrinsic gradient of a function f is defined as $\mathfrak{X}f = (e_r(f), e_\theta(f))$. Note that $\mathfrak{X}f = R(\theta)\nabla f$, where $R(\theta)$ is the rotation by θ and ∇f is the gradient in cartesian coordinates.

The dual basis to $\{e_r, e_\theta\}$ is $\{dr, rd\theta\}$ and the volume element is $dA = rdrd\theta$. The adjoints e_r^* and e_θ^* of e_r and e_θ are defined as

$$\int_{\mathbb{D}} e_r(u)v\, dA = \int_{\mathbb{D}} ue_r^*(v)\, dA \quad \text{ for all } u, v \in C_0^\infty(\mathbb{D})$$

and similarly for e_θ. They turn out to be

$$e_r^* = -\frac{1}{r}e_r(r\cdot), \qquad e_\theta^* = -e_\theta.$$

The divergence of a vector field $F = (F^1, F^2)$ is defined as

$$\operatorname{div}_{\mathfrak{X}} F = -\big(e_r^*(F^1) + e_\theta^*(F^2)\big) = \frac{1}{r}e_r(rF^1) + e_\theta(F^2),$$

and one verifies that $\Delta_p u = 0$ in $\mathbb{D}$ is equivalent to

$$\operatorname{div}_{\mathfrak{X}}(|\mathfrak{X}u|^{p-2}\mathfrak{X}u) = 0 \quad \text{ in } \mathbb{D}.$$

6.1. **Lemma.** *Let $k \in \mathbb{R}$, $k \geq 1$. The function $f_k(r,\theta) = r^k a_k(\theta)$ satisfies $\Delta_p f_k = 0$ in $\mathbb{D}$, if $a_k\colon \mathbb{R} \to \mathbb{R}$ is 2π-periodic and satisfies*[4]

$$a_{\theta\theta} + V(a, a_\theta)a = 0, \tag{50}$$

where

$$V(a, a_\theta) = \frac{\big((2p-3)k^2 - (p-2)k\big)a_\theta^2 + \big((p-1)k^2 - (p-2)k\big)k^2a^2}{(p-1)a_\theta^2 + k^2a^2}.$$

This is the same separation equation that is obtained in [2] using polar coordinates.

[4] The index k is henceforth dropped from a and f. Also, a has to be 2π-periodic for the statement to make sense; the periodicity of a is not proved until in Lemma 6.3.

Proof. We have

$$e_r(f) = kr^{k-1}a, \qquad e_\theta(f) = r^{k-1}a_\theta,$$

so that

$$|\mathfrak{X}f| = r^{k-1}(a_\theta^2 + k^2a^2)^{\frac{1}{2}}.$$

Thus we obtain

$$\begin{aligned}
|\mathfrak{X}f|^{p-2} &= r^{(k-1)(p-2)}(a_\theta^2 + k^2a^2)^{\frac{p-2}{2}},\\
|\mathfrak{X}f|^{p-2}e_r(f) &= r^{(k-1)(p-1)}(a_\theta^2 + k^2a^2)^{\frac{p-2}{2}}ka,\\
|\mathfrak{X}f|^{p-2}e_\theta(f) &= r^{(k-1)(p-1)}(a_\theta^2 + k^2a^2)^{\frac{p-2}{2}}a_\theta,
\end{aligned}$$

and further, denoting $\gamma = (p-1)k^2 - (p-2)k$,

$$\begin{aligned}
&\operatorname{div}_{\mathfrak{X}}(|\mathfrak{X}f|^{p-2}\mathfrak{X}f) = \frac{1}{r}e_r\left(r|\mathfrak{X}f|^{p-2}e_r(f)\right) + e_\theta\left(|\mathfrak{X}f|^{p-2}e_\theta(f)\right)\\
&= r^{(k-1)(p-1)-1}\left\{\gamma(a_\theta^2 + k^2a^2)^{\frac{p-2}{2}}a + \frac{\partial}{\partial\theta}\left((a_\theta^2 + k^2a^2)^{\frac{p-2}{2}}a_\theta\right)\right\},
\end{aligned}$$

where

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\left((a_\theta^2 + k^2a^2)^{\frac{p-2}{2}}a_\theta\right)\\
&= (a_\theta^2 + k^2a^2)^{\frac{p-4}{2}}\left((a_\theta^2 + k^2a^2)a_{\theta\theta} + (p-2)a_\theta^2(a_{\theta\theta} + k^2a)\right).
\end{aligned}$$

Hereby

$$\operatorname{div}_{\mathfrak{X}}(|\mathfrak{X}f|^{p-2}\mathfrak{X}f) = 0$$

if a verifies

$$\left((a_\theta^2 + k^2a^2) + (p-2)a_\theta^2\right)a_{\theta\theta} + \left(\gamma(a_\theta^2 + k^2a^2) + (p-2)k^2a_\theta^2)\right)a = 0.$$

The lemma follows. $\square$

6.2. **Lemma.** *The equation (50) has a unique solution $a \in C^\infty(\mathbb{R})$ with given initial data $a(0), a_\theta(0)$.*

Proof. Denote

$$\begin{aligned}
\gamma &= (p-1)k^2 - (p-2)k,\\
\beta &= (2p-3)k^2 - (p-2)k,
\end{aligned}$$

so that

$$|V| \le \frac{\max\{\gamma, \beta\}}{\min\{p-1, 1\}} \le C(p,k). \tag{51}$$

Written as a system, (50) reads

$$(a, a_\theta)_\theta = \overline{f}(a, a_\theta), \tag{52}$$

where $\overline{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the function

$$\overline{f}(x,y) = (y, -V(x,y)x) \quad \text{for } (x,y) \neq (0,0),$$

and, by (51), we may define $\overline{f}(0,0) = 0$. To obtain the existence and uniqueness of a local solution to (50), it suffices to observe that the function $\overline{f}$ is 1-homogeneous and thereby Lipschitz.

Multiplying (50) by a_θ, we obtain

$$|(a_\theta^2)_\theta| \leq C_{p,k}|(a^2)_\theta|. \tag{53}$$

taking antiderivatives and square roots of both sides of (53) implies that the logarithmic derivative of a remains bounded. Thus $|a|$ remains bounded, except perhaps when $\theta \to \infty$. It follows from (50) that $|a_{\theta\theta}|$ is similarly bounded, and thereby the same must hold also for $|a_\theta|$. Since both $|a|$ and $|a_\theta|$ are bounded like this, the existence and uniqueness of solutions follows for $-\infty < \theta < \infty$.

Since the system is C^∞ except at points where $a = a_\theta = 0$, also the solutions are C^∞ except at such points. But such points are ruled out (for nonconstant a) by uniqueness. □

6.3. **Lemma.** *There exists an increasing sequence $\{k_j\}_{j=1}^\infty$ of positive real numbers so that the following holds:*

For each k in the sequence, let $a = a_k$ be the corresponding solution to (50) (i.e. a is such that $r^k a(\theta)$ is p-harmonic in $\mathbb{D}$) with $a(0) = 0$ and $a_\theta(0) = 1$. Then a has a period $\lambda = \lambda(p,k) > 0$ dividing 2π such that $\lambda \to 0$ as $k \to \infty$. Moreover, each such a satisfies

$$a(\theta + \frac{\lambda}{2}) = -a(\frac{\lambda}{2} - \theta) \quad \textit{for all } \theta,$$

in particular

$$\int_0^{2\pi} a(\theta)d\theta = 0.$$

Proof. We know that a is positive in $\{\theta\colon 0 < \theta < \varepsilon\}$ for some $\varepsilon > 0$. We claim that a does not remain positive forever, i.e. that $a(\theta_0) = 0$ for some $\theta_0 > 0$. Assume, on the contrary, that $a > 0$ in $\mathbb{R}_+$. Since (for $p > 2$)

$$V \geq \frac{\min\{\gamma, \beta\}}{\max\{p-1, 1\}} = k^2 - \frac{p-2}{p-1}k = \tau > 0,$$

a satisfies

$$a_{\theta\theta} + \tau a \leq 0 \quad \text{in } \mathbb{R}_+.$$

Let b be the solution of

$$b_{\theta\theta} + \tau b = 0 \quad \text{in } \mathbb{R}_+$$

with $b(0) = 0$ and $b_\theta(0) = 1$, i.e. let

$$b(\theta) = \frac{\sin(\sqrt{\tau}\theta)}{\sqrt{\tau}},$$

and consider the function $c = a - b$. It satisfies

$$c_{\theta\theta} + \tau c = f \leq 0 \quad \text{in } \mathbb{R}_+$$

with $c(0) = 0$, $c_\theta(0) = 0$. We have

$$c(\theta) = \int_0^\theta f(t-\theta)\frac{\sin(\sqrt{\tau}t)}{\sqrt{\tau}}dt,$$

so that $c \leq 0$ for $0 \leq \theta \leq \frac{\pi}{\sqrt{\tau}}$, i.e. $a \leq b$ for $0 \leq \theta \leq \frac{\pi}{\sqrt{\tau}}$. In particular,

$$a(\frac{\pi}{\sqrt{\tau}}) \leq 0,$$

contrary to our assumption of positivity. Thus there must exist a number $\theta_0 > 0$ such that $a(\theta_0) = 0$.

