On different ways of constructing relevant invariant measures

Tapani Tolonen

List of included articles

This dissertation consists of an introductory part and the following publications:

(A) Esa Järvenpää & Tapani Tolonen: *Relations between natural and observable measures*, Nonlinearity 18 (2005), 897-912.

(B) Tapani Tolonen: *Dynamics of sectional mappings*, Preprint 333, Department of Mathematics and Statistics, University of Jyväskylä.

The author of this dissertation has actively taken part in research of the paper (A).

1 Introduction

1.1 Invariant measures

The theory of dynamical systems has a close relationship with physics and natural sciences in general, and, therefore, the idea of physical relevance plays an important role in it. Although the setting and the methods used are purely mathematical, one is eager to see them having observable counterparts in real physical systems. In ergodic theory, especially in the theory of invariant measures, this desire has lead to the concept - or, more accurately, concepts - of the Sinai-Ruelle-Bowen (SRB) measure. The ideas behind the SRB measures were developed in papers and monographs by Yakov Sinai, David Ruelle and Rufus Bowen. For example [12], [11], [2]. (See also [14] for more references.)

Let X be some suitable measure space and let $T : X \to X$ be a mapping. The pair forms a dynamical system that is studied in the ergodic theory, usually with some restrictions imposed on the function or space. For a given measure μ on X, the function T defines the image measure $T_*\mu$, which is defined by

$$T_*\mu(B) = \mu(T^{-1}(B)),$$

for every $B \subset X$. We are interested in the measures that are invariant with respect to T, that is, the measures μ for which $T_*\mu = \mu$. In a sense, invariant measures describe the dynamical behaviour of T by giving its equilibrium states. Let us think that the density of a measure on the unit interval is represented by piles of objects, for example, boxes, such that if the pile is high in some spot, then the measure of this spot is large. If the measure is invariant, then the mapping T does not change this distribution, though it may move the boxes. The ones taken away from one pile are compensated b!
y others taken from the other piles and moved into this one. The system is in equilibrium. Thus, once we know the invariant measure, we know the system more or less. Invariant measures usually have their supports in attractors, but while an attractor is only a set, with all points equal in importance, the invariant measure distinguishes between the different points in the attractor and therefore describes the behaviour of the function better. Indeed, if the invariant measure of a set is large, then we know that most of the points stay long in this set when iterated. The search for invariant measures is thus motivated. If the measure of the space where we are operating is finite, we are usually interested in invariant *probability* measures, that is, measures that are scaled so that the measure of the whole space is 1. This is because the positive multiples of any invariant measure are also invariant, but in a sense they are still the same.

However, dynamical systems often have also many invariant probability measures, and only some of them are interesting. For example, we know that every continuous function from the unit interval to itself has a fixed point, and, therefore, the Dirac measure at this point is invariant for the function. Since this same measure is invariant for all the functions with the same fixed point, it does not necessarily give us good information of the function. Of course, if the fixed point is also an attractor, we are quite happy with this. Since all the points tend towards it, the fixed point - and therefore the measure supported on it - is somehow representative. The points and measures that are not representative are in a sense *exceptional*. The idea of "exception" has a lot to do with physical relevance.

When we try to grasp real phenomena by mathematical models, there are often some anomalous features in the model that are possible but rarely or never experienced in nature. This is usually expressed by labelling them mere "theoretical possibilities". The repellent fixed point is a case in point. To call it exceptional is to say that it has no physical relevance, since in the real situation we would not even observe it. This is where the measure theory comes in handy: We have some back-ground measure, usually the Lebesgue measure, and we use it to estimate the importance of points. If the exceptional behaviour is present only in a set with zero measure, then it is irrelevant and does not have to be taken seriously. As stated earlier, there are exceptional or unrepresentative measures as well as points, and the quest for SRB measures is exactly the quest for physically relevant, representative and unexceptional measures. It can be achieved in many ways and, therefore, there! have been many different candidates for SRB measures. For an important class of mappings, namely, hyperbolic systems, these seemingly different definitions actually give the same measure. However, in [1] it was noticed that the definitions of the SRB measure may differ considerably for more general dynamical systems. I will introduce three candidates for SRB measures: natural and observable measures and zero-noise limits, but first I state some definitions.

