RENORMALIZED SOLUTIONS ON QUASI OPEN SETS WITH NONHOMOGENEOUS BOUNDARY VALUES

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Contents

1. Introduction

Let $U \subset \mathbb{R}^N$, $N \geq 2$, be a bounded, (quasi)open set, and consider a problem

$$\begin{cases} -\mathrm{div}\,(a(x, \nabla u)) = \mu & \text{in } U, \\ \qquad\qquad\quad u = g & \text{on } \partial U, \end{cases} \tag{1.1}$$

where μ is a bounded Radon measure, $g \in W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and the mapping $u \mapsto -\mathrm{div}\,(a(x, \nabla u))$ operates on some special set of functions defined in U, namely $\widehat{W}^{1,p}(U)$ (see Definition 3.6), $p > 1$. The mapping $u \mapsto -\mathrm{div}\,(a(x, \nabla u))$ is assumed to be similar to the p-Laplacian, $-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$ (see chapter 2 for a detailed definition of $a(\cdot, \cdot)$).

If $p > N$, then the solution to this problem is well known. In this case the space of measures with bounded variation in U is a subset of the dual space $W^{-1,p'}(U)$, and the existence and uniqueness of the solution follows from classical results, see [14]. The case $1 < p \leq N$ is, however, more complicated, and this is the case we consider here.

The existence of the solution (with zero boundary values) in the class $W_0^{1,q}(U)$, $q < \frac{N(p-1)}{N-1}$, satisfying

$$\int_U a(x, \nabla u) \cdot \nabla\varphi \, dx = \int_U \varphi \, d\mu \qquad \text{for every } \ \varphi \in C_0^\infty(U) \ ,$$

was proved in [2] and [3] using a sequence of regular functions approximating the measure in question. This gives a sequence of solutions for the easier problem described above, and it was then shown that this sequence converges to the solution for the original problem. This definition is, however, valid only in the case $p > 2 - 1/N$, and the solution may not even belong to the space $L^1(U)$ otherwise, see Example 2.16 in [5]. This solution also fails to be unique, as is seen from the example of Serrin in [15]. Thus, some alternative definition for the solution is needed.

The fact that the solutions might fall out of the class $W_0^{1,1}(U)$ suggests that the base set for solutions needs to be larger than a usual Sobolev space. It is, however, true that the truncations at level $k > 0$ of the solution obtained by the approximation method belong to the Sobolev space $W_0^{1,p}(U)$. This is the motivation for the base set $\widehat{W}^{1,p}(U)$ we use here. The introduction of this special set of functions and also the uniqueness problem was considered in [1], where the definition of *entropy solutions* is introduced (see also [4] and [12]). In order to obtain the uniqueness of entropy solutions, the so called *entropy condition* (see Equation (3.5) in [1]) was introduced.

Entropy solutions, however, only apply to measures that are absolutely continuous with respect to the p-capacity. For general bounded Radon measures this definition may not be well defined. To overcome this obstacle, the definition of *renormalized solutions* was introduced in [5]. The main idea was to accept a larger set of test functions than the class $C_0^\infty(U)$, which is used in the distributional setting. In [5], they essentially accept bounded Sobolev functions as test functions and call the solutions obtained this way renormalized solutions. The existence (and uniqueness when the measure is absolutely continuous with respect to p-capacity) of these solutions was then shown. Some extra conditions which assure the uniqueness also for the general bounded Radon measures were also introduced. However, the general case of the uniqueness of renormalized solutions with bounded Radon measure, and, more importantly, the uniqueness of distributional solutions in the class $\widehat{W}_0^{1,p}$ still remain as open questions. In the special case $p = N$ the uniqueness problem has been solved in [8], [9] and [6] by using some additional conditions. In the first two the solution belongs to the so called grand Sobolev space $W_0^{1,N)}$, and in [6] there are some additional assumptions on the regularity of the domain as well as on the regularity of the gradient of the solution.

Here we concentrate on the definition of a renormalized solution and generalize this definition for quasi open sets and non-homogeneous boundary values. Using the methods obtained this way, we then show some special versions of the strong comparison principle for renormalized solutions.

Chapters 2 to 4 are dedicated to introducing all the necessary definitions and basic tools to handle the problem (1.1) in a quasi open setting. In chapter 2 we discuss only basic mathematical concepts, while in chapters 3 and 4 we cover the properties of quasi open sets, examine some measure theory and study the fine properties of functions, namely, sets of finite perimeter.

In chapter 5 we define our solution to the problem (1.1) when U is a quasi open set. This will be called a renormalized solution, and the definition follows the lead from [5]. In this chapter we also generalize all the necessary integral and level estimates for the quasi open setting and nonhomogeneous boundary values. The chapter ends with the existence and uniqueness proof for renormalized solutions in this setting. While the quasi openness of the set does not play any major role in this proof and the boundary condition offers only technical problems, it should be noticed that this proof uses only the definition of the renormalized solution. The uniqueness of the renormalized solution in open sets when the measure is in $\mathcal{M}_0(\Omega)$ is already known (see [5], Remark 2.17), but the proof relies on the uniqueness proof for entropy solutions (see [1]) and the fact that in this case these two solution classes are actually one and the same. The uniqueness proof using only the definition has not been given (without additional restraints, see [5], Remark 10.7, see also [10]).