Next, let

$$a(\theta + \theta_0) = g(\theta),$$
$$-a(\theta_0 - \theta) = h(\theta).$$

We have

$$g(0) = h(0) = 0,$$
$$g_\theta(0) = h_\theta(0) = a_\theta(\theta_0).$$

Hence, by the uniqueness of solutions,

$$a(\theta + \theta_0) = -a(\theta_0 - \theta)$$

for every θ. With $\lambda = 2\theta_0$, the Lemma follows, except that λ must divide 2π, and therefore most values of k are ruled out. Conversely, given $j \in \mathbb{N}$, we can fix $k = k(p,j) > 0$ (not necessarily an integer) such that the solution a_k has period $\lambda = 2\pi/j$. An explicit formula for k is calculated in [36]; there it is shown that

$$k(p,j) \sim \frac{p}{2(p-1)}j + \frac{p^2-4}{4p(p-1)} + O\left(\frac{1}{j}\right).$$

We refer the reader to [2] and [36] for more details. $\square$

From now on we assume that k belongs to the sequence $\{k_j\}_{j=1}^{\infty}$ in Lemma 6.3.

6.4. **Lemma.** *Let k be fixed, let a be the corresponding function as in Lemma 6.3, and let $f(r,\theta) = r^k a(\theta)$. Then*

$$-\frac{d}{d\varepsilon}|_{\varepsilon=0}\Delta_p(f+\varepsilon v) = 0$$

reduces to

$$-\operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}v) = 0,$$

where

(54)

$$\begin{aligned} &A(r,\theta) \\ &= r^{(p-2)(k-1)}(a_\theta^2 + k^2a^2)^{\frac{p-4}{2}} \begin{pmatrix} a_\theta^2 + (p-1)k^2a^2 & (p-2)kaa_\theta \\ (p-2)kaa_\theta & k^2a^2 + (p-1)a_\theta^2 \end{pmatrix}. \end{aligned}$$

Proof. Since

$$\begin{aligned} &\frac{d}{d\varepsilon}\left(|\mathfrak{X}f + \varepsilon\mathfrak{X}v|^{p-2}(\mathfrak{X}f + \varepsilon\mathfrak{X}v)\right) \\ &= |\mathfrak{X}f + \varepsilon\mathfrak{X}v|^{p-2}\mathfrak{X}v \\ &+ (p-2)(\mathfrak{X}f\cdot\mathfrak{X}v + \varepsilon|\mathfrak{X}v|^2)|\mathfrak{X}f + \varepsilon\mathfrak{X}v|^{p-4}(\mathfrak{X}f + \varepsilon\mathfrak{X}v), \end{aligned}$$

the expression

$$-\frac{d}{d\varepsilon}|_{\varepsilon=0}\operatorname{div}_{\mathfrak{X}}\left(|\mathfrak{X}f + \varepsilon\mathfrak{X}v|^{p-2}(\mathfrak{X}f + \varepsilon\mathfrak{X}v)\right)$$

becomes

$$\begin{aligned} &-\operatorname{div}_{\mathfrak{X}}\left(|\mathfrak{X}f|^{p-2}\mathfrak{X}v + (p-2)|\mathfrak{X}f|^{p-4}(\mathfrak{X}f\cdot\mathfrak{X}v)\mathfrak{X}f\right) \\ &= -\operatorname{div}_{\mathfrak{X}}\left(|\mathfrak{X}f|^{p-2}\mathfrak{X}v + (p-2)|\mathfrak{X}f|^{p-4}(\mathfrak{X}f\otimes\mathfrak{X}f)\mathfrak{X}v\right) \\ &= -\operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}v), \end{aligned}$$

where

$$A = |\mathfrak{X}f|^{p-4}\left(|\mathfrak{X}f|^2 I + (p-2)(\mathfrak{X}f\otimes\mathfrak{X}f)\right).$$

Inserting

$$|\mathfrak{X}f| = r^{k-1}(a_\theta^2 + k^2a^2)^{\frac{1}{2}}$$

yields (54). □

6.5. **Lemma.** *Let A be as in (54). Then*

$$r^{(p-2)(k-1)}(a_\theta^2+k^2a^2)^{\frac{p-2}{2}}|\xi|^2 \le A\xi\cdot\xi \le (p-1)r^{(p-2)(k-1)}(a_\theta^2+k^2a^2)^{\frac{p-2}{2}}|\xi|^2$$

for all $\xi \in \mathbb{R}^2$.

Proof. The eigenvalues λ of the matrix

$$B = \begin{pmatrix} a_\theta^2 + (p-1)k^2a^2 & (p-2)kaa_\theta \\ (p-2)kaa_\theta & k^2a^2 + (p-1)a_\theta^2 \end{pmatrix}$$

are

$$\begin{aligned}\lambda &= \frac{\operatorname{tr} B \pm \sqrt{(\operatorname{tr} B)^2 - 4\det B}}{2} \\ &= \frac{pa_\theta^2 + pk^2a^2 \pm \sqrt{p^2(a_\theta^2 + k^2a^2)^2 - 4\det B}}{2}.\end{aligned}$$

Since

$$\det B = (p-1)(a_\theta^2 + k^2a^2)^2,$$

we obtain

$$\lambda = \frac{(p \pm (p-2))\,(a_\theta^2 + k^2a^2)}{2}.$$

The claim follows. □

Since a_θ and a never vanish simultaneously, A is elliptic and degenerates like $r^{(p-2)(k-1)}$ at the origin. We will look for solutions v to the equation

$$Tv = -\operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}v) = 0 \quad \text{ in } \mathbb{D}$$

in the weighted Sobolev space

$$Y_1 = \left\{ v \in W^{1,2}_{\mathrm{loc}}(\mathbb{D}^*) : \int_{\mathbb{D}} |v|^2 r^{2\beta} + |\nabla v|^2 r^{2\alpha}\, dA < \infty \right\},$$

where $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, $\alpha = (p-2)(k-1)/2$ and $\beta > \alpha$.

6.6. **Lemma.** *Let k be fixed, let A be as in (54), and let $M > 0$. There exists a solution $v \in Y_1 \cap C^\infty(\overline{\mathbb{D}} \setminus \{0\}) \cap W^{1,\infty}(\mathbb{D})$ to*

$$Tv = -\operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}v) = 0 \quad \textit{in } \mathbb{D}$$

such that

$$\int_0^{2\pi} \frac{dv}{d\nu}(1,\theta)d\theta = M.$$

Proof. The conormal on $\partial\mathbb{D}$ with respect to T is

$$\nu^*(\theta) = A^t(1,\theta)\begin{pmatrix}1\\0\end{pmatrix} = (a^2 + k^2a_\theta^2)^{\frac{p-4}{2}} \begin{pmatrix} a_\theta^2 + (p-1)k^2a^2 \\ (p-2)kaa_\theta \end{pmatrix}.$$

With

$$\omega = (a_\theta^2 + k^2a^2)^{\frac{p-4}{2}},$$

we have

$$\nu^* = \omega\left(a_\theta^2 + (p-1)k^2a^2\right)\begin{pmatrix}1\\0\end{pmatrix} + \omega(p-2)kaa_\theta\begin{pmatrix}0\\1\end{pmatrix},$$

so that

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\omega(a_\theta^2+(p-1)k^2a^2)}\left(\nu^* - (p-2)\omega k a a_\theta \begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= q\left(\nu^* + \tau\begin{pmatrix}0\\1\end{pmatrix}\right),$$

where

$$\begin{aligned} q(\theta) &= \frac{(a_\theta^2+k^2a^2)^{\frac{4-p}{2}}}{a_\theta^2+(p-1)k^2a^2}, \\ \tau(\theta) &= -(a_\theta^2+k^2a^2)^{\frac{p-4}{2}}(p-2)kaa_\theta. \end{aligned} \tag{55}$$

It follows that

$$\frac{\partial}{\partial\nu} = q(\frac{\partial}{\partial\nu^*} + \tau\frac{\partial}{\partial\theta}), \tag{56}$$

and in particular that the equation

$$\frac{\partial v}{\partial\nu} = \psi$$

is equivalent to the equation

$$\frac{\partial v}{\partial\nu^*} + \tau\frac{\partial v}{\partial\theta} = \frac{\psi}{q}. \tag{57}$$

Thus we want to solve the oblique derivative problem

$$\begin{cases} Tv & = 0 \qquad \text{in } \mathbb{D} \\ \frac{\partial v}{\partial\nu^*} + \tau\frac{\partial v}{\partial\theta} & = \frac{\psi}{q} \qquad \text{on } \partial\mathbb{D}, \end{cases} \tag{58}$$

with the additional constraint

$$\int_0^{2\pi}\psi(\theta)d\theta = M.$$

We postpone the proof of the following Lemma to the next section:

