Self-improving properties of generalized Orlicz-Poincaré inequalities

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List of included articles

This thesis consists of an introductory part and of the following three articles:

[A] T. Heikkinen: **Sharp self-improving properties of generalized Orlicz-Poincaré inequalities in connected metric measure spaces**, Preprint 327, Department of Mathematics and Statistics, University of Jyväskylä, 2006.

[B] T. Heikkinen: **Self-improving properties of generalized Orlicz-Poincaré inequalities in metric measure spaces**, Preprint 328, Department of Mathematics and Statistics, University of Jyväskylä, 2006.

[C] T. Heikkinen: **Characterizations of Orlicz-Sobolev spaces in terms of generalized Orlicz-Poincaré inequalities**, Preprint 329, Department of Mathematics and Statistics, University of Jyväskylä, 2006.

1 Introduction

1.1 Classical Sobolev embeddings

Let $B \subset \mathbb{R}^n$ be a ball. The classical Sobolev embedding theorem [17] states that the Sobolev space $W^{1,p}(B)$, $1 \le p < n$, consisting of functions $u \in L^p(B)$ having weak (distributional) partial derivatives in $L^p(B)$, is continuously embedded in $L^{p^*}(B)$, where $p^* = np/(n-p)$. Moreover, there is a constant C depending only on dimension n such that the Sobolev-Poincaré inequality

$$\|u - u_B\|_{L^{p^*}(B)} \le C \| |\nabla u| \|_{L^p(B)}, \tag{1}$$

where $u_B = \fint_B u\,dx = \frac{1}{|B|}\int u\,dx$, holds for each $u \in W^{1,p}(B)$. If $p > n$, then $u \in W^{1,p}(B)$ has a representative, which is Hölder continuous with exponent $1-n/p$:

$$\sup_{x,y \in B} |u(x) - u(y)| \le C|x-y|^{1-n/p} \| |\nabla u| \|_{L^p(B)}. \tag{2}$$

This result originates from the work of Morrey [14]. In the critical case $p = n$, the space $W^{1,p}(B)$ is continuously embedded in the Orlicz space $\exp L^{n'}(B)$, where $n' = n/(n-1)$, and $u \in W^{1,n}(B)$ satisfies the Trudinger inequality

$$\|u - u_B\|_{\exp L^{n'}(B)} \le C \| |\nabla u| \|_{L^n(B)}, \tag{3}$$

where $\|\cdot\|_{\exp L^{n'}(B)}$ is the Luxemburg norm generated by the Young function $\Phi(t) = \exp(t^{n'}) - 1$. The Trudinger inequality (3) was independently obtained by Yudovich [20], Pokhozhaev [15] and Trudinger [18].

1.2 Sobolev spaces and Poincaré inequalities in metric spaces

Recently there have been attempts to generalize the theory of Sobolev spaces to the setting of a metric space equipped with a measure. Motivation for such a generalization comes from several examples. These include the study of the Carnot-Caratheodory metric generated by a family of vector fields, theory of quasiconformal mappings on Loewner spaces, analysis on topological manifolds, potential theory on infinite graphs and analysis on fractals, see [9] and the references therein.

One of the possible definitions of a Sobolev space on a metric measure space is based on upper gradients. A Borel function $g : X \to [0, \infty]$ is an upper gradient of a function $u : X \to \overline{\mathbb{R}}$, if for all rectifiable curves $\gamma : [0, l] \to X$,

$$|u(\gamma(0)) - u(\gamma(l))| \le \int_\gamma g\,ds \tag{4}$$

whenever both $u(\gamma(0))$ and $u(\gamma(l))$ are finite, and $\int_\gamma g\,ds = \infty$ otherwise. The concept of an upper gradient was introduced by Heinonen and Koskela in [11]. The

(Newton-)Sobolev space $N^{1,p}(X)$, consisting of functions $u \in L^p(X)$ having an upper gradient $g \in L^p(X)$, was introduced and studied by Shanmugalingam [16].

To obtain a reasonable theory one has to make some assumptions on the underlying space $X = (X, d, \mu)$. The basic assumptions are that the measure μ is doubling, and that the space X supports a p-Poincaré inequality for some $p \geq 1$. The doubling property of the measure means that there exists a constant C_d such that

$$\mu(2B) \leq C_d \mu(B), \tag{5}$$

whenever B is a ball. It is easy to see that the doubling property is equivalent to the existence of constants s and C_s such that

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \geq C_s^{-1} \left(\frac{r}{r_0}\right)^s \tag{6}$$

holds, whenever $x \in B(x_0, r_0)$ and $r \leq r_0$. The space X supports a p-Poincaré inequality, if there exist constants C_P and $\tau \geq 1$ such that

$$⨍_B |u - u_B| \, d\mu \leq C_P r \left(⨍_{\tau B} g^p \, d\mu \right)^{1/p} \tag{7}$$

whenever $B = B(x,r) \subset X$ is a ball, $u \in L^1_{\mathrm{loc}}(X)$ and g is an upper gradient of u.

The doubling spaces supporting the 1-Poincaré inequality (and so the p-Poincaré inequality for every $p \geq 1$) include Riemannian manifolds with nonnegative Ricci curvature, Q-regular orientable topological manifolds satisfying the local linear contractability condition, Carnot groups and more general Carnot-Carathéodory spaces associated with a system of vector fields satisfying Hörmander's condition, as well as more exotic spaces constructed by Bourdon and Pajot, Laakso, and Hanson and Heinonen, see [9] and the references therein. There exist also spaces that support the p-Poincare inequality for fixed $p > 1$ but not for any smaller exponent. In such spaces the usual representation theorems in terms of Riesz potentials may not be available.

Under the assumption that μ is doubling and that X supports the p-Poincaré inequality, versions of inequalities (1), (2) and (3) hold for functions in $N^{1,p}$. This is a consequence of a more general result, due to Hajłasz and Koskela [8, 9], concerning the self-improving properties of (7) for a general pair (u, g): Assume that μ satisfies (6), $B \subset X$ is a ball, $\delta > 0$, and that a pair (u, g), where $u \in L^1_{\mathrm{loc}}(X)$ and $0 \leq g \in L^p_{\mathrm{loc}}(X)$ satisfies the p-Poincaré inequality (7) for every ball B' such that $\tau B' \subset \hat{B} = (1+\delta)\tau B$. There exists a constant $C = C(C_s, s, C_P, \tau, \delta)$ such that the following holds.

1) If $p < s$, then

$$\sup_{t>0} t \left(\frac{\mu(\{x : |u(x) - u_B| > t\}}{\mu(B)} \right)^{1/p_s} \leq C r_B \left(⨍_{\hat{B}} g^p \, d\mu \right)^{1/p}, \tag{8}$$

where $p_s = \frac{sp}{s-p}$. Consequently, for $q < p_s$, we have

$$\left(\fint_B |u - u_B|^q \, d\mu \right)^{1/q} \le C' r_B \left(\fint_{\tau' B} g^p \, d\mu \right)^{1/p}, \tag{9}$$

where C' depends on C and q. In general, (7) does not yield (9) with $q = p_s$. However, if a pair (u, g) has the truncation property, which means that for every $b \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$ and $\varepsilon \in \{-1, 1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1 < v \le t_2\}})$, where $v = \varepsilon(u - b)$ and $v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}$, satisfies the p-Poincaré inequality, then we have (9) with $q = p_s$.

2) If $p = s$, then

$$\|u - u_B\|_{Ł^\Phi(B)} \le C r_B \|g\|_{Ł^s(\hat{B})}, \tag{10}$$

where $\|\cdot\|_{Ł^\Phi(B)}$ is the normalized Luxemburg norm generated by the function $\Phi(t) = \exp t - 1$. Moreover, if X is connected and $s > 1$, then (10) holds with $\Phi(t) = \exp(t^{s'}) - 1$, where $s' = \frac{s}{s-1}$.

3) If $p > s$, then u has a locally Hölder continuous representative, for which

$$|u(x) - u(y)| \le C r_B^{s/p} d(x,y)^{1-s/p} \left(\fint_{\hat{B}} g^p \, d\mu \right)^{1/p} \tag{11}$$

for $x, y \in B$.

1.3 Generalized Poincaré inequalities

Franchi, Pérez and Wheeden [5, 6] and MacManus and Pérez [12, 13] studied the self-improving properties of inequalities of type

$$\fint_B |u - u_B| \, d\mu \le \|u\|_a a(\tau B), \tag{12}$$

where $\|u\|_a > 0$, $\tau \ge 1$ and $a : \{B \subset X : B \text{ is a ball}\} \to [0, \infty)$ is a functional that satisfies certain discrete summability conditions. In [12] MacManus and Pérez showed that if $\delta > 0$ is fixed, and the functional a satisfies condition

$$\sum a(B_i)^r \mu(B_i) \le c^r a(B)^r \mu(B), \tag{13}$$

whenever the balls B_i are disjoint and contained in the ball B, then the Poincaré-type inequality (12) improves to

$$\sup_{\lambda > 0} \lambda \left(\frac{\mu(\{x \in B : |u(x) - u_B| > \lambda\})}{\mu(B)} \right)^{1/r} \le C\|a\| \|u\|_a a(\hat{B}), \tag{14}$$

where $\|a\|$ is the minimum of the constants c so that (13) holds and $\hat{B} = (1+\delta)\tau B$. In [13], they proved that if X is connected, $r > 1$, and a satisfies the stronger condition

$$\sum a(B_i)^r \leq c^r a(B)^r, \tag{15}$$

whenever the balls B_i are disjoint and contained in the ball B, then

$$\|u - u_B\|_{\mathcal{L}^\Phi(B)} \leq Ca(\hat{B}), \tag{16}$$

where $\Phi(t) = \exp(t^{r'}) - 1$ and $r' = \frac{r}{r-1}$.

To see that the results of MacManus and Pérez generalize those of Hajłasz and Koskela, simply note that if μ satisfies (6), then the functional

$$a(B) = r_B \left(\fint_B g^p \right)^{1/p},$$

where $0 \leq g \in L^p_{\mathrm{loc}}(X)$, satisfies condition (13) with $r = sp/(s-p)$, if $p < s$, and condition (15) with $r = s$, if $p = s$.

1.4 Self-improving properties of Orlicz-Poincaré inequalities in connected spaces

A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if it has the form

$$\Phi(t) = \int_0^t \phi(s)\, ds,$$

where $\phi : [0, \infty) \to [0, \infty]$ is increasing, left-continuous function, which is neither identically zero nor identically infinite on $(0, \infty)$. The purpose of this thesis is to study the self-improving properties of the following Φ-Poincaré inequality, introduced recently in [19] in connection with the study of the Orlicz-Sobolev space $N^{1,\Phi}(X)$ consisting of functions $u \in L^\Phi(X)$ having an upper gradient $g \in L^\Phi(X)$.

Definition 1.1 *Let Φ be a Young function. A pair (u, g) of measurable functions, $u \in L^1_{loc}(X)$ and $g \geq 0$, satisfies the Φ-Poincaré inequality (in an open set U), if there are constants C_P and τ such that*

$$\fint_B |u - u_B|\, d\mu \leq C_P r_B \Phi^{-1} \left(\fint_{\tau B} \Phi(g)\, d\mu \right) \tag{17}$$

for every ball $B \subset X$ (such that $\tau B \subset U$).

Notice that, in $\mathbb{R}^n$, a pair $(u, |\nabla u|)$ of a weakly differentiable function and the length of its weak gradient satisfies the 1-Poincaré inequality, and so, by Jensen's inequality, the Φ-Poincaré inequality for every Young function Φ.

An optimal embedding theorem for the Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^n)$ was recently proved by Cianchi [1, 2], see also [3]. Our first and main goal is to extend this result to the metric setting.

Before stating (a version of) Cianchi's result and its generalization we have to introduce some notation. Let $s > 1$. For a Young function Φ satisfying

$$\int_0^1 \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty \quad \text{and} \quad \int_0^\infty \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt = \infty, \tag{18}$$

define

$$\Phi_s = \Phi \circ \Psi_s^{-1}, \tag{19}$$

where

$$\Psi_s(r) = \left(\int_0^r \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt\right)^{1/s'}. \tag{20}$$

If

$$\int^\infty \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty, \tag{21}$$

define

$$\omega_s(t) = (t\Theta^{-1}(t^{s'}))^{s'}, \tag{22}$$

where Θ^{-1} is the left-continuous inverse of the function given by

$$\Theta(r) = s' \int_r^\infty \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt \tag{23}$$

and $\hat{\Phi}$ is the conjugate of Φ.

We will state Cianchi's result only for balls, but it actually holds for much more general domains: Let $s \geq 2$, let $B \subset \mathbb{R}^s$ be a ball, and let u be a weakly differentiable function such that $|\nabla u| \in L^\Phi(B)$. Then there is a constant C depending only on s such that

1) If (18) holds, then

$$\|u - u_B\|_{L^{\Phi_s}(B)} \leq C \||\nabla u|\|_{L^\Phi(B)}.$$

Moreover, $L^{\Phi_s}(B)$ is the smallest Orlicz space into which $W^{1,\Phi}(B)$ can be continuously embedded.

2) If (21) holds, then u has a continuous representative for which

$$|u(x) - u(y)| \leq C \||\nabla u|\|_{L^\Phi(B)} \omega_s^{-1}(|x-y|^{-s}),$$

for $x, y \in B$.

The main result of this thesis is the following.

Theorem 1.2 ([A]) *Assume that X is connected, μ satisfies* (6) *with $1 < s < \infty$, $B \subset X$ is a ball, $\delta > 0$, $\hat{B} = (1+\delta)\tau B$, $g \in L^\Phi(\hat{B})$, and that a pair $(\hat{u}, \hat{g})$, where $\hat{u} = u/\|g\|_{L^\Phi(\hat{B})}$ and $\hat{g} = g/\|g\|_{L^\Phi(\hat{B})}$, satisfies the Φ-Poincaré inequality in $\hat{B}$.*

1) If (18) *holds, then*

$$\|u - u_B\|_{L_w^{\Phi_s}(B)} \le C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}, \tag{24}$$

where Φ_s is defined by (19)-(20)*. Moreover, if the pair $(\hat{u}, \hat{g})$ has the truncation property, then*

$$\|u - u_B\|_{L^{\Phi_s}(B)} \le C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}. \tag{25}$$

2) If (21) *holds, then, for Lebesgue points $x, y \in B$ of u,*

$$|u(x) - u(y)| \le C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \omega_s^{-1}(r_B^s \mu(B)^{-1} d(x,y)^{-s}), \tag{26}$$

where ω_s is defined by (22)-(23).

Here, $C = C(C_s, s, C_P, \tau, \delta)$.

As a consequence, we obtain an optimal embedding theorem for the space $N^{1,\Phi}(X)$.

Corollary 1.3 *Assume that (X, d, μ) is a doubling metric measure space that supports the Φ-Poincaré inequality and satisfies* (6) *with $s > 1$. Let B be a ball, $\delta > 0$ and $\hat{B} = (1+\delta)\tau B$.*

1) If Φ satisfies (18)*, then $N^{1,\Phi}(\hat{B}) \subset L^{\Phi_s}(B)$, where the embedding is continuous. Moreover, each $u \in N^{1,\Phi}(\hat{B})$ and every upper gradient g of u satisfy* (25).

2) If Φ satisfies (21)*, then $N^{1,\Phi}(\hat{B}) \subset C(B)$. Moreover, each $u \in N^{1,\Phi}(\hat{B})$ and every upper gradient g of u satisfy* (26).

Apart from the case $X = \mathbb{R}^n$, Theorem 1.2 and Corollary 1.3 seem to be new even if the Φ-Poincaré inequality in the assumptions is replaced by the 1-Poincaré inequality.

The following example gives concrete expressions for the "Sobolev conjugate" Φ_s.

Example 1.4 *Let Φ be equivalent to the function $t^p \log^q t$ near infinity, where either $p = 1$ and $q \ge 0$ or $p > 1$ and $q \in \mathbb{R}$. Then Φ_s is equivalent near infinity to*

$$\begin{cases} t^{sp/(s-p)} (\log t)^{sq/(s-p)} & \textit{if } 1 \le p < s \\ \exp(t^{s/(s-1-q)}) & \textit{if } p = s,\ q < s-1 \\ \exp(\exp(t^{s/(s-1)})) & \textit{if } p = s,\ q = s-1. \end{cases}$$

1.5 Self-improving properties of Orlicz-Poincaré inequalities in the general case

It is essential in Theorem 1.2 that the underlying space X is connected. In [B] we investigate the general case. Instead of assuming that Φ is a Young function, we assume the following:

(Φ-1) $\Phi : [0,\infty) \to [0,\infty)$ is an increasing bijection.

(Φ-2) The function $t \mapsto \frac{\Phi(t)}{t^{s/(s+1)}}$ is increasing.

Notice that (Φ-2) allows Φ to increase essentially more slowly than any Young function. The results concerning such Φ are new also for connected spaces.

Theorem 1.5 ([B]) *Assume that Φ satisfies (Φ-1) and (Φ-2), μ satisfies* (6) *with $0 < s < \infty$, $B \subset X$ is a ball, $\delta > 0$, $\tau \geq 1$, $\hat{B} = (1+\delta)\tau B$, and that a pair $(\hat{u}, \hat{g})$, where $\hat{u} = \|g\|^{-1}_{L^\Phi(\hat{B})} u$ and $\hat{g} = \|g\|^{-1}_{L^\Phi(\hat{B})} g$, satisfies the Φ-Poincaré inequality in $\hat{B}$.*

1) If

$$\int_0^1 \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt < \infty \quad \text{and} \quad \int_0^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt = \infty, \tag{27}$$

then

$$\|u - u_B\|_{L_w^{\tilde{\Phi}_s}(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}, \tag{28}$$

where

$$\tilde{\Phi}_s^{-1}(r) = \int_0^r \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt. \tag{29}$$

Moreover, if the pair $(\hat{u}, \hat{g})$ has the truncation property, then

$$\|u - u_B\|_{L^{\tilde{\Phi}_s}(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}. \tag{30}$$

2) If

$$\int^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt < \infty, \tag{31}$$

then, for Lebesgue points $x, y \in B$ of u,

$$|u(x) - u(y)| \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \tilde{\omega}_s(\mu(B)^{-1} r_B^s d(x,y)^{-s}),$$

where

$$\tilde{\omega}_s(r) = \int_r^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt. \tag{32}$$

Here, $C = C(C_s, s, C_P, \tau, \delta)$.

If Φ is "close" to the function $t \mapsto t^s$, the conclusion of Theorem 1.5 is weaker than that of Theorem 1.2.

Example 1.6 *Let Φ be equivalent near infinity to the function $t^s \log^q t$. Then the function Φ_s is equivalent near infinity to*

$$\begin{cases} \exp(t^{s/(s-1-q)}) & \text{if } q < s-1 \\ \exp(\exp(t^{s/(s-1)})) & \text{if } q = s-1, \end{cases}$$

and the function $\tilde{\Phi}_s$ is equivalent near infinity to

$$\begin{cases} \exp(t^{s/(s-q)}) & \text{if } q < s \\ \exp(\exp(t)) & \text{if } q = s. \end{cases}$$

On the other hand, if Φ is a Young function such that the function $t \mapsto \Phi(t)/t^p$ is either decreasing for some $p < s$, or increasing for some $p > s$, then Theorem 1.5 is equivalent to Theorem 1.2 ([B, Theorem 1.5]).

1.6 Self-improving properties of generalized Orlicz-Poincaré inequalities

In both [A] and [B], we also prove some abstract self-improving results which can be formulated in terms of spaces $A_\tau^{\Phi,s}(\Omega)$ defined as follows.

Definition 1.7 *Let Ω be an open set, Φ a Young function, $\tau \geq 1$ and $0 < s \leq \infty$. Denote*

$$\mathcal{B}_\tau(\Omega) = \{\{B_i\} : \text{balls } \tau B_i \text{ are disjoint and contained in } \Omega\}$$

and

$$\|u\|_{A_\tau^{\Phi,s}(\Omega)} = \sup_{\mathcal{B} \in \mathcal{B}_\tau(\Omega)} \| \sum_{B \in \mathcal{B}} \left(\mu(B)^{-1/s} \fint_B |u - u_B| \, d\mu \right) \chi_B \|_{L^\Phi(\Omega)}.$$

Then $A_\tau^{\Phi,s}(U)$ consists of all locally integrable functions u for which the number $\|u\|_{A_\tau^{\Phi,s}(U)}$ is finite.

