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PROJECTED MEASURES ON MANIFOLDS AND INCOMPLETE PROJECTION FAMILIES

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List of included articles

This dissertation consists of an introductory part and the following publications:

(A) Esa Järvenpää, Maarit Järvenpää, Mika Leikas: *(Non)regularity of Projections of Measures Invariant Under Geodesic Flow*, Comm. Math. Phys. 254 (2005), no. 3, 695–717.

(B) Mika Leikas, *Packing dimensions, transversal mappings and geodesic flows*, Ann. Acad. Sci. Fenn. Mathematica, Volumen 29 (2004), 489-500.

(C) Esa Järvenpää, Maarit Järvenpää, François Ledrappier and Mika Leikas, *One-dimensional families of projections*.

The author of this dissertation has actively taken part in research of the papers (A) and (C).

Introduction

1 Projected sets and measures on $\mathbb{R}^n$

The behaviour of dimensions of sets and measures under projections has been studied for decades. The first theorems on this topic concentrated on the Hausdorff dimension of projections of sets onto lines in the plane. Later, these results were generalized to higher-dimensional spaces, and, finally, corresponding theorems were proven for the Hausdorff dimension of measures. Similar results were later proven for the packing dimension of sets and measures.

The following theorem is the most fundamental of the projection results. It was first proven for planar sets by Marstrand [Mar] in the 1950's. Kaufman [K] reproved Marstrand's theorem, and, finally, Mattila [Mat1] generalized it to higher dimensions. Here $\dim_H$ stands for the Hausdorff dimension, n and m are integers with $0 < m < n$, $G(n,m)$ is the Grassmann manifold of all m-dimensional subspaces of $\mathbb{R}^n$, $P_V\colon \mathbb{R}^n \to V$ is the orthogonal projection onto $V \in G(n,m)$, and $\gamma_{n,m}$ is the unique orthogonally invariant Radon probability measure on $G(n,m)$.

Theorem 1.1. *When $A \subset \mathbb{R}^n$ is a Borel set,*

$$\dim_H P_V(A) = \min\{\dim_H A,\, m\}$$

for $\gamma_{n,m}$- almost all $V \in G(n,m)$.

Marstrand's proof for Theorem 1.1 in the planar case was geometric. Later, Kaufman [K] proved the same result by using potential-theoretic methods. Also Mattila's proof for the general case is based on the potential-theoretic approach.

One of the key concepts in potential theory is the energy of a measure. The α-energy of a finite Borel measure μ on a metric space (X,d) is

$$I_\alpha(\mu) := \int_X \int_X \frac{1}{d(x,y)^\alpha}\, d\mu(x)\, d\mu(y).$$

The Hausdorff dimension of a set and the energies of measures supported on it are tied together; by Frostman's lemma (see [Mat2], Theorem 8.9.) we know that for a Borel set $A \subset \mathbb{R}^n$

$$\dim_H A = \sup\{s : \text{there is a compactly supported measure } \mu \text{ on } A \text{ such that } 0 < \mu(A) < \infty \text{ and } I_s(\mu) < \infty\}.$$

The concepts of the energy and the Hausdorff dimension of a measure are also closely related. For $f\colon X \to [-\infty, \infty]$ we set

$$\mu\text{- }\operatorname*{ess\,inf}_{x \in X} f(x) = \sup\{s \in [-\infty, \infty] : f(x) \geq s \text{ for } \mu\text{- almost all } x \in X\},$$

and $B(x, r)$ is the closed ball with radius $r > 0$ centered at $x \in X$. We define the Hausdorff dimension of a finite Borel measure μ on a metric space (X, d) to be

$$\begin{aligned}\dim_H \mu &= \mu\text{- }\operatorname*{ess\,inf}_{x \in X} \Big(\liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \Big) \\ &= \inf\{\dim_H A : A \text{ is a Borel set with } \mu(A) > 0\}.\end{aligned}$$

From the definitions we immediately get that if $I_\alpha(\mu) < \infty$ for some $\alpha > 0$, then $\dim_H \mu \geq \alpha$. On the other hand, if there is a constant $C > 0$ such that $\mu(B(x, r)) \leq Cr^\alpha$ for all $x \in X$ and $r > 0$, which is a rather strong condition that implies $\dim_H \mu \geq \alpha$, then $I_\alpha(\mu) < \infty$.