6.7. **Lemma.** *The oblique derivative problem (58) has a solution* $v \in Y_1 \cap C^\infty(\overline{\mathbb{D}}\setminus\{0\}) \cap W^{1,\infty}(\mathbb{D})$ *if*

$$\int_0^{2\pi}\psi(\theta)q(\theta)^{-1}g(\theta)\,d\theta = 0 \qquad \text{for all } g\in E,$$

where E *is the finite-dimensional set consisting of all the boundary values* $g(\theta) = F(1,\theta)$ *of solutions* $F \in Y_1 \cap C^\infty(\overline{\mathbb{D}}\setminus\{0\})$ *to the homogeneous adjoint problem*

$$\begin{cases} TF & = 0 \quad \text{in } \mathbb{D} \\ \frac{\partial F}{\partial\nu^*} - \frac{\partial}{\partial\theta}(\tau F) & = 0 \quad \text{on } \partial\mathbb{D}. \end{cases} \tag{59}$$

□

Hence what we need is

$$\text{(60)} \qquad \int_0^{2\pi} \frac{\psi}{q}\, g\, d\theta = 0 \quad \text{for all } g \in E, \quad \text{and} \quad \int_0^{2\pi} \psi\, d\theta = M.$$

Writing $\varphi = \psi/q$ and using the bracket notation, (60) reads

$$\langle \varphi, g\rangle = 0 \quad \text{for all } g \in E \quad \text{and} \quad \langle \varphi, q\rangle = M.$$

A necessary condition clearly is $q \notin E$, but it is also sufficient: if $q \notin E$, we write $q = q_E + q_\perp$, where $q_E \in E$ and $q_\perp \perp E$. Then

$$\langle \varphi, q\rangle = \langle \varphi, q_\perp\rangle,$$

so if $q \notin E$, we can pick any function ψ such that $\varphi = \frac{\psi}{q} \notin E$ in order to have $\langle \varphi, q\rangle \neq 0$, and then multiply by a constant to obtain $\langle \varphi, q\rangle = M$.

In order to finish the proof, we still need to show that $q \notin E$. Suppose, on the contrary, that $q \in E$, i.e. $q(\theta) = F(1, \theta)$ for some solution $F \in Y_1 \cap C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ to

$$\begin{cases} TF & = 0 & \text{in } \mathbb{D} \\ \frac{\partial F}{\partial \nu^*}(1,\theta) & = \frac{d}{d\theta}\left(\tau(\theta) q(\theta)\right) & \text{on } \partial\mathbb{D}. \end{cases}$$

Let θ_0 be a global minimum point of q. (Such a point exists since $q \in C^\infty(\partial\mathbb{D})$.) By the maximum principle (see Lemma 7.6 on page 60), $(1, \theta_0)$ is a minimum point of F on $\overline{\mathbb{D}}$. By (56),

$$\frac{\partial F}{\partial \nu^*}(1,\theta) = \frac{1}{q(\theta)} \frac{\partial F}{\partial \nu}(1,\theta) - \tau(\theta) \frac{\partial F}{\partial \theta}(1,\theta).$$

At the minimum point $(1, \theta_0)$, the last term on the right hand side equals zero, and the outer normal derivative of F has to be nonpositive. Since $q > 0$, we obtain

$$\frac{\partial F}{\partial \nu^*}(1, \theta_0) \leq 0,$$

i.e. by (59),

$$\frac{d}{d\theta}|_{\theta=\theta_0}\big(\tau(\theta) q(\theta)\big) \leq 0.$$

But this is impossible by the following very important Lemma. $\square$

6.8. **Lemma.** *Let $k \geq 2$, let q and τ be as in (55), and let θ_0 be a minimum point of q. Then*

$$(\tau q)_\theta(\theta_0) > 0.$$

Proof. Recall that

$$q = \left(a_\theta^2 + k^2a^2\right)^{\frac{4-p}{2}} \left(a_\theta^2 + (p-1)k^2a^2\right)^{-1},$$
$$\tau q = -\left(a_\theta^2 + (p-1)k^2a^2\right)^{-1}(p-2)kaa_\theta,$$

and that a satisfies $a_{\theta\theta} = -Va$, where

$$V = \frac{\big((2p-3)k^2 - (p-2)k\big)a_\theta^2 + \big((p-1)k^2 - (p-2)k\big)k^2a^2}{(p-1)a_\theta^2 + k^2a^2}.$$

We start with the simpler case $p = 4$, where

$$q = (a_\theta^2 + 3k^2a^2)^{-1},$$
$$\tau q = -(a_\theta^2 + 3k^2a^2)^{-1}2kaa_\theta,$$
$$V = \frac{(5k^2 - 2k)a_\theta^2 + (3k^2 - 2k)k^2a^2}{3a_\theta^2 + k^2a^2}.$$

Now

$$q_\theta = -(a_\theta^2 + 3k^2a^2)^{-2}2(a_\theta a_{\theta\theta} + 3k^2aa_\theta)$$
$$= -2aa_\theta(a_\theta^2 + 3k^2a^2)^{-2}(3k^2 - V),$$

which is zero only when $a = 0$ or $a_\theta = 0$ or $V = 3k^2$. The last alternative leads to

$$a_\theta^2(4k^2 + 2k) = k^2a^2(-2k),$$

which is impossible. Next, the case $a = 0$ turns out to be a local maximum for q, since denoting $A = 2(a_\theta^2 + 3k^2a^2)^{-2}$ and differentiating q_θ yields

$$q_{\theta\theta} = \big(-A(3k^2 - V)\big)_\theta aa_\theta - A(3k^2 - V)(aa_\theta)_\theta,$$

which, when $a = 0$, equals

$$-a_\theta^2A(3k^2 - V),$$

and thus has the same sign as $V - 3k^2$. But when $a = 0$, $V = (5k^2 - 2k)/3$, and $3k^2 - V > 0$. Hence $q_{\theta\theta} < 0$ when $a = 0$, so the local maximum of q must occur when $a_\theta = 0$. At such a point,

$$(\tau q)_\theta = 2k(3k^2)^{-2}V,$$

i.e. the sign of $(\tau q)_\theta$ equals the sign of V, which is always positive.

Now consider the case $p \neq 4$. We calculate

$$(\tau q)_\theta = (p-2)\left(a_\theta^2 + (p-1)k^2a^2\right)^{-2}\left((p-1)k^2a^2 - a_\theta^2\right)\left(Va^2 + a_\theta^2\right),$$

which is positive when $a_\theta = 0$ or when $a_\theta^2 < (p-1)k^2a^2$. Also,

$$q_\theta = (a_\theta^2 + k^2a^2)^{\frac{2-p}{2}} \left(a_\theta^2 + (p-1)k^2a^2\right)^{-2} (2-p)aa_\theta J, \tag{61}$$

where

$$J = (3k^2 - V)a_\theta^2 + \big((p-1)k^2 - (p-3)V\big)\, k^2a^2. \tag{62}$$

Thus the extremal points of q are the points where $a = 0$ or $a_\theta = 0$ or $J = 0$. Rewriting (61) as

$$q_\theta = AB(2-p)aa_\theta J,$$

and differentiating, yields

$$q_{\theta\theta} = (2-p)\big(aa_\theta(ABJ)_\theta + (aa_\theta)_\theta ABJ\big).$$

Whenever $a = 0$ or $a_\theta = 0$,

$$q_{\theta\theta} = (2-p)(a_\theta^2 - Va^2)ABJ,$$

and we conclude: $q_{\theta\theta}$ has the sign of $-J$ when $a = 0$, and $q_{\theta\theta}$ has the sign of $+J$ when $a_\theta = 0$.

Next, inserting the formula for V in (62), expanding, and disregarding positive terms, yields that J has the same sign as

$$\begin{aligned} &\big((k+1)p-2\big)a_\theta^2 + \big((-k+1)p^2 + (5k-5)p - 4k + 6\big)k^2a^2 \\ =&\big((k+1)p-2\big)a_\theta^2 + \big((-k+1)(p-p_1)(p-p_2)\big)k^2a^2, \end{aligned} \tag{63}$$

where

$$p_{1,2} = \frac{5}{2} \pm \frac{1}{2}\sqrt{\frac{9k-1}{k-1}}$$

with the convention $p_1 < p_2$. Further, factoring out $k-1$ in (63) yields that J has the same sign as

$$\left(\frac{k+1}{k-1}p - \frac{2}{k-1}\right) a_\theta^2 - (p-p_1)(p-p_2)k^2a^2.$$

We note that the coefficient of a_θ^2 is positive and that $p_1 < 1$. Thus the sign of J is positive whenever $p < p_2$. In this case we observe that the local minimum of q occurs when $a_\theta = 0$, and we have $(\tau q)_\theta > 0$ as desired.