Theorem 1.8 below was proved in [A]. Similar results in the general setting were obtained in [B]. Notice that here the number $1 < s < \infty$ is *any* number and need not have anything to do with (6).

Theorem 1.8 *Let X be connected, μ doubling, Φ a Young function, $B \subset X$ a ball, $1 < s < \infty$, $\tau \geq 1$ and $\delta > 0$. Denote $\hat{B} = (1+\delta)\tau B$.*

1) If (18) *holds, then*

$$\|u - u_B\|_{L_w^{\Phi_s}(B)} \leq C \|u\|_{A_\tau^{\Phi,s}(\hat{B})}, \tag{33}$$

where Φ_s is defined by (19)-(20).

2) If (21) *holds, then, for Lebesgue points $x, y \in B$ of u,*

$$|u(x) - u(y)| \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\omega_s^{-1}(\mu(B_{xy})^{-1}), \tag{34}$$

where $B_{xy} = B(x, 2d(x,y))$, and ω_s is defined by (22)-(23).

Here, $C = C(C_d, \tau, \delta)$.

It is easy to see that the first part of Theorem 1.2 is a consequence of inequality (33). Also, when $\Phi(t) = t^s$, (33) easily implies the generalized Trudinger inequality (16) of MacManus and Peréz.

1.7 Characterizations of Orlicz-Sobolev spaces in terms of (generalized) Orlicz-Poincaré inequalities

The results from [A] and [B] discussed above deal with improved regularity of functions that satisfy a Φ-Poincare inequality. In the euclidean setting, Orlicz-Sobolev functions satisfy such inequalities. It is then natural to ask if functions that satisfy such an inequality have a nice generalized gradient. The main result of [C] is the following.

Theorem 1.9 *Assume that Φ is a doubling Young function, μ is doubling, $\Omega \subset X$ is open, $u, g \in L^\Phi(\Omega)$, and that the pair $(\hat{u}, \hat{g})$, where $\hat{u} = u/\|g\|_{L^\Phi(\Omega)}$ and $\hat{g} = g/\|g\|_{L^\Phi(\Omega)}$, satisfies the Φ-Poincaré inequality in Ω. Then a representative of u has a Φ-weak upper gradient g_u such that $\|g_u\|_{L^\Phi(\Omega)} \le C(C_d, C_P, \tau)\|g\|_{L^\Phi(\Omega)}$.*

In the case $\Phi(t) = t^p$, $p \ge 1$, the result was essentially proven in [4], see [7].

It follows from Theorem 1.9 that, for an open set $\Omega \subset \mathbb{R}^n$, the Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ can be characterized in terms of the Φ-Poincaré inequality: $u \in L^\Phi(\Omega)$ belongs to $W^{1,\Phi}(\Omega)$ if and only if there exists $g \in L^\Phi(\Omega)$ such that the pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ-Poincaré inequality in Ω.

If both Φ and its conjugate $\hat{\Phi}$ are doubling, the assumptions of Theorem 1.9 can be relaxed. In order to conclude that a representative of $u \in L^\Phi(\Omega)$ is in $N^{1,\Phi}(\Omega)$, it suffices to assume that the number

$$\|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)} = \sup_{\mathcal{B} \in \mathcal{B}_\tau(\Omega)} \|\sum_{B \in \mathcal{B}} \big(r_B^{-1} \fint_B |u - u_B|\, d\mu\big)\chi_B\|_{L^\Phi(\Omega)}, \tag{35}$$

where

$$\mathcal{B}_\tau(\Omega) = \big\{\{B_i\} : \text{balls } \tau B_i \text{ are disjoint and contained in } \Omega\big\},$$

is finite. Notice that $\|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)} \le \lambda$ if and only if there is a functional $\nu : \{B \subset \Omega : B \text{ is a ball}\} \to [0, \infty)$ such that

$$\sum_i \nu(B_i) \le 1, \tag{36}$$

whenever the balls $B_i \in \mathcal{B}$ are disjoint, and that the generalized Φ-Poincaré inequality

$$\fint_B |u - u_B|\, d\mu \leq \lambda r_B \Phi^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right) \tag{37}$$

holds whenever $\tau B \subset \Omega$.

The spaces $\mathcal{A}_\tau^{1,\Phi}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) : \|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)} < \infty\}$, for $\Phi(t) = t^p$, were studied in [10]. Theorem 1.10 below is a generalization of [10, Theorem 1.1].

Theorem 1.10 ([C]) *Assume that both Φ and $\hat{\Phi}$ are doubling, μ is doubling, and that $\Omega \subset X$ is an open set. Then a representative of $u \in \mathcal{A}_\tau^{1,\Phi}(\Omega) \cap L^\Phi(\Omega)$ has a Φ-weak upper gradient g with $\|g\|_{L^\Phi(\Omega)} \leq C(C_d, \tau)\|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)}$.*

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Included articles

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Sharp self-improving properties of generalized Orlicz-Poincaré inequalities in connected metric measure spaces

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Sharp self-improving properties of generalized Orlicz-Poincaré inequalities in connected metric measure spaces

Toni Heikkinen[*]

Abstract

We study the self-improving properties of (generalized) Φ-Poincaré inequalities in connected metric spaces equipped with a doubling measure. Our results are optimal and generalize some of the results of Cianchi [1, 2], Hajłasz and Koskela [5, 6], and MacManus and Pérez [12].

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1 Introduction and main results

Let $X = (X, d, \mu)$ be a metric measure space with μ a Borel regular outer measure satisfying $0 < \mu(U) < \infty$, whenever U is nonempty, open and bounded. Suppose further that μ is doubling, that is, there exists a constant C_d such that

$$\mu(2B) \leq C_d \mu(B), \tag{1}$$

whenever B is a ball. It is easy to see that the doubling property is equivalent to the existence of constants s and C_s such that

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \geq C_s^{-1} \left(\frac{r}{r_0}\right)^s \tag{2}$$

holds, whenever $x \in B(x_0, r_0)$ and $r \leq r_0$.

A pair (u, g) of measurable functions, $g \geq 0$, satisfies the p-Poincaré inequality, if there exist constants C_P and $\tau \geq 1$ such that

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \left(\fint_{\tau B} g^p \, d\mu\right)^{1/p} \tag{3}$$

for every ball $B = B(x, r) \subset X$. Hajłasz and Koskela [5, 6] proved the following self-improving properties of (3): Assume that μ satisfies (2), and that a pair (u, g), where $g \in L^p_{\mathrm{loc}}(X)$ satisfies the p-Poincaré inequality (3). Let $\delta > 0$ and $\hat{B} = (1 + \delta)\tau B$. There exists a constant $C = C(C_s, s, C_P, \tau, \delta)$ such that the following holds.

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1) If $p < s$, then

$$\sup_{t>0} t\left(\frac{\mu(\{x : |u(x) - u_B| > t\})}{\mu(B)}\right)^{1/p_s} \leq C r_B \left(\fint_{\hat{B}} g^p \, d\mu\right)^{1/p}, \tag{4}$$

where $p_s = \frac{sp}{s-p}$. Consequently, for $q < p_s$, we have

$$\left(\fint_B |u - u_B|^q \, d\mu\right)^{1/q} \leq C' r_B \left(\fint_{\tau' B} g^p \, d\mu\right)^{1/p}, \tag{5}$$

where C' depends on C and q. In general, (3) does not yield (5) with $q = p_s$. However, if a pair (u, g) has the truncation property, which means that for every $b \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$ and $\varepsilon \in \{-1, 1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1 < v \leq t_2\}})$, where $v = \varepsilon(u - b)$ and $v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}$, satisfies the p-Poincaré inequality, then we have (5) with $q = p_s$.

2) If $p = s > 1$ and X is connected, then

$$\|u - u_B\|_{Ł^\Phi(B)} \leq C r_B \|g\|_{Ł^s(\hat{B})}, \tag{6}$$

where $\|\cdot\|_{Ł^\Phi(B)}$ is the normalized Luxemburg norm generated by the function $\Phi(t) = \exp(t^{s'}) - 1$ (see Section 2) and $s' = \frac{s}{s-1}$.

3) If $p > s$, then u has a locally Hölder continuous representative, for which

$$|u(x) - u(y)| \leq C r_B^{s/p} d(x, y)^{1-s/p} \left(\fint_{\hat{B}} g^p \, d\mu\right)^{1/p} \tag{7}$$

for $x, y \in B$.

Franchi, Pérez and Wheeden [4] and MacManus and Pérez [11, 12] studied the self-improving properties of inequalities of type

$$\fint_B |u - u_B| \, d\mu \leq \|u\|_a a(\tau B), \tag{8}$$

where $\|u\|_a > 0$, $\tau \geq 1$ and $a : \{B \subset X : B \text{ is a ball}\} \to [0, \infty)$ is a functional that satisfies certain discrete summability conditions. In [11] MacManus and Pérez showed that if $\delta > 0$ is fixed, and the functional a satisfies condition

$$\sum a(B_i)^r \mu(B_i) \leq c^r a(B)^r \mu(B), \tag{9}$$

whenever the balls B_i are disjoint and contained in the ball B, then the Poincaré-type inequality (8) improves to

$$\sup_{\lambda>0} \lambda \left(\frac{\mu(\{x \in B : |u(x) - u_B| > \lambda\})}{\mu(B)}\right)^{1/r} \leq C\|a\|\|u\|_a a(\hat{B}), \tag{10}$$

where $\|a\|$ is the minimum of the constants c so that (9) holds and $\hat{B} = (1 + \delta)\tau B$. In [12], they proved that if X is connected, $r > 1$, and a satisfies the stronger condition

$$\sum a(B_i)^r \leq c^r a(B)^r, \tag{11}$$

whenever the balls B_i are disjoint and contained in the ball B, then

$$\|u - u_B\|_{L^\Phi(B)} \leq Ca(\hat{B}), \tag{12}$$

where $\Phi(t) = \exp(t^{r'}) - 1$ and $r' = \frac{r}{r-1}$.

To see that the results of MacManus and Pérez generalize those of Hajłasz and Koskela, simply note that if μ satisfies (2), then the functional

$$a(B) = r_B \left(\fint_B g^p \, d\mu \right)^{1/p},$$

where $0 \leq g \in L^p_{\mathrm{loc}}(X)$, satisfies condition (9) with $r = sp/(s-p)$, if $p < s$, and condition (11) with $r = s$, if $p = s$.

In this paper we are interested in the self-improving properties of the following Φ-Poincaré inequality, introduced recently in [14]. For the definition and properties of Young functions and Orlicz spaces, see Section 2.

Definition 1.1 *Let Φ be a Young function. A pair (u, g) of measurable functions, $u \in L^1_{loc}(X)$ and $g \geq 0$, satisfies the Φ-Poincaré inequality (in an open set U), if there are constants C_P and τ such that*

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \Phi^{-1} \left(\fint_{\tau B} \Phi(g) \, d\mu \right) \tag{13}$$

for every ball $B \subset X$ (such that $\tau B \subset U$).

Assuming that the underlying space is connected, we obtain results which are sharp in the sense that they reproduce a version of Cianchi's optimal embedding theorem for Orlicz-Sobolev spaces on $\mathbb{R}^n$ [1, 2]. Notice that a pair $(u, |\nabla u|)$ of a weakly differentiable function and the length of its weak gradient satisfies the 1-Poincaré inequality, and so, by Jensen's inequality, the Φ-Poincaré inequality for every Young function Φ.

Let $s > 1$. For a Young function Φ satisfying

$$\int_0^1 \left(\frac{t}{\Phi(t)} \right)^{s'-1} dt < \infty \quad \text{and} \quad \int_0^\infty \left(\frac{t}{\Phi(t)} \right)^{s'-1} dt = \infty, \tag{14}$$

define

$$\Phi_s = \Phi \circ \Psi_s^{-1}, \tag{15}$$

where

$$\Psi_s(r) = \left(\int_0^r \left(\frac{t}{\Phi(t)} \right)^{s'-1} dt \right)^{1/s'}. \tag{16}$$

If

$$\int^\infty \left(\frac{t}{\Phi(t)} \right)^{s'-1} dt < \infty, \tag{17}$$

define

$$\omega_s(t) = (t\Theta^{-1}(t^{s'}))^{s'}, \tag{18}$$

where Θ^{-1} is the left-continuous inverse of the function given by

$$\Theta(r) = s' \int_r^\infty \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt \tag{19}$$

and $\hat{\Phi}$ is the conjugate of Φ. We wish to point out that, under (14), functions Φ, Ψ_s and Φ_s are bijections. Notice also that one can modify any Young function Φ near zero so that the condition

$$\int_0^1 \left(\frac{t}{\tilde{\Phi}(t)}\right)^{s'-1} dt < \infty$$

is satisfied for the modified function $\tilde{\Phi}$ and that $L^{\tilde{\Phi}}_{\mathrm{loc}}(X) = L^{\Phi}_{\mathrm{loc}}(X)$.

We will state Cianchi's result only for balls, but it actually holds for much more general domains (see [1, 2, 3]): *Let $s \geq 2$, let $B \subset \mathbb{R}^s$ be a ball, and let u be a weakly differentiable function such that $|\nabla u| \in L^\Phi(B)$. Then there is a constant C depending only on s such that*

1) If (14) *holds, then*

$$\|u - u_B\|_{L^{\Phi_s}(B)} \leq C \||\nabla u|\|_{L^\Phi(B)}.$$

Moreover, $L^{\Phi_s}(B)$ is the smallest Orlicz space into which $W^{1,\Phi}(B)$ can be continuously embedded.

2) If (17) *holds, then u has a continuous representative for which*

$$|u(x) - u(y)| \leq C \||\nabla u|\|_{L^\Phi(B)} \omega_s^{-1}(|x-y|^{-s}),$$

for $x, y \in B$.

Theorems 1.2 and 1.4 below generalize the result of Cianchi.

Theorem 1.2 *Assume that X is connected, μ satisfies* (2) *with $1 < s < \infty$, $B \subset X$ is a ball, $\delta > 0$, $\hat{B} = (1+\delta)\tau B$, $g \in L^\Phi(\hat{B})$, and that a pair $(\hat{u}, \hat{g})$, where $\hat{u} = \|g\|^{-1}_{L^\Phi(\hat{B})} u$ and $\hat{g} = \|g\|^{-1}_{L^\Phi(\hat{B})} g$, satisfies the Φ-Poincaré inequality in $\hat{B}$.*

1) If (14) *holds, then*

$$\|u - u_B\|_{L^{\Phi_s}_w(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}, \tag{20}$$

where Φ_s is defined by (15)-(16).

2) If (17) *holds, then, for Lebesgue points $x, y \in B$ of u,*

$$|u(x) - u(y)| \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \omega_s^{-1}(r_B^s \mu(B)^{-1} d(x,y)^{-s}), \tag{21}$$

where ω_s is defined by (18)-(19).

Here, $C = C(C_s, s, C_P, \tau, \delta)$.

If the Φ-Poincaré inequality is stable under truncations, the weak estimate (20) turns into a strong one.

Definition 1.3 *A pair (u,g) has the truncation property, if for every $b \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$ and $\varepsilon \in \{-1,1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1<v\le t_2\}})$, where $v = \varepsilon(u-b)$ and*

$$v_{t_1}^{t_2} = \min\{\max\{0, v-t_1\}, t_2-t_1\},$$

satisfies the Φ-Poincaré inequality (with fixed constants).

A weakly differentiable function u on $\mathbb{R}^n$ satisfies $|\nabla v_{t_1}^{t_2}| = |\nabla u|\chi_{\{t_1<v\le t_2\}}$, which implies that the pair $(u, |\nabla u|)$ has the truncation property.

Theorem 1.4 *Suppose that the assumptions of Theorem 1.2 are in force,* (14) *holds, and that the pair $(\hat{u}, \hat{g})$ has the truncation property. Then*

$$\|u-u_B\|_{L^{\Phi_s}(B)} \le C r_B \mu(B)^{-1/s} \|g\|_{L^{\Phi}(\hat{B})}, \tag{22}$$

where Φ_s is defined by (15)-(16) *and $C = C(C_s, s, C_P, \tau, \delta)$.*

The following example gives concrete expressions for the "Sobolev conjugate" Φ_s.

Example 1.5 *Let Φ be equivalent to the function $t^p \log^q t$ near infinity, where either $p=1$ and $q \ge 0$ or $p>1$ and $q \in \mathbb{R}$. Then Φ_s is equivalent near infinity to*

$$\begin{cases} t^{sp/(s-p)}(\log t)^{sq/(s-p)} & \textit{if } 1 \le p < s \\ \exp(t^{s/(s-1-q)}) & \textit{if } p = s,\ q < s-1 \\ \exp(\exp(t^{s/(s-1)})) & \textit{if } p = s,\ q = s-1. \end{cases}$$

In a general metric space we cannot talk about partial derivatives, but the concept of an upper gradient has turned out to be a useful substitute for the length of a gradient.

Definition 1.6 ([10]) *A Borel function $g : X \to [0,\infty]$ is an upper gradient of a function $u : X \to \overline{\mathbb{R}}$, if for all rectifiable curves $\gamma : [0,l] \to X$,*

$$|u(\gamma(0)) - u(\gamma(l))| \le \int_\gamma g\,ds \tag{23}$$

whenever both $u(\gamma(0))$ and $u(\gamma(l))$ are finite, and $\int_\gamma g\,ds = \infty$ otherwise.

More generally, g is a Φ-weak upper gradient of u, if the family of rectifiable curves for which (23) does not hold has zero Φ-modulus (see Section 2). The Orlicz-Sobolev space $N^{1,\Phi}(X)$ consisting of functions $u \in L^\Phi(X)$ having a Φ-weak upper gradient $g \in L^\Phi(X)$ was recently studied by Tuominen [14]. We say that X supports the Φ-Poincaré inequality, if the Φ-Poincaré inequality holds for all locally integrable functions and their upper gradients. If X supports the Φ-Poincaré inequality, then any pair (u,g) of a locally integrable function and its Φ-weak upper gradient $g \in L^{\Phi}(X)$ has the truncation property (Lemma 2.4). Thus, we obtain an optimal embedding theorem for the space $N^{1,\Phi}(X)$.

Theorem 1.7 *Assume that (X, d, μ) is a doubling metric measure space that supports the Φ-Poincaré inequality and satisfies* (2) *with $s > 1$. Let B be a ball, $\delta > 0$ and $\hat{B} = (1+\delta)\tau B$.*

1) *If Φ satisfies* (14)*, then $N^{1,\Phi}(\hat{B}) \subset L^{\Phi_s}(B)$, where Φ_s is defined by* (15)-(16)*. Moreover, for every $u \in N^{1,\Phi}(\hat{B})$ and for every Φ-weak upper gradient g of u, we have*

$$\|u - u_B\|_{L^{\Phi_s}(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^{\Phi}(\hat{B})}.$$

2) *If Φ satisfies* (17)*, then every $u \in N^{1,\Phi}(\hat{B})$ has a locally uniformly continuous representative. Moreover, for every Φ-weak upper gradient g of u, we have*

$$|u(x) - u(y)| \leq C r_B \mu(B)^{-1/s} \|g\|_{L^{\Phi}(\hat{B})} \omega_s^{-1}(r_B^s \mu(B)^{-1} d(x,y)^{-s}),$$

for $x, y \in B$, where ω_s is defined by (18)-(19)*.*

Here, $C = C(C_s, s, C_P, \tau, \delta)$.

Apart from the case $X = \mathbb{R}^n$, theorems 1.2, 1.4 and 1.7 seem to be new even if the Φ-Poincaré inequality in the assumptions is replaced by the 1-Poincaré inequality. The spaces supporting the 1-Poincaré inequality include Riemannian manifolds with nonnegative Ricci curvature, Q-regular orientable topological manifolds satisfying the local linear contractability condition, Carnot groups and more general Carnot-Carathéodory spaces associated with a system of vector fields satisfying Hörmander's condition, as well as more exotic spaces constructed by Bourdon and Pajot, Laakso, and Hanson and Heinonen, see [6] and the references therein.

Our next result is an embedding theorem for the space $A_{\tau}^{\Phi,s}(U)$ defined as follows.