1.2 Some definitions

A probability measure μ, which is not necessarily invariant, is *ergodic* with respect to the mapping $f : X \to X$, if $\mu(E) = 1$ or $\mu(E) = 0$ for every set $E \subset X$ for which $f^{-1}(E) = E$. The function f is *non-singular*, if $\mathcal{L}(f^{-1}(E)) = 0$ for every $E \subset X$, provided that $\mathcal{L}(E) > 0$. This means that big sets are not mapped into small sets. Here $\mathcal{L}$ is the Lebesgue measure. It could also be replaced by other reference measures. A measure μ is *absolutely continuous* with respect to another measure ν if $\nu(A) = 0$ implies $\mu(A) = 0$.

1.3 Observable and natural measure

The observable measure is nowadays, to my knowledge, generally called *the* SRB measure. It formalizes the above-mentioned idea of typical point behaviour. Let X be a topological space, equipped with a measure λ such that $\lambda(U) > 0$ for every open $U \subset X$ and let T be a mapping from X to itself. We say that a measure μ is *observable* if there exists an open $U \subset X$ such that for λ-almost every $x \in U$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_*^k \delta_x = \mu,$$

where the limit is taken in the normal weak* sense. Here δ_x is the usual Dirac measure in x. It gives measure 1 to every set that contains the point x and 0 to all the other sets. Usually λ is taken to be the Lebesgue measure, since it is considered the most natural. The measure obtained above is called observable because there is a "large" set, contained in the open set U, consisting of points that all behave ! the same way. The physical counterparts of these points can therefore in principle be observed in a real physical test.

The measure μ is called *natural* if there exists an open set $U \subset X$ such that for every measure ν that is absolutely continuous with respect to λ and with support in U, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_*^k \nu = \mu.$$

This definition is in a sense a modification of the former one: Instead of taking only the time-average of one point, one also takes a space-average over a large set of points.

1.4 Zero-noise limits

There is still another interesting way of achieving physical relevance for the invariant measure. The idea is as follows (see also [7], and [14]): Given a dynamical system, one can perturb it a little by moving each point randomly after each iteration. Then one can take the invariant measure of this random process, if it exists. When the size of the random perturbation tends to zero, these invariant measures of the random processes tend towards an invariant measure of the unperturbed function in the weak* topology. This is a way to create an invariant measure and also to estimate the relevance of a given invariant measure. If it coincides with the zero-noise limit, it is called *stochastically stable*. It is physically relevant, because adding random noise in the process simulates the necessary imprecision and indefiniteness of any real physical test measurement. Like Young puts it: "If one accepts that the world is intrinsically a little nois! y, then zero-noise limits are the observable invariant measures.“ [14, p.736].

1.5 The relationship between natural and observable measures

The difference between the definitions of natural and observable measures was especially noticed by Michael Blank and Leonid Bunimovich, who showed in [1] that though an observable measure is always also natural, the converse is not true in general. Moreover, they gave some requirements under which the converse does hold. In [4] we wanted to study the relationship more closely and, especially, construct a function that has no observable measure but still has a natural measure. Blank and Bunimovich had already constructed such a function, but their natural measure was not ergodic, and we wanted to modify it to get this property.

Blank and Bunimovich's function was an example of a larger class of mappings that were studied by Tomoki Inoue in [5]. They are defined on the unit interval, have one point of discontinuity, two indifferent fixed points and under iteration most of the points wander between these two in an unstable manner. Gerhard Keller used an Inoue function in [6] to show that a stochastic attractor (measure) is not always an SRB-measure, and the selfsame function worked also for the purposes of Blank and Bunimovich. Originally, we also tried to modify some Inoue function but ended up taking only the main idea of wandering points and used a completely different method of construction. I try to clarify the idea a little without going into details.