To state the analogous result to Theorem 1.1 for measures we have to define some new concepts. If μ is a measure on X, its image under a mapping $f\colon X \to Y$ is denoted by $f\mu$, and for $A \subset Y$ $f\mu(A) = \mu(f^{-1}(A))$. When μ and ν are measures, $\mu \ll \nu$ means that μ is absolutely continuous with respect to ν. Finally, $\mathcal{H}^m|_V$ is the restriction of $\mathcal{H}^m$ to a subset $V \subset \mathbb{R}^n$, and $\mathcal{H}^m|_V(A) = \mathcal{H}^m(V \cap A)$ for $A \subset \mathbb{R}^n$.

Theorem 1.2. *Let μ be a finite Borel measure on $\mathbb{R}^n$. Then*

$$\dim_H P_V\mu = \min\{\dim_H \mu,\, m\}$$

for $\gamma_{n,m}$- almost all $V \in G(n, m)$. In addition, if $\dim_H \mu > m$, then

$$P_V\mu \ll \mathcal{H}^m|_V$$

for $\gamma_{n,m}$- almost all V. Moreover, if $I_m(\mu) < \infty$, then

$$P_V\mu \ll \mathcal{H}^m|_V \quad \textit{with the Radon-Nikodym derivative in } L^2(V, \mathcal{H}^m|_V)$$

for $\gamma_{n,m}$- almost all V.

Theorem 1.2 was proven by Hu and Taylor in [HT].

The behaviour of the packing dimension under projections is not as straightforward as that of the Hausdorff dimension. Whilst the Hausdorff dimension of a set or a measure is preserved under almost all projections, the packing dimension may decrease for almost all of them. However, in [FH] Falconer and

Howroyd proved that the packing dimension of the projected set or measure will be the same for almost all projections. To show this, they introduced a new packing-type dimension, a k-dimensional packing dimension in $\mathbb{R}^n$. The usual (lower) packing dimension, $\dim_p$, of a finite Borel measure μ on a metric space (X, d) is defined to be

$$\begin{aligned}\dim_p \mu &= \mu\text{- }\operatorname*{ess\,inf}_{x\in X} \Big(\limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \Big) \\ &= \inf\{\dim_p A : A \text{ is a Borel set with } \mu(A) > 0\}.\end{aligned}$$

The new dimension, $\dim_k$, is defined by means of the convolution of the measure with a k-dimensional kernel. For a finite Borel measure μ on a metric space (X, d) and $k \in \mathbb{N}$, let

$$F_k^\mu(x, r) := \int_X \min\{1, r^k d(x, y)^{-k}\}\, d\mu(y) = kr^k \int_r^\infty \frac{\mu(B(x, h))}{h^{k+1}}\, dh.$$

Then we define

$$\dim_k \mu = \mu\text{- }\operatorname*{ess\,inf}_{x\in X} \Big(\limsup_{r\to 0} \frac{\log F_k^\mu(x,r)}{\log r} \Big).$$

For Borel sets $A \subset \mathbb{R}^n$ we set

$$\dim_k A := \sup\{\dim_k \mu : \mu \text{ is a compactly supported, finite Borel measure on } A\}.$$

The projection results by Falconer and Howroyd state the following:

Theorem 1.3. *Let μ be a finite Borel measure on $\mathbb{R}^n$, and let $A \subset \mathbb{R}^n$ be a Borel set. Then*

$$\dim_p P_V \mu = \dim_m \mu \quad \textit{and} \quad \dim_p P_V A = \dim_m A$$

for $\gamma_{n,m}$- almost all $V \in G(n, m)$.