Definition 1.8 *Let U be an open set, Φ a Young function, $\tau \geq 1$ and $0 < s \leq \infty$. Denote*

$$\mathcal{B}_{\tau}(U) = \{\{B_i\} : \textit{balls } \tau B_i \textit{ are disjoint and contained in } U\}$$

and

$$\|u\|_{A_{\tau}^{\Phi,s}(U)} = \sup_{\mathcal{B} \in \mathcal{B}_{\tau}(U)} \| \sum_{B \in \mathcal{B}} \left(\mu(B)^{-1/s} \fint_B |u - u_B| \, d\mu \right) \chi_B \|_{L^{\Phi}(U;\mu)}.$$

Then $A_{\tau}^{\Phi,s}(U)$ consists of all locally integrable functions u for which the number $\|u\|_{A_{\tau}^{\Phi,s}(U)}$ is finite.

Notice that below $1 < s < \infty$ is *any* number and need not have anything to do with (2).

Theorem 1.9 *Let X be connected, μ doubling, Φ a Young function, $B \subset X$ a ball, $1 < s < \infty$, $\tau \geq 1$ and $\delta > 0$. Denote $\hat{B} = (1+\delta)\tau B$.*

1) If (14) *holds, then*

$$\|u - u_B\|_{L_w^{\Phi_s}(B)} \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}, \tag{24}$$

where Φ_s is defined by (15)-(16).

2) If (17) *holds, then, for Lebesgue points $x, y \in B$ of u,*

$$|u(x) - u(y)| \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\omega_s^{-1}(\mu(B_{xy})^{-1}), \tag{25}$$

where $B_{xy} = B(x, 2d(x,y))$, and ω_s is defined by (18)-(19).

Here, $C = C(C_d, \tau, \delta)$.

It is easy to see that the first part of Theorem 1.2 is a consequence of inequality (24). In Section 4 we will show that it also implies the generalized Trudinger inequality (12) of MacManus and Peréz.

The results in this paper deal with connected spaces. The setting of a disconnected space will be investigated in the forthcoming paper [8].

2 Preliminaries

2.1 Metric measure spaces

Throughout this paper $X = (X, d, \mu)$ is a metric space equipped with a measure μ. By a measure we mean Borel regular outer measure satisfying $0 < \mu(U) < \infty$ whenever U is open and bounded.

Open and closed balls of radius r centered at x will be denoted by $B(x,r)$ and $\overline{B}(x,r)$. Sometimes we denote the radius of a ball B by r_B. For a positive number λ, we define $\lambda B(x,r) := B(x, \lambda r)$.

Recall from the introduction that the doubling property of a measure implies a lower decay estimate (2) for the measure of a ball. In connected spaces we can estimate the measure of a ball also from above.

Lemma 2.1 *Let X be connected and μ doubling. Then there are constants $\alpha > 0$ and $C \geq 1$ depending only on C_d such that*

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \leq C\left(\frac{r}{r_0}\right)^\alpha, \tag{26}$$

whenever $x \in B(x_0, r_0)$ and $r \leq r_0$.

For a proof, see for example [12].

2.2 Young functions and Orlicz spaces

In this subsection we give a brief review of Young functions and Orlicz spaces. A more detailed treatment of the subject can be found for example in [13].

A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if it has the form

$$\Phi(t) = \int_0^t \phi(s)\, ds,$$

where $\phi : [0,\infty) \to [0,\infty]$ is increasing, left-continuous function, which is neither identically zero nor identically infinite on $(0,\infty)$. A Young function is convex and, in particular, satisfies

$$\Phi(\varepsilon t) \leq \varepsilon \Phi(t) \tag{27}$$

for $0 < \varepsilon \leq 1$ and $0 \leq t < \infty$.

The right-continuous generalized inverse of a Young function Φ is

$$\Phi^{-1}(t) = \inf\{s : \Phi(s) > t\}.$$

We have that

$$\Phi(\Phi^{-1}(t)) \leq t \leq \Phi^{-1}(\Phi(t))$$

for $t \geq 0$.

The conjugate of a Young function Φ is the Young function defined by

$$\hat{\Phi}(t) = \sup\{ts - \Phi(s) : s > 0\}$$

for $t \geq 0$.

Let Φ be a Young function. The Orlicz space $L^\Phi(X)$ is the set of all measurable functions u for which there exists $\lambda > 0$ such that

$$\int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) < \infty.$$

The Luxemburg norm of $u \in L^\Phi(X)$ is

$$\|u\|_{L^\Phi(X)} = \|u\|_{L^\Phi(X;\mu)} = \inf\{\lambda > 0 : \int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) \leq 1\}.$$

If $\|u\|_{L^\Phi(X)} \neq 0$, we have that

$$\int_X \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(X)}}\right) d\mu(x) \leq 1.$$

The following generalized Hölder inequality holds for Luxemburg norms:

$$\int_X u(x)v(x)\, d\mu(x) \leq 2\|u\|_{L^\Phi(X)}\|v\|_{L^{\hat{\Phi}}(X)}.$$

The weak Orlicz space $L_w^\Phi(X)$ is defined to be the set of all those measurable functions for which the weak Luxemburg norm

$$\|u\|_{L_w^\Phi(X)} = \inf\{\lambda > 0 : \sup_{t>0} \Phi(t)\mu(\{x \in X : \frac{|u(x)|}{\lambda} > t\}) \leq 1\}$$

is finite. If $\|u\|_{L_w^\Phi(X)} \neq 0$, it follows that

$$\sup_{t>0} \Phi(t)\mu(\{x \in X : \frac{|u(x)|}{\|u\|_{L_w^\Phi(X)}} > t\}) \leq 1.$$

The normalized (weak) Luxemburg norm, that is, the (weak) Luxemburg norm taken with respect to measure $\mu(X)^{-1}\mu$, will be denoted by $\|\cdot\|_{Ł^\Phi(X)}$ ($\|\cdot\|_{Ł_w^\Phi(X)}$).

A function Φ dominates a function Ψ globally (resp. near infinity), if there is a constant C such that

$$\Psi(t) \leq \Phi(Ct)$$

for all $t \geq 0$ (resp. for t larger than some t_0).

Functions Φ and Ψ are equivalent globally (near infinity), if each dominates the other globally (near infinity).

If $\mu(X) < \infty$ and Φ dominates Ψ near infinity, we have that

$$\|u\|_{Ł^\Psi(X)} \leq C(\Phi, \Psi)\|u\|_{Ł^\Phi(X)}. \tag{28}$$

2.3 Φ-weak upper gradients

Let Φ be a Young function. The Φ-modulus of a curve family Γ is

$$\mathrm{Mod}_\Phi(\Gamma) = \inf\left\{\|g\|_{L^\Phi(X)} : \int_\gamma g\,ds \geq 1 \text{ for all } \gamma \in \Gamma\right\}.$$

If X supports the Φ-Poincaré inequality, then (13) holds for functions and their Φ-weak upper gradients. This is an immediate consequence of the following lemma ([14], Lemma 4.3).

Lemma 2.2 *Let Φ be a Young function and let $g \in L^\Phi(X)$ be a Φ-weak upper gradient of a function u. Then there is a decreasing sequence (g_i) of upper gradients of u such that $g_i \to g$ in $L^\Phi(X)$.*

An important property of Φ-weak upper gradients is the following ([14],Lemma 4.11).

Lemma 2.3 *Let Φ be a Young function. Assume that $u \in ACC_\Phi(X)$ and that the functions v and w have Φ-weak upper gradients $g_v, g_w \in L^\Phi(X)$. If E is a Borel set such that $u|_E = v$ and $u|_{X\setminus E} = w$, then the function*

$$g = g_v\chi_E + g_w\chi_{X\setminus E}$$

is a Φ-weak upper gradient of u.

Here "$u \in ACC_\Phi(X)$" means that the family Γ of rectifiable curves for which $u \circ \gamma$ is not absolutely continuous on $[0, l(\gamma)]$ has zero Φ-modulus.

It follows from the lemma above that if $g \in L^\Phi(X)$ is a Φ-weak upper gradient of a measurable function v, then $g\chi_{\{t_1 < v \leq t_2\}}$ is a Φ-weak upper gradient of the function $v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}$. Thus, we have the following.

Lemma 2.4 *If X supports the Φ-Poincaré inequality, then every pair (u, g) of a locally integrable function and its Φ-weak upper gradient $g \in L^\Phi(X)$ has the truncation property.*

3 Proofs of main theorems

The proof of Theorem 1.9 requires several lemmas. In the first three lemmas equivalent representations of conditions (14) and (17) and of functions Φ_s and ω_s are given. The proofs of lemmas 3.1 and 3.2 can be found in [3], and the proof of 3.3 in [1].

Lemma 3.1 *Let Φ be a Young function. We have*

$$\int_0 \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt < \infty \quad \text{if and only if} \quad \int_0 \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty \tag{29}$$

and

$$\int^\infty \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt < \infty \quad \text{if and only if} \quad \int^\infty \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty. \tag{30}$$

Moreover, the function Φ_s is globally equivalent to the function D_s given by

$$D_s(t) = (tJ^{-1}(t^{s'}))^{s'} \tag{31}$$

for $t \geq 0$, where J^{-1} is the left-continuous inverse of the function given by

$$J(r) = s' \int_0^r \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt. \tag{32}$$

Lemma 3.2 *Let Φ be a Young function. Then $\|r^{-1/s'}\|_{L^{\hat{\Phi}}(t,\infty)} < \infty$ for every $t > 0$, if and only if*

$$\int_0 \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt < \infty. \tag{33}$$

Moreover,

$$\|r^{-1/s'}\|_{L^{\hat{\Phi}}(t,\infty)} = D_s^{-1}(1/t) \tag{34}$$

for $t > 0$, where D_s^{-1} is the right-continuous inverse of D_s.

$$\tag{35}$$

Lemma 3.3 *Let Φ be a Young function. Then $\|r^{-1/s'}\|_{L^{\hat{\Phi}}(0,t)} < \infty$ for every $t > 0$, if and only if*

$$\int^{\infty} \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt < \infty. \tag{36}$$

Moreover,

$$\|r^{-1/s'}\|_{L^{\hat{\Phi}}(0,t)} = \omega_s^{-1}(1/t) \tag{37}$$

for $t > 0$, where ω_s^{-1} is the right-continuous inverse of ω_s.

It is easy to see that, for $C \geq 1$,

$$D_s^{-1}(Ct) \leq C D_s^{-1}(t) \tag{38}$$

and

$$\omega_s^{-1}(C^{-1}t) \leq C\omega_s^{-1}(t). \tag{39}$$

Lemma 3.4 *Let Φ be a Young function. Then*

$$\Phi(r)^{-1/s} r \leq \Phi_s^{-1}(\Phi(r))$$

for $r \geq 0$.

Proof. Since Φ is convex, the function $t \mapsto t/\Phi(t)$ is decreasing. Hence

$$\Phi_s^{-1}(\Phi(r)) = \left(\int_0^r \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt\right)^{1/s'} \geq \left(r\left(\frac{r}{\Phi(r)}\right)^{s'-1}\right)^{1/s'} = \Phi(r)^{-1/s} r.$$

□

The next lemma is the part of the proofs of theorems 1.9 and 1.2, where the connectedness of the space comes into play.

Lemma 3.5 *Assume that X is connected, μ doubling, $\tau \geq 1$ and $\delta > 0$. Let B be a ball, $x \in B$ and $0 < r < \delta r_B$. Then there is a sequence $\{B_0, \ldots, B_k\}$ of balls contained in $(1+\delta)B$ such that $\mu(B_0)$ is comparable to $\mu(B)$, $\mu(B_k)$ is comparable to $\mu(B(x,r))$, $\{B_1, \ldots, B_k\} \in \mathcal{B}_\tau(\hat{B})$,*

$$2\mu(B_{i+1}) \leq \mu(B_i) \leq C\mu(B_{i+1}), \tag{40}$$

for $1 \leq i < k$, and

$$|u_{B(x,r)} - u_{B_0}| \leq C \sum_{i=1}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu, \tag{41}$$

where $C = C(C_d, \tau, \delta)$.

Proof. Fix $x \in B$ and $0 < r < \delta r_B$. Let $\mathcal{C}_j$ be a cover of $A_j = B(x, 2^{-j}\delta r_B) \setminus B(x, 2^{-j-1}\delta r_B)$ by balls of radius $(20\tau)^{-1}2^{-j}\delta r_B$ centered at A_j such that the balls $\frac{1}{2}D$, $D \in \mathcal{C}_j$, are disjoint. It follows easily from the doubling property of μ that $\#\mathcal{C}_j \le C$. Since X is connected, there must be a sequence $\{B'_0, \dots, B'_{k-1}\} \subset \cup_{j=1}^m \mathcal{C}_j$ so that $B'_0 \in \mathcal{C}_1$, $B'_i \cap B'_{i+1} \ne \emptyset$ for all i, $B'_{k-1} \subset B(x,r)$ and $\mu(B'_{k-1})$ is comparable to $\mu(B(x,r))$. Denote $B_0 = B'_0$, $B_k = B'_k = B(x,r)$ and $B_i := 5B'_i$ for $1 \le i < k$. Then $B'_i \subset B_{i+1}$, and so

$$|u_{B'_i} - u_{B'_{i+1}}| \le |u_{B'_i} - u_{B_{i+1}}| + |u_{B_{i+1}} - u_{B'_{i+1}}| \le C \fint_{B_{i+1}} |u - u_{B_{i+1}}| \, d\mu.$$

Thus

$$|u_{B(x,r)} - u_{B_0}| \le \sum_{i=0}^{k-1} |u_{B'_i} - u_{B'_{i+1}}| \le C \sum_{i=1}^{k} \fint_{B_i} |u - u_{B_i}| \, d\mu.$$

We will show that $\{B_i\}$ has a subsequence that belongs to $\mathcal{B}_\tau(\hat{B})$ and satisfies (40) and (41). For $1 \le j \le m$, choose $D_j \in \{B_i\}$ centered at A_j such that

$$\fint_{D_j} |u - u_{D_j}| \, d\mu = \max \left\{ \fint_{B_i} |u - u_{B_i}| \, d\mu : x_{B_i} \in A_j \right\},$$

where x_{B_i} denotes the center of B_i. Then

$$|u_{B(x,r)} - u_{B_0}| \le C \sum_{j=1}^{m} \fint_{D_j} |u - u_{D_j}| \, d\mu.$$

If $|i-j| \ge 2$, then $\tau D_i \cap \tau D_j = \emptyset$.

By (2) and (26) there are constants $\alpha > 0$ and $\beta > 0$ depending on C_d such that

$$C^{-1} 2^{-\beta n} \le \frac{\mu(D_{j+n})}{\mu(D_j)} \le C 2^{-\alpha n} \tag{42}$$

for $j, n \ge 1$. Let $n \ge 2$ be such that $C2^{-\alpha n} \le 2^{-1}$. For $p + (i-1)n \le m$, denote $B_i^p = D_{p+(i-1)n}$. Then the sequence $\{B_1^p, B_2^p \dots\}$ satisfies (40) and belongs to $\mathcal{B}_\tau(\hat{B})$. By choosing $1 \le p < n$ such that

$$\sum_i \fint_{B_i^p} |u - u_{B_i^p}| \, d\mu = \max_{1 \le q < n} \sum_i \fint_{B_i^q} |u - u_{B_i^q}| \, d\mu,$$

we obtain

$$|u(x) - u_{B_0}| \le C \sum_i \fint_{B_i^p} |u - u_{B_i^p}| \, d\mu.$$

The proof is complete. $\square$

We need one more lemma, a weak-type estimate for a sharp fractional maximal function defined by

$$M^{\#}_{s,B_0} u(x) = \sup_{x \in B \subset B_0} \mu(B)^{-1/s} \fint_B |u - u_B| \, d\mu, \tag{43}$$

for a ball $B_0 \subset X$, $u \in L^1(B_0)$ and $0 < s \le \infty$.

Lemma 3.6 *Let Φ be a Young function. Then*

$$\|M^{\#}_{s,B}u\|_{L^{\Phi}_{w}(B)} \le C(C_d,\tau)\|u\|_{A^{\Phi,s}_{\tau}(\tau B)}.$$

Proof. We may assume that $\|u\|_{A^{\Phi,s}_{\tau}(\tau B)} = 1$. Let $x \in B$ such that $M^{\#}_{s,B}u(x) > \lambda$. By the definition of $M^{\#}_{s,B}u$, there is a ball $B_x \subset B$ containing x such that

$$\mu(B_x)^{-1/s} \fint_{B_x} |u - u_{B_x}|\, d\mu > \lambda.$$

So,

$$\mu(B_x) \le \Phi(\lambda)^{-1}\Phi\Big(\mu(B_x)^{-1/s} \fint_{B_x} |u - u_{B_x}|\, d\mu\Big)\mu(B_x). \tag{44}$$

By the standard $5r$-covering lemma ([9, Theorem 1.16]), we can cover the set

$$\{x \in B : M^{\#}_{s,B}(x) > \lambda\}$$

by balls $5\tau B_i$ such that the balls τB_i are disjoint and that each B_i is contained in B and satisfies (44). Using the doubling property of μ, estimate (44), inequality (27), and the fact that $\{B_i\} \in \mathcal{B}_{\tau}(\tau B)$, we obtain

$$\begin{aligned}
&\mu(\{x \in B : M^{\#}_{s,B}u(x) > \lambda\}) \le \sum_i \mu(5\tau B_i) \le C(C_d,\tau)\sum_i \mu(B_i)\\
&\le C(C_d,\tau)\Phi(\lambda)^{-1}\sum_i \Phi\Big(\mu(B_i)^{-1/s}\fint_{B_i} |u - u_{B_i}|\, d\mu\Big)\mu(B_i)\\
&\le \Phi\left(\frac{\lambda}{C(C_d,\tau)}\right)^{-1}\sum_i \Phi\Big(\mu(B_i)^{-1/s}\fint_{B_i} |u - u_{B_i}|\, d\mu\Big)\mu(B_i)\\
&\le \Phi\left(\frac{\lambda}{C(C_d,\tau)}\right)^{-1}.
\end{aligned}$$

The claim follows by the definition of $\|\cdot\|_{L^{\Phi}_{w}}$. □

Proof of Theorem 1.9. 1) Denote $B' = (1+\delta)B$. It suffices to show that the pointwise inequality

$$|u(x) - u_B| \le C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B})}\Phi_s^{-1}\left(\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_{w}(B')}}\right)\right) \tag{45}$$

holds for Lebesgue points $x \in B$. Indeed, if (45) holds, then

$$\begin{aligned}
\mu\left(x \in B : \frac{|u(x) - u_B|}{C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B})}} > t\right) &\le \mu\left(x \in B : \Phi_s^{-1}\circ\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_{w}(B')}}\right) > t\right)\\
&\le \mu\left(x \in B : \frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_{w}(B')}} > \Phi^{-1}\circ\Phi_s(t)\right)\\
&\le \Phi_s(t)^{-1}.
\end{aligned}$$

Fix a Lebesgue point $x \in B$ of u and $0 < r \leq \delta r_B$. Let $\{B_0, \dots B_k\}$ be the chain from Lemma 3.5 corresponding to x and r. Since the balls B_i, $i \geq 1$, are disjoint, we have that

$$\sum_{i=1}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu = \| \sum_{i=1}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu \frac{\chi_{B_i}}{\mu(B_i)} \|_{L^1(X)}$$

and

$$\sum_{i=1}^{k} \fint_{B_i} |u-u_{B_i}|\, d\mu \frac{\chi_{B_i}}{\mu(B_i)} = \sum_{i=1}^{k} \mu(B_i)^{-1/s} \fint_{B_i} |u-u_{B_i}|\, d\mu \chi_{B_i} \cdot \sum_{i=1}^{k} \mu(B_i)^{-1/s'} \chi_{B_i}.$$

Hence, by the Hölder inequality,

$$\begin{aligned} &\sum_{i=1}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu \\ &\leq 2 \| \sum_{i=1}^{k} \mu(B_i)^{-1/s} \fint_{B_i} |u - u_{B_i}|\, d\mu \chi_{B_i} \|_{L^\Phi(X)} \cdot \| \sum_{i=1}^{k} \mu(B_i)^{-1/s'} \chi_{B_i} \|_{L^{\hat{\Phi}}(X)} \\ &\leq 2 \|u\|_{A_\tau^{\Phi,s}(\hat{B})} \cdot \| \sum_{i=1}^{k} \mu(B_i)^{-1/s'} \chi_{B_i} \|_{L^{\hat{\Phi}}(X)}. \end{aligned}$$