The observable measure is more or less concentrated in the set where every point stays under iteration most of the time. If there is no set where the point stays longer and longer, then there is no observable measure. However, there may still be a natural measure, because it is constructed by also averaging over many points at any time period. A natural measure is concentrated in a set where most of the points stay most of the time. The difference is this: At any moment the majority of points may be in some set, but if one takes any *one* point and observes its orbit, it oscillates between the neighbourhoods of two points, A and B, in a very unstable way: first it stays long in the vicinity of A, then much longer close to B and so on, so that there does not exist any average location of the point. And all the time, however, the absolute majority of points is close - and closer every moment - to A. Every point will leave this majority sooner or later, but only to be replaced by other points coming back from their wanderings, so that at any moment *most of them* are close to A. Therefore, the Dirac measure on A will be natural and also ergodic for this mapping.

We were able to construct such a function on the unit interval, but it was highly discontinuous. (The Inoue functions are also discontinuous but only at one point.) By modifying the domain space we managed to make it continuous and even to make the domain space connected. Later, Michał Misiurewicz in [10] showed that there are such continuous functions even on the n-dimensional torus. In 2006, Victor Kleptsyn [8] proved that there are such smooth functions in the plane. His example is also based on wandering between two points.

After constructing the counter-example, we turned our attention to positive results and managed to prove the theorem 2.4 in [4], in which we show what is to be required of a natural measure to obtain other properties, for example, observability and ergodicity. Roughly speaking, the results were as follows. (Please consult [4] for the precise formulation and full generality.)

1. If the natural measure is invariant and absolutely continuous (with respect to the Lebesgue measure), then it is also ergodic.

2. If the function is non-singular and its natural measure is invariant, then the latter is absolutely continuous.

3. If the function is non-singular and the Lebesgue measure is absolutely continuous with respect to the invariant natural measure, then the latter is ergodic, observable and absolutely continuous.

We also constructed counter-examples showing that these requirements were necessary.

1.6 Sectional mappings

The problem behind [13] was natural: to describe the relationship between the so called sectional dynamics and the dynamics of the function itself. I try to clarify these concepts.

For any mapping in a linear space of dimension at least two, one can define sectional mappings in the following way. Consider, for example, a function f from the two-dimensional unit square to itself. This f can be presented as $f = (f_1, f_2)$, where f_1 and f_2 are from the unit square to the unit interval. For a fixed $a \in [0, 1]$ we can define two sectional functions, namely, $H_a(x) = f_1(x, a)$ and $V_a(y) = f_2(a, y)$. These functions are from the unit interval to itself and thus they have their own dynamics, for instance, their own observable measures, and one may ask whether these dynamics have something to do with the dynamics of f. More precisely: If all the sections, both horizontal and vertical, have, say, chaotic dynamics, does f necessarily display somehow chaotic behaviour, and vice versa? And if the dynamics of all the sections is contracting, that is, the sectional mappings take everything into one point, does this mean that f also takes everything into one point, or could it be chaotic nevertheless?

It turns out that, in general, the sectional dynamics and the "global" dynamics may not have anything to do with each other. Even if the function is quite good, that is, smooth, it can happen that sectional chaos leads to global contraction and sectional contraction to global chaos. I showed in [13] that all the invariant measures obtained in it were also stochastically stable and, therefore, physically relevant. The mappings considered were as good as one could hope for, namely smooth and hyperbolic, so this behaviour is not likely to be merely exceptional.

In my opinion it is obvious that if one wants to have a connection between sectional dynamics and the "global" dynamics, one has to require some dynamical properties of the functions, but I do not see how this could be done. In the two-dimensional setting the "global chaos but sectional contraction"-situation is easily achieved by rotating the unit square. If these rotations are prohibited in one way or the other, one could perhaps prevent this, but I do not have any other ideas.