The remaining case to check is $p \geq p_2$ and $J \leq 0$, where $J \leq 0$ reads

$$\big((k+1)p-2\big)a_\theta^2 \leq \big((k-1)p^2 + (-5k+5)p + (4k-6)\big)k^2a^2,$$

i.e.

$$a_\theta^2 \leq \frac{(k-1)p^2 + (-5k+5)p + (4k-6)}{(k+1)p-2}k^2a^2.$$

Denote the fraction on the right hand side by F. It suffices to show (for $k \geq 2$) that $F < p-1$ when $p \geq p_2$, since $a_\theta^2 < (p-1)k^2a^2$ yielded $(\tau q)_\theta > 0$ as desired. Now $F < p-1$ precisely when

$$p^2 + (2k-4)p - 2k + 4 > 0,$$

in particular whenever

$$p > \sqrt{k^2 - 2k} - k + 2.$$

But for $k \geq 2$ we have $\sqrt{k^2-2k} - k + 2 < 4$, while $p \geq p_2 > 4$. This completes the proof. □

With our solution v as in Lemma 6.6, we obtain a number $b > 0$ and numbers $r_2, r_1 < 1$, all depending on k and p, such that

$$|\int_0^{2\pi} v(r_2,\theta) - v(r_1,\theta)\, d\theta| > b > 0,$$

as described in the previous section, see page 34.

At this point, pick $\varepsilon > 0$ and denote

$$q_\varepsilon(r,\theta) = (\widehat{f+\varepsilon v})(r,\theta) - (f+\varepsilon v)(r,\theta), \tag{64}$$

where $(\widehat{f+\varepsilon v})$ denotes the unique p-harmonic function in $\mathbb{D}$ with

$$(\widehat{f+\varepsilon v}) - (f+\varepsilon v) \in W_0^{1,p}(\mathbb{D}).$$

We need the following estimate which employs only the facts that $\Delta_p f = 0$ and $Tv = 0$:

6.9. **Lemma.**

$$\int_{\mathbb{D}\cap\{r>\frac{1}{2}\}} |\nabla q_\varepsilon| dA \leq C\varepsilon^\tau, \tag{65}$$

where $\tau > 1$ *and* $C = C(p, ||\nabla f||_{L^\infty(\mathbb{D})}, ||\nabla v||_{L^\infty(\mathbb{D})}) > 0$.

Proof. Let $\varphi \in W_0^{1,p}(\mathbb{D})$, and estimate the integral

$$\begin{aligned}\Big|\int_{\mathbb{D}} &|\mathfrak{X}(f+\varepsilon v)|^{p-2}\mathfrak{X}(f+\varepsilon v)\cdot\mathfrak{X}\varphi - |\mathfrak{X}f|^{p-2}\mathfrak{X}f\cdot\mathfrak{X}\varphi \\ &-\varepsilon\frac{d}{dt}|_{t=0}\left(|\mathfrak{X}(f+tv)|^{p-2}\mathfrak{X}(f+tv)\cdot\mathfrak{X}\varphi\right) dA\Big|\end{aligned} \tag{66}$$

using the inequalities

$$\left||x+\varepsilon y|^{p-2}(x+\varepsilon y) - |x|^{p-2}x - \varepsilon\frac{d}{dt}|_{t=0}\left(|x+ty|^{p-2}(x+ty)\right)\right|$$

$$\leq \begin{cases} C|\varepsilon y|^{p-1} & \text{for } 2 < p \leq 3, \\ C|\varepsilon y|^2(|\varepsilon y|^{p-3} + |x|^{p-3}) & \text{for } p > 3, \end{cases}$$

valid for vectors $x, y \in \mathbb{R}^2$. Since the second and third term in (66) are zero, we obtain

$$\left| \int_{\mathbb{D}} |\mathfrak{X}(f+\varepsilon v)|^{p-2} \mathfrak{X}(f+\varepsilon v) \cdot \mathfrak{X}\varphi \, dA \right|$$

$$\leq \begin{cases} C\varepsilon^{p-1} \int_{\mathbb{D}} |\nabla v|^{p-1} |\nabla \varphi| \, dA & (2 < p \leq 3), \\ C\varepsilon^2 \int_{\mathbb{D}} |\nabla v|^2 |\nabla \varphi| (\varepsilon^{p-3} |\nabla v|^{p-3} + |\nabla f|^{p-3}) \, dA & (p > 3). \end{cases}$$

Since $|\nabla v|$ and $|\nabla f|$ are in L^∞, we have

$$\left| \int_{\mathbb{D}} |\mathfrak{X}(f+\varepsilon v)|^{p-2} \mathfrak{X}(f+\varepsilon v) \cdot \mathfrak{X}\varphi \, dA \right| \leq C\varepsilon^\sigma ||\varphi||_{1,p}, \tag{67}$$

where $\sigma = \min\{2, p-1\} > 1$ and $C = C(p, ||\nabla f||_\infty, ||\nabla v||_\infty)$.

Next, use $\varphi = q_\varepsilon$ in (67) together with Young's inequality to obtain

$$\begin{aligned} &\int_{\mathbb{D}} |\nabla(f+\varepsilon v)|^p \, dA \\ &= \int_{\mathbb{D}} |\mathfrak{X}(f+\varepsilon v)|^{p-2} \mathfrak{X}(f+\varepsilon v) \cdot \mathfrak{X}(\widehat{f+\varepsilon v}) \, dA \\ &\qquad + \int_{\mathbb{D}} |\mathfrak{X}(f+\varepsilon v)|^{p-2} \mathfrak{X}(f+\varepsilon v) \cdot \mathfrak{X} q_\varepsilon \, dA \\ &\leq \frac{1}{p} ||(\widehat{f+\varepsilon v})||_{1,p}^p + \frac{p-1}{p} ||f+\varepsilon v||_{1,p}^p + C\varepsilon^\sigma ||q_\varepsilon||_{1,p}. \end{aligned} \tag{68}$$

Finally, use the inequality

$$|x+y|^p - |x|^p - p|x|^{p-2} x \cdot y \geq C_p |y|^2 (|y| + |x+y|)^{p-2},$$

valid for $x, y \in \mathbb{R}^2$ and $p \geq 2$. With $x = \nabla(\widehat{f+\varepsilon v})$ and $y = \nabla q_\varepsilon$, this becomes

$$\begin{aligned} &\int_{\mathbb{D}} |\nabla(f+\varepsilon v)|^p \, dA - \int_{\mathbb{D}} |\nabla(\widehat{f+\varepsilon v})|^p \, dA \\ &\geq C \int_{\mathbb{D}} |\nabla q_\varepsilon|^2 \left(|\nabla q_\varepsilon| + |\nabla(f+\varepsilon v)| \right)^{p-2} \, dA \\ &\geq C \left(\int_{\mathbb{D}} |\nabla q_\varepsilon|^p \, dA + \int_{\mathbb{D}} |\nabla q_\varepsilon|^2 |\nabla(f+\varepsilon v)|^{p-2} \, dA \right), \end{aligned}$$

since $\Delta_p(\widehat{f+\varepsilon v}) = 0$. Combine this with (68) to obtain

$$\int_{\mathbb{D}} |\nabla q_\varepsilon|^p \, dA + \int_{\mathbb{D}} |\nabla q_\varepsilon|^2 |\nabla(f+\varepsilon v)|^{p-2} \, dA \leq C\varepsilon^\sigma ||q_\varepsilon||_{1,p}, \tag{69}$$

in particular

$$\int_{\mathbb{D}} |\nabla q_\varepsilon|^p \, dA \leq C\varepsilon^\sigma ||q_\varepsilon||_{1,p},$$

i.e. (since $q_\varepsilon \in W_0^{1,p}(\mathbb{D})$)

$$||q_\varepsilon||_{1,p} \leq C\varepsilon^{\frac{\sigma}{p-1}}.$$

Inserting this in the right hand side of (69) yields

$$\int_{\mathbb{D}} |\nabla q_\varepsilon|^2 |\nabla(f+\varepsilon v)|^{p-2} \, dA \leq C\varepsilon^{\frac{\sigma p}{p-1}}.$$

Since $|\nabla f| > \rho > 0$ when $r > \frac{1}{2}$ and since $|\nabla v|$ is bounded from above,

$$\int_{\mathbb{D}\cap\{r>\frac{1}{2}\}} |\nabla q_\varepsilon|^2 \, dA \leq C\varepsilon^{\frac{\sigma p}{p-1}},$$

and Schwarz's inequality yields the lemma with

$$\tau = \frac{\sigma p}{2(p-1)} > 1.$$

□

The rest is similar to the half-plane case. By the previous estimate and the fact that

$$\int_0^{2\pi} f(r,\theta)\, d\theta = 0$$

for each r, we have

$$\begin{aligned}
&\left| \int_0^{2\pi} (\widehat{f+\varepsilon v})(r_2,\theta) - (\widehat{f+\varepsilon v})(r_1,\theta) d\theta \right| \\
&\geq \varepsilon \left| \int_0^{2\pi} v(r_2,\theta) - v(r_1,\theta) d\theta \right| - \left| \int_0^{2\pi} q(r_2,\theta) - q(r_1,\theta) d\theta \right| \\
&\geq \varepsilon b - C\varepsilon^\tau.
\end{aligned}$$

Since $\tau > 1$, this is greater than zero for ε small enough. For such an ε, we have a gap[5] between

$$\int_0^{2\pi} (\widehat{f+\varepsilon v})(r_2,\theta)\, d\theta$$

and

$$\int_0^{2\pi} (\widehat{f+\varepsilon v})(r_1,\theta)\, d\theta.$$

Hence one of the functions

$$\Phi(r,\theta) = (\widehat{f+\varepsilon v})(r_2 r,\theta) - (\widehat{f+\varepsilon v})(0)$$

[5]This gap is the η_k that we would like to estimate as $k \to \infty$.

or

$$\Phi(r,\theta) = (\widehat{f+\varepsilon v})(r_1 r,\theta) - (\widehat{f+\varepsilon v})(0)$$

must fail the mean value principle as required.