Chapter 6 is the main chapter in this work, and here we show the comparison principle for the renormalized solution when the measure in question is absolutely continous with respect to the p-capacity. Results of the same type have already been studied, for example, for so called entropy solutions, but the methods used here are completely different since here we use only the definition or renormalized solutions. Although these results do not apply straight away to the case of general bounded Radon measures, the calculations in Theorem 6.3 are mostly valid also in the general case. These calculations may thus prove to be useful also in the study of the problem (1.1) in the case of general measures.

2. Preliminaries and notations

In this chapter we introduce some important and frequently used notations and also recall some essential definitions.

In the following, Ω always means an open and bounded subset of $\mathbb{R}^N$. On the other hand, U always means a *quasi open* (see Definition 3.1) and bounded subset of $\mathbb{R}^N$.

The function $a(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ in the problem (1.1) satisfies the following conditions for some constants $0 < \alpha, \gamma < \infty$:
the function $x \mapsto a(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^N$,
the function $\xi \mapsto a(x, \xi)$ is continuous for almost every $x \in \mathbb{R}^N$,

(M1) $\quad a(x,\xi)\cdot\xi \geq \alpha|\xi|^p \qquad$ for a.e. $x \in \mathbb{R}^N$ and $\forall\, \xi \in \mathbb{R}^N$,

(M2) $\quad |a(x,\xi)| \leq \gamma\,|\xi|^{p-1} \qquad \forall\, \xi \in \mathbb{R}^N$,

(M3) $\quad (a(x,\xi) - a(x,\xi'))\cdot(\xi-\xi') > 0 \qquad$ for a.e. $x \in \mathbb{R}^N$ and $\forall\, \xi, \xi' \in \mathbb{R}^N,\ \xi \neq \xi'$.

For any $E \subset \mathbb{R}^N$ we define the *p-capacity of E* as

$$\mathrm{cap}_p(E) = \inf_{u\in\mathcal{F}(E)} \int_{\mathbb{R}^N} (|u|^p + |\nabla u|^p)\, dx\ ,$$

where

$$\mathcal{F}(E) = \{u \in W^{1,p}(\mathbb{R}^N) : 0 \leq u \leq 1,\ u=1 \text{ in an open set containing } E\}\ .$$

There are many other definitions for the p-capacity, see, for example 1.1, in [11]. The choice of the p-capacity is not too important here;

we only need to know that the sets of zero p-capacity do not change if the definition changes. Remember also that the p-capacity is an outer measure (see [7], Theorem 1 in section 4.7). In the following, the monotonicity of the p-capacity is frequently used, that is, if $A \subset B$, then $\mathrm{cap}_p(A) \leq \mathrm{cap}_p(B)$.

We use notation $\mathcal{M}_b(U)$ for the space of bounded Radon measures, and $\mathcal{M}_0(U)$ for the measures $\mu \in \mathcal{M}_b(U)$ that are absolutely continuous with respect to the p-capacity. That is, for $\mu \in \mathcal{M}_0(U)$ and $E \subset U$ we have $\mu(E) = 0$ if $\mathrm{cap}_p(E) = 0$. Results obtained in this paper mainly concern only measures in $\mathcal{M}_0(U)$.

When talking about measures, we are usually not interested in those parts of the set which have zero measures. These zero sets are simply omitted by saying that the result holds μ -almost everywhere (abbreviated by μ-a.e.), where μ is the measure in question. If the measure is not specified, we assume it is an n-dimensional Lebesgue-measure, $\mathcal{L}^N$. In the case of the ($p-$)capacity we say *quasi everywhere*, abbreviated by q.e.

For $k > 0$ and $s \in \mathbb{R}$, we define a function $T_k(s) : R \to \mathbb{R}$ by

$$T_k(s) = \max\{-k, \min\{k, s\}\} \ ,$$

in other words, truncation at levels k and $-k$.

The space of L^p-functions in Ω, $L^p(\Omega)$, is the space of Lebesgue measurable functions $u : \Omega \to \mathbb{R}^N$ whose $L^p(\Omega)$-norm

$$\|u\|_{L^p} = \Big(\int_\Omega |u|^p \, dx \Big)^{1/p}$$

is finite.
Moreover, the space of Sobolev functions, $W^{1,p}(\Omega)$, consists of those $L^p(\Omega)$-functions whose first-order weak partial derivatives $D_i u$ also belong to $L^p(\Omega)$ (see [7], chapter 4). The Sobolev norm of u is defined by

$$\|u\|_{1,p} = \Big(\int_\Omega |u|^p + |\nabla u|^p \, dx \Big)^{1/p} \ ,$$

where $\nabla u = (D_1 u, ..., D_n u)$ is the weak gradient of u. Later on, in chapter 3, we define these for quasi open sets.

When talking about L^p-spaces, we frequently need the concept of conjugate exponents. We use notations p and p' for this purpose, that is, $1/p + 1/p' = 1$.

Constants used in calculations are usually marked by C, or C_i, $i = 1, 2, ...$, if the same calculations involve more than one constant. These constanst are usually different and do not depend on each other in any way. If the constant depends on some important feature (say, dimension N or p or ε), then this is noted by subscripts, for example, C_ε.

3. Quasi topology and Sobolev functions

It is a well known fact that if we have an open set E and a continuous function f, then the set $f^{-1}(E)$ is also open. Later on, we are typically dealing with the situation where the functions are not continuous but only Sobolev functions. Then, for example, the level sets $\{u > t\}$ which play an important role in the following, cannot be expected to be open but only *quasi open*.