Of course, the idea of sectional mapping makes sense also in any dimension greater than two, but then there are many ways to do the sectioning: In the three-dimensional space one can intersect the space by orthogonal lines or planes, and the possibilities are only multiplied when we move to an infinite-dimensional setting, for example, to the Hilbert cube. In this setting we define the sectional dynamics to be chaotic (respectively, contracting) if the dynamics for every finite-dimensional section is chaotic (contracting).

The reason why one wants to study infinite-dimensional spaces is, perhaps surprisingly, physical. The theory of statistical mechanics of lattice gases deals with infinite-dimensional systems, the lattice models. It is known that the set of equilibrium states of the whole system is the closed convex hull of limits of the equilibrium states of finite subsystems with different boundary conditions. Therefore, in a sense, the sectional systems determine the behaviour of the whole system. This interplay between an infinite-dimensional space and finite-dimensional subsystems is important, since in any real physical test, one can observe only finite quantities, as was pointed out in Esa and Maarit Järvenpää's article [3]. This is also important in the theory of coupled map lattices.

Though the equilibrium states are not given as invariant measures of some dynamical systems, one can nevertheless think that there is some "dynamics of nature" behind them. One purpose of [13] was to show that this dynamics must be of a rather special kind, since no standard smoothness assumptions made in the theory of dynamical systems imply this kind of behaviour. This was done by generalizing the above-mentioned two-dimensional mappings in infinite dimensions, that is, constructing mappings for which the invariant measures of the finite subsystems with fixed boundary values did not have anything in common with the respective measure of the whole system. These examples cannot be ignored as merely arbitrary exceptions, as they also have most of the nice properties one could ask for, namely, smoothness and hyperbolicity.

References

[1] Blank, Michael & Bunimovich, Leonid:
Multicomponent dynamical systems: SRB measures and phase transitions,
Nonlinearity 16 (2003), 387-401.

[2] Bowen, Rufus:
Markov partitions for Axiom A diffeomorphisms,
American Journal of Mathematics 92 (1970), 725-747.

[3] Järvenpää, Esa & Järvenpää, Maarit:
On the Definition of SRB-Measures for Coupled Map Lattices,
Communications in Mathematical Physics, 220, 1-12 (2001).

[4] Järvenpää, Esa & Tolonen, Tapani:
Relations between natural and observable measures,
Nonlinearity, 18 (2005),897-912.

[5] Inoue, Tomoki:
Sojourn times in small neighborhoods of indifferent fixed points of one-dimensional dynamical systems,
Ergodic Theory and Dynamical Systems, (2000), 20, 241-257.

[6] Keller, Gerhard:
Completely mixing maps without limit measure,
Colloquium Mathematicum, 100 (2004), no 1, 73-76.

[7] Kifer, Yuri :
Random perturbations of dynamical systems,
Birkhäuser, Boston, 1988.

[8] Kleptsyn, Victor:
An example of non-coincidence of minimal and statistical attractors,
Ergodic Theory and Dynamical Systems, 26 (2006),759-768.

[9] Kolyada, S. & Manin, Y. & Ward, T. (eds.):
Algebraic and Topological Dynamics (Contemporary Mathematics 385),
American Mathematical Society, Providence, RI (2005).

[10] Misiurewicz, Michał:
Ergodic natural measures,
in [9], 1-6.

[11] Ruelle, David:
Thermodynamic Formalism, Encyclopedia of Mathematics and its Applications, Vol 5,
Addison-Wesley, Reading, MA, 1978.

[12] Sinai, Yakov:
Markov Partitions and C-diffeomorphisms,
Functional Analysis and its Applications 2 (1968), 64-89.

[13] Tolonen, Tapani:
Dynamics of the sectional mappings
preprint, Department of Mathematics and Statistics, University of Jyväskylä, 2007.

[14] Young, Lai-Sang:
What are SRB Measures and Which Dynamical Systems Have Them,
Journal of Statistical Physics, Vol. 108, Nos 5/6, September 2002.