7. The Oblique Derivative Problem

In this section we prove Lemma 6.7. Let us start by restating the ingredients. Our function spaces are the weighted L^2 and $W^{1,2}$ spaces, denoted by Y_0 and Y_1, respectively, where $\alpha = (p-2)(k-1)/2$ and $\beta > \alpha$:

$$
\begin{aligned}
Y_0 &= \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{D}^*) : \int_{\mathbb{D}} |f(r,\theta)|^2 r^{2\beta}\, dA < \infty \right\}, \\
Y_1 &= \left\{ f \in W^{1,2}_{\mathrm{loc}}(\mathbb{D}^*) : \int_{\mathbb{D}} |f(r,\theta)|^2 r^{2\beta} + |\nabla f(r,\theta)|^2 r^{2\alpha}\, dA < \infty \right\}.
\end{aligned}
$$

The inner product in Y_0 is defined as

$$(f \mid g)_{Y_0} = \int_{\mathbb{D}} f(r,\theta) r^{\beta}\, g(r,\theta) r^{\beta}\, dA,$$

and the inner products in Y_1, and in

$$Y_0^* = \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{D}^*) : \int_{\mathbb{D}} |f(r,\theta)|^2 r^{-2\beta}\, dA < \infty \right\},$$

are defined accordingly. The dual pairing between $f \in Y_0$ and $g \in Y_0^*$ is

$$\langle f | g \rangle = \int_{\mathbb{D}} f g\, dA.$$

The operator $T : Y_1 \to Y_0^*$ is

$$Tu = -\operatorname{div}_{\mathfrak{X}}(A\, \mathfrak{X} u), \tag{70}$$

where A is the (degenerate) elliptic and symmetric matrix with entries in $C^\infty(\overline{\mathbb{D}} \setminus \{0\})$, as defined in (54), page 42.

First, we want to find a solution $w \in Y_1$ to

$$
\begin{cases}
Tw &= 0 \qquad \text{in } \mathbb{D} \\
\frac{\partial w}{\partial \nu^*} + \tau \frac{\partial w}{\partial \theta} &= h \qquad \text{on } \partial\mathbb{D},
\end{cases} \tag{71}
$$

where τ and h are 2π-periodic functions in the class $C^\infty(\mathbb{R})$. The function τ is fixed as in (55), page 44; our goal is to find a condition for h such that the problem (71) admits a solution.

The strategy is to first consider the problem

$$
\begin{cases}
Tu &= g \qquad \text{in } \mathbb{D} \\
\frac{\partial u}{\partial \nu^*} + \tau \frac{\partial u}{\partial \theta} &= 0 \qquad \text{on } \partial\mathbb{D},
\end{cases} \tag{72}
$$

and find a suitable condition for $g \in Y_0^*$ such that problem (72) admits a solution. Thereafter the problem (71) is reduced to the problem (72), and the desired condition for h is obtained.

We start by constructing the Dirichlet form for the oblique derivative problem, following Folland [10], pages 240–241. Our form $D : Y_1 \times Y_1 \to \mathbb{R}$ should satisfy

$$D(v,u) - \langle v \,|\, Tu \rangle = \int_{\partial\mathbb{D}} v \left(\frac{\partial u}{\partial \nu^*} + \tau \frac{\partial u}{\partial \theta} \right) d\theta,$$

so that the condition

$$D(v,u) = \langle v \,|\, g \rangle \quad \text{for all } v \in Y_1$$

guarantees that $u \in Y_1$ is a weak solution to (72).

First, we calculate

(73)

$$\begin{aligned}
\operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}u) &= \frac{1}{r} e_r\big(r\,[A\mathfrak{X}u]_1\big) + e_\theta\big([A\mathfrak{X}u]_2\big) \\
&= \frac{1}{r}\big\{ r e_r\big([A\mathfrak{X}u]_1\big) + [A\mathfrak{X}u]_1 \big\} + e_\theta\big([A\mathfrak{X}u]_2\big) \\
&= \frac{1}{r}\big\{ r e_r\big(a_{11}e_r(u) + a_{12}e_\theta(u)\big) + a_{11}e_r(u) + a_{12}e_\theta(u) \big\} + e_\theta\big([A\mathfrak{X}u]_2\big) \\
&= a_{11}e_re_r(u) + e_r(a_{11})e_r(u) + a_{12}e_re_\theta(u) + e_r(a_{12})e_\theta(u) \\
&\quad + \frac{1}{r}a_{11}e_r(u) + \frac{1}{r}a_{12}e_\theta(u) \\
&\quad + a_{21}e_\theta e_r(u) + e_\theta(a_{21})e_r(u) + a_{22}e_\theta e_\theta(u) + e_\theta(a_{22})e_\theta(u) \\
&= a_{11}e_re_r(u) + a_{12}e_re_\theta(u) + a_{21}e_\theta e_r(u) + a_{22}e_\theta e_\theta(u) \\
&\quad + \left(e_r(a_{11}) + \frac{1}{r}a_{11} + e_\theta(a_{21}) \right) e_r(u) \\
&\quad + \left(e_r(a_{12}) + \frac{1}{r}a_{12} + e_\theta(a_{22}) \right) e_\theta(u).
\end{aligned}$$

Now

$$\begin{aligned}
e_re_\theta(u) = e_r\Big(\frac{1}{r}\frac{\partial u}{\partial \theta}\Big) &= -\frac{1}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r}e_r\Big(\frac{\partial u}{\partial \theta}\Big) \\
&= -\frac{1}{r^2}e_\theta(u) + \frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial \theta}\Big) = -\frac{1}{r^2}e_\theta(u) + \frac{1}{r}\frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial r}\Big) \\
&= -\frac{1}{r^2}e_\theta(u) + e_\theta e_r(u),
\end{aligned}$$

so

$$a_{12}e_re_\theta(u) = a_{12}e_\theta e_r(u) - \frac{1}{r}a_{12}e_\theta(u),$$

and (73) reads

(74)
$$\begin{aligned}
&\operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}u)\\
&= a_{11}e_re_r(u) + \big(a_{12}+a_{21}\big)e_\theta e_r(u) + a_{22}e_\theta e_\theta(u)\\
&\quad + \left(e_r(a_{11}) + \frac{1}{r}a_{11} + e_\theta(a_{21})\right) e_r(u) + \Big(e_r(a_{12}) + e_\theta(a_{22})\Big)e_\theta(u).
\end{aligned}$$

Next, if
$$\mathcal{C} = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}$$
is any matrix with $c \in C^\infty(\overline{\mathbb{D}})$ (to be specified later), and if we denote $B = (b_{ij}) = A + \mathcal{C}$, then

(75)
$$\begin{aligned}
&\operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}u)\\
&= b_{11}e_re_r(u) + \big(b_{12}+b_{21}\big)e_\theta e_r(u) + b_{22}e_\theta e_\theta(u)\\
&\quad + \left(e_r(a_{11}) + \frac{1}{r}a_{11} + e_\theta(a_{21})\right) e_r(u) + \Big(e_r(a_{12}) + e_\theta(a_{22})\Big)e_\theta(u).
\end{aligned}$$

Next, we need a Stokes' Theorem.

7.1. **Lemma.**
$$\int_{\partial\mathbb{D}} v\frac{\partial u}{\partial \nu_B^*}d\theta = \int_{\mathbb{D}} v\operatorname{div}_{\mathfrak{X}}(B\mathfrak{X}u)\,dA + \int_{\mathbb{D}} \mathfrak{X}v\cdot B\mathfrak{X}u\,dA, \tag{76}$$
for each $u, v \in Y_1$.