In this section we introduce the notions of quasi open sets and quasi continuous functions. We also define the class of Sobolev functions in quasi open sets and then introduce a slightly larger set of functions, $\widehat{W}^{1,p}(U)$, which we use as a base class for solutions of the problem (1.1). Since quasi openness and quasi continuity are linked to the p-capacity, we also introduce some necessary tools and definitions concerning capacity and some important sets. For a deeper understanding of these so called fine properties of sets, the reader should consult [11], where most of the definitions and results of this section are from.

We start out this section by defining quasi open sets and quasi continuous functions:

Definition 3.1. A set $U \subset \mathbb{R}^N$ is said to be *quasi open* if for every $\varepsilon > 0$ there exists an open set G such that $U \cup G$ is open and $\mathrm{cap}_p(G) < \varepsilon$. A function $f : U \to \overline{\mathbb{R}}$ is *quasi continuous* if for every $\varepsilon > 0$ there is an open set $G \subset \mathbb{R}^N$ such that $\mathrm{cap}_p(G) < \varepsilon$ and the restriction $f|_{U\setminus G}$ is finite valued and continuous.

Remark 3.2. If $f : U \to \mathbb{R}^N$ is quasi continuous, then the level sets $\{f < k\}$ and $\{f > k\}$ are quasi open for every $k \in \mathbb{R}$ (see [11], Theorem 1.4). Observe also that a quasi continous function is finite quasi everywhere.

Since the class of Sobolev functions, $W^{1,p}(E)$, is originally defined only for open sets E, we need to do some modifications concerning the case where the base set is only quasi open. Difficulties arise especially in the case when the functions do not vanish on the boundary of the base set. In this case the notion of a generalized derivative requires some deeper concern.

Definition 3.3. A family $\mathcal{B}$ of quasi open sets is called a *quasi covering* of a set E if there exists a countable union B of sets from $\mathcal{B}$ such that $\mathrm{cap}_p(E\setminus B) = 0$.

Definition 3.4. Let $U \subset \mathbb{R}^N$ be a quasi open set. Suppose that u is a restriction to U of some function $v \in W^{1,p}(V)$, where $V \subset \mathbb{R}^N$ is an open set containing U. Define the *fine gradient* of u as a restriction of ∇v to U, that is

$$\nabla u := (\nabla v)_{|U} \ .$$

Generalizing this idea, we define $\mathcal{F}(U)$ to be the set of functions $u : U \to \mathbb{R}$ for which there exists a quasi covering $\mathcal{B}$ of U such that for every $B \in \mathcal{B}$ there is an open set V_B (containing B) and a function $v_B \in W^{1,p}(V_B)$ such that $u_{|B} = (v_B)_{|B}$. The *fine gradient* of a function $u \in \mathcal{F}(U)$ is defined in every $B \in \mathcal{B}$ by

$$(\nabla u)_{|B} := (\nabla v_B)_{|B} \ .$$

It follows that ∇u is defined a.e. on U and it is independent of the choice of the quasi covering $\mathcal{B}$ (see [11], Definition 2.1.).

Definition 3.5. The *Sobolev space* $W^{1,p}(U)$ in the quasi open set U is defined by

$$W^{1,p}(U) := \{u \in \mathcal{F}(U) : \ \|u\|_{1,p} < \infty\} \ ,$$

where

$$\|u\|_{1,p} := \Big(\int_U |u|^p + |\nabla u|^p \, dx \Big)^{1/p}$$

is the $(1,p)$-norm of u.
The Sobolev space $W_0^{1,p}(U)$ in the quasi open set U is defined by

$$W_0^{1,p}(U) := \bigcap \{W_0^{1,p}(G) : \ G \text{ open}, \, G \supset U\} \ .$$

Now we are ready to introduce the class of truncated Sobolev functions, $\widehat{W}^{1,p}(U)$, which is the main class of functions from now on when we are dealing with the solutions of the problem (1.1).

Definition 3.6. For every $1 \leq p < \infty$, we define for measurable and q.e. finite functions u

$$\widehat{W}^{1,p}(U) := \{u : U \to \overline{\mathbb{R}} \ : \ T_k(u) \in W^{1,p}(U) \text{ for every } k > 0\} \ .$$

Similarly,

$$\widehat{W}_0^{1,p}(U) := \{u : U \to \overline{\mathbb{R}} \ : \ T_k(u) \in W_0^{1,p}(U) \text{ for every } k > 0\} \ .$$

Since the function $u \in \widehat{W}^{1,p}(U)$ does not necessarily belong to any normal Sobolev space, we need to define the gradient of u in some appropriate sense. This is done using the same technique as in Definition 3.4:

Definition 3.7. Let $u \in \widehat{W}^{1,p}(U)$, where U is quasi open, and $\mathcal{B}$ is a quasi covering of U. For each $k > 0$ we have $T_k(u) \in W^{1,p}(U)$. Moreover, for each $B \in \mathcal{B}$ there exists an open set G_B containing B and a function $w_{k,B} \in W^{1,p}(G_B)$ such that $(T_k u)_{|B} = (w_{k,B})_{|B}$.

For $l > k$ we have a function $w_{l,B} \in W^{1,p}(G_B)$ such that $(T_l u)_{|B} = (w_{l,B})_{|B}$, and in the set $\{|u| < k\}$ we have

$$(w_{l,B})_{|B} = (T_l u)_{|B} = (T_k u)_{|B} = (w_{k,B})_{|B}$$

for every $l > k$. Now we define $v_{k,B} := \nabla w_{k,B}$ in the set $\{|u| < k\} \cap B$. In this set we have a.e. equality $\nabla w_{k,B} = \nabla w_{l,B}$ for every $l > k$, and thus we get almost everywhere in B defined function v_B by going to the limit with respect to k.