In the late 90's the behaviour of different dimensions of measures under more general families of mappings was studied. For $q > 1$ the q-dimension of a Borel measure μ on $\mathbb{R}^n$ is defined to be

$$D_q(\mu) = \sup\Big\{ s : \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^s} \Big)^{q-1} d\mu(x) < \infty \Big\}.$$

When $q = 2$, the integral in the formula is just $I_s(\mu)$. In [HK] Hunt and Kaloshin showed that if μ is a compactly supported Borel probability measure on $\mathbb{R}^n$ and $1 < q \le 2$, then

$$D_q(L\mu) = \min\{m, D_q(\mu)\}$$

for almost every linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ (in the sense of the Lebesgue measure on the space of $m \times n$ matrices). Sauer and Yorke proved the preservation of the 2-dimension and the Hausdorff dimension for more general function families. Their results in [SY] state that for compactly supported Borel probability measures μ on $\mathbb{R}^n$,

$$\begin{aligned} D_2(g\mu) = D_2(\mu),\ \text{if } D_2(\mu) \le m \text{ and} \\ \dim_H g\mu = \dim_H \mu,\ \text{if } \dim_H \mu \le m \end{aligned}$$

for almost every C^1 function $g\colon \mathbb{R}^n \to \mathbb{R}^m$. One difficulty in proving such results as this is the fact that there is no simple substitute for the Lebesgue measure in the infinite-dimensional space $C^1(\mathbb{R}^n, \mathbb{R}^m)$. Sauer and Yorke use the notion of *prevalence* to handle this problem. A Borel set $A \subset X$ is called prevalent if there exists a probability measure μ on X such that $\mu(X \setminus A + x) = 0$ for every $x \in X$. In particular, prevalent sets are also dense.

The aforementioned theorems 1.1, 1.2 and 1.3 are the foundation of the measure-theoretic projection theory. In this dissertation we will generalize these results to more general spaces and projection families.

2 Projected measures on manifolds and transversality

2.1 The work of Ledrappier and Lindenstrauss

In [LL] Ledrappier and Lindenstrauss studied the behaviour of the Hausdorff dimension under the projection from the unit tangent bundle of a two-dimensional Riemann manifold onto the base manifold. They were able to prove the following theorem:

Theorem 2.1. *Let M be a compact two-dimensional Riemann manifold, let μ be a Radon probability measure on the unit tangent bundle SM, and let $\Pi\colon SM \to M$ be the natural projection. If μ is invariant under the geodesic flow, then the following statements are true:*

(a) *If* $\dim_H \mu \le 2$, *then* $\dim_H \Pi\mu = \dim_H \mu$.

(b) *If* $\dim_H \mu > 2$, *then* $\Pi\mu \ll \mathcal{H}^2$.

Moreover, if $I_\alpha(\mu) < \infty$ for some $\alpha > 2$, then the Radon-Nikodym derivative of $\Pi\mu$ is in $L^2(M, \mathcal{H}^2)$.

This theorem looks pretty similar to Theorem 1.2. The interesting difference is the fact that Theorem 1.2 says nothing about any specific projection, it just tells that something is true for almost all of them. Projection results are usually of this nature. On the other hand, the theorem by Ledrappier and Lindenstrauss gives us the information on one particular projection. At first glance this is somewhat surprising, but further study of the situation shows that we can prove the theorem by investigating a suitable family of generalized projections, that is, mappings that are members of a parametrized, tranversal family.

2.2 Transversality

Generalized projections were studied by Peres and Schlag in [PS]. The characteristic that is common to all projection-type mappings is transversality. In this work we use the following definition of transversality. Let $0 < m \leq k, n-1$ be integers and let $Q \subset \mathbb{R}^k$ be an open and connected set, the set of the parameters. If $P\colon Q \times \mathbb{R}^n \to \mathbb{R}^m$, $(\lambda, x) \mapsto P_\lambda(x)$ is a continuous mapping, we define for all $x \neq y$

$$\Phi_{x,y}(\lambda) = \frac{P_\lambda(x) - P_\lambda(y)}{|x-y|}.$$

We assume that all partial derivatives $\partial_\lambda^\eta P$ and $\partial_\lambda^\eta \Phi_{x,y}$ are bounded by constants depending only on η for every $\eta = (\eta_1, \ldots, \eta_k) \in \mathbb{N}^k$. However, the most important requirement is the existence of a constant $C_T > 0$ such that for every $\lambda \in Q$ and x, $y \in \mathbb{R}^n$, $x \neq y$

$$\det(D_\lambda \Phi_{x,y}(\lambda) D_\lambda \Phi_{x,y}(\lambda)^T) \geq C_T^2, \text{ whenever } |\Phi_{x,y}(\lambda)| \leq C_T. \tag{2.1}$$