Proof. By definition,
$$\int_{\mathbb{D}} v\operatorname{div}_{\mathfrak{X}} U\,dA = -\int_{\mathbb{D}} U\cdot\mathfrak{X}v\,dA \tag{77}$$
for each $U \in C^1(\mathbb{D};\mathbb{R}^2)$ and $v \in C_0^\infty(\mathbb{D})$. When v is not compactly supported, we multiply it by φ_ε, a standard radial function in $C_0^\infty(\mathbb{D})$ satisfying $\varphi_\varepsilon \to \chi_{\mathbb{D}}$ as $\varepsilon \to 0$. Then, by (77),
$$\begin{aligned}
\int_{\mathbb{D}} \varphi_\varepsilon v\operatorname{div}_{\mathfrak{X}} U\,dA &= -\int_{\mathbb{D}} U\cdot\mathfrak{X}(\varphi_\varepsilon v)\,dA\\
&= -\int_{\mathbb{D}} U\cdot(v\mathfrak{X}\varphi_\varepsilon)\,dA - \int_D U\cdot(\varphi_\varepsilon\mathfrak{X}v)\,dA.
\end{aligned}$$
Letting $\varepsilon \to 0$ yields, since $\mathfrak{X}\varphi_\varepsilon \to (-\delta_1, 0)$ as $\varepsilon \to 0$,
$$\int_{\mathbb{D}} v\operatorname{div}_{\mathfrak{X}} U\,dA = \int_{\partial\mathbb{D}} U_1 v\,d\theta - \int_{\mathbb{D}} U\cdot\mathfrak{X}v\,dA.$$

With $U = B\mathfrak{X}u$, we have $U_1 = b_{11}e_r(u) + b_{12}e_\theta(u)$, and

$$\frac{\partial u}{\partial \nu_B^*} = B^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathfrak{X}u = B \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathfrak{X}u = U_1,$$

which yields the Lemma[6]. □

Now, replacing the divergence term in (76) by the form calculated in (74), yields

$$\begin{aligned}
\int_{\partial\mathbb{D}} v \frac{\partial u}{\partial \nu_B^*} d\theta &= \int_{\mathbb{D}} \mathfrak{X}v \cdot B\mathfrak{X}u \, dA \\
&+ \int_{\mathbb{D}} v \left\{ b_{11}e_r e_r(u) + \big(b_{12} + b_{21}\big)e_\theta e_r(u) + b_{22}e_\theta e_\theta(u) \right\} dA \\
&+ \int_{\mathbb{D}} v \left\{ \left(e_r(b_{11}) + \frac{1}{r}b_{11} + e_\theta(b_{21}) \right) e_r(u) \right. \\
&\quad \left. + \Big(e_r(b_{12}) + e_\theta(b_{22})\Big) e_\theta(u) \right\} dA,
\end{aligned} \tag{78}$$

and replacing the middle term on the right-hand side of (78) by the form calculated in (75) yields

$$\begin{aligned}
\int_{\partial\mathbb{D}} v \frac{\partial u}{\partial \nu_B^*} d\theta = \int_{\mathbb{D}} \mathfrak{X}v \cdot B\mathfrak{X}u \, dA &+ \int_{\mathbb{D}} v \operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}u) \, dA \\
+ \int_{\mathbb{D}} v &\left\{ \left(e_r(b_{11}) + \frac{1}{r}b_{11} + e_\theta(b_{21}) \right) e_r(u) \right. \\
&\quad \left. + \Big(e_r(b_{12}) + e_\theta(b_{22})\Big) e_\theta(u) \right\} dA \\
- \int_{\mathbb{D}} v &\left\{ \left(e_r(a_{11}) + \frac{1}{r}a_{11} + e_\theta(a_{21}) \right) e_r(u) \right. \\
&\quad \left. + \Big(e_r(b_{12}) + e_\theta(b_{22})\Big) e_\theta(u) \right\} dA.
\end{aligned}$$

Thus we finally obtain (since $c_{ij} = b_{ij} - a_{ij}$ and since $c_{11} = c_{22} = 0$)

$$\begin{aligned}
\int_{\partial\mathbb{D}} v \frac{\partial u}{\partial \nu_B^*} d\theta &= \int_{\mathbb{D}} \mathfrak{X}v \cdot B\mathfrak{X}u \, dA + \int_{\mathbb{D}} v \operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}u) \, dA \\
&+ \int_{\mathbb{D}} v \big\{ e_\theta(c_{21})e_r(u) + e_r(c_{12})e_\theta(u) \big\} \, dA.
\end{aligned}$$

The Dirichlet form is now defined such that

$$\int_{\partial\mathbb{D}} v \frac{\partial u}{\partial \nu_B^*} d\theta = \int_{\mathbb{D}} v \operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}u) \, dA + D(v, u),$$

[6]*Note added in proof.* This proves the Lemma only for $u, v \in C^1(\mathbb{D})$. To actually obtain the Lemma for $u, v \in Y_1$, we need to take care of the origin. We repeat the stated proof for an annulus, and obtain an extra boundary term that vanishes as the inner radius of the annulus approaches zero. This is proved in a similar manner as (89), page 61.

i.e.

$$\begin{aligned} D(v,u) &= \int_{\partial\mathbb{D}} v\frac{\partial u}{\partial \nu_B^*}d\theta - \int_{\mathbb{D}} v \operatorname{div}_{\mathfrak{X}}(A\mathfrak{X}u)\,dA \\ &= \int_{\mathbb{D}} \mathfrak{X}v\cdot B\mathfrak{X}u\,dA \\ &\quad + \int_{\mathbb{D}} v\big\{e_\theta(c_{21})e_r(u)+e_r(c_{12})e_\theta(u)\big\}\,dA. \end{aligned}$$

Finally, we need to choose the matrix $\mathcal{C}$ such that

$$\frac{\partial u}{\partial \nu_B^*} = \frac{\partial u}{\partial \nu_A^*} + \tau(\theta)\frac{\partial u}{\partial \theta}.$$

Since

$$\frac{\partial u}{\partial \nu_B^*} = (A+\mathcal{C})^t\begin{pmatrix}1\\0\end{pmatrix}\cdot\mathfrak{X}u = \frac{\partial u}{\partial \nu_A^*} + \mathcal{C}^t\begin{pmatrix}1\\0\end{pmatrix}\cdot\mathfrak{X}u,$$

and since

$$\mathcal{C}^t\begin{pmatrix}1\\0\end{pmatrix}\cdot\mathfrak{X}u = \begin{pmatrix}0 & c\\ -c & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}e_r(u)\\ e_\theta(u)\end{pmatrix} = -c\,e_\theta(u) = -c\frac{\partial u}{\partial\theta},$$

we choose c to be any function in $C^\infty(\overline{\mathbb{D}})$ such that $-c(1,\theta)=\tau(\theta)$, and such that $c(r,\theta)=0$ for $r<1/2$, say. This latter condition is needed to obtain coercivity:

7.2. **Lemma.**

$$|D(u,u)| \ge C_1||u||_{Y_1}^2 - C_2||u||_{Y_0}^2.$$

Proof. First we estimate

(79)
$$|D(u,u)| \ge \left|\int_{\mathbb{D}}\mathfrak{X}u\cdot B\mathfrak{X}u\,dA\right| - \left|\int_{\mathbb{D}} u\big\{e_\theta(c_{21})e_r(u)+e_r(c_{12})e_\theta(u)\big\}\,dA\right|.$$

Since $\mathfrak{X}u\cdot B\mathfrak{X}u = \mathfrak{X}u\cdot A\mathfrak{X}u$ and since A is elliptic, i.e. $A\xi\cdot\xi \ge Cr^{2\alpha}|\xi|^2$, we have

$$\int_{\mathbb{D}}\mathfrak{X}u\cdot B\mathfrak{X}u\,dA \ge C\int_{\mathbb{D}} r^{2\alpha}|\mathfrak{X}u|^2\,dA = C\big(||u||_{Y_1}^2 - ||u||_{Y_0}^2\big).$$

For the second term on the right-hand side of (79), we first have

$$\int_{\mathbb{D}} u\big\{e_\theta(c_{21})e_r(u)+e_r(c_{12})e_\theta(u)\big\}\,dA \le C\int_{\{r\ge\frac12\}} |u\big(e_r(u)+e_\theta(u)\big)|\,dA,$$

since the functions c_{ij} are in the class $C^\infty(\overline{\mathbb{D}})$ and are supported in the annulus $\{r \geq 1/2\}$. Then, we estimate using Young's inequality:

$$\begin{aligned}
&\int_{\{r\geq\frac{1}{2}\}} |u\big(e_r(u)+e_\theta(u)\big)|\,dA \leq C\int_{\{r\geq\frac{1}{2}\}} |u\mathfrak{X}u|\,dA \\
&\leq C\varepsilon\int_{\{r\geq\frac{1}{2}\}} |\mathfrak{X}u|^2\,dA + C\frac{1}{\varepsilon}\int_{\{r\geq\frac{1}{2}\}} |u|^2\,dA \\
&\leq C\varepsilon||u||^2_{Y_1} + C\frac{1}{\varepsilon}||u||^2_{Y_0}.
\end{aligned}$$