By the definition of Luxemburg norm

$$\| \sum_{i=1}^{k} \mu(B_i)^{-1/s'} \chi_{B_i} \|_{L^{\hat{\Phi}}(X)} = \inf\{\lambda > 0 : \sum_{i=1}^{k} \hat{\Phi}\left(\frac{\mu(B_i)^{-1/s'}}{\lambda}\right) \mu(B_i) \leq 1\}.$$

For each i, we have that

$$\begin{aligned} \hat{\Phi}\left(\frac{\mu(B_i)^{-1/s'}}{\lambda}\right) \mu(B_i) &\leq 2 \int_{\frac{\mu(B_i)}{2}}^{\mu(B_i)} \hat{\Phi}\left(\frac{t^{-1/s'}}{\lambda}\right) dt \\ &\leq \int_{\frac{\mu(B_i)}{2}}^{\mu(B_i)} \hat{\Phi}\left(\frac{2t^{-1/s'}}{\lambda}\right) dt, \end{aligned}$$

where the first inequality follows from the fact that the function

$$t \mapsto \hat{\Phi}(t^{-1/s'}/\lambda)$$

is decreasing, and the second from (27). Since

$$\mu(B_{i+1}) \leq \frac{\mu(B_i)}{2},$$

we obtain

$$\sum_{i=1}^{k} \hat{\Phi}\left(\frac{\mu(B_i)^{-1/s'}}{\lambda}\right) \mu(B_i) \leq \int_{\frac{\mu(B_k)}{2}}^{\mu(B_1)} \hat{\Phi}\left(\frac{2t^{-1/s'}}{\lambda}\right) dt,$$

which implies that

$$\|\sum_{i=1}^{k}\mu(B_i)^{-1/s'}\chi_{B_i}\|_{L^{\hat\Phi}(X)} \le \inf\{\lambda>0:\int_{\frac{\mu(B_k)}{2}}^{\mu(B_1)}\hat\Phi\left(\frac{2t^{-1/s'}}{\lambda}\right)dt\le 1\}$$
$$= 2\|t^{-1/s'}\|_{L^{\hat\Phi}(\frac{\mu(B_k)}{2},\mu(B_1))}.$$

Thus

$$\sum_{i=1}^{k}\fint_{B_i}|u-u_{B_i}|\,d\mu \le C\|u\|_{A_\tau^{\Phi,s}(\hat B)}\|t^{-1/s'}\|_{L^{\hat\Phi}(\frac{\mu(B_k)}{2},\mu(B_1))}. \tag{46}$$

By similar reasoning,

$$\begin{aligned}|u_{B_0}-u_B| &\le |u_{B_0}-u_{B'}|+|u_{B'}-u_B| \le C\fint_{B'}|u-u_{B'}|\,d\mu\\ &\le C\|u\|_{A_\tau^{\Phi,s}(\hat B)}\|t^{-1/s'}\|_{L^{\hat\Phi}(\frac{\mu(B')}{2},\mu(B'))}.\end{aligned} \tag{47}$$

It follows from estimates (46) and (47) that

$$|u_{B(x,r)}-u_B| \le C\|u\|_{A_\tau^{\Phi,s}(\hat B)}\|t^{-1/s'}\|_{L^{\hat\Phi}(C^{-1}\mu(B(x,r)),C\mu(B))}. \tag{48}$$

Hence, by lemmas 3.1 and 3.2, and by (38),

$$|u_{B(x,r)}-u_B| \le C\|u\|_{A_\tau^{\Phi,s}(\hat B)}\Phi_s^{-1}(\mu(B(x,r))^{-1}). \tag{49}$$

Next, we will estimate $|u(x)-u_{B(x,r)}|$ in terms of maximal function (43). For $i\ge 0$, denote $B_i=B(x,2^{-i}r)$. By the Lebesgue differentiation theorem ([9, Theorem 1.8]), $u_{B_i}\to u(x)$, as $i\to\infty$. Thus, by (1) and (26),

$$\begin{aligned}|u(x)-u_{B(x,r)}| &\le \sum_{i\ge 0}|u_{B_i}-u_{B_{i+1}}|\\ &\le C\sum_{i\ge 0}\fint_{B_i}|u-u_{B_i}|\,d\mu\\ &\le C\sum_{i\ge 0}\mu(B_i)^{1/s}M^{\#}_{s,B'}u(x)\\ &\le C\mu(B(x,r))^{1/s}M^{\#}_{s,B'}u(x).\end{aligned}$$

So, by Lemma 3.6,

$$|u(x)-u_{B(x,r)}| \le C\|u\|_{A_\tau^{\Phi,s}(\hat B)}\mu(B(x,r))^{1/s}\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}. \tag{50}$$

Combining the above estimates, we obtain

$$|u(x)-u_B| \le C\|u\|_{A_\tau^{\Phi,s}(\hat B)}\left(\Phi_s^{-1}(\mu(B_r)^{-1})+\mu(B_r)^{1/s}\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right),$$

where $B_r = B(x,r)$. If $\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right) \geq \mu(B_{\delta r_B})^{-1}$, we can choose $r \leq \delta r_B$ such that

$$\mu(B_r)^{-1} \leq \Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right) \leq C\mu(B_r)^{-1}.$$

Then

$$\begin{aligned}\mu(B_r)^{1/s}\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}} &\leq C\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right)^{-1/s}\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}} \\ &\leq C\Phi_s^{-1}\left(\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right)\right),\end{aligned} \tag{51}$$

where the last inequality comes from Lemma 3.4. Thus, we obtain (45).

If

$$\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right) < \mu(B_{\delta r_B})^{-1},$$

it suffices to combine estimate (50), where $r = \delta r_B$, with the estimate

$$\begin{aligned}|u_{B_{\delta r_B}} - u_B| &\leq C\fint_{B'}|u - u_{B'}|\,d\mu \\ &\leq C\mu(B')^{1/s}M^{\#}_{s,B'}u(x) \\ &\leq C\|u\|_{A^{\Phi,s}_\tau(\hat{B})}\mu(B_{\delta r_B})^{1/s}\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\end{aligned}$$

and argue as in (51).

2) Letting r tend to zero in (48) and using Lemma 3.3 and (39), we obtain

$$|u(x) - u_B| \leq C\|u\|_{A^{\Phi,s}_\tau(\hat{B})}\omega_s^{-1}(\mu(B)^{-1}). \tag{52}$$

Let $x, y \in B$ be Lebesgue points of u. Denote $B_{xy} = B(x, 2d(x,y))$. If $d(x,y) > \frac{1}{3}\delta r_B$, then $\mu(B) \leq C\mu(B_{xy})$. So

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_B| + |u(y) - u_B| \\ &\leq C\|u\|_{A^{\Phi,s}_\tau(\hat{B})}\omega_s^{-1}(\mu(B)^{-1}) \\ &\leq C\|u\|_{A^{\Phi,s}_\tau(\hat{B})}\omega_s^{-1}(\mu(B_{xy})^{-1}).\end{aligned}$$

If $d(x,y) \leq \frac{1}{3}\delta r_B$, then (52), applied to the ball B_{xy}, yields

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_{B_{xy}}| + |u(y) - u_{B_{xy}}| \\ &\leq C\|u\|_{A^{\Phi,s}_\tau(\hat{B}_{xy})}\omega_s^{-1}(\mu(B_{xy})^{-1}).\end{aligned}$$

Since we may assume that $\delta < 1/2$, it follows that $\hat{B}_{xy} \subset \hat{B}$. Hence

$$|u(x) - u(y)| \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\omega_s^{-1}(\mu(B_{xy})^{-1}).$$

$\square$

Proof of Theorem 1.2. 1) By Theorem 1.9, it suffices to show that

$$\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \le Cr_B\mu(B)^{-1/s}\|g\|_{L^\Phi(\hat{B})}. \tag{53}$$

We may assume that $\|g\|_{L^\Phi(\hat{B})} = 1$. Let D be a ball such that $\tau D \subset \hat{B}$. Then, by (13) and (2),

$$\begin{aligned}\fint_D |u - u_D|\, d\mu &\le C_p r_D \Phi^{-1}\left(\fint_{\tau D} \Phi(g)\, d\mu\right) \\ &\le Cr_B\mu(B)^{-1/s}\mu(D)^{1/s}\Phi^{-1}\left(\fint_{\tau D} \Phi(g)\, d\mu\right).\end{aligned}$$

Hence, for $\mathcal{D} \in \mathcal{B}_\tau(\hat{B})$,

$$\sum_{D\in\mathcal{D}} \Phi\left(\frac{\mu(D)^{-1/s}\fint_D |u - u_D|\, d\mu}{Cr_B\mu(B)^{-1/s}}\right)\mu(D) \le \sum_{D\in\mathcal{D}}\int_{\tau D}\Phi(g)\, d\mu \le \int_{\hat{B}}\Phi(g)\, d\mu \le 1,$$

which implies that

$$\|\sum_{D\in\mathcal{D}}\left(\mu(D)^{-1/s}\fint_D |u - u_D|\, d\mu\right)\chi_D\|_{L^\Phi(\hat{B})} \le Cr_B\mu(B)^{-1/s}.$$

By taking supremum over $\mathcal{B}_\tau(\hat{B})$, we obtain (53).

2) We may assume that $\delta < 1/2$. Let D be a ball centered at B so that $\hat{D} = (1+\delta)\tau D \subset \hat{B}$. Fix a Lebesgue point $x \in D$, $0 < r < \delta r_D$ and let $\{B_i\}$ be the chain from Lemma 3.5 corresponding to D, x and r. Clearly, the chain can be chosen so that $r_{B_{i+1}} \le \frac{r_{B_i}}{2}$. Since the balls B_i, $i \ge 1$, are disjoint, we have that

$$\sum_{i=1}^k \fint_{B_i} |u - u_{B_i}|\, d\mu = \|\sum_{i=1}^k \mu(B_i)^{-1}\fint_{B_i} |u - u_{B_i}|\, d\mu\chi_{B_i}\|_{L^1(X)}$$

and

$$\sum_{i=1}^k \mu(B_i)^{-1}\fint_{B_i} |u - u_{B_i}|\, d\mu\chi_{B_i} = \sum_{i=1}^k r_i^{-1}\fint_{B_i} |u - u_{B_i}|\, d\mu\chi_{B_i}\cdot\sum_{i=1}^k r_i\mu(B_i)^{-1}\chi_{B_i}.$$

So, by the Hölder inequality,

$$\begin{aligned}&\sum_{i=1}^k \fint_{B_i} |u - u_{B_i}|\, d\mu \\ &\le 2\|\sum_{i=1}^k r_i^{-1}\fint_{B_i} |u - u_{B_i}|\, d\mu\chi_{B_i}\|_{L^\Phi(X)}\cdot\|\sum_{i=1}^k r_i\mu(B_i)^{-1}\chi_{B_i}\|_{L^{\hat{\Phi}}(X)}.\end{aligned}$$

Since the pair $\|g\|^{-1}_{L^\Phi(\hat{B})}(u,g)$ satisfies the Φ-Poincaré inequality in $\hat{B}$ and $\{B_i\} \in \mathcal{B}_\tau(\hat{D}) \subset \mathcal{B}_\tau(\hat{B})$, we have that

$$\|\sum_{i=1}^{k} r_i^{-1} \fint_{B_i} |u - u_{B_i}|\, d\mu \chi_{B_i}\|_{L^\Phi(X)} \leq C\|g\|_{L^\Phi(\hat{B})}.$$

By the definition of Luxemburg norm

$$\|\sum_{i=1}^{k} r_i \mu(B_i)^{-1}\chi_{B_i}\|_{L^{\hat{\Phi}}(X)} = \inf\{\lambda > 0 : \sum_{i=1}^{k} \hat{\Phi}\left(\frac{r_i\mu(B_i)^{-1}}{\lambda}\right)\mu(B_i) \leq 1\}.$$

By (2),

$$\mu(B_i)^{-1} \leq (C_B r_i)^{-s},$$

where $C_B = Cr_B^{-1}\mu(B)^{1/s}$. Since the function $t \mapsto \hat{\Phi}(at)/t$ is increasing, for every $a > 0$, we have

$$\hat{\Phi}\left(\frac{r_i\mu(B_i)^{-1}}{\lambda}\right)\mu(B_i) \leq \hat{\Phi}\left(\frac{r_i(C_B r_i)^{-s}}{\lambda}\right)(C_B r_i)^s = \hat{\Phi}\left(\frac{t_i^{-1/s'}}{C_B\lambda}\right)t_i,$$

where $t_i = (C_B r_i)^s$. It follows that

$$\begin{aligned}
\|\sum_{i=1}^{k} r_i \mu(B_i)^{-1}\chi_{B_i}\|_{L^{\hat{\Phi}}(X)} &\leq C_B^{-1}\inf\{\lambda > 0 : \sum_{i=1}^{k}\hat{\Phi}\left(\frac{t_i^{-1/s'}}{\lambda}\right)t_i \leq 1\} \\
&\leq 2C_B^{-1}\inf\{\lambda > 0 : \int_0^{t_1}\hat{\Phi}\left(\frac{t^{-1/s'}}{\lambda}\right) \leq 1\} \\
&= 2C_B^{-1}\|t^{-1/s'}\|_{L^{\hat{\Phi}}(0,t_1)} \\
&\leq Cr_B\mu(B)^{-1/s}\omega_s^{-1}(\mu(B)^{-1}r_B^s r_D^{-s}),
\end{aligned}$$

where the last inequality comes from Lemma 3.3 and from (39). Thus

$$\begin{aligned}
|u(x) - u_{B_0}| &= \lim_{r\to 0}|u_{B(x,r)} - u_{B_0}| \\
&\leq Cr_B\mu(B)^{-1/s}\|g\|_{L^\Phi(\hat{B})}\omega_s^{-1}(\mu(B)^{-1}r_B^s r_D^{-s}).
\end{aligned}$$

By similar reasoning,

$$\begin{aligned}
|u_{B_0} - u_D| \leq |u_{D'} - u_{B_0}| + |u_{B_0} - u_{D'}| &\leq C\fint_{D'}|u - u_{D'}|\, d\mu \\
&\leq Cr_B\mu(B)^{-1/s}\|g\|_{L^\Phi(\hat{B})}\omega_s^{-1}(\mu(B)^{-1}r_B^s r_D^{-s}).
\end{aligned}$$

So,

$$|u(x) - u_D| \leq Cr_B\mu(B)^{-1/s}\|g\|_{L^\Phi(\hat{B})}\omega_s^{-1}(\mu(B)^{-1}r_B^s r_D^{-s}). \tag{54}$$

Let $x, y \in B$ be Lebesgue points of u. If $d(x,y) > \frac{1}{3}\delta r_B$, then (54) with $D = B$ yields

$$\begin{aligned} |u(x) - u(y)| &\le |u(x) - u_B| + |u(y) - u_B| \\ &\le C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \omega_s^{-1}(\mu(B)^{-1} r_B^s d(x,y)^{-s}). \end{aligned}$$

If $d(x,y) \le \frac{1}{3}\delta r_B$, then $\hat{D} \subset \hat{B}$, for the ball $D = B(x, 2d(x,y))$, and so by (54) and (39),

$$\begin{aligned} |u(x) - u(y)| &\le |u(x) - u_D| + |u(y) - u_D| \\ &\le C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \omega_s^{-1}(\mu(B)^{-1} r_B^s d(x,y)^{-s}). \end{aligned}$$

□

Remark 3.7 *As shown above, the first part of Theorem 1.2 is a consequence of Theorem 1.9. More generally, suppose that* (2) *holds, and that a function u satisfies an inequality of type*

$$\fint_D |u - u_D|\, d\mu \le \|u\|_\nu r_D^\alpha \Phi^{-1}\left(\frac{\nu(\tau D)}{\mu(\tau D)}\right), \tag{55}$$

where $\alpha > 0$, and $\nu : \{B : B \text{ is a ball }\} \to [0, \infty)$ satisfies $\sum \nu(B_i) \le 1$, whenever the balls B_i are disjoint and contained in $\hat{B}$. Then, an argument similar to the proof of (53)*, shows that*

$$\|u\|_{A_\tau^{\Phi, s/\alpha}(\hat{B})} \le C r_B^\alpha \mu(B)^{-\alpha/s} \|u\|_\nu. \tag{56}$$

Thus, if (14) *holds, with s/α in place of s, Theorem 1.9 yields*

$$\|u - u_B\|_{L_w^{\Phi_{s/\alpha}}(B)} \le C r_B^\alpha \mu(B)^{-\alpha/s} \|u\|_\nu.$$

The properties of functions satisfying inequalities of type (55) *with $\Phi(t) = t^p$ were studied in [7].*

Remark 3.8 *Suppose that* (2) *and* (14) *hold, and that a pair (u, g), where $0 < \int_{\hat{B}} \Phi(g)\, d\mu < \infty$, satisfies the Φ-Poincaré inequality in $\hat{B}$. Then, for the measure $\hat{\mu} = \left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1} \mu$, we have that $\|g\|_{L^\Phi(\hat{B};\hat{\mu})} = 1$. Since* (2) *and* (13) *trivially hold for $\hat{\mu}$ with the same constants as they hold for μ, Theorem 1.2, for the measure $\hat{\mu}$, yields*

$$\|u - u_B\|_{L_w^{\Phi_s}(B;\hat{\mu})} \le C r_B \hat{\mu}(B)^{-1/s},$$

which is equivalent to

$$\sup_{t>0} \Phi_s\left(\frac{t}{C r \mu(B)^{-1/s}} \left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1/s}\right) \mu(\{|u - u_B| > t\}) \le \int_{\hat{B}} \Phi(g)\, d\mu, \tag{57}$$

where $\{|u - u_B| > t\} = \{x \in B : |u(x) - u_B| > t\}$.

Proof of Theorem 1.4. Suppose that (2) and (14) hold, and that a pair (u, g), where $0 < \int_{\hat{B}} \Phi(g)\, d\mu < \infty$, has the truncation property. Choose b such that

$$\mu(\{u \geq b\}) \geq \mu(B)/2 \quad \text{and} \quad \mu(\{u \leq b\}) \geq \mu(B)/2.$$

Let $v_+ = \max\{u - b, 0\}$ and $v_- = -\min\{u - b, 0\}$. We need the following elementary lemma.

Lemma 3.9 *Let ν be a finite measure on Y. If $w \geq 0$ is a ν-measurable function such that $\nu(\{w = 0\}) \geq \nu(Y)/2$, then, for $t > 0$,*

$$\nu(\{w > t\}) \leq 2 \inf_{c \in \mathbb{R}} \nu(\{|w - c| > t/2\}).$$

Proof. If $|c| \leq t/2$, then $\{w > t\} \subset \{|w - c| > t/2\}$. Otherwise, $\{w = 0\} \subset \{|w - c| > t/2\}$, and so

$$\nu(\{w > t\}) \leq \nu(Y) \leq 2\nu(\{w = 0\}) \leq 2\nu(\{|w - c| > t/2\}).$$

$\square$

Let v denote either v_+ or v_-. For $k \in \mathbb{Z}$, denote $v_k = v_{2^{k-1}}^{2^k}$ and $g_k = g\chi_{\{2^{k-1} < v \leq 2^k\}}$. Then

$$\mu(\{v > 2^k\}) \leq \mu(\{v_k > 2^{k-2}\}) \leq 2\mu(\{|v_k - (v_k)_B| > 2^{k-3}\}) \tag{58}$$

for $k \in \mathbb{Z}$. Let $C = 2^5 C_0$, where C_0 is the constant from inequality (57). Using (58) and (57) for the pair (v_k, g_k) we obtain

$$\begin{aligned}
&\int_B \Phi_s\left(\frac{v}{Cr\mu(B)^{-1/s}}\left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1/s}\right) d\mu \\
&\leq \sum_{k \in \mathbb{Z}} \int_{\{2^k < v \leq 2^{k+1}\}} \Phi_s\left(\frac{v}{Cr\mu(B)^{-1/s}}\left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1/s}\right) d\mu \\
&\leq \sum_{k \in \mathbb{Z}} \Phi_s\left(\frac{2^{k+1}}{Cr\mu(B)^{-1/s}}\left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1/s}\right) \mu(\{v > 2^k\}) \\
&\leq \sum_{k \in \mathbb{Z}} \Phi_s\left(\frac{2^{k-3}}{C_0 r\mu(B)^{-1/s}}\left(\int_{\hat{B}} \Phi(g_k)\, d\mu\right)^{-1/s}\right) \mu(\{|v_k - (v_k)_B| > 2^{k-3}\}) \\
&\leq \sum_{k \in \mathbb{Z}} \int_{\hat{B}} \Phi(g_k)\, d\mu \\
&\leq \int_{\hat{B}} \Phi(g)\, d\mu.
\end{aligned}$$

Thus

$$\inf_{b \in \mathbb{R}} \int_B \Phi_s\left(\frac{|u - b|}{Cr\mu(B)^{-1/s}}\left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1/s}\right) d\mu \leq \int_{\hat{B}} \Phi(g)\, d\mu. \tag{59}$$

This, for the pair $\|g\|^{-1}_{L^\Phi(\hat{B})}(u,g)$ in place of (u,g), yields

$$\inf_{b\in\mathbb{R}} \|u-b\|_{L^{\Phi_s}(B)} \le Cr\mu(B)^{-1/s}\|g\|_{L^\Phi(\hat{B})}.$$

Since $\|u-u_B\|_{L^\Phi(A)} \le 2\inf_{b\in\mathbb{R}}\|u-b\|_{L^\Phi(A)}$ for any set A of finite measure, the proof is complete. $\square$

4 Strong inequalities without truncation

In this section we will show how the weak estimate (24) implies strong ones. We begin with an easy lemma.