Here $D_\lambda \Phi_{x,y}$ is the matrix of the derivative of $\Phi_{x,y}$, A^T is the transpose of a matrix A, and det stands for the determinant. This condition guarantees that if points x and y are mapped close to each other compared to their original distance, then the images $P_\lambda(x)$ and $P_\lambda(y)$ will recede fast enough when we change the parameter λ. A family $\{P_\lambda\colon \lambda \in Q\}$ satisfying the above conditions is called transversal.

In [PS] Peres and Schlag defined a more general notion of transversality, the so-called β-transversality. A family $\{P_\lambda\colon \lambda \in Q\}$ is β-transversal if on top of the boundedness of the derivatives there exists a constant $C_\beta > 0$ such that for every $\lambda \in Q$ and distinct x, $y \in \mathbb{R}^n$ the inequality

$$|\Phi_{x,y}(\lambda)| \leq C_\beta |x-y|^\beta$$

implies that

$$\det(D_\lambda \Phi_{x,y}(\lambda) D_\lambda \Phi_{x,y}(\lambda)^T) \geq C_\beta^2 |x-y|^{2\beta}.$$

The condition we are using is a special case of the β-transversality, corresponding to the choice $\beta = 0$. Peres and Schlag proved several results concerning the Sobolev dimension of a measure when it is mapped with a transversal family. The Fourier transform of a finite Borel measure μ on $\mathbb{R}^n$ is denoted by $\hat{\mu}$ and defined as

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\, d\mu(x).$$

Then the Sobolev $(2, \gamma)$ norm of μ is

$$\|\mu\|_{2,\gamma} := \Big(\int_{\mathbb{R}^n} |\hat{\mu}(\xi)|^2 |\xi|^{2\gamma}\, d\xi \Big)^{1/2},$$

and the Sobolev dimension of the measure is

$$\dim_s \mu := \sup \Big\{ \alpha : \int_{\mathbb{R}^n} |\hat{\mu}(\xi)|^2 (1 + |\xi|)^{\alpha - n}\, d\xi < \infty \Big\}.$$

If $0 < \dim_s \mu < n$, then $\dim_s \mu = \sup\{\alpha : I_\alpha(\mu) < \infty\}$, and thus $\dim_H \mu \geq \dim_s \mu$. A simplified version of the main result in [PS] can be stated as follows.

Theorem 2.2. *Let $\{P_\lambda : \lambda \in Q\}$ be a β-transversal family, and let μ be a finite Borel measure on $\mathbb{R}^n$. There is a constant $c_0 > 0$ depending only on m and k such that if $I_\alpha(\mu) < \infty$ for some $\alpha > 0$, then for every compact $Q' \subset Q$*

$$\int_{Q'} \|P_\lambda \mu\|_{2,\gamma}^2\, d\lambda \leq C_\gamma I_\alpha(\mu),$$

provided that $0 < (m + 2\gamma)(1 + c_0\beta) \leq \alpha$. Moreover, if $0 < \sigma \leq \min\{\alpha, m\}$, then

$$\dim\{\lambda \in Q : \dim_s \mu \leq \sigma\} \leq k + \sigma - \frac{\alpha}{1 + c_0\beta}.$$

All the projection results for the Hausdorff dimension in Section 1 follow as corollaries from these more general results. The connections between the original projection results and the results by Peres and Schlag are discussed in the extensive survey paper [Mat3].