As in Definition 3.4, we now obtain an a.e. in U defined function v, which we may say to be the gradient of $u \in \widehat{W}^{1,p}(U)$, write $\nabla u = v$.

Remark 3.8. The function $u \in \widehat{W}^{1,p}(U)$ has a cap_p-quasi continuous representation. This can be seen by choosing for each $k > 0$ an open set G_k (containing the infinity points of u) such that $\mathrm{cap}_p(G_k) < 2^{-k}\varepsilon$ and $T_k(u)$ is continuous in G_k^C. Now the set $G := \cup G_k$ is open and $\mathrm{cap}_p(G) < \varepsilon$. Moreover,

$$G^C = \cap G_k^C \subset G_k^C \quad \forall\, k > 0 \ .$$

Now, if $x_0 \in G^C$, we choose k such that $u(x_0) < k$. Then we have $\limsup_{x \to x_0} u(x) < k$, since otherwise $\limsup_{x \to x_0} T_k u(x) = k$, which is a contradiction. Thus,

$$\limsup_{x \to x_0} u(x) = \limsup_{x \to x_0} T_k u(x) = T_k u(x_0) = u(x_0) \ .$$

The same deduction for $\liminf_{x \to x_0} u(x)$ shows that u is continuous in G^C, that is, u is quasi continuous.

In the following we are always considering this quasi continuous version of u.

Next we point out some important properties of the level sets of $\widehat{W}^{1,p}$-functions. These estimates of measure and capacity are well known for open sets, but the quasi openness imposes some technical details which we bring forth here.

Lemma 3.9. *Let* $u \in \widehat{W}^{1,p}(U)$ *and* $1<p\leq N$. *Then*

$$\mathrm{cap}_p\big(K \cap \{|u| > k\}\big) \to 0 \quad \textit{as} \quad k \to \infty$$

for any compact set $K \subset\subset U$.

Proof. Let $\varepsilon > 0$. By Remark 3.8 we know that u is cap_p-quasi continuous. Thus there exists an open set V_ε such that $\mathrm{cap}_p(V_\varepsilon) < \varepsilon$ and u is continuous on $U \backslash V_\varepsilon$. Since u is continuous also on $K \backslash V_\varepsilon$ and $K \backslash V_\varepsilon$ is closed and bounded, we know that there exists $k_\varepsilon \in \mathbb{R}$ such that

$|u| \le k_\varepsilon$ on $K \backslash V_\varepsilon$. Moreover, for $k \ge k_\varepsilon$ we have $K \cap \{|u| > k\} \subset V_\varepsilon$ so that

$$\mathrm{cap}_p\big(K \cap \{|u| > k\}\big) \le \mathrm{cap}_p(V_\varepsilon) < \varepsilon \qquad \text{for every } k \ge k_\varepsilon \ .$$

□

Lemma 3.10. *If $u \in \widehat{W}^{1,p}(U)$, then*

$$\mathcal{H}^{N-1}(\{|u| = \infty\}) = 0 \ .$$

Proof. From the definition of $\widehat{W}^{1,p}(U)$ it follows that $\mathrm{cap}_p(\{|u|{=}\infty\}){=}0$, and thus we have $\mathcal{H}^s(\{|u| = \infty\}) = 0$ for every $s > N - p$ (see [7], Theorem 4 in section 4.7). □

Remark 3.11. Notice that the compact set K in Lemma 3.9 is only a technicality, which can be removed after we define renormalized solutions, see Corollary 5.3. On the other hand, when we use Lemma 3.9 in chapter 6, the setting is always as in Lemma 3.9, that is, the compact set K is present.

Finally, we introduce a sort of generalization of Sobolev's embedding theorem. We define a set $\widetilde{W}_g^{1,p}(U)$, which is needed to assure that u can be extended properly to a larger open set Ω, and this way we obtain the right to use the usual version of the Sobolev theorem. Later, when we introduce renormalized solutions, one should notice that renormalized solutions (or truncations of them at high levels, to be precise) indeed fulfill this property. The size of Ω is not important here; we only need to know that Ω is bounded and the distance from U to the boundary of Ω is large enough to assure the existence of φ introduced in the lemma. This lemma is used in Theorem 5.5 for renormalized solutions, but we state the result already here since this result is true for more general functions than just renormalized solutions.

Lemma 3.12. *Let $U \subset \mathbb{R}^N$ be a bounded quasi open set, $\Omega \subset \mathbb{R}^N$ an open and bounded set, $U \subset\subset \Omega$, $1 \le p < N$ and*

$$u \in \widetilde{W}_g^{1,p}(U) \ \ := \{u \in W^{1,p}(\Omega) \ : \ u = g \ \ \textit{q.e. on } \Omega \backslash U\} \ ,$$

for some $g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, that is, a function u can be extended quasi continuously to a bounded function g outside of U.
Then there exists $C = C_{p,q,N,U,\Omega} > 0$ such that

$$\|u\|_{L^q(U)} \le C\|\nabla(u\,\varphi)\|_{L^p(\Omega)} \qquad \textit{for every } \ q \in [1, p^*] \ ,$$

where $p^* = \frac{Np}{N-p}$ and $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ on $\overline{U}$.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ on $\overline{U}$. Now $u\,\varphi \in W_0^{1,p}(\Omega)$, and the result follows from the usual Sobolev embedding theorem for Ω as

$$\|u\|_{L^q(U)} \le \|u\,\varphi\|_{L^q(\Omega)} \le C\|\nabla(u\,\varphi)\|_{L^p(\Omega)} \ .$$

□

4. Measure theory and fine properties of functions

Level sets $\{u > t\}$, $t \in \mathbb{R}$, of Sobolev functions play a very important role in the theory that will be developed in chapter 6. As noted earlier, these level sets are not open but only quasi open sets. This is, however, not enough to verify the existence of some important tools, namely, the Gauss-Green Theorem 4.9. For this purpose we need to study the so called fine properties of functions, which include definitions of functions of bounded variation and sets of finite perimeter. Here we recall only the basic properties of these functions and sets; a more detailed representation can be found, for example, from [7].