Lemma 4.1 *Let $\mu(X)<\infty$, and let Φ and Ψ be Young functions such that*

$$\int_1^\infty \frac{\Psi'(t)}{\Phi(t)}\,dt<\infty. \tag{60}$$

Then $L^\Phi_w(X)\subset L^\Psi(X)$ and there is a constant $C=C(\Psi,\Phi)$ such that

$$\|u\|_{\text{Ł}^\Psi(X)} \le C\|u\|_{\text{Ł}^\Phi_w(X)}. \tag{61}$$

Proof. Assume $\|u\|_{L^\Phi_w(X)}=1$. Denoting $\tilde{\mu}=\mu(X)^{-1}\mu$, we obtain

$$\begin{aligned}\int_X \Psi(|u|)\,d\tilde{\mu} &= \int_0^\infty \Psi'(t)\tilde{\mu}(\{x\in X: |u|>t\})\,dt\\ &\le \Psi(1)+\int_1^\infty \Psi'(t)\tilde{\mu}(\{x\in X: |u|>t\})\,dt\\ &\le \Psi(1)+\int_1^\infty \frac{\Psi'(t)}{\Phi(t)}\,dt =: C',\end{aligned}$$

which implies (61) with $C=\max\{C',1\}$. $\square$

For a measure ν on X, denote

$$\|u\|_{A^{\Phi,s}_\tau(U;\nu)} = \sup_{\mathcal{B}\in\mathcal{B}_\tau(U)} \Big\|\sum_{B\in\mathcal{B}}\left(\nu(B)^{-1/s}\fint_B |u-u_B|\,d\nu\right)\chi_B\Big\|_{L^\Phi(U;\nu)}.$$

For a ball $B\subset X$, denote $\mu_B=\mu(B)^{-1}\mu$.

Theorem 4.2 *Suppose that the assumptions of Theorem 1.9 are in force,* (14) *holds, and that Ψ is a Young function satisfying*

$$\int_1^\infty \frac{\Psi'(t)}{\Phi_s(t)}\,dt<\infty. \tag{62}$$

Then

$$\|u-u_B\|_{\text{Ł}^\Psi(B)} \le C\|u\|_{A^{\Phi,s}_\tau(\hat{B};\mu_{\hat{B}})}, \tag{63}$$

where $C = C(C_d, s, \tau, \delta, \Phi, \Psi)$. Moreover, if μ satisfies (2), $g \in L^{\Phi}(\hat{B})$, and a pair $\|g\|^{-1}_{L^{\Phi}(\hat{B})}(u, g)$ satisfies the Φ-Poincaré inequality in $\hat{B}$, then

$$\|u - u_B\|_{\text{Ł}^{\Psi}(B)} \leq C r_B \|g\|_{\text{Ł}^{\Phi}(\hat{B})}, \tag{64}$$

where $C = C(C_s, s, C_P, \tau, \delta, \Phi, \Psi)$.

Proof. Theorem 1.9, applied to the measure $\mu_B = \mu(B)^{-1}\mu$, yields

$$\|u - u_B\|_{\text{Ł}^{\Phi_s}_w(B)} \leq C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B};\mu_B)}.$$

So, by Lemma 4.1,

$$\|u - u_B\|_{\text{Ł}^{\Psi}(B)} \leq C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B};\mu_B)}.$$

Since

$$\|\cdot\|_{L^{\Phi}(\hat{B};\mu_B)} \leq C\|\cdot\|_{L^{\Phi}(\hat{B};\mu_{\hat{B}})},$$

it follows that

$$\|u\|_{A^{\Phi,s}_{\tau}(\hat{B};\mu_B)} \leq C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B};\mu_{\hat{B}})}.$$

Inequality (64) follows from inequalities (63) and (53). □

Notice that if Φ_s increases quickly enough, condition (62) is satisfied with $\Psi(t) = \Phi_s(t/2)$, and we have

$$\|u - u_B\|_{\text{Ł}^{\Phi_s}(B)} \leq C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B};\mu_{\hat{B}})}. \tag{65}$$

In particular, this is the case when Φ is equivalent to $t \mapsto t^s$ near infinity.

Suppose now that (8) holds with a functional a satisfying (11), and that Φ is equivalent to $t \mapsto t^s$ near infinity. Then

$$\begin{aligned}
\|u\|_{A^{\Phi,s}_{\tau}(\hat{B};\mu_{\hat{B}})} &= \sup_{\mathcal{B}\in\mathcal{B}_{\tau}(\hat{B})} \Big\| \sum_{B\in\mathcal{B}} \left(\mu_{\hat{B}}(B)^{-1/s} \fint_B |u - u_B|\,d\mu\right)\chi_B \Big\|_{\text{Ł}^{\Phi}(\hat{B})} \\
&\leq C \sup_{\mathcal{B}\in\mathcal{B}_{\tau}(\hat{B})} \Big\| \sum_{B\in\mathcal{B}} \left(\mu_{\hat{B}}(B)^{-1/s} \fint_B |u - u_B|\,d\mu\right)\chi_B \Big\|_{\text{Ł}^{s}(\hat{B})} \\
&\leq C \sup_{\mathcal{B}\in\mathcal{B}_{\tau}(\hat{B})} \left(\sum_{B\in\mathcal{B}} \left(\fint_B |u - u_B|\,d\mu\right)^s\right)^{1/s} \\
&\leq C\|u\|_a \sup_{\mathcal{B}\in\mathcal{B}_{\tau}(\hat{B})} \left(\sum_{B\in\mathcal{B}} a(\tau B)^s\right)^{1/s} \\
&\leq C\,\|u\|_a a(\hat{B}),
\end{aligned}$$

where the first inequality comes from (28). Thus (65) implies the generalized Trudinger inequality (12).

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Self-improving properties of generalized Orlicz-Poincaré inequalities in metric measure spaces

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Self-improving properties of generalized Orlicz-Poincaré inequalities in metric measure spaces

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Abstract

The sharp self-improving properties of generalized Φ-Poincaré inequalities in *connected* metric measure spaces were recently obtained in [6]. In this paper we investigate the general setting. We also include the case where Φ increases essentially more slowly than the function $t \mapsto t$. Our results generalize some results of Hajłasz and Koskela [4, 5] and MacManus and Pérez [8].

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1 Introduction and main results

Let $X = (X, d, \mu)$ be a metric measure space with μ a Borel regular outer measure satisfying $0 < \mu(U) < \infty$, whenever U is nonempty, open and bounded. Suppose further that μ is doubling, that is, there exists a constant C_d such that

$$\mu(2B) \leq C_d \mu(B), \tag{1}$$

whenever B is a ball. It is easy to see that the doubling property is equivalent to the existence of constants s and C_s such that

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \geq C_s^{-1} \left(\frac{r}{r_0}\right)^s \tag{2}$$

holds, whenever $x \in B(x_0, r_0)$ and $r \leq r_0$.

Definition 1.1 ([10]) *Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing bijection. A pair (u, g) of measurable functions, $u \in L^1_{loc}(X)$ and $g \geq 0$, satisfies the Φ-Poincaré inequality (in an open set U), if there are constants C_P and τ such that*

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \Phi^{-1} \left(\fint_{\tau B} \Phi(g) \, d\mu \right) \tag{3}$$

for every ball $B \subset X$ (such that $\tau B \subset U$).

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The following sharp self-improving result for the Φ-Poincaré inequality was recently proved in [6].

Theorem A *Assume that Φ is a Young function, X is connected, μ satisfies* (2) *with $1 < s < \infty$, $B \subset X$ is a ball, $\delta > 0$, $\tau \geq 1$, $\hat{B} = (1+\delta)\tau B$, and that a pair $(\hat{u}, \hat{g})$, where $\hat{u} = \|g\|^{-1}_{L^\Phi(\hat{B})} u$ and $\hat{g} = \|g\|^{-1}_{L^\Phi(\hat{B})} g$, satisfies the Φ-Poincaré inequality in $\hat{B}$.*

1) If

$$\int_0^1 \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty \quad \text{and} \quad \int_0^\infty \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt = \infty, \tag{4}$$

then

$$\|u - u_B\|_{L^{\Phi_s}_w(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}, \tag{5}$$

where

$$\Phi_s = \Phi \circ \Psi_s^{-1}, \tag{6}$$

$$\Psi_s(r) = \left(\int_0^r \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt\right)^{1/s'} \tag{7}$$

and $s' = s/(s-1)$.

2) If

$$\int^\infty \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty, \tag{8}$$

then, for Lebesgue points $x, y \in B$ of u,

$$|u(x) - u(y)| \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \omega_s(\mu(B)^{-1} r_B^s d(x,y)^{-s}), \tag{9}$$

where

$$\omega_s^{-1}(t) = (t\Theta^{-1}(t^{s'}))^{s'} \tag{10}$$

and Θ^{-1} is the left-continuous inverse of the function given by

$$\Theta(r) = s' \int_r^\infty \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt. \tag{11}$$

Here, $C = C(C_s, s, C_P, \tau, \delta)$.

Let $U \subset X$ be open, $0 < s \leq \infty$ and $\tau \geq 1$. Denote

$$\mathcal{B}_\tau(U) = \{\{B_i\} : \text{balls } \tau B_i \text{ are disjoint and contained in } U\},$$

$$\|u\|_{A^{\Phi,s}_\tau(U)} = \sup_{\mathcal{B} \in \mathcal{B}_\tau(U)} \| \sum_{B \in \mathcal{B}} \left(\mu(B)^{-1/s} \fint_B |u - u_B|\, d\mu\right) \chi_B \|_{L^\Phi(X)}$$

and

$$A^{\Phi,s}_\tau(U) = \{u \in L^1(U) : \|u\|_{A^{\Phi,s}_\tau(U)} < \infty\}.$$

It is easy to see that $\|u\|_{A_\tau^{\Phi,s}(U)} \le \lambda$, if and only if there is a functional $\nu : \{B \subset U :\ B \text{ is a ball }\} \to [0,\infty)$ such that

$$\sum \nu(B_i) \le 1, \tag{12}$$

whenever the balls B_i are disjoint, and that

$$\fint_B |u - u_B|\, d\mu \le \lambda \mu(B)^{1/s} \Phi^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right), \tag{13}$$

whenever $\tau B \subset U$. The self-improving properties of abstract Poincaré-type inequalities similar to (13), for $\Phi(t) = t^p$, were studied by Franchi, Pérez and Wheeden [2, 3], and MacManus and Pérez [8, 9].

If μ satisfies (2) and a pair $(\hat{u}, \hat{g})$, where $\hat{u} = \|g\|^{-1}_{L^\Phi(B)} u$ and $\hat{g} = \|g\|^{-1}_{L^\Phi(B)} g$, satisfies the Φ-Poincaré inequality in a ball B, then

$$\|u\|_{A_\tau^{\Phi,s}(B)} \le C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(B)}. \tag{14}$$

Thus, the first case of Theorem A is a consequence of the following embedding theorem for the space $A_\tau^{\Phi,s}(U)$.

Theorem B [6, Theorem 1.9] *Let X be connected, μ doubling, Φ a Young function, $B \subset X$ a ball, $1 < s < \infty$, $\tau \ge 1$ and $\delta > 0$. Denote $\hat{B} = (1+\delta)\tau B$.*

1) If (4) holds, then

$$\|u - u_B\|_{L_w^{\Phi_s}(B)} \le C \|u\|_{A_\tau^{\Phi,s}(\hat{B})},$$

where Φ_s is defined by (6)-(7).

2) If (8) holds, then, for Lebesgue points $x, y \in B$ of u,

$$|u(x) - u(y)| \le C \|u\|_{A_\tau^{\Phi,s}(\hat{B})} \omega_s(\mu(B_{xy})^{-1}),$$

where $B_{xy} = B(x, 2d(x,y))$, and ω_s is defined by (10)-(11).

Here, $C = C(C_d, \tau, \delta)$.

It is essential in the above theorems that the underlying space X is connected. In this paper we investigate the general case. Instead of assuming that Φ is a Young function, we assume the following:

(Φ-1) $\Phi : [0,\infty) \to [0,\infty)$ is an increasing bijection.

(Φ-2) The function $t \mapsto \frac{\Phi(t)}{t^{s/(s+1)}}$ is increasing.

Notice that (Φ-2) allows Φ to increase essentially more slowly than any Young function. The results concerning such Φ are new also for connected spaces.

Our first result is a counterpart of Theorem A in the general setting. It extends the results of Hajłasz and Koskela [4, 5].

Theorem 1.2 *Assume that* Φ *satisfies* (Φ*-1) and* (Φ*-2),* μ *satisfies* (2) *with* $0 < s < \infty$, $B \subset X$ *is a ball,* $\delta > 0$, $\tau \geq 1$, $\hat{B} = (1+\delta)\tau B$, *and that a pair* $(\hat{u}, \hat{g})$, *where* $\hat{u} = \|g\|^{-1}_{L^\Phi(\hat{B})} u$ *and* $\hat{g} = \|g\|^{-1}_{L^\Phi(\hat{B})} g$, *satisfies the* Φ*-Poincaré inequality in* $\hat{B}$.

1) If

$$\int_0^1 \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt < \infty \quad \text{and} \quad \int_0^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt = \infty, \tag{15}$$

then

$$\|u - u_B\|_{L_w^{\tilde{\Phi}_s}(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}, \tag{16}$$

where

$$\tilde{\Phi}_s^{-1}(r) = \int_0^r \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt. \tag{17}$$

2) If

$$\int^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt < \infty, \tag{18}$$

then, for Lebesgue points $x, y \in B$ *of* u,

$$|u(x) - u(y)| \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \tilde{\omega}_s(\mu(B)^{-1} r_B^s d(x,y)^{-s}),$$

where

$$\tilde{\omega}_s(r) = \int_r^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt. \tag{19}$$

Here, $C = C(C_s, s, C_P, \tau, \delta)$.

If the Φ-Poincaré inequality is stable under truncations, the weak estimate (5) turns into a strong one. We say that a pair (u, g) has the truncation property, if for every $b \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$ and $\varepsilon \in \{-1, 1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1 < v \leq t_2\}})$, where $v = \varepsilon(u - b)$ and

$$v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\},$$

satisfies the Φ-Poincaré inequality (with fixed constants).

Theorem 1.3 *Suppose that the assumptions of Theorem 1.2 are in force,* (15) *holds, and that the pair* $(\hat{u}, \hat{g})$ *has the truncation property. Then*

$$\|u - u_B\|_{L^{\tilde{\Phi}_s}(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}, \tag{20}$$

where Φ_s *is defined by* (6)-(7) *and* $C = C(C_s, s, C_P, \tau, \delta)$.

How good is Theorem 1.2 compared to Theorem A? If Φ is "close" to the function $t \mapsto t^s$, then $\tilde{\Phi}_s$ increases essentially more slowly than Φ_s.

Example 1.4 *Let Φ be equivalent near infinity to the function $t^s \log^q t$. Then the function Φ_s is equivalent near infinity to*

$$\begin{cases} \exp(t^{s/(s-1-q)}) & \text{if } q < s-1 \\ \exp(\exp(t^{s/(s-1)})) & \text{if } q = s-1, \end{cases}$$

and the function $\tilde{\Phi}_s$ is equivalent near infinity to

$$\begin{cases} \exp(t^{s/(s-q)}) & \text{if } q < s \\ \exp(\exp(t)) & \text{if } q = s. \end{cases}$$

If Φ is a Young function such that the function $t \mapsto \Phi(t)/t^p$ is either decreasing for some $p < s$, or increasing for some $p > s$, then Theorem 1.2 gives the same result as Theorem A. In these cases the Sobolev conjugate Φ_s and the function ω_s can be represented in a very simple form.

Theorem 1.5 *(1) Suppose that Φ satisfies (Φ-1) and (Φ-2) and that the function $t \mapsto \Phi(t)/t^p$ is decreasing for some $p < s$. Then $\tilde{\Phi}_s$ is globally equivalent to the function Φ_s^* whose inverse is given by*

$$(\Phi_s^*)^{-1}(r) = \Phi^{-1}(r) r^{-1/s}.$$

If Φ is a Young function, then also Φ_s is globally equivalent to Φ_s^*.

(2) If Φ is a Young function such that $\Phi(t)/t^p$ is increasing for some $p > s$, then both ω_s and $\tilde{\omega}_s$ are comparable the function ω_s^* given by

$$\omega_s^*(r) = \Phi^{-1}(r) r^{-1/s}.$$

Let us now turn to the results concerning the embeddings of spaces $A_\tau^{\Phi,s}(U)$. We begin with the case $s = \infty$. Theorem 1.6 below extends (the non-weighted version of) the result of MacManus and Pérez [8].

Theorem 1.6 *Assume that μ is doubling, $s = \infty$, Φ is doubling and satisfies (Φ-1) and (Φ-2), $B \subset X$ is a ball, $\tau \geq 1$ and $\delta > 0$. Then*

$$\|u - u_B\|_{L_w^\Phi(B)} \leq C \|u\|_{A_\tau^{\Phi,\infty}(\hat{B})},$$

where $\hat{B} = (1+\delta)\tau B$ and $C = C(C_d, \tau, \delta, \Phi)$.

Since, by Lemma 3.3,

$$\|u\|_{A_\tau^{\tilde{\Phi}_s,\infty}(U)} \leq \|u\|_{A_\tau^{\Phi,s}(U)}, \tag{21}$$

we have the following.

Theorem 1.7 *Assume that μ is doubling, Φ satisfies (Φ-1) and (Φ-2),* (15) *holds and that $\tilde{\Phi}_s$ is doubling. Let $B \subset X$ a ball, $\tau \geq 1$ and $\delta > 0$. Then*

$$\|u - u_B\|_{L_w^{\tilde{\Phi}_s}(B)} \leq C \|u\|_{A_\tau^{\Phi,s}(\hat{B})}, \tag{22}$$

where $\hat{B} = (1+\delta)\tau B$ and $C = C(C_d, \tau, \delta, s, \Phi)$.

Under the extra assumption that singletons have zero measure, (22) holds also for non-doubling $\tilde{\Phi}_s$.

Theorem 1.8 *Assume that μ is doubling, $\mu(\{x\}) = 0$ for $x \in X$, $0 < s < \infty$, and that Φ satisfies (Φ-1) and (Φ-2). Let $B \subset X$ be a ball, $\tau \geq 1$ and $\delta > 0$. Denote $\hat{B} = (1+\delta)\tau B$.*

1) If (15) holds, then

$$\|u - u_B\|_{L_w^{\tilde{\Phi}_s}(B)} \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})},$$

where $\tilde{\Phi}_s$ is defined by (17).

2) If (18) holds, then, for Lebesgue points $x, y \in B$ of u,

$$|u(x) - u(y)| \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\tilde{\omega}_s(\mu(B_{xy})^{-1}),$$

where $B_{xy} = B(x, 2d(x,y))$ and $\tilde{\omega}_s$ is defined by (19).

Here, $C = C(C_d, \tau, \delta, s)$.

2 Preliminaries

Throughout this paper $X = (X, d, \mu)$ is a metric space equipped with a measure μ. By a measure we mean a Borel regular outer measure satisfying $0 < \mu(U) < \infty$ whenever U is open and bounded.