2.3 Improvements to the work of Ledrappier and Lindenstrauss

In their paper, Ledrappier and Lindenstrauss asked if something more could be said about the smoothness of the projected measure $\Pi\mu$ when $I_\alpha(\mu) < \infty$

for some $\alpha > 2$ and if one could generalize the preservation theorem to higher-dimensional manifolds.

With Esa and Maarit Järvenpää we were able to answer these questions in [JJLe]. When proving Theorem 2.1, Ledrappier and Lindenstrauss used the geometry of the plane, and their proof does not give any clues as to what will happen when the dimension of the base manifold is higher than two. By approaching the problem from a different direction we reproved Theorem 2.1 using the methods of Peres and Schlag.

The most important single consequence of the invariance of the measure μ with respect to the geodesic flow is that suitable normalized restrictions $\tilde{\mu}$ of the original measure μ can be written as

$$\tilde{\mu} = \psi(\nu \times \mathcal{L}^1),$$

where ψ is a diffeomorphism from $[0,1]^3$ to $\psi([0,1]^3) \subset SM$, ν is a probability measure on $[0,1]^2$ and $\mathcal{L}^1$ is the one-dimensional Lebesgue measure. With this representation we were able to show that when the projection of $\tilde{\mu}$ is mapped to the plane with a chart φ, the measure $\varphi(\Pi\tilde{\mu})$ behaves in the following way: For all non-negative Borel functions $f\colon \mathbb{R}^2 \to [0,\infty]$,

$$\int_{\mathbb{R}^2} f(x,t)\, d(\varphi(\Pi\widetilde{\mu}))(x,t) \asymp \int_0^1 \int_{\mathbb{R}} f(x,t)\, d(P_t\nu)(x)\, d\mathcal{L}^1(t), \tag{2.2}$$

where

$$\left\{P_t\colon [0,1]^2 \to \mathbb{R} : t \in [0,1]\right\}$$

is a family of transversal mappings. Here $A \asymp B$ means that there is a constant $c > 0$ such that $\frac{1}{c}A \le B \le cA$. Then the result concerning the dimension of the projected measure follows from the theorems of Peres and Schlag.

This approach gives also a hint on why the dimension may decrease in higher dimensions. Regardless of the dimension n of the base manifold, the invariance of the original measure always produces a one-dimensional family of projection-type mappings onto $(n-1)$-dimensional planes in $2(n-1)$-dimensional space. But since the dimension of the collection of $(n-1)$-planes in $2(n-1)$-dimensional space is $(n-1)^2 > 1$, if $n > 2$, the family of projections is incomplete, and the theorem fails. We also give some easy examples to verify the failure of the preservation of the dimension in higher dimensions.

For a refined smoothness result on the two dimensional case we have to modify the theorems of Peres and Schlag. We want to show that when we map the projection of the above mentioned restriction of μ to the plane with a chart, the resulting measure will have fractional derivatives in L^2 provided that $I_\alpha(\tilde{\mu}) < \infty$ for some $\alpha > 2$. The aim is to estimate the Sobolev norm of the projected measure by the energy of the original measure; we want to find a constant $C > 0$ such that

$$\|\varphi(\Pi\tilde{\mu})\|_{2,\gamma}^2 \leq CI_\alpha(\tilde{\mu}) \tag{2.3}$$

for all $\gamma < (\alpha-2)/2$. This means that the projected measure has fractional derivatives of order γ in L^2 for all $\gamma < (\alpha-2)/2$. To get the desired estimate we use the Littlewood-Paley composition. Lemma 4.1. from [PS] gives a Schwartz function $\psi\colon \mathbb{R}^n \to \mathbb{R}$ such that ψ has vanishing moments of all orders and for any finite measure ν in $\mathbb{R}^n$ and for any $\gamma \in \mathbb{R}$

$$\|\nu\|_{2,\gamma}^2 \asymp \sum_{j=-\infty}^{\infty} 2^{2j\gamma} \int_{\mathbb{R}^n} (\psi_{2^{-j}} * \nu)(x)\, d\nu(x),$$

where $\psi_{2^{-j}}(x) = 2^{jn}\psi(2^j x)$. Here $\psi_{2^{-j}} * \nu$ is the convolution of $\psi_{2^{-j}}$ and measure ν, defined by

$$(\psi_{2^{-j}} * \nu)(x) = \int_{\mathbb{R}^n} \psi_{2^{-j}}(x-y)\, d\nu(y).$$

When we plug the projected measure in this formula and use the properties of ψ several times we are able to get the estimate in (2.3) after longish calculations.