We start this section by recalling the properties of measures we use throughout this paper. Remember that the measures we are considering are bounded Radon measures and absolutely continuous with respect to the p-capacity, write $\mu_0 \in \mathcal{M}_0(U)$. This means that $\mu_0(E) = 0$ for any measurable set $E \subset \mathbb{R}^N$ such that $\operatorname{cap}_p(E) = 0$. It is well known that measures of this type can be written as a sum of $L^1(\Omega)$ and $L^{p'}(\Omega)$ functions when Ω is an open set (see [4]). First we show that this is true also for quasi open sets.

Theorem 4.1. *Let U be a quasi open set and $\mu_0 \in \mathcal{M}_0(U)$. Then there exist functions $F \in L^1(U)$ and $G \in L^{p'}(U)$ such that*

$$\int_U v \, d\mu_0 = \int_U Fv \, dx + \int_U G \cdot \nabla v \, dx \qquad \forall\, v \in W_0^{1,p}(U) \cap L^\infty(U),$$

that is, $\mu_0 = F - \operatorname{div}(G)$ in this sense.

Proof. First we choose some open set $V \subset R^N$ such that $U \subset V$ and define a new measure μ^* in V by

$$\mu^*(E) = \begin{cases} \mu_0(E) & \text{for any Borel set } E \subset U \\ 0 & \text{for any Borel set } E \subset V \setminus U \,. \end{cases}$$

Thus we have $\mu^* \in \mathcal{M}_0(V)$. Now let $v \in W_0^{1,p}(U) \cap L^\infty(U)$. Since the claim is true for open sets (see [5], Proposition 2.5 or [4], Theorem 2.1), there exist functions $F \in L^1(V)$ and $G \in L^{p'}(V)$ for which we may calculate

$$\begin{aligned}
&\Big| \int_U Fv\,dx + \int_U G\cdot\nabla v\,dx - \int_U v\,d\mu_0 \Big| \\
&= \Big| \int_V Fv\,dx + \int_V G\cdot\nabla v\,dx - \int_{V\setminus U} Fv\,dx - \int_{V\setminus U} G\cdot\nabla v\,dx - \int_U v\,d\mu_0 \Big| \\
&= \Big| \int_V v\,d\mu_0 - \int_U v\,d\mu_0 \Big| \\
&= \Big| \int_U v\,d\mu_0 + \int_{V\setminus U} v\,d\mu_0 - \int_U v\,d\mu_0 \Big| = 0
\end{aligned}$$

since $v = 0$ and $\nabla v = 0$ quasi everywhere outside of U. $\square$

Next we introduce the functions of bounded variation and sets of finite perimeter and one important property that we get from the finiteness of the perimeter.

Definition 4.2. Let $\Omega \subset \mathbb{R}^N$ be an open set. A function $f \in L^1(\Omega)$ is of *bounded variation* in Ω if

$$\sup\Big\{ \int_\Omega f \operatorname{div}\varphi\,dx \;:\; \varphi \in C_0^\infty(\Omega;\mathbb{R}^N),\ |\varphi| \le 1 \Big\} < \infty\ . \tag{4.1}$$

In this case we write $f \in BV(\Omega)$.

Definition 4.3. An $\mathcal{L}^N$-measurable subset $E \subset \mathbb{R}^N$ has *finite perimeter* in Ω if $\chi_{_E} \in BV(\Omega)$.

Example 4.4. If $u \in W^{1,1}(\Omega)$, then $u \in BV(\Omega)$. This follows from the integration by parts formula for Sobolev functions since

$$\int_\Omega f \operatorname{div}\varphi\,dx = -\int_\Omega \nabla f\cdot\varphi\,dx \le \int_\Omega |\nabla f|\,dx < \infty\ .$$

Moreover, the level set $E_t := \{u > t\}$ of u has finite perimeter for a.e. $t \in \mathbb{R}$. See [7], Theorem 1 in section 5.5 for proof.

Theorem 4.5. *Let $E \subset \mathbb{R}^N$ be $\mathcal{L}^N$-measurable. Then E has locally finite perimeter if and only if*

$$\mathcal{H}^{N-1}(K \cap \partial_* E) < \infty$$

for every compact set $K \subset \mathbb{R}^N$.

Proof. See [7], Theorem 5.11.1. $\square$

The main goal of this section is to achieve a Gauss-Green type equality for sets of finite perimeter. Gauss-Green type inequalities involve the concept of outer normal on the boundary of the set. Defining this in the sets of finite perimeter takes some very careful examination of measure theoretic properties of the boundary sets. Here we only point out the existence of such an outer normal and use it to obtain the required results. A complete representation of the theory can be found, for example, from [7].