Open and closed balls of radius r centered at x will be denoted by $B(x,r)$ and $\overline{B}(x,r)$. Sometimes we denote the radius of a ball B by r_B. For a positive number λ we define $\lambda B(x,r) := B(x, \lambda r)$.

Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing bijection. Denote by $L^\Phi(X)$ the set of all measurable functions u for which there exists $\lambda > 0$ such that

$$\int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) < \infty.$$

For $u \in L^\Phi(X)$, define

$$\|u\|_{L^\Phi(X)} = \inf\{\lambda > 0 : \int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) \leq 1\}.$$

If Φ is convex, the functional $\|\cdot\|_{L^\Phi(X)}$ is a norm on $L^\Phi(X)$.

If $\|u\|_{L^\Phi(X)} \neq 0$, we have that

$$\int_X \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(X)}}\right) d\mu(x) \leq 1.$$

Denote by $L_w^\Phi(X)$ the set of all measurable functions for which the number

$$\|u\|_{L_w^\Phi(X)} = \inf\{\lambda > 0 : \sup_{t>0} \Phi(t)\mu(\{x \in X : \frac{|u(x)|}{\lambda} > t\}) \leq 1\}$$

is finite. If $\|u\|_{L^\Phi_w(X)} \neq 0$, it follows that

$$\sup_{t>0} \Phi(t)\mu(\{x \in X : \frac{|u(x)|}{\|u\|_{L^\Phi_w(X)}} > t\}) \leq 1.$$

A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if it has the form

$$\Phi(t) = \int_0^t \phi(s)\, ds,$$

where $\phi : [0,\infty) \to [0,\infty]$ is increasing, left-continuous function, which is neither identically zero nor identically infinite on $(0,\infty)$. A Young function is convex and, in particular, satisfies

$$\Phi(\varepsilon t) \leq \varepsilon \Phi(t) \tag{23}$$

for $0 < \varepsilon \leq 1$ and $0 \leq t < \infty$.

The right-continuous generalized inverse of a Young function Φ is

$$\Phi^{-1}(t) = \inf\{s : \Phi(s) > t\}.$$

We have that

$$\Phi(\Phi^{-1}(t)) \leq t \leq \Phi^{-1}(\Phi(t))$$

for $t \geq 0$.

The conjugate of a Young function Φ is the Young function defined by

$$\hat{\Phi}(t) = \sup\{ts - \Phi(s) : s > 0\}$$

for $t \geq 0$.

We have that

$$t \leq \Phi^{-1}(t)\hat{\Phi}^{-1}(t) \leq 2t \tag{24}$$

for $t \geq 0$.

A function Φ dominates a function Ψ globally (resp. near infinity), if there is a constant C such that

$$\Psi(t) \leq \Phi(Ct)$$

for all $t \geq 0$ (resp. for t larger than some t_0).

Functions Φ and Ψ are equivalent globally (near infinity), if each dominates the other globally (near infinity).

If Φ dominates Ψ near infinity and Φ and Ψ are not equivalent near infinity, then Ψ increases essentially more slowly that Φ.

Φ is doubling, if there is a constant C such that

$$\Phi(2t) \leq C\Phi(t)$$

for all t.

3 Proofs

Lemma 3.1 *Suppose that Φ satisfies (Φ-1) and (Φ-2). Then, for $0 \le \varepsilon \le 1$ and $t \ge 0$,*

$$\Phi(\varepsilon t) \le \varepsilon^{s/(s+1)}\Phi(t), \tag{25}$$

$$\tilde{\Phi}_s(\varepsilon t) \le \varepsilon \tilde{\Phi}_s(t), \tag{26}$$

and

$$\tilde{\omega}_s(\varepsilon t) \le \varepsilon^{-1/s}\tilde{\omega}_s(t), \tag{27}$$

Proof. We have

$$\Phi(\varepsilon t) = \frac{\Phi(\varepsilon t)}{(\varepsilon t)^{s/(s+1)}}(\varepsilon t)^{s/(s+1)} \le \frac{\Phi(t)}{t^{s/(s+1)}}(\varepsilon t)^{s/(s+1)} = \varepsilon^{s/(s+1)}\Phi(t).$$

By (Φ-2), the function $\Phi^{-1}(t)/t^{1+1/s}$ is decreasing. Hence

$$\begin{aligned}\tilde{\Phi}_s^{-1}(\varepsilon^{-1}r) &= \int_0^{\varepsilon^{-1}r} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt = \varepsilon^{-1}\int_0^r \frac{\Phi^{-1}(\varepsilon^{-1}t)}{(\varepsilon^{-1}t)^{1+1/s}}\,dt \\ &\le \varepsilon^{-1}\int_0^r \frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt = \varepsilon^{-1}\tilde{\Phi}_s^{-1}(r),\end{aligned}$$

which is equivalent to (26). Since Φ^{-1} is increasing, we have

$$\tilde{\omega}_s(\varepsilon r) = \int_{\varepsilon r}^{\infty} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt = \varepsilon\int_r^{\infty} \frac{\Phi^{-1}(\varepsilon t)}{(\varepsilon t)^{1+1/s}}\,dt \le \varepsilon^{-1/s}\tilde{\omega}_s(r).$$

□

Let $U \subset X$ be open and let $v \in L^1(U)$. The maximal function of v is

$$M_U v(x) = \sup_{x\in B\subset U} \fint_B |v|\,d\mu.$$

It is well known that there is a constant $C = C(C_d)$ such that

$$\|M_U v\|_{L^1_w(U)} \le C\|v\|_{L^1(U)}. \tag{28}$$

Proof of Theorem 1.2. We may assume that $\|g\|_{L^\Phi(\hat{B})} = 1$.

1) It suffices to show that the pointwise inequality

$$|u(x) - u_B| \le C r_B \mu(B)^{-1/s}\tilde{\Phi}_s^{-1}\Big(M_{\hat{B}}\Phi(g)(x)\Big), \tag{29}$$

holds for Lebesgue points $x \in B$ of u. Indeed, if (29) holds, then by (28),

$$\left\|\tilde{\Phi}_s\left(\frac{|u-u_B|}{C r_B\mu(B)^{-1/s}}\right)\right\|_{L^1_w(B)} \le \left\|M_{\hat{B}}\Phi(g)\right\|_{L^1_w(B)} \le C\left\|\Phi(g)\right\|_{L^1(\hat{B})} \le C,$$

and the claim follows by (26).

Fix a Lebesgue point x of u. For $i \geq 1$, let $B_i = B(x, 2^{-i/s}\delta)$. By the Lebesgue differentiation theorem, $\lim_{i\to\infty} u_{B_i} = u(x)$. So

$$|u(x) - u_{B_1}| \leq \sum_{i=1}^{\infty} |u_{B_i} - u_{B_{i+1}}| \leq C \sum_{i=1}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu.$$

By denoting $B_0 = (1+\delta)B$,

$$|u_B - u_{B_1}| \leq |u_B - u_{B_0}| + |u_{B_0} - u_{B_1}| \leq C \fint_{B_0} |u - u_{B_0}|\, d\mu.$$

Thus

$$|u(x) - u_B| \leq C \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu.$$

By (3) and (2),

$$\begin{aligned} \fint_{B_i} |u - u_{B_i}|\, d\mu &\leq C r_i \Phi^{-1}(\mu(B_i)^{-1}) \\ &\leq C r_i \Phi^{-1}((C_B r_i)^{-s}), \end{aligned}$$

where $C_B = r_B^{-1}\mu(B)^{1/s}$. Hence, by denoting $t_i = (C_B r_i)^{-s}$,

$$\begin{aligned} \sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu &\leq C C_B^{-1} \sum_{i=0}^{k} t_i^{-1/s} \Phi^{-1}(t_i) \\ &= C r_B \mu(B)^{-1/s} \sum_{i=0}^{k} t_i \frac{\Phi^{-1}(t_i)}{t_i^{1+1/s}}. \end{aligned}$$

Since the function $t \mapsto \frac{\Phi^{-1}(t)}{t^{1+1/s}}$ is decreasing, we have that

$$t_i \frac{\Phi^{-1}(t_i)}{t_i^{1+1/s}} \leq 2 \int_{\frac{1}{2}t_i}^{t_i} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt,$$

for $i \geq 0$. So, by summing and noting that $t_i \leq \frac{1}{2}t_{i+1}$, we obtain

$$\sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu \leq C r_B \mu(B)^{-1/s} \int_{\frac{1}{2}t_0}^{t_k} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt. \tag{30}$$

Thus

$$\sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu \leq C r_B \mu(B)^{-1/s} \tilde{\Phi}_s^{-1}(t_k). \tag{31}$$

The remaining part of the series will be estimated in terms of $M\Phi(g)$:

$$\begin{aligned} \sum_{i=k}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu &\leq C \sum_{i=k}^{\infty} r_i \Phi^{-1}\Big(M_{\hat{B}}\Phi(g)(x)\Big) \\ &\leq C r_k \Phi^{-1}\Big(M_{\hat{B}}\Phi(g)(x)\Big) \\ &= C r_B \mu(B)^{-1/s} t_k^{-1/s} \Phi^{-1}\Big(M_{\hat{B}}\Phi(g)(x)\Big). \end{aligned} \tag{32}$$

By combining (31) and (32), we obtain

$$|u(x)-u_B| \le C r_B \mu(B)^{-1/s}\left(\tilde{\Phi}_s^{-1}(t_k)+t_k^{-1/s}\Phi^{-1}\Big(M_{\hat{B}}\Phi(g)(x)\Big)\right). \tag{33}$$

Since, by (Φ-2), the function $\Phi^{-1}(t)/t^{1+1/s}$ is decreasing, we have that

$$\tilde{\Phi}_s^{-1}(r)=\int_0^r \frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt \ge r\frac{\Phi^{-1}(r)}{r^{1+1/s}}=r^{-1/s}\Phi^{-1}(r). \tag{34}$$

If $M_{\hat{B}}\Phi(g)(x)\ge t_0$, we choose k such that

$$t_k \le M_{\hat{B}}\Phi(g)(x)\le Ct_k.$$

Then, by (33) and (34), we obtain (29). If $M_{\hat{B}}\Phi(g)(x)<t_0$, it suffices to use (32), with $k=0$, and (34).

2) We may assume that $\delta<1/2$. Let D be a ball centered at B so that $\hat{D}=(1+\delta)\tau D\subset\hat{B}$. Fix a Lebesgue point $x\in D$ of u. Let $B_0=(1+\delta)D$ and $B_i=B(x,2^{-i/s}\delta)$ for $i\ge 1$. By the same argument that led to (30), we have that

$$\sum_{i=0}^{\infty}\fint_{B_i}|u-u_{B_i}|\,d\mu \le Cr_B\mu(B)^{-1/s}\int_{C^{-1}\mu(B)^{-1}r_B^s r_D^{-s}}^{\infty}\frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt.$$

Thus

$$|u(x)-u_D|\le Cr_B\mu(B)^{-1/s}\tilde{\omega}_s(C^{-1}\mu(B)^{-1}r_B^s r_D^{-s}). \tag{35}$$

Let $x,y\in B$ be Lebesgue points of u. If $d(x,y)>\frac{1}{3}\delta r_B$, then (35), with $D=B$, yields

$$\begin{aligned}|u(x)-u(y)| &\le |u(x)-u_B|+|u(y)-u_B|\\ &\le Cr_B\mu(B)^{-1/s}\tilde{\omega}_s(C^{-1}\mu(B)^{-1}r_B^s d(x,y)^{-s}).\end{aligned}$$

If $d(x,y)\le\frac{1}{3}\delta r_B$, then $\hat{D}\subset\hat{B}$, for the ball $D=B(x,2d(x,y))$, and so, by (35),

$$\begin{aligned}|u(x)-u(y)| &\le |u(x)-u_D|+|u(y)-u_D|\\ &\le Cr_B\mu(B)^{-1/s}\tilde{\omega}_s(C^{-1}\mu(B)^{-1}r_B^s d(x,y)^{-s}).\end{aligned}$$

Thus, the claim follows by (27). □

The proof of Theorem 1.3 is completely analogous to the proof of Theorem 1.4 in [6]. We will not repeat the details.

Proof of Theorem 1.5 (1) By (34),

$$\tilde{\Phi}_s^{-1}(r)\ge r^{-1/s}\Phi^{-1}(r).$$

If $t\mapsto\Phi(t)/t^p$ is decreasing, then $t\mapsto\Phi^{-1}(t)/t^{1/p}$ is increasing. Hence, if $p<s$,

$$\tilde{\Phi}_s^{-1}(r)=\int_0^r\frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt\le\frac{\Phi^{-1}(r)}{r^{1/p}}\int_0^r t^{1/p-1/s-1}\,dt=(1/p-1/s)^{-1}\frac{\Phi^{-1}(r)}{r^{1/s}}.$$

Thus $\tilde{\Phi}_s$ and Φ_s^* are globally equivalent. Let Φ be a Young function. Then the function $t \mapsto t/\Phi(t)$ is decreasing and so

$$\Psi_s(r) = \left(\int_0^r \left(\frac{t}{\Phi(t)} \right)^{s'-1} dt \right)^{1/s'} \geq \left(r \left(\frac{r}{\Phi(r)} \right)^{s'-1} \right)^{1/s'} = \Phi(r)^{-1/s} r.$$

On the other hand, if $t \mapsto \Phi(t)/t^p$ is decreasing, with $p < s$, then

$$\begin{aligned} \Psi_s(r) &= \left(\int_0^r \left(\frac{t}{\Phi(t)} \right)^{\frac{1}{s-1}} dt \right)^{\frac{s-1}{s}} \leq \frac{r^{p/s}}{\Phi(r)^{1/s}} \left(\int_0^r t^{\frac{s-p}{s-1}-1} dt \right)^{\frac{s-1}{s}} \\ &= \frac{r^{p/s}}{\Phi(r)^{1/s}} \left(\frac{s-1}{s-p} r^{\frac{s-p}{s-1}} \right)^{\frac{s-1}{s}} = \left(\frac{s-1}{s-p} \right)^{\frac{s-1}{s}} \frac{r}{\Phi(r)^{1/s}}. \end{aligned}$$

Since $\Phi_s^{-1} = \Psi_s \circ \Phi^{-1}$, we have that

$$\frac{\Phi^{-1}(r)}{r^{1/s}} \leq \Phi_s^{-1}(r) \leq \left(\frac{s-1}{s-p} \right)^{\frac{s-1}{s}} \frac{\Phi^{-1}(r)}{r^{1/s}}.$$

(2) Let $p > s$ and let $\Phi(t)/t^p$ be increasing. Then $\Phi^{-1}(t)/t^{1/p}$ is decreasing and so

$$\tilde{\omega}_s(r) = \int_r^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}} \, dt \leq \frac{\Phi^{-1}(r)}{r^{1/p}} \int_r^\infty t^{1/p-1/s-1} \, dt = (1/s - 1/p)^{-1} \frac{\Phi^{-1}(r)}{r^{1/s}}.$$

On the other hand,

$$\tilde{\omega}_s(r) \geq \int_r^{2r} \frac{\Phi^{-1}(t)}{t^{1+1/s}} \, dt \geq r \frac{\Phi^{-1}(2r)}{(2r)^{1+1/s}} \geq 2^{-1-1/s} \frac{\Phi^{-1}(r)}{r^{1/s}}.$$

It was shown in [1] that

$$\omega_s(r) = \|t^{-1/s'}\|_{L^{\hat{\Phi}}((0,1/r))}. \tag{36}$$

Since

$$\int_0^{1/r} \hat{\Phi}(t^{-1/s'}/\lambda) \, dt \geq r^{-1} \hat{\Phi}(r^{1/s'}/\lambda),$$

it follows that

$$\omega_s(r) \geq r^{1/s'} \hat{\Phi}^{-1}(r)^{-1}.$$

Hence, by (24),

$$\omega_s(r) \geq \frac{1}{2} \Phi^{-1}(r) r^{-1/s}.$$

Assume that $\Phi(t)/t^p$ is increasing. Let $t > t'$. By using (24), the fact that $\Phi^{-1}(t)/t^{1/p}$ is decreasing, and (24) again, we obtain

$$\hat{\Phi}(t) \leq \Phi^{-1}(\hat{\Phi}(t)) t \leq \Phi^{-1}(\hat{\Phi}(t')) \left(\frac{\hat{\Phi}(t)}{\hat{\Phi}(t')} \right)^{1/p} t \leq 2 \frac{\hat{\Phi}(t')}{t'} \left(\frac{\hat{\Phi}(t)}{\hat{\Phi}(t')} \right)^{1/p} t,$$

which implies that

$$\frac{\hat{\Phi}(t)}{t^{p'}} \le 2^{p'} \frac{\hat{\Phi}(t')}{t'^{p'}}.$$

If $p > s$, then $p' < s'$. So,

$$\begin{aligned}\int_0^{1/r} \hat{\Phi}(t^{-1/s'}/\lambda)\, dt &\le 2^{p'} \frac{\hat{\Phi}(r^{1/s'}/\lambda)}{r^{p'/s'}} \int_0^{1/r} t^{-p'/s'}\, dt \\ &= Cr^{-1}\hat{\Phi}(r^{1/s'}/\lambda) \\ &\le r^{-1}\hat{\Phi}(Cr^{1/s'}/\lambda),\end{aligned}$$

where the last inequality comes from (23) and $C = 2^{p'}(1-p'/s')^{-1}$. Hence, by (36) and (24),

$$\omega_s(r) \le Cr^{1/s'}\hat{\Phi}^{-1}(r)^{-1} \le C\Phi^{-1}(r)r^{-1/s}.$$

□

For a ball $B_0 \subset X$, $u \in L^1(B_0)$ and $0 < s \le \infty$, define

$$M^{\#}_{s,B_0}u(x) = \sup_{x\in B\subset B_0} \mu(B)^{-1/s} ⨍_B |u - u_B|\, d\mu. \tag{37}$$

Lemma 3.2 *Let Φ satisfy (Φ-1) and (Φ-2). Then*

$$\|M^{\#}_{s,B}u\|_{L^{\Phi}_w(B)} \le C\|u\|_{A^{\Phi,s}_{\tau}(\tau B)},$$

where $C = C(C_d, \tau, s)$.