Finally, choose $\varepsilon > 0$ small enough such that $C\varepsilon \leq 1/2$ to obtain

$$\begin{aligned}
&|D(u,u)| \geq \left|\int_{\mathbb{D}} \mathfrak{X}u\cdot B\mathfrak{X}u\,dA\right| - \left|\int_{\mathbb{D}} u\big\{e_\theta(c_{21})e_r(u)+e_r(c_{12})e_\theta(u)\big\}\,dA\right| \\
&\geq C||u||^2_{Y_1} - \frac{1}{2}||u||^2_{Y_1} - C_2||u||^2_{Y_0} = C_1||u||^2_{Y_1} - C_2||u||^2_{Y_0}
\end{aligned}$$

as wanted. □

Next, consider the adjoint Dirichlet form

$$D^*(v,u) = D(u,v).$$

7.3. **Lemma.** *If $u \in Y_1$ satisfies, for some $f \in Y_0^*$,*

$$D^*(v,u) = \langle v \mid f\rangle \qquad \textit{for all } v \in Y_1,$$

then u is a weak solution to the boundary value problem

$$\begin{cases} Tu & = f \qquad \textit{in } \mathbb{D} \\ \frac{\partial u}{\partial\nu^*} - \frac{\partial}{\partial\theta}(\tau u) & = 0 \qquad \textit{on } \partial\mathbb{D}. \end{cases} \tag{80}$$

Proof. Since A is symmetric, Stokes' Theorem yields

$$\int_{\mathbb{D}} vTu - uTv\,dA = \int_0^{2\pi} u(1,\theta)\frac{\partial v}{\partial\nu^*}(1,\theta) - v(1,\theta)\frac{\partial u}{\partial\nu^*}(1,\theta)\,d\theta. \tag{81}$$

By definition of $D(v,u)$,

$$D(v,u) = \int_{\mathbb{D}} vTu\,dA + \int_0^{2\pi} v(1,\theta)\left(\frac{\partial u}{\partial\nu^*}(1,\theta) + \tau(\theta)\frac{\partial u}{\partial\theta}(1,\theta)\right)\,d\theta,$$

and

$$D^*(v,u) = \int_{\mathbb{D}} uTv\,dA + \int_0^{2\pi} u(1,\theta)\left(\frac{\partial v}{\partial\nu^*}(1,\theta) + \tau(\theta)\frac{\partial v}{\partial\theta}(1,\theta)\right)\,d\theta,$$

so combined with (81),

$$D^*(v,u) - D(v,u) = \int_0^{2\pi} u(1,\theta)\tau(\theta)\frac{\partial v}{\partial\nu^*}(1,\theta) - v(1,\theta)\tau(\theta)\frac{\partial u}{\partial\nu^*}(1,\theta)\,d\theta.$$

Thus $D^*(v,u)$ and $D(v,u)$ differ only on the boundary, and the boundary condition for D^* is

$$\int_0^{2\pi} v\frac{\partial u}{\partial \nu^*} + u\tau\frac{\partial v}{\partial \theta}\,d\theta = \int_0^{2\pi} v\left(\frac{\partial}{\partial \nu^*} - \frac{\partial}{\partial \theta}(\tau u)\right)\,d\theta.$$

□

7.4. **Lemma.** *The imbedding $id\colon Y_1 \to Y_0$ is compact.*

Proof. Let $\varepsilon > 0$ be small, and denote $id = id_1 + id_\varepsilon$, where id_1 is the imbedding restricted to the annulus $A_1 = \{(r,\theta)\colon \varepsilon < r < 1, 0 \le \theta < 2\pi\}$ and id_ε is the imbedding restricted to the annulus $A_\varepsilon = \{(r,\theta)\colon 0 < r < \varepsilon, 0 \le \theta < 2\pi\}$. The mapping id_1 is compact (since the imbedding $W^{1,2} \to L^2$ is compact), so it suffices to show for $u \in Y_1$ that

$$||u||^2_{Y_0(A_\varepsilon)} \le C(\varepsilon)||u||^2_{Y_1},$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Indeed, this guarantees that id can be made arbitrary close to the compact operator id_1, which yields (see e.g. Folland, [10, Theorem 0.34]) that id itself is compact.

For $u \in Y_1$ and $0 < r < 1$, we use the estimate

$$|u(r,\theta)| \le \int_r^1 |\nabla u(s,\theta)|\,ds + ⨍_{(1/2,1)} |u(s,\theta)|\,ds$$

along with the Cauchy-Schwartz inequality

$$\begin{aligned}\int_r^1 |\nabla u(s,\theta)|\,ds &\le \left(\int_r^1 |\nabla u(s,\theta)|^2 s^{2\alpha+1}\,ds\right)^{\frac{1}{2}}\left(\int_r^1 s^{-(2\alpha+1)}\,ds\right)^{\frac{1}{2}}\\ &\le Cr^{-\alpha}\left(\int_r^1 |\nabla u(s,\theta)|^2 s^{2\alpha+1}\,ds\right)^{\frac{1}{2}},\end{aligned}$$

to obtain

$$\begin{aligned}\int_{A_\varepsilon} u^2 r^{2\beta}\,dA &\le C\int_0^{2\pi}\int_0^{\varepsilon} r^{2\beta} r^{-2\alpha}\int_r^1 |\nabla u(s,\theta)|^2 s^{2\alpha}\,sds\,rdr\,d\theta\\ &\quad + C\int_{A_\varepsilon}\left(⨍_{(1/2,1)} |u(s,\theta)|\,ds\right)^2 dA\\ &\le C||u||^2_{Y_1}\int_0^{\varepsilon} r^{2\beta-2\alpha+1}\,dr + C|A_\varepsilon|\\ &= C\varepsilon^{2(\beta-\alpha+1)}||u||^2_{Y_1} + C\varepsilon^2\end{aligned}$$

as wanted, since $\beta > \alpha$. □

Next, denote

$$\mathcal{W} = \{u \in Y_1 : D^*(v,u) = 0 \text{ for each } v \in Y_1\},$$

i.e.

$$\mathcal{W} = \{v \in Y_1 : D(v,u) = 0 \text{ for each } u \in Y_1\}.$$

Finally, since our Dirichlet form is coercive and since the injection $Y_1 \to Y_0$ is compact, the standard Fredholm-Riesz-Schauder theory (see e.g. [10], Theorem 7.21, pages 250–251) yields that $\mathcal{W}$ is finite-dimensional in Y_0, and that the problem (72) admits a solution whenever

$$\langle g \,|\, v\rangle = \int_{\mathbb{D}} gv\, dA = 0$$

for each $v \in \mathcal{W}$. This is the condition for g, the remaining step is to find the condition for h in the original problem (71).

7.5. **Lemma.** *The problem (71) admits a solution whenever*

$$\int_0^{2\pi} h(\theta)v(1,\theta)d\theta = 0 \tag{82}$$

for each $v \in \mathcal{W}$.

Proof. Let $h \in C^\infty(\partial\mathbb{D})$ satisfy (82). Let $H \in Y_1 \cap C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ be such that

$$\frac{\partial H}{\partial \nu^*}(1,\theta) + \tau(\theta)\frac{\partial H}{\partial \theta}(1,\theta) = h(\theta).$$

We claim that

$$\int_{\mathbb{D}} v\, TH\, dA = 0 \qquad \text{for all } v \in \mathcal{W}. \tag{83}$$

Indeed, let $v \in \mathcal{W}$, i.e. $D(v,u) = 0$ for each $u \in Y_1$. Now

$$D(v,H) - \int_{\mathbb{D}} vTH\, dA = \int_0^{2\pi} v \left(\frac{\partial H}{\partial \nu^*} + \tau\frac{\partial H}{\partial \theta}\right) d\theta,$$

so the claim follows.

Since (83) holds, we can solve the problem (72) with $g = -TH$, obtaining a weak solution u to

$$\begin{cases} Tu & = -TH \quad \text{in } \mathbb{D} \\ \frac{\partial u}{\partial \nu^*} + \tau\frac{\partial u}{\partial \theta} & = 0 \quad \text{on } \partial\mathbb{D}. \end{cases}$$

Now the function $w = u + H$ solves (71). $\square$

With the desired existence part done, we will next prove regularity of solutions to $Tu = 0$ in $\mathbb{D}$. Since the coefficients of the matrix A are in the class $C^\infty(\overline{\mathbb{D}} \setminus \{0\})$, also the solutions are automatically in this class. The task is to prove Sobolev regularity at the origin.

We start with a maximum principle that corresponds to Wolff's Lemma 3.8:

7.6. **Lemma.** *If $u \in Y_1 \cap C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ satisfies $Tu = 0$ in $\mathbb{D}$, then $u \in L^\infty(\mathbb{D})$; in fact,*

$$u(r,\theta) \leq \max_{0 \leq \theta < 2\pi} u(1,\theta)$$

for all $(r,\theta) \in \mathbb{D}$.

Proof. For $0 < R < 1$, let

$$A_R = \{(r,\theta) \colon R < r < 1,\ 0 \leq \theta < 2\pi\} \subset \mathbb{D}^*.$$

We make two claims.