2.4 The behaviour of the packing dimension

After the results for the Hausdorff dimension the interest turned to the packing dimension of the projected measure in the two dimensional case. For this purpose we need to generalize the result by Falconer and Howroyd (Theorem 1.3) to manifolds. We have to choose our setting as follows to be able to get any estimates. Let (L, d_L) be a smooth, bounded l-dimensional Riemannian manifold equipped with the distance function d_L induced by the Riemannian metric, let (N, d_N) be a smooth n-dimensional Riemannian manifold, and let (M, d_M) be a smooth m-dimensional Riemannian manifold. We suppose that $l, n \geq m$ so that in our setting L corresponds to $G(n,m)$, and furthermore, N and M correspond to $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $P\colon L \times N \to M$ be a continuous function such that for all $j \in \{0, 1, \dots\}$ there exists a constant C_j such that whenever $k_1 + \cdots + k_l = j$,

$$\|\partial_{\lambda_1}^{k_1} \dots \partial_{\lambda_l}^{k_l} P(\lambda, x)\| \leq C_j$$

for all $(\lambda, x) \in L \times N$. We later use the notation $P_\lambda(x) = P(\lambda, x)$. The basic assumptions we need are the following:

(1) There are finite collections $\{\phi, V\}$ and $\{\varphi, U\}$ of charts on L and M, respectively, with the following property: there exists $R > 0$ such that for all $\lambda \in L$ and $u \in M$

$$B(\lambda, R) \subset V \text{ and } B(u, R) \subset U$$

for some V and U.

(2) The Lipschitz constants of the mappings $\varphi, \varphi^{-1}, \phi$, and ϕ^{-1} are uniformly bounded from above by a positive constant K.

(3) Mapping $T : \{(x, y, \lambda) \in N^2 \times L : x \neq y,\ d_M(P_\lambda(x), P_\lambda(y)) \leq R\} \to \mathbb{R}^m$,

$$T_{x,y}(\lambda) = \frac{\varphi \circ P_\lambda(x) - \varphi \circ P_\lambda(y)}{d_N(x, y)}$$

is transversal, i.e., there exists a constant $C_T > 0$ such that

$$\det(D_\lambda T_{x,y}(\lambda)(D_\lambda T_{x,y}(\lambda))^T) \geq C_T \quad \text{whenever} \quad |T_{x,y}(\lambda)| \leq C_T.$$

We refer to this property also by saying that P is transversal. Moreover, we assume that

$$|\partial_{\lambda_j} \partial_{\lambda_k} (T_{x,y})_i(\lambda)| \leq L < \infty$$

for all $j, k \in \{1, \dots, l\}$, $i \in \{1, \dots, m\}$, $x, y \in N$, and $\lambda \in L$.

With these restrictions we are able to use Lemma 2.1 in [JJN] to estimate the size of the set of parameters that map two points close to each other and we get

Theorem 2.3. *Let (L, d_L), (N, d_N) and (M, d_M) be as above, and let $P \colon L \times N \to M$ be transversal. If μ is a finite, compactly supported Borel measure on N, then*

$$\dim_p P_\lambda \mu = \dim_m \mu$$

for $\mathcal{H}^l$-almost all $\lambda \in L$.

The proof of Theorem 2.3 can be found from [L]. From this result it follows by straightforward calculations that if the situation is like in Theorem 2.1, then

$$\dim_p \Pi\mu = \dim_2 \mu.$$

2.5 Lower dimensional families of projections

As mentioned before, in the setting of Ledrappier and Lindenstrauss the invariance of the original measure always produces a one-dimensional family of projection-type mappings. This leads to the study of incomplete families of projections. The dimension of the family of all projections into m-dimensional subspaces in $\mathbb{R}^n$ is $m(n-m)$. In [JJLL] our aim was to find out how much the dimension of an invariant measure may drop under the projection on higher dimensional manifolds, and thus we had to study one-dimensional projection families from $\mathbb{R}^n$ to $\mathbb{R}^m$. Simultaneously, we were able to get dimension estimates for k-dimensional families of projections from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ and from $\mathbb{R}^n$ to $\mathbb{R}$.