Proof. We may assume that $\|u\|_{A^{\Phi,s}_{\tau}(\tau B)} = 1$. Let $x \in B$ such that $M^{\#}_{s,B}u(x) > \lambda$. By the definition of $M^{\#}_{s,B}u$, there is a ball $B_x \subset B$ containing x such that

$$\mu(B_x)^{-1/s} ⨍_{B_x} |u - u_{B_x}|\, d\mu > \lambda.$$

This implies that

$$\mu(B_x) \le \Phi(\lambda)^{-1}\Phi\Big(\mu(B_x)^{-1/s} ⨍_{B_x} |u - u_{B_x}|\, d\mu\Big)\mu(B_x). \tag{38}$$

By the standard $5r$-covering lemma ([7, Theorem 1.16]), we can cover the set $\{x \in B : M^{\#}_{s,B}(x) > \lambda\}$ by balls $5\tau B_i$ such that the balls τB_i are disjoint and that each B_i is contained in B and satisfies (38). Using the doubling property of μ, estimate (38), inequality (25), and the fact that $\{B_i\} \in \mathcal{B}_\tau(\tau B)$, we obtain

$$\begin{aligned}&\mu(\{x \in B : M^{\#}_{s,B}u(x) > \lambda\}) \le \sum_i \mu(5\tau B_i) \le C(C_d,\tau)\sum_i \mu(B_i) \\ &\le C(C_d,\tau)\Phi\,(\lambda)^{-1}\sum_i \Phi\Big(\mu(B_i)^{-1/s} ⨍_{B_i} |u - u_{B_i}|\, d\mu\Big)\mu(B_i) \\ &\le \Phi\left(\frac{\lambda}{C(C_d,\tau,s)}\right)^{-1} \sum_i \Phi\Big(\mu(B_i)^{-1/s} ⨍_{B_i} |u - u_{B_i}|\, d\mu\Big)\mu(B_i) \\ &\le \Phi\left(\frac{\lambda}{C(C_d,\tau,s)}\right)^{-1}.\end{aligned}$$

The claim follows by the definition of $\|\cdot\|_{L^\Phi_w}$. $\square$

Proof of Theorem 1.6. Fix a ball $B_0 \subset X$. Denote $B_0' = (1+\delta)B_0$, $\mathcal{B} = \{B : x_B \in B_0 \text{ and } r_B \le \delta r_{B_0}\}$, $\mathcal{B}^+ = \mathcal{B} \cup \{B_0'\}$,

$$M_{\mathcal{B}} u(x) = \sup_{x\in B\in\mathcal{B}} \fint_B |u|\,d\mu \quad \text{and} \quad M^{\#}_{\mathcal{B}^+} u(x) = \sup_{x\in B\in\mathcal{B}^+} \fint_B |u-u_B|\,d\mu.$$

For $\lambda > 0$, let $\Omega_\lambda = \{x : M_{\mathcal{B}}u(x) > \lambda\}$ and $\Sigma_\lambda = \{x : M^{\#}_{\mathcal{B}^+}u(x) > \lambda\}$. The following *good λ inequality* was proved by MacManus and Pérez in [8]: There are constants C_0 and ε_0 such that for all $\lambda > 0$ and $0 < \varepsilon \le \varepsilon_0$,

$$\mu(\Omega_{C_0\lambda}) \le C_0\varepsilon\mu(\Omega_\lambda) + C_0\mu(\Sigma_{\varepsilon\lambda}). \tag{39}$$

We will show that (39) implies

$$\|M_{\mathcal{B}}u\|_{L^\Phi_w(B_0)} \le C\|M^{\#}_{\mathcal{B}^+}u\|_{L^\Phi_w(B_0)}. \tag{40}$$

We may assume that $\|M^{\#}_{\mathcal{B}^+}u\|_{L^\Phi_w(B)} = 1$. Then $\Phi(t)\mu(\Sigma_t) \le 1$ for $t > 0$. Since Φ is doubling, there is a constant C_1 such that $\Phi(C_0\lambda) \le C_1\Phi(\lambda)$ for $\lambda > 0$. By setting

$$\varepsilon = \min\{\varepsilon_0, (2C_0C_1)^{-1}\}$$

in (39), we obtain

$$\begin{aligned}\Phi(C_0\lambda)\mu(\Omega_{C_0\lambda}) &\le \frac{1}{2}\Phi(\lambda)\mu(\Omega_\lambda) + C\Phi(\varepsilon\lambda)\mu(\Sigma_{\varepsilon\lambda}) \\ &\le \frac{1}{2}\sup_{\lambda>0}\Phi(\lambda)\mu(\Omega_\lambda) + C\end{aligned}$$

for $\lambda > 0$. Hence

$$\sup_{\lambda>0}\Phi(\lambda)\mu(\Omega_\lambda) \le C.$$

By (25), we obtain (40). Denote $u_0 = u - u_{B_0}$. By the Lebesgue differentiation theorem

$$u_0(x) \le M_{\mathcal{B}}u_0(x)$$

for almost every $x \in B_0$. Hence

$$\|u - u_{B_0}\|_{L^\Phi_w(B_0)} \le \|M_{\mathcal{B}}u_0\|_{L^\Phi_w(B_0)} \le C\|M^{\#}_{\mathcal{B}^+}u_0\|_{L^\Phi_w(B_0)}.$$

Since $M^{\#}_{\mathcal{B}^+}u_0 = M^{\#}_{\mathcal{B}^+}u \le M^{\#}_{\infty,B_0'}u$, the claim follows from Lemma 3.2. $\square$

Theorem 1.7 follows from Theorem 1.6 via the following lemma.

Lemma 3.3 *Assume that $U \subset X$ is open, Φ satisfies (Φ-1) and (Φ-2) and that* (15) *holds. Then*

$$\|u\|_{A^{\tilde\Phi_s,\infty}_\tau(U)} \le \|u\|_{A^{\Phi,s}_\tau(U)}.$$

Proof. Since

$$\tilde{\Phi}_s^{-1}(t) \geq t^{-1/s}\Phi^{-1}(t),$$

we have, in (13), that

$$\begin{aligned} \mu(B)^{1/s}\Phi^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right) &= \nu(\tau B)^{1/s}\left(\frac{\nu(\tau B)}{\mu(B)}\right)^{-1/s}\Phi^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right) \\ &\leq \tilde{\Phi}_s^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right). \end{aligned}$$

This implies the claim. □

Proof of Theorem 1.8. 1) It suffices to show that the pointwise inequality

$$|u(x) - u_B| \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\tilde{\Phi}_s^{-1}\left(\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B)}}\right)\right), \tag{41}$$

where $B' = (1+\delta)B$, holds for Lebesgue points $x \in B$. Fix such a point x and choose balls B_i, $i \geq 1$, centered at x, so that $B_1 \subset B'$, and

$$r_{B_{i+1}} = \sup\{r > 0 : \mu(B(x,r)) \leq \frac{1}{2}\mu(B_i)\}.$$

This is possible because $\lim_{r\to 0}\mu(B(x,r)) = \mu(\{x\}) = 0$. By (1), we have that

$$2\mu(B_{i+1}) \leq \mu(B_i) \leq C\mu(B_{i+1}) \tag{42}$$

for all i.

By the Lebesgue differentiation theorem, $\lim_{i\to\infty} u_{B_i} = u(x)$. So

$$|u(x) - u_{B_1}| \leq \sum_{i=1}^{\infty}|u_{B_i} - u_{B_{i+1}}| \leq C\sum_{i=1}^{\infty}\fint_{B_i}|u - u_{B_i}|\,d\mu.$$

By denoting $B_0 = B'$,

$$|u_B - u_{B_1}| \leq |u_B - u_{B_0}| + |u_{B_0} - u_{B_1}| \leq C\fint_{B_0}|u - u_{B_0}|\,d\mu.$$

Thus

$$|u(x) - u_B| \leq C\sum_{i=0}^{\infty}\fint_{B_i}|u - u_{B_i}|\,d\mu.$$

For every i, we have

$$\fint_{B_i}|u - u_{B_i}|\,d\mu \leq \|u\|_{A_\tau^{\Phi,s}(\hat{B})}\mu(B_i)^{1/s}\Phi^{-1}(\mu(B_i)^{-1}).$$

Hence

$$\begin{aligned} \sum_{i=0}^{k}\fint_{B_i}|u - u_{B_i}|\,d\mu &\leq \|u\|_{A_\tau^{\Phi,s}(\hat{B})}\sum_{i=0}^{k}\mu(B_i)^{1/s}\Phi^{-1}(\mu(B_i)^{-1}) \\ &= \|u\|_{A_\tau^{\Phi,s}(\hat{B})}\sum_{i=0}^{k}\mu(B_i)^{-1}\frac{\Phi^{-1}(\mu(B_i)^{-1})}{(\mu(B_i)^{-1})^{1+1/s}}. \end{aligned}$$

Since the function $t \mapsto \frac{\Phi^{-1}(t)}{t^{1+1/s}}$ is decreasing, we have that

$$\mu(B_i)^{-1}\frac{\Phi^{-1}(\mu(B_i)^{-1})}{(\mu(B_i)^{-1})^{1+1/s}} \le 2\int_{\frac{1}{2}\mu(B_i)^{-1}}^{\mu(B_i)^{-1}} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt,$$

for $i \ge 0$. So, by summing and noting that $\mu(B_i)^{-1} \le \frac{1}{2}\mu(B_{i+1})^{-1}$, we obtain

$$\sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\,d\mu \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \int_{\frac{1}{2}\mu(B_0)^{-1}}^{\mu(B_k)^{-1}} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt. \tag{43}$$

Thus

$$\sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\,d\mu \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \tilde{\Phi}_s^{-1}(\mu(B_k)^{-1}). \tag{44}$$

The remaining part of the series will be estimated in terms of the sharp fractional maximal function (37). Using (42), we obtain

$$\begin{aligned}\sum_{i=k}^{\infty} \fint_{B_i} |u - u_{B_i}|\,d\mu &\le C\sum_{i=k}^{\infty} \mu(B_i)^{1/s} M^{\#}_{s,B'}u(x) \\ &\le C\mu(B_k)^{1/s} M^{\#}_{s,B'}u(x).\end{aligned}$$

So, by Lemma 3.2,

$$\sum_{i=k}^{\infty} \fint_{B_i} |u - u_{B_i}|\,d\mu \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \mu(B_k)^{1/s} \frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}. \tag{45}$$

Combining the estimates (44) and (45), we obtain

$$|u(x) - u_B| \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \left(\tilde{\Phi}_s^{-1}(\mu(B_k)^{-1}) + \mu(B_k)^{1/s} \frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}} \right).$$

If $\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right) \ge \mu(B_0)^{-1}$, we choose k such that

$$\mu(B_k)^{-1} \le \Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right) \le C\mu(B_k)^{-1}.$$

Since, by (34),

$$\Phi(r)^{-1/s} r \le \tilde{\Phi}_s^{-1}(\Phi(r)), \tag{46}$$

we obtain (41).

If $\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right) < \mu(B_0)^{-1}$, it suffices use (45) and (46).

2) Letting k tend to infinity in (43), yields

$$|u(x) - u_B| \le \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}|\,d\mu \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \tilde{\omega}_s(\mu(B)^{-1}). \tag{47}$$

Let $x, y \in B$ be Lebesgue points of u. Denote $B_{xy} = B(x, 2d(x, y))$. If $d(x, y) > \frac{1}{3}\delta r_B$, then $\mu(B) \leq C\mu(B_{xy})$. So, by (47) and (27),

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_B| + |u(y) - u_B| \\ &\leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\tilde{\omega}_s(\mu(B)^{-1}) \\ &\leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\tilde{\omega}_s(\mu(B_{xy})^{-1}).\end{aligned}$$

If $d(x, y) \leq \frac{1}{3}\delta r_B$, then (47), applied to the ball B_{xy}, yields

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_{B_{xy}}| + |u(y) - u_{B_{xy}}| \\ &\leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B}_{xy})}\tilde{\omega}_s(\mu(B_{xy})^{-1}).\end{aligned}$$

Since we may assume that $\delta < 1/2$, it follows that $\hat{B}_{xy} \subset \hat{B}$. Hence

$$|u(x) - u(y)| \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\tilde{\omega}_s(\mu(B_{xy})^{-1}).$$

□

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Characterizations of Orlicz-Sobolev spaces in terms of generalized Orlicz-Poincaré inequalities

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Characterizations of Orlicz-Sobolev spaces in terms of generalized Orlicz-Poincaré inequalities

Toni Heikkinen[*]

Abstract

We show that the Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^n)$ can be characterized in terms of the (generalized) Φ-Poincaré inequality. We also prove similar results in the general metric space setting.

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1 Introduction

Let Φ be a Young function and let $\Omega \subset \mathbb{R}^n$ be open. A pair (u, g) of measurable functions, $u \in L^1_{\mathrm{loc}}(\Omega)$ and $g \geq 0$, satisfies the Φ-Poincaré inequality in Ω, if there are constants $C_P \geq 1$ and $\tau \geq 1$ such that

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \Phi^{-1} \left(\fint_{\tau B} \Phi(g) \, d\mu \right) \tag{1}$$

for every ball $B = B(x, r_B)$ such that $\tau B \subset \Omega$. Here, $u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u$ and $\tau B = B(x, \tau r_B)$. It is well known that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies the 1-Poincaré inequality

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \fint_B |\nabla u| \, d\mu$$

for every ball $B \subset \Omega$. Thus, by Jensen's inequality, (1) holds with $\tau = 1$ and $g = |\nabla u|$. Our first result says that also the converse holds: If $u \in L^\Phi(\Omega)$ and there exists $g \in L^\Phi(\Omega)$ such that (1) holds (for the normalized pair), then u belongs to the Sobolev class $W^{1,\Phi}(\Omega)$.

Theorem 1.1 *Suppose that Φ is an N-function, $\Omega \subset \mathbb{R}^n$ is open, $u, g \in L^\Phi(\Omega)$ and that the pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ-Poincaré inequality in Ω. Then $u \in W^{1,\Phi}(\Omega)$ and $\||\nabla u|\|_{L^\Phi(\Omega)} \leq C(C_P, \tau, n)\|g\|_{L^\Phi(\Omega)}$.*

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For the definitions of Young and N-functions, see Section 2.1 below. Theorem 1.1 was proven by Hajłasz in [4] for $\Phi(t) = t^p$, $p \geq 1$, and by Tuominen in [13] for a doubling Φ whose conjugate is also doubling.

Our second result is a counterpart of Theorem 1.1 in the general metric setting. Let $X = (X, d, \mu)$ be a metric measure space with μ a Borel regular outer measure satisfying $0 < \mu(U) < \infty$, whenever U is nonempty, open and bounded. Suppose further that μ is doubling, that is, there exists a constant C_d such that

$$\mu(2B) \leq C_d \mu(B), \tag{2}$$

whenever B is a ball.

Our substitute for the usual Sobolev class $W^{1,\Phi}$ is based on upper gradients. We call a Borel function $g : X \to [0, \infty]$ an upper gradient of a function $u : X \to \overline{\mathbb{R}}$, if

$$|u(\gamma(0)) - u(\gamma(l))| \leq \int_\gamma g \, ds \tag{3}$$

for all rectifiable curves $\gamma : [0, l] \to X$. The concept of an upper gradient was introduced in [8]; also see [9]. Further, g as above is called a Φ-weak upper gradient if (3) holds for all curves γ except for a family of Φ-modulus zero, see Section 2.2 below. The Sobolev space $N^{1,\Phi}(X)$ consists of all functions in $L^\Phi(X)$ that have a (Φ-weak) upper gradient that belongs to $L^\Phi(X)$.

Theorem 1.2 *Suppose that Φ is a doubling Young function, $\Omega \subset X$ is open, $u, g \in L^\Phi(\Omega)$, and that the pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ-Poincaré inequality in Ω. Then a representative of u has a Φ-weak upper gradient g_u such that $\|g_u\|_{L^\Phi(\Omega)} \leq C(C_d, C_P, \tau)\|g\|_{L^\Phi(\Omega)}$.*

In the case $\Phi(t) = t^p$, $p \geq 1$, the result was essentially proven in [3], see [5].

In many important settings, including Riemannian manifolds with nonnegative Ricci curvature and Carnot-Carathéodory spaces associated with a system of vector fields satisfying Hörmander's condition, the Φ-Poincaré inequality holds for pairs (u, g), where $u \in N^{1,\Phi}(X)$ and g is an upper gradient of u, see [6]. In these settings Theorem 1.2 gives a characterization for $N^{1,\Phi}(X)$.

If both Φ and its conjugate are doubling, then the assumptions of Theorem 1.2 can be relaxed. In order to conclude that a representative of $u \in L^\Phi(\Omega)$ is in $N^{1,\Phi}(\Omega)$, it suffices to assume that the number

$$\|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)} = \sup_{\mathcal{B} \in \mathcal{B}_\tau(\Omega)} \Big\| \sum_{B \in \mathcal{B}} \big(r_B^{-1} \fint_B |u - u_B| \, d\mu\big) \chi_B \Big\|_{L^\Phi(\Omega)}, \tag{4}$$

where

$$\mathcal{B}_\tau(\Omega) = \big\{\{B_i\} : \text{balls } \tau B_i \text{ are disjoint and contained in } \Omega\big\},$$

is finite. Notice that $\|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)} \leq \lambda$ if and only if there is a functional $\nu : \{B \subset \Omega : B \text{ is a ball}\} \to [0, \infty)$ such that

$$\sum_i \nu(B_i) \leq 1, \tag{5}$$

whenever the balls B_i are disjoint, and that the generalized Φ-Poincaré inequality

$$\fint_B |u - u_B|\, d\mu \le \lambda r_B \Phi^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right) \tag{6}$$

holds whenever $\tau B \subset \Omega$. In particular, if a pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ-Poincaré inequality in Ω, then

$$\|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)} \le C_P \|g\|_{L^\Phi(\Omega)}. \tag{7}$$

The spaces $\mathcal{A}_\tau^{1,\Phi}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) : \|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)} < \infty\}$, for $\Phi(t) = t^p$, were studied in [7]. Theorem 1.3 below is a generalization of [7, Theorem 1.1].

Theorem 1.3 *Let $\Omega \subset X$ be an open set and let Φ be a doubling Young function whose conjugate is doubling. Then a representative of $u \in \mathcal{A}_\tau^{1,\Phi}(\Omega) \cap L^\Phi(\Omega)$ has a Φ-weak upper gradient g with $\|g\|_{L^\Phi(\Omega)} \le C(C_d, \tau)\|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)}$.*

If the assumptions of Theorem 1.3 are in force and the space X supports the Φ-Poincaré inequality (that is, (1) holds for pairs (u, g), where $u \in N^{1,\Phi}(X)$ and g is an upper gradient of u), then $A_\tau^{1,\Phi}(\Omega) \cap L^\Phi(\Omega)$ is isomorphic to $N^{1,\Phi}(\Omega)$ and the norms $\|\cdot\|_{L^\Phi(\Omega)} + \|\cdot\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)}$ and $\|\cdot\|_{N^{1,\Phi}(\Omega)}$ are equivalent.

2 Preliminaries

Throughout this paper C will denote a positive constant whose value is not necessarily the same at each occurrence. By writing $C = C(\lambda_1, \dots, \lambda_n)$ we indicate that the constant depends only on $\lambda_1, \dots, \lambda_n$.

2.1 Young functions and Orlicz spaces

In this subsection we recall the basic facts about Young functions and Orlicz spaces. An exhaustive treatment of the subject is [11].

A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if it has the form

$$\Phi(t) = \int_0^t \phi(s)\, ds,$$

where $\phi : [0, \infty) \to [0, \infty]$ is an increasing, left-continuous function, which is neither identically zero nor identically infinite on $(0, \infty)$.

If, in addition, $\phi(0) = 0$, $0 < \phi(t) < \infty$ for $t > 0$ and $\lim_{t\to\infty} \phi(t) = \infty$, then Φ is called an N-function.

A Young function is convex and, in particular, satisfies

$$\Phi(\varepsilon t) \le \varepsilon \Phi(t) \tag{8}$$

for $0 < \varepsilon \le 1$ and $0 \le t < \infty$.

If Φ is a Young function and $\mu(X) < \infty$, then Jensen's inequality

$$\Phi\left(\fint_X u\,d\mu\right) \le \fint_X \Phi(u)\,d\mu \tag{9}$$

holds for $0 \le u \in L^1(X)$.

The right-continuous generalized inverse of a Young function Φ is

$$\Phi^{-1}(t) = \inf\{s : \Phi(s) > t\}.$$

We have that

$$\Phi(\Phi^{-1}(t)) \le t \le \Phi^{-1}(\Phi(t))$$

for $t \ge 0$.

The conjugate of a Young function Φ is the Young function defined by

$$\hat{\Phi}(t) = \sup\{ts - \Phi(s) : s > 0\}$$

for $t \ge 0$.

Let Φ be a Young function. The Orlicz space $L^\Phi(X)$ is the set of all measurable functions u for which there exists $\lambda > 0$ such that

$$\int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) < \infty.$$

The Luxemburg norm of $u \in L^\Phi(X)$ is

$$\|u\|_{L^\Phi(X)} = \inf\{\lambda > 0 : \int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) \le 1\}.$$

If $\|u\|_{L^\Phi(X)} \ne 0$, we have that

$$\int_X \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(X)}}\right) d\mu(x) \le 1.$$

The following generalized Hölder inequality holds for Luxemburg norms:

$$\int_X u(x)v(x)\,d\mu(x) \le 2\|u\|_{L^\Phi(X)}\|v\|_{L^{\hat{\Phi}}(X)}.$$

Let $E^\Phi(X)$ denote the closure of the space of bounded, boundedly supported functions in $L^\Phi(X)$.

Lemma 2.1 *Let Φ be an N-function.*

(a) The dual of $E^\Phi(X)$ is isomorphic to $L^{\hat{\Phi}}(X)$; For every $F \in (E^\Phi(X))^*$, there exists $v \in L^{\hat{\Phi}}(X)$ such that

$$F(u) = \int uv\,d\mu.$$

Moreover,

$$\|v\|_{L^{\hat{\Phi}}(X)} \le \|F\| \le 2\|v\|_{L^{\hat{\Phi}}(X)}.$$

(b) If $\Omega \subset \mathbb{R}^n$ is open, then $C_0^\infty(\Omega)$ is dense in $E^\Phi(\Omega)$.

A Young function Φ is doubling, if there exists a constant $C_\Phi \geq 1$ such that

$$\Phi(2t) \leq C_\Phi \Phi(t)$$

for $t \geq 0$.