Definition 4.6. Let $x \in \mathbb{R}^N$. We say $x \in \partial_* E$, the *measure theoretic boundary* of E if

$$\limsup_{r\to 0} \frac{\mathcal{L}^N(B(x,r)\cap E)}{r^N} > 0$$

and

$$\limsup_{r\to 0} \frac{\mathcal{L}^N(B(x,r)\cap E^C)}{r^N} > 0 \ .$$

Remark 4.7. Notice that the measure theoretic boundary is a subset of the usual topological boundary of the set. When we are dealing with the sets of finite perimeter, it is usually sufficient to consider only those boundary points that belong to the measure theoretic boundary. Our setting in chapter 6 is such. That is, we may omit those topological boundary points that fall out of the measure theoretical boundary (if any).

Theorem 4.8. *For $\mathcal{H}^{N-1}$-a.e. $x \in \partial_* E$ there exists a unit vector $\nu_E(x)$ such that*

$$\lim_{r\to 0} \frac{\mathcal{L}^N(B(x,r)\cap E \cap H^+(x))}{r^N} = 0$$

and

$$\lim_{r\to 0} \frac{\mathcal{L}^N(B(x,r)\cap E^C \cap H^-(x))}{r^N} = 0,$$

where

$$H^+(x) := \{y \in \mathbb{R}^N \ : \ \nu_E(x)\cdot(y-x) \geq 0\},$$
$$H^-(x) := \{y \in \mathbb{R}^N \ : \ \nu_E(x)\cdot(y-x) \leq 0\}$$

are half-spaces defined by $\nu_E(x)$. Such a vector is called the measure *theoretic unit outer normal to E at x.*

Proof. See [7], sections 5.7 and 5.8. □

Theorem 4.9 (Generalized Gauss-Green Theorem)**.** *Let $E \subset \mathbb{R}^N$ have locally finite perimeter. For $\mathcal{H}^{N-1}$-a.e. $x \in \partial_* E$ there is a unique measure theoretic unit outer normal $\nu_{_E}(x)$ such that*

$$\int_E \operatorname{div}\varphi \, dx = \int_{\partial_* E} \varphi \cdot \nu_{_E} \, d\mathcal{H}^{N-1}$$

for all $\varphi \in C_0^1(\mathbb{R}^N)$.

Proof. See [7], section 5.8, Theorem 1. □

5. Renormalized solutions on quasi open sets

With the proper quasi topological setting introduced, we are now ready to define our solutions for the problem (1.1). Renormalized solutions have already been introduced, for example, in [5] but only for open sets and zero boundary values. Here we extend the concept of the definition for quasi open sets and nonhomogeneous boundary values. The main purpose for this is to gain some necessary tools for chapter 6 (mainly for Lemma 6.4), but for the sake of completeness we also show the existence and uniqueness of renormalized solutions in this setting.

It is known that the renormalized solution with zero boundary data is unique if the right-hand side measure is absolutely continuous with respect to the p-capacity. This has been shown, for example, in [5] and it also follows from the uniqueness of *entropy solutions* shown, for example, in [1], [4] and [12] since, in this case, these two solution classes are one and the same (see [5], Remark 2.17). However, in [5], some extra assumptions on the operator $a(x, \nabla u)$ (namely, Hölder continuity) are used, and other proofs use the definition of the entropy solution.

The proof shown here differs from the above mentioned ones in such a way that here we use only the definition of the renormalized solution and do not make any extra assumptions on the operator $a(x, \nabla u)$. The proof also includes the necessary technical modifications generated by the boundary function g. The quasi openness of the base set does not play any major role in the proof, since all the necessary adjustments in the definitions and calculations have already been made in the earlier chapters.

We start with the definition of renormalized solutions.

Definition 5.1. Let $U \subset \mathbb{R}^N$ be a quasi open set, μ a measure in $\mathcal{M}_0(U)$, $1<p\leq N$ and $g \in W^{1,p}(V) \cap L^\infty(V)$ for some open $V \supset\supset U$. A *renormalized solution* of the problem (1.1) is a measurable and almost everywhere finite function $u \in \widehat{W}^{1,p}(U)$ such that

$$(1) \qquad |\nabla u|^{p-1} \in L^r(U) \qquad \text{for every} \quad r < \frac{N}{N-1}$$

$$(2) \qquad \begin{cases} \displaystyle\int_U a(x, \nabla u) \cdot \nabla w \, dx = \int_U w \, d\mu \\ u = g \quad \text{a.e. on } U^C \quad \text{and} \quad u - g \in \widehat{W}_0^{1,p}(U) \end{cases}$$

whenever $w \in W_0^{1,p}(U) \cap L^\infty(U)$ is a function for which there exist $w^{+\infty}, w^{-\infty} \in W^{1,s}(U) \cap L^\infty(U)$ such that

$$\begin{cases} w = w^{+\infty} & \text{almost everywhere in the set } \{u > k\} \\ w = w^{-\infty} & \text{almost everywhere in the set } \{u < -k\} \end{cases}$$

for some $k > 0$ and $s > N$.

One of the most important aspects in the theory of renormalized solutions is to control the integrals of the gradient of truncated solutions or, equivalently, integrals of the gradient over the set where the solution is bounded. The following lemmas introduce these estimates in our setting. The main difficulty here is generated by the boundary function g. Otherwise, these proofs would be similar to those in [5] and [1].