To get any reasonable estimates for the lower bound of the Hausdorff dimension of a projected measure under an incomplete projection family, we have to assure that the mapping is changing fast enough when we change the parameter. This is guaranteed by assuming that the incomplete family is *full*, that is, the partial derivatives of the image spaces $P_\lambda(\mathbb{R}^n)$ with respect to the components of the parameter λ span a parallelopiped whose k-dimensional volume is bounded from above and below by uniform constants at every point of the parameter space.

With such an assumption we were able to calculate the desired lower bounds in the above mentioned cases.

Theorem 2.4. *Let $k \in \{1, \dots, m(n-m)-1\}$, let $\Lambda \subset \mathbb{R}^k$ be open and connected, and let μ be a finite, compactly supported Radon measure on $\mathbb{R}^n$. Let $\{P_\lambda \colon \mathbb{R}^n \to \mathbb{R}^m : \lambda \in \Lambda\}$ be a full family of orthogonal projections, where $m \in \{1, \dots, n-1\}$. Then,*

1) if $m = n-1$,

$$\dim_H P_\lambda\mu \geq \begin{cases} \dim_H \mu, & \text{if } \dim_H \mu < k \\ k, & \text{if } k \leq \dim_H \mu < k+1 \\ \dim_H \mu - 1, & \text{if } \dim_H \mu \geq k+1 \end{cases}$$

for $\mathcal{L}^k$-almost all $\lambda \in \Lambda$,

2) if $k = 1$,

$$\dim_H P_\lambda\mu \geq \begin{cases} \max\{0, \dim_H \mu - (n-m-1)\}, & \text{if } \dim_H \mu < n-m \\ 1, & \text{if } n-m \leq \dim_H \mu < n-m+1 \\ \dim_H \mu - (n-m), & \text{if } \dim_H \mu \geq n-m+1 \end{cases}$$

for $\mathcal{L}$-almost all $\lambda \in \Lambda$, and

3) if $m = 1$,

$$\dim_H P_\lambda \mu \geq \begin{cases} \max\{0, \dim_H \mu - (n-k-1)\}, & \text{if } \dim_H \mu < n-k \\ 1, & \text{if } \dim_H \mu \geq n-k \end{cases}$$

for $\mathcal{L}^k$*-almost all* $\lambda \in \Lambda$.

As usual, to get such lower bounds for the dimension of the projected measure, we have to estimate the size of the set of parameters which map a given point close to the origin. If we can show that there is a constant $C > 0$ such that for any $z \in \mathbb{R}^n$ and $\delta > 0$

$$\mathcal{L}^k\{\lambda \in \Lambda : |P_\lambda(z)| \leq \delta\} \leq C\frac{\delta^d}{|z|^d},$$

then it follows by Fubini's theorem that the Hausdorff dimension of any measure μ is preserved under almost all projections of the family whenever $\dim_H \mu \leq d$.

The first case in Theorem 2.4 follows by using a quantitative version of the inverse function theorem from Peres and Schlag [PS, Lemma 7.6]. The second one requires slightly more work. The key is to correctly expand the one-dimensional collection of the m-dimensional image spaces $P_\lambda(\mathbb{R}^n)$ of the original family into an $(n-m)$-dimensional family of hyperplanes and use the estimates from the first case for the projections onto these hyperplanes. Then the claim for the original collection follows straightforwardly by Fubini's theorem. In the last case we also have to expand the original family; we embed the image lines $P_\lambda(\mathbb{R}^n)$ into a $k(n-k)$-collection of $(n-k)$-dimensional subspaces. For this complete family the dimensional bounds follow from the original projection results in Section 1. Fubini's theorem then again leads to the estimate 3).

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