Claim 1. Fix $R, \rho \in (0,1)$ such that $R < \rho$. Then there exists a function $\psi \colon \overline{A_R} \to \mathbb{R}$, continuous and such that

$$\text{for all } \theta,\ \psi(R,\theta) = R^{-\alpha} \ \text{ and } \ \psi(1,\theta) = 0, \tag{84}$$

$$T\psi = 0 \text{ in } A_R, \tag{85}$$

$$\psi(r,\theta) \leq C(\rho) \text{ when } \rho \leq r < 1. \tag{86}$$

To prove this, define $\psi_0 \colon \overline{A_R} \to \mathbb{R}$ as

$$\psi_0(r,\theta) = \psi_0(r) = (r^{-2\alpha} - 1)R^\alpha.$$

Then ψ_0 satisfies (84) and

$$\int_R^1 |\psi_0'(r)|^2 r^{2\alpha} r dr \leq CR^{2\alpha} < C$$

uniformly in R. By Dirichlet's principle, there is a function $\psi \colon \overline{A_R} \to \mathbb{R}$ satisfying (84), (85) and

$$\int_{A_R} |\nabla\psi(r,\theta)|^2 r^{2\alpha}\, dA < C \tag{87}$$

uniformly in R. Then (86) follows by elliptic regularity, T being uniformly elliptic on (say) $A_{R/2}$.

Claim 2. If $u \in Y_1$ and $Tu = 0$ in $\mathbb{D}^*$, then

$$\lim_{r\to 0}\left(\max_{0\le\theta<2\pi} r^\alpha u(r,\theta)\right) = 0.$$

To prove this, define the dyadic annuli

$$A_k = \left\{(r,\theta)\colon 2^{-(k+1)} < r < 2^{-(k-1)},\, 0\le\theta<2\pi\right\}.$$

for $k\in\mathbb{N}$, $k\ge 1$. Then, define $r_k = 2^{-k}$ along with the functions

$$M_k = \max_{0\le\theta<2\pi} u(r_k,\theta),$$

and

$$m_k = \min_{0\le\theta<2\pi} u(r_k,\theta).$$

Since the equation $Tu = 0$ is invariant under scaling, Harnack's inequality is valid in each annulus A_k with uniform bounds. Applying the weak Harnack's inequality in A_k yields

$$(M_k - m_k)^2 \le C\int_{A_k} |\nabla u|^2\, dA,$$

which readily yields

$$r_k^{2\alpha}(M_k - m_k)^2 \le C\int_{A_k} r_k^{2\alpha}|\nabla u|^2\, dA.$$

Next, we sum over k to obtain

$$\sum_{k=1}^\infty r_k^{2\alpha}(M_k - m_k)^2 \le C||u||_{Y_1}^2 < \infty,$$

especially

$$r_k^\alpha(M_k - m_k) \to 0 \quad \text{as } k\to\infty. \tag{88}$$

Computations using only that $u\in Y_1$ now give

$$\limsup_{k\to\infty} r_k^\alpha m_k \le 0. \tag{89}$$

Indeed, were (89) not true, there would exist an $\varepsilon_0 > 0$ and a subsequence of r_k (still call it r_k) such that $m_k \ge \varepsilon_0 r_k^{-\alpha}$ for each k. Fix k_1 and choose k_2 large enough such that M_{k_1} becomes negligible compared to m_{k_2}. Without loss of generality, we may assume that u is a radial function satisfying $u(r_{k_1}) = 0$ and $u(r_{k_2}) = M \ge \varepsilon_0 r_{k_2}^{-\alpha}$. We have

$$\int_{r_{k_2}}^{r_{k_1}} u'(r)\, dr = M,$$

and we want to estimate

$$\int_{r_{k_2}}^{r_{k_1}} |u'(r)|^2 r^{2\alpha} r dr$$

from below. Let $v(r) = u'(r)r^{\alpha+1/2}$, so that we want to estimate

$$\int_{r_{k_2}}^{r_{k_1}} v(r)^2 \, dr = ||v||_2^2$$

from below, under the condition

$$\int_{r_{k_2}}^{r_{k_1}} v(r) r^{-(\alpha+1/2)} \, dr = \langle v, r^{-(\alpha+1/2)} \rangle = M.$$

By elementary geometry, the smallest value of $||v||_2$ under $\langle v, g \rangle = M$ is attained when v is parallel to g, i.e. $v = gM/||g||_2$. In our case, $g(r) = r^{-(\alpha+1/2)}$ and

$$\int_{r_{k_2}}^{r_{k_1}} |u'(r)|^2 r^{2\alpha} r dr \geq \frac{M^2}{\int_{r_{k_2}}^{r_{k_1}} r^{-(2\alpha+1)} dr} = \frac{M^2}{r_{k_2}^{-2\alpha} - r_{k_1}^{-2\alpha}}$$

$$\geq \varepsilon_0 \frac{r_{k_2}^{-2\alpha}}{r_{k_2}^{-2\alpha} - r_{k_1}^{-2\alpha}} \geq C\varepsilon_0^2.$$

Summing over k, we obtain $||u||_{Y_1} = \infty$, a contradiction.

Thus (89) must hold, and by (88), also $\limsup_{k\to\infty} r_k^\alpha M_k \leq 0$. If we similarly consider $-u$, we have Claim 2.

Suppose now (r, θ) is given. Fix $\varepsilon > 0$. Let R be small and consider $u - \varepsilon\psi$, with ψ as in Claim 1 (taking $\rho = r$). Then $T(u - \varepsilon\psi) = 0$ in A_R, and

$$\max_\theta (u - \varepsilon\psi)(1, \theta) = \max_\theta u(1, \theta),$$

since $\psi(1, \theta) = 0$. By Claim 2,

$$\max_\theta (u - \varepsilon\psi)(R, \theta) \to -\infty \text{ as } R \to 0.$$

Choose R small enough such that

$$\max_\theta (u - \varepsilon\psi)(R, \theta) < \min_\theta (u - \varepsilon\psi)(1, \theta).$$

Then, by the maximum principle applied to $u - \varepsilon\psi$ on A_R, we have

$$(u - \varepsilon\psi)(r, \theta) \leq \max_\theta (u - \varepsilon\psi)(1, \theta).$$

Since $\psi(r, \theta) \leq C(r)$ and $\max_\theta (u - \varepsilon\psi)(1, \theta) = \max_\theta u(1, \theta)$, we finally obtain

$$u(r, \theta) \leq \varepsilon\psi(r, \theta) + \max_\theta u(1, \theta) \leq \varepsilon C(r) + \max_\theta u(1, \theta).$$

Now let $\varepsilon \to 0$. $\square$

The following corresponds to Wolff's Lemma 3.12.

7.7. **Lemma.** *If $u \in Y_1 \cap C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ and $Tu = 0$ in $\mathbb{D}$, then*

$$\max_\theta u(r,\theta) - \min_\theta u(r,\theta) \leq Cr^\gamma \tag{90}$$

for some $\gamma > 0$. Consequently,

$$\int_{\mathbb{D}} |\nabla u|^q \, dA < \infty$$

for any $q \in (0, \infty]$.

Proof. Let the dyadic annuli A_k along with r_k, M_k and m_k be as in the proof of Claim 2 in Lemma 7.6. Lemma 7.6 implies that M_k decreases and m_k increases with k. Hence the solutions $M_k - u$ and $u - m_k$ are positive in the annulus A_{k+1}. Harnack's inequality (applied on the ring of radius r_{k+1}) yields for $M_k - u$ that

$$\max_\theta \big(M_k - u(r_{k+1}, \theta)\big) \leq C \min_\theta \big(M_k - u(r_{k+1}, \theta)\big),$$

i.e.

$$M_k - m_{k+1} \leq C\big(M_k - M_{k+1}\big), \tag{91}$$

and similarly for $u - m_k$,

$$M_{k+1} - m_k \leq C\big(m_{k+1} - m_k\big). \tag{92}$$

Adding (91) and (92) yields for $\omega_k = M_k - m_k$,

$$\omega_k + \omega_{k+1} \leq C\big(\omega_k - \omega_{k+1}\big),$$

i.e.

$$\omega_{k+1} \leq \frac{C-1}{C+1}\omega_k = A\omega_k,$$

where $0 < A < 1$. This yields, by iteration as in Gilbarg-Trudinger [12, Lemma 8.23], the statement (90) with the number $\gamma > 0$ satisfying $(1/2)^\gamma = A$.

Again, since T is scaling invariant, the coefficients of T are smooth uniformly in each annulus. Elliptic regularity gives for any $q > 0$,

$$\int_{A_{k+1}} |\nabla u|^q \, dA \leq C\big(M_k - m_k\big)^q \leq Cr_k^{q\gamma}.$$

Now

$$\int_{\mathbb{D}^*} |\nabla u|^q \, dA \leq C \sum_{k=1}^{\infty} \int_{A_k} |\nabla u|^q \, dA \leq C \sum_{k=1}^{\infty} (r_{k-1})^{q\gamma} < \infty,$$

which implies the Lemma. □

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