Lemma 5.2. *Let $\mu \in \mathcal{M}_0(U)$, $1 < p \leq N$ and $u \in \widehat{W}^{1,p}(U)$ a renormalized solution to* (1.1). *Then, for every $k > 0$ we have the gradient estimate*

$$\int_U |\nabla T_k(u)|^p \, dx \leq C\, k \ . \tag{5.1}$$

Proof. Define $h := \|g\|_{L^\infty}$ and let $k > 0$. Using $w := T_k(u - T_h(u)) \in W_0^{1,p}(U) \cap L^\infty(U)$ as a test function in Equation 5.1 (2) (notice that $w = k$ when $u > k + h$, $w = -k$ when $u < -k - h$ and

$$\nabla T_k(u - T_h(u)) = \begin{cases} \nabla u & \text{when } h \leq |u| \leq k + h \\ 0 & \text{otherwise} \end{cases} \quad),$$

we get

$$\int_U T_k(u - T_h(u)) \, d\mu = \int_U a(x, \nabla u) \cdot \nabla T_k(u - T_h(u)) \, dx$$
$$= \int_{\{h \leq |u| \leq k+h\}} a(x, \nabla u) \cdot \nabla u \, dx \ .$$

Thus,

$$\int_{\{|u| \leq k+h\}} a(x, \nabla u) \cdot \nabla u \, dx = \int_{\{|u| < h\}} a(x, \nabla u) \cdot \nabla u \, dx + \int_{\{h \leq |u| \leq k+h\}} a(x, \nabla u) \cdot \nabla u \, dx$$
$$\leq \int_{\{|u| < h\}} a(x, \nabla u) \cdot \nabla u \, dx + k\mu(U) \ . \tag{5.2}$$

Using the Hölder inequality with the assumption (M2) we then estimate

$$\begin{aligned}
\int\limits_{\{|u|<h\}} a(x,\nabla u)\cdot\nabla u\,dx &\le \int\limits_{\{|u|<h\}} |a(x,\nabla u)\cdot\nabla u|\,dx \\
&\le \Big(\int\limits_{\{|u|<h\}} |a(x,\nabla u)|^{p'}\,dx\Big)^{1/p'}\Big(\int\limits_{\{|u|<h\}} |\nabla u|^{p}\,dx\Big)^{1/p} \\
&\le \Big(\int\limits_{\{|u|<h\}} |\nabla u|^{(p-1)p'}\,dx\Big)^{1/p'}\Big(\int\limits_{\{|u|<h\}} |\nabla u|^{p}\,dx\Big)^{1/p} \\
&\le \int\limits_{U} |\nabla T_h u|^p\,dx < C\ ,
\end{aligned} \tag{5.3}$$

where C does not depend on the choice of k since h is a fixed constant. Combining Equations (5.2) and (5.3) and using (M1), we finally have

$$\begin{aligned}
\int\limits_U |\nabla T_k u|^p\,dx &\le \int\limits_U |\nabla T_{k+h} u|^p\,dx \\
&\le \frac{1}{\alpha}\int\limits_U a\big(x,\nabla T_{k+h}(u)\big)\cdot\nabla T_{k+h}(u)\,dx \le C_1\,k
\end{aligned}$$

since the first integral is increasing with respect to k. $\square$

Now we may rephrase Lemma (3.9) for renormalized solutions without intersecting with the compact set:

Corollary 5.3. *Let $u\in\widehat{W}^{1,p}(U)$ be a renormalized solution and $1<p\le N$. Then*

$$\operatorname{cap}_p\big(\{|u|>k\}\big)\to 0 \quad \textit{as} \quad k\to\infty\ .$$

Proof. See Remark 2.11 in [5]. $\square$

Corollary 5.4. *Let $\mu\in\mathcal{M}_0(U)$, $1<p\le N$ and $u\in\widehat{W}^{1,p}(U)$ a renormalized solution to* (1.1)*. Then we have*

$$\lim_{k\to\infty}\frac{1}{k}\int\limits_U |\nabla T_k(u)|^p\,dx = 0\ . \tag{5.4}$$

Proof. First we notice that, for $h:=\|g\|_{L^\infty}$, we have $\frac{1}{k}T_k(u-T_h(u))\to 0$ μ-almost everywhere as $k\to\infty$. This is true since $\operatorname{cap}_p(\{|u|=\infty\})=0$ by the definition of $\widehat{W}^{1,p}(U)$ and since μ is absolutely continuous with respect to the p-capacity. Thus we have

$$\lim_{k\to\infty}\frac{1}{k}\int\limits_U T_k(u-T_h(u))\,d\mu = 0,$$

and now we can use the proof of Lemma 5.2 with the test function $\frac{1}{k}T_k(u - T_h(u))$ to obtain our result. □

It is equally important to control the measure of the level sets of the solution and its gradient. The next lemma gives us estimates for them, and using these estimates, we can, for example, deduce that $\mathcal{L}^N(\{|u| > k\}) \to 0$ as $k \to \infty$. Notice that an estimate like this is already proved for the p-capacity in Lemma 3.9 but only locally. The estimate given here is global. One should also notice that in this lemma we have limit $p < N$, but the result is essentially true also for $p = N$. There is only a slight difference in the constants and the power of k, see [5], Theorem 4.1.

Lemma 5.5. *Let $u \in \widehat{W}^{1,p}(U)$ be a renormalized solution and $1<p<N$. Then for every $k > 0$*

$$\mathcal{L}^N\big(\{|u| > k\}\big) \le C_1 k^{-\frac{N(p-1)}{N-p}} \qquad \textit{and} \tag{5.5}$$

$$\mathcal{L}^N\big(\{|\nabla u| > k\}\big) \le C_2 k^{-\frac{N(p-1)}{N-1}} \tag{5.6}$$

for some C_1 and C_2 independent of u.

Proof. Let $k > 0$. Since $\{|u| \ge k\} = \{|T_k(u)| \ge k\}$, we can calculate

$$\begin{aligned}\mathcal{L}^N\big(\{|u| \ge k\}\big) &\le \int\limits_{\{|T_k(u) \ge k|\}} \frac{|T_k(u)|}{k}\, dx \le \int\limits_U \left(\frac{|T_k(u)|}{k}\right)^{p^*} dx \\ &= k^{-p^*} \|T_k(u)\|^{p^*}_{L^{p^*}(U)} .\end{aligned}$$

Now let Ω and $\varphi \in C_0^\infty(\Omega)$ be as in Lemma 3.12. Using this lemma, we then have

$$k^{-p^*} \|T_k(u)\|^{p^*}_{L^{p^*}(U)} \le C_3 k^{-p^*} \|\nabla\big(T_k(u)\,\varphi\big)\|^{p^*}_{L^p(\Omega)} .$$

From Minkowski's inequality we get

$$\|\nabla\big(T_k(u)\,\varphi\big)\|_{L^p(\Omega)} \le \|T_k(u)\,\nabla\varphi\|_{L^p(\Omega)} + \|\varphi\,\nabla T_k(u)\|_{L^p(\Omega)} ,$$

and we can continue estimating these terms separately. First,

$$\Big(\int\limits_\Omega |T_k(u)\,\nabla\varphi|^p\,dx\Big)^{1/p} = \Big(\int\limits_{\Omega\setminus U} |T_k(g)\,\nabla\varphi|^p\,dx + \int\limits_U |T_k(u)\,\nabla\varphi|^p\,dx\Big)^{1/p} \le C_4$$

since $\varphi \equiv 1$ on U. Secondly,

$$\begin{aligned}\Big(\int\limits_\Omega |\varphi\,\nabla T_k(u)|^p\,dx\Big)^{1/p} &= \Big(\int\limits_{\Omega\setminus U} |\varphi\,\nabla T_k(g)|^p\,dx + \int\limits_U |\varphi\,\nabla T_k(u)|^p\,dx\Big)^{1/p} \\ &\le \Big(C_5 + C_6 \int\limits_U |\nabla T_k(u)|^p\,dx\Big)^{1/p} .\end{aligned}$$

By combining the above equations and using the estimate (5.1), we now have

$$\|\nabla\big(T_k(u)\,\varphi\big)\|^{p^*}_{L^p(\Omega)} \le \big(C_4 + C_8\,k^{1/p}\big)^{p^*} \le C_9\,k^{p^*/p}\ ,$$

and the claim (5.5) follows. The claim (5.6) follows from (5.5) as in [1], Lemma 4.2. □

Using the previous lemma, we now get another estimate in the spirit of Lemma 5.2 and Corollary 5.4. This lemma is specially crafted for the uniqueness proof of renormalized solutions.

Lemma 5.6. *Let* $1<p\le N$, $u\in \widehat{W}^{1,p}(U)$ *a renormalized solution and* $k>0$. *Then*

$$\lim_{h\to\infty}\int\limits_{\{h<|u|<h+k\}} |\nabla u|^p\,dx = 0\ .$$

Proof. Let $\varepsilon>0$, $k>0$ and $\mu\in\mathcal{M}_0(U)$. Here we use the decomposition $\mu = F-\mathrm{div}\,(G)$ as introduced in Theorem 4.1. Choose $h>\|g\|_{L^\infty}$ such that

$$\int\limits_{\{|u|>h\}} k\,F + |G|^{p'}\,dx < \varepsilon\ .$$

This is possible because of Lemma 5.5 and the absolute continuity of integrals with respect to the measure since $kF+|G|^{p'}\in L^1(U)$ by definition. Now we use a test function $w := T_k(u-T_h(u))\in W^{1,p}_0(U)\cap L^\infty(U)$ with Definition 5.1 (notice that $w=k$ when $u>k+h$, $w=-k$ when $u<-k-h$ and

$$\nabla T_k(u-T_h(u)) = \begin{cases}\nabla u & \text{when } h\le |u|\le k+h\\ 0 & \text{otherwise}\end{cases}\qquad)$$

to get (using (M1)

$$\begin{aligned}\int\limits_{\{h<|u|<h+k\}} |\nabla u|^p\,dx &\le \frac{1}{\alpha}\int\limits_{\{h<|u|<h+k\}} a(x,\nabla u)\cdot\nabla u\,dx\\ &= \frac{1}{\alpha}\int\limits_{\Omega} a(x,\nabla u)\cdot\nabla T_k(u-T_h(u))\,dx\\ &= \frac{1}{\alpha}\int\limits_{\Omega} T_k(u-T_h(u))\,d\mu\\ &= \frac{1}{\alpha}\Bigg(\int\limits_{\{|u|>h\}} T_k(u-T_h(u))\,F\,dx\\ &\qquad + \int\limits_{\{h<|u|<h+k\}} G\cdot\nabla T_k(u-T_h(u))\,dx\Bigg)\ .\end{aligned}$$

From Young's inequality we now get

$$\int\limits_{\{h<|u|<h+k\}} |\nabla u|^p \, dx \le \frac{1}{\alpha}\Bigg(\int\limits_{\{|u|>h\}} k\, F \, dx + C_{\alpha,p,p'} \int\limits_{\{h<|u|<h+k\}} |G|^{p'} \, dx + \frac{\alpha}{2} \int\limits_{\{h<|u|<h+k\}} |\nabla u|^p \, dx \Bigg),