Lemma 2.2 *Let Φ be doubling a Young function.*

1) *The space $C_0(X)$ of bounded, boundedly supported continuous functions is dense in $L^\Phi(X)$.*
2) *The modular convergence and the norm convergence are equivalent, that is,*

$$\|f_j - f\|_{L^\Phi(X)} \to 0,$$

if and only if

$$\int_X \Phi(|f_j - f|)\, d\mu \to 0.$$

Lemma 2.3 *Suppose that Φ is a doubling Young function and that $\{g_i\} \subset L^\Phi(X)$ satisfies*

$$\sup_i \|g_i\|_{L^\Phi(X)} < \infty$$

and

$$\lim_{\mu(A)\to 0} \sup_i \int_A \Phi(g_i)\, d\mu = 0.$$

Then there exists a subsequence (g_{i_j}) of (g_i) and $g \in L^\Phi(X)$ such that $g_{i_j} \to g$ weakly in $L^\Phi(X)$.

Lemma 2.3 easily follows from [11, p.144, Corollary 2].

If both Φ and $\hat{\Phi}$ are doubling, then $L^\Phi(X)$ is reflexive, and so every bounded sequence in $L^\Phi(X)$ admits a weakly converging subsequence.

2.2 Sobolev spaces on metric measure spaces

The Φ-modulus of a curve family Γ is

$$\mathrm{Mod}_\Phi(\Gamma) = \inf\Big\{ \|g\|_{L^\Phi(X)} : \int_\gamma g\, ds \geq 1 \text{ for all } \gamma \in \Gamma \Big\}. \tag{10}$$

The Sobolev space $N^{1,\Phi}(X)$, defined by Tuominen in [12], consists of the functions $u \in L^\Phi(X)$ having a Φ-weak upper gradient $g \in L^\Phi(X)$. The space $N^{1,\Phi}(X)$ is a Banach space with the norm

$$\|u\|_{N^{1,\Phi}(X)} = \|u\|_{L^\Phi(X)} + \inf \|g\|_{L^\Phi(X)},$$

where the infimum is taken over Φ-weak upper gradients $g \in L^\Phi(X)$ of u.

We need the following lemma from [12].

Lemma 2.4 ([12], Theorem 4.17) *Suppose that $u_i \to u \in L^\Phi(X)$ and $g_i \to g \in L^\Phi(X)$ weakly in $L^\Phi(X)$ and that g_i is a Φ-weak upper gradient of u_i. Then g is a Φ-weak upper gradient of a representative of u.*

If Φ is doubling and $\Omega \subset \mathbb{R}^n$ is an open set, then $N^{1,\Phi}(\Omega)$ is isomorphic to $W^{1,\Phi}(\Omega)$ [12, Theorem 6.19]. As usual, $W^{1,\Phi}(\Omega)$ is the space of functions $u \in L^\Phi(\Omega)$ having weak partial derivatives in $L^\Phi(\Omega)$. A function $\partial u/\partial x_i \in L^1_{\mathrm{loc}}(\Omega)$ is a weak partial derivative of u (w.r.t. x_i) if

$$\int u\frac{\partial \varphi}{\partial x_i} = -\int \frac{\partial u}{\partial x_i}\varphi$$

for all $\varphi \in C_0^\infty(\Omega)$.

2.3 Lipschitz functions

A function $u : X \to \mathbb{R}$ is L-Lipschitz if $|u(x) - u(y)| \leq L\,\mathrm{d}(x,y)$ for all $x, y \in X$. The lower and upper pointwise Lipschitz constants of a locally Lipschitz function u are

$$\operatorname{lip} u(x) = \liminf_{r\to 0}\frac{L(u,x,r)}{r} \quad \text{and} \quad \operatorname{Lip} u(x) = \limsup_{r\to 0}\frac{L(u,x,r)}{r},$$

where

$$L(u,x,r) = \sup_{\mathrm{d}(x,y)\leq r}|u(x) - u(y)|.$$

The lower Lipschitz constant $\operatorname{lip} u$, and hence also $\operatorname{Lip} u$, is an upper gradient of a locally Lipschitz function u (cf. [1]).

3 Proofs

Proof of Theorem 1.1 We may assume that $\|g\|_{L^\Phi(\Omega)} = 1$. By Lemma 2.1, it suffices to show that the functional $\frac{\partial u}{\partial x_i} : C_0^\infty(\Omega) \to \mathbb{R}$;

$$\frac{\partial u}{\partial x_i}[\varphi] := -\int u\frac{\partial \varphi}{\partial x_i}$$

is bounded with respect to the norm $\|\cdot\|_{L^{\hat\Phi}(\Omega)}$ and satisfies $\|\frac{\partial u}{\partial x_i}\| \leq C$. Choose $0 \leq \psi \in C_0^\infty(B(0,1))$ such that $\int \psi = 1$ and let $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$ for $\varepsilon > 0$. Then

$$\frac{\partial u}{\partial x_i}[\varphi] = -\lim_{\varepsilon\to 0}\int (u * \psi_\varepsilon)\frac{\partial \varphi}{\partial x_i} = \lim_{\varepsilon\to 0}\int\left(u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right)\varphi.$$

By the Hölder inequality,

$$\left|\frac{\partial u}{\partial x_i}[\varphi]\right| \leq 2\liminf_{\varepsilon\to 0}\left\|u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right\|_{L^\Phi(\mathrm{supp}\ \varphi)}\|\varphi\|_{L^{\hat\Phi}(\mathrm{supp}\ \varphi)}.$$

Since $\int \frac{\partial \psi_\varepsilon}{\partial x_i} = 0$, we have that

$$\left(u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right)(x) = \left((u - u_{B(x,\varepsilon)}) * \frac{\partial \psi_\varepsilon}{\partial x_i}\right)(x).$$

Thus

$$\left|u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right|(x) \le C\varepsilon^{-n-1} \int_{B(x,\varepsilon)} |u(y) - u_{B(x,\varepsilon)}|\, dy \le C\Phi^{-1}\left(⨍_{B(x,\tau\varepsilon)} \Phi(g(y))\, dy\right).$$

Let $K = \operatorname{supp} \varphi$ and let $\varepsilon > 0$ be such that $K_{\tau\varepsilon} = \{x \in \mathbb{R}^n : d(x, K) < \tau\varepsilon\} \subset \Omega$. Then, by Fubini's theorem,

$$\begin{aligned}
\int_K \Phi\left(C^{-1}\left|u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right|(x)\, dx\right) &\le \int_K ⨍_{B(x,\tau\varepsilon)} \Phi(g(y))\, dy\, dx \\
&= \int_{K_{\tau\varepsilon}} \Phi(g(y)) \int_{B(y,\tau\varepsilon)\cap K} |B(x,\tau\varepsilon)|^{-1}\, dx\, dy. \\
&\le \int_{K_{\tau\varepsilon}} \Phi(g(y))\, dy \\
&\le 1.
\end{aligned}$$

Thus

$$\liminf_{\varepsilon\to 0} \left\|u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right\|_{L^\Phi(\operatorname{supp} \varphi)} \le C,$$

which completes the proof. □

For the proofs of Theorem 1.2 and Theorem 1.3, which are based on approximation by discrete convolutions, we need a couple of lemmas. Lemma 3.1 follows from a Whitney type covering result for doubling metric measure spaces, see [2, Theorem III.1.3], [10, Lemma 2.9]. For the proof of Lemma 3.2, we refer to [10, Lemma 2.16].

Lemma 3.1 *Let $\Omega \subset X$ be open. Given $\varepsilon > 0$, $\lambda \ge 1$, there is a cover $\{B_i = B(x_i, r_i)\}$ of Ω with the following properties:*

(1) $r_i \le \varepsilon$ for all i,

(2) $\lambda B_i \subset \Omega$ for all i,

(3) if λB_i meets λB_j, then $r_i \le 2r_j$,

(4) each ball λB_i meets at most $C = C(C_d, \lambda)$ balls λB_j.

A collection $\{B_i\}$ as above is called an (ε, λ)-covering of Ω. Clearly, an (ε, λ)-cover is an (ε', λ')-cover provided $\varepsilon' \ge \varepsilon$ and $\lambda' \le \lambda$.

Lemma 3.2 *Let $\Omega \subset X$ be open, and let $\mathcal{B} = \{B_i = B(x_i, r_i)\}$ be an $(\infty, 2)$-cover of Ω. Then there is a collection $\{\varphi_i\}$ of functions $\Omega \to \mathbb{R}$ such that*

1) each φ_i is $C(C_d) r_i^{-1}$-Lipschitz.

2) $0 \le \varphi_i \le 1$ for all i,

3) $\varphi_i(x) = 0$ for $x \in X \setminus 2B_i$ for all i, *4)* $\sum_i \varphi_i(x) = 1$ *for all* $x \in \Omega$.

A collection $\{\varphi_i\}$ *as above is called a partition of unity with respect to* $\mathcal{B}$.

Let $\mathcal{B} = \{B_i\}$ be as in the lemma above, and let $\{\varphi_i\}$ be a partition of unity with respect to $\mathcal{B}$. For a locally integrable function u on Ω, define

$$u_{\mathcal{B}}(x) = \sum_i u_{B_i} \varphi_i(x). \tag{11}$$

The following lemma describes the most important properties of $u_{\mathcal{B}}$.

Lemma 3.3

1) The function $u_{\mathcal{B}}$ *is locally Lipschitz. Moreover, for each* $x \in B_i$,

$$\operatorname{Lip} u_{\mathcal{B}}(x) \le C(C_d) r_{B_i}^{-1} \fint_{5B_i} |u - u_{5B_i}| \, d\mu.$$

2) Let Φ *be a doubling Young function and let* $u \in L^\Phi(\Omega)$*. If* $\mathcal{B}_k$ *is an* $(\varepsilon_k, 2)$*-cover of* Ω *and* $\varepsilon_k \to 0$ *as* $k \to \infty$*, then* $u_{\mathcal{B}_k} \to u$ *in* $L^\Phi(\Omega)$.

Proof. 1) Let $x, y \in B_i$, and let $J = \{j : 2B_j \cap 2B_i \ne \emptyset\}$. Then $\#J \le C(C_d)$ and $B_j \subset 5B_i$ for each $j \in J$. Using the properties of the functions φ_i, we have that

$$\begin{aligned}
|u_{\mathcal{B}}(x) - u_{\mathcal{B}}(y)| &= \Big| \sum_{j \in J} (u_{B_j} - u_{B_i}) \big(\varphi_j(x) - \varphi_j(y)\big) \Big| \\
&\le C(C_d) r_{B_i}^{-1} \operatorname{d}(x,y) \max_{j \in J} |u_{B_j} - u_{B_i}| \\
&\le C(C_d) r_{B_i}^{-1} \operatorname{d}(x,y) \fint_{5B_i} |u - u_{5B_i}| \, d\mu,
\end{aligned}$$

and the first claim follows.

2) We begin by showing that, for every $w \in L^\Phi(\Omega)$,

$$\|w_{\mathcal{B}}\|_{L^\Phi(\Omega)} \le C(C_d) \|w\|_{L^\Phi(\Omega)}. \tag{12}$$

We may assume that $\|w\|_{L^\Phi(\Omega)} = 1$. By Jensen's inequality $\Phi(|w_{\mathcal{B}}|) \le (\Phi(|w|))_{\mathcal{B}}$. Hence, by the properties of the functions φ_i,

$$\begin{aligned}
\int_\Omega \Phi(|w_{\mathcal{B}}|) \, d\mu &\le \int_\Omega (\Phi(|w|))_{\mathcal{B}} \, d\mu \le \sum_i \int_\Omega (\Phi(|w|))_{B_i} \varphi_i \, d\mu \\
&\le \sum_i \int_{2B_i} \Phi(|w|)_{B_i} \, d\mu \le C_d \sum_i \int_{B_i} \Phi(|w|) \, d\mu \\
&= C_d \int_\Omega \Phi(|w|) \sum_i \chi_{B_i} \, d\mu \le C(C_d) \int_\Omega \Phi(|w|) \, d\mu \\
&\le C(C_d).
\end{aligned}$$

Thus, by (8), we obtain (12).

Let $u \in L^\Phi(\Omega)$ and $\varepsilon > 0$. By Lemma 2.2 (1), there exists $v \in C_0(\Omega)$ such that $\|u - v\|_{L^\Phi(\Omega)} < \varepsilon$. Then, by (12), we obtain

$$\|u_{\mathcal{B}} - v_{\mathcal{B}}\|_{L^\Phi(\Omega)} = \|(u-v)_{\mathcal{B}}\|_{L^\Phi(\Omega)} \le C(C_d)\|u - v\|_{L^\Phi(\Omega)} < C(C_d)\varepsilon,$$

and so

$$\begin{aligned}\|u_{\mathcal{B}} - u\|_{L^\Phi(\Omega)} &\le \|u_{\mathcal{B}} - v_{\mathcal{B}}\|_{L^\Phi(\Omega)} + \|v_{\mathcal{B}} - v\|_{L^\Phi(\Omega)} + \|v - u\|_{L^\Phi(\Omega)} \\ &< \|v_{\mathcal{B}} - v\|_{L^\Phi(\Omega)} + C(C_d)\varepsilon.\end{aligned}$$

Therefore it suffices to show that $\|v_{\mathcal{B}_k} - v\|_{L^\Phi(\Omega)} \to 0$ as $\varepsilon_k \to 0$. Now $|v_{\mathcal{B}_k} - v| \le 2\sup|v|$, and for all x we have that

$$|v_{\mathcal{B}_k}(x) - v(x)| \le \sum_{2B_i \ni x} \fint_{B_i} |v(y) - v(x)|\,d\mu(y) \le C(C_d)\fint_{B(x,5\varepsilon_k)} |v(y) - v(x)|\,d\mu(y),$$

which converges to 0 as $\varepsilon_k \to 0$ by the continuity of v. Thus, by the dominated convergence theorem,

$$\int_\Omega \Phi(|v_{\mathcal{B}_k} - v|)\,d\mu \to 0,$$

and so, by Lemma 2.2 (2), $\|v_{\mathcal{B}_k} - v\|_{L^\Phi(\Omega)} \to 0$. □

Proof of Theorem 1.3. Let $u \in \mathcal{A}^{1,\Phi}_\tau(\Omega) \cap L^\Phi(\Omega)$. For $j \in \mathbb{N}$, let $\mathcal{B}_j$ be a $(j^{-1}, 5\tau)$-cover (and hence also a $(j^{-1}, 2)$-cover) of Ω. Then, by Lemma 3.3 (2), $u_j := u_{\mathcal{B}_j} \to u$ in $L^\Phi(\Omega)$. Let us show that

$$\|\operatorname{Lip} u_j\|_{L^\Phi(\Omega)} \le C(C_d, \tau)\|u\|_{\mathcal{A}^{1,\Phi}_\tau(\Omega)}. \tag{13}$$

By Lemma 3.3 (1),

$$\operatorname{Lip} u_j \le C(C_d) \sum_{B \in \mathcal{B}_j} r_B^{-1} \fint_{5B} |u - u_{5B}|\,d\mu\,\chi_B.$$

It follows from Lemma 3.1 (4) that $\mathcal{B}_j$ can be divided into $k = C(C_d, \tau)$ subfamilies $\mathcal{B}_{j,1}, \ldots, \mathcal{B}_{j,k}$ so that each of the families $5\tau\mathcal{B}_{j,l}$ consists of disjoint balls. Since the families $5\mathcal{B}_{j,1}, \ldots, 5\mathcal{B}_{j,k}$ belong to $\mathcal{B}_\tau(\Omega)$, we have that

$$\begin{aligned}\|\operatorname{Lip} u_j\|_{L^\Phi(\Omega)} &\le C(C_d) \sum_{l=1}^{k} \Big\| \sum_{B \in \mathcal{B}_{j,l}} r_B^{-1} \fint_{5B} |u - u_{5B}|\,d\mu\,\chi_B \Big\|_{L^\Phi(\Omega)} \\ &\le C(C_d) \sum_{l=1}^{k} \Big\| \sum_{B \in 5\mathcal{B}_{j,l}} r_B^{-1} \fint_{B} |u - u_{B}|\,d\mu\,\chi_B \Big\|_{L^\Phi(\Omega)} \\ &\le C(C_d, \tau)\|u\|_{\mathcal{A}^{1,\Phi}_\tau(\Omega)}.\end{aligned}$$

Since Φ and $\hat{\Phi}$ are doubling, $L^\Phi(\Omega)$ is reflexive. Thus the bounded sequence $(\operatorname{Lip} u_j)$ has a subsequence, also denoted by $(\operatorname{Lip} u_j)$, that converges weakly to some $g \in L^\Phi(\Omega)$. By Lemma 2.4, g is a Φ-weak upper gradient of a representative of u. As a weak limit g satisfies

$$\|g\|_{L^\Phi(\Omega)} \leq \liminf_{j\to\infty} \| \operatorname{Lip} u_j\|_{L^\Phi(\Omega)} \leq C(C_d,\tau)\|u\|_{\mathcal{A}^{1,\Phi}_\tau(\Omega)}.$$

Proof of Theorem 1.2 We may assume that $\|g\|_{L^\Phi(\Omega)} = 1$. Define the functions u_j as in the proof of Theorem 1.3. By (13) and (7), we have that

$$\|\operatorname{Lip} u_j\|_{L^\Phi(\Omega)} \leq C(C_d, C_P, \tau).$$

Let us show that

$$\lim_{\mu(E)\to 0} \sup_j \int_E \Phi(\operatorname{Lip} u_j)\, d\mu = 0. \tag{14}$$

By Lemma 3.3 (1) and by the Φ-Poincaré inequality,

$$\begin{aligned}
\operatorname{Lip} u_j &\leq C(C_d) \sum_{B\in\mathcal{B}_j} r_B^{-1} \fint_{5B} |u - u_{5B}|\, d\mu \chi_B \\
&\leq C(C_d, C_P) \sum_{B\in\mathcal{B}_j} \Phi^{-1}\left(\fint_{5\tau B} \Phi(g)\, d\mu\right) \chi_B.
\end{aligned}$$

Thus

$$\int_E \Phi(\operatorname{Lip} u_j)\, d\mu \leq C(C_d, C_P, C_\Phi) \sum_{B\in\mathcal{B}_j} \frac{\mu(E\cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g)\, d\mu.$$

Since B_j can be divided into $k = C(C_d,\tau)$ subfamilies $\mathcal{B}_{j,1}, \ldots, \mathcal{B}_{j,k}$ so that each of the families $5\tau\mathcal{B}_{j,l}$ consists of disjoint balls, it suffices to show that, for $1 \leq l \leq k$,

$$\lim_{\mu(E)\to 0} \sum_{B\in\mathcal{B}_{j,l}} \frac{\mu(E\cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g)\, d\mu = 0.$$

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\int_A \Phi(g) < \varepsilon$ whenever $\mu(A) < \delta$. Denote by $\mathcal{B}$ the family of those balls B in $\mathcal{B}_{j,l}$ for which

$$\frac{\mu(E\cap B)}{\mu(5\tau B)} < \varepsilon.$$

Also, let $\mathcal{B}' = \mathcal{B}_{j,l} \setminus \mathcal{B}$. Now, if $\mu(E) < \varepsilon\delta$, we have that $\mu(\cup_{B\in\mathcal{B}'} 5\tau B) \leq \varepsilon^{-1}\mu(E) < \delta$. Thus

$$\begin{aligned}
&\sum_{B\in\mathcal{B}_{j,l}} \frac{\mu(E\cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g)\, d\mu \\
&= \sum_{B\in\mathcal{B}} \frac{\mu(E\cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g)\, d\mu + \sum_{B\in\mathcal{B}'} \frac{\mu(E\cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g)\, d\mu \\
&\leq \varepsilon \int_\Omega \Phi(g)\, d\mu + \int_{\cup_{B\in\mathcal{B}'} 5\tau B} \Phi(g)\, d\mu \\
&\leq 2\varepsilon.
\end{aligned}$$

This completes the proof of (14).

By Lemma 2.3, a subsequence of $(\operatorname{Lip} u_j)$ converges weakly to some $g_u \in L^{\Phi}(\Omega)$, which, by Lemma 2.4, is a Φ-weak upper gradient of a representative of u. Moreover, as a weak limit, g_u satisfies

$$\|g_u\|_{L^{\Phi}(\Omega)} \leq \liminf_{j\to\infty} \|\operatorname{Lip} u_j\|_{L^{\Phi}(\Omega)} \leq C(C_d, C_P, \tau).$$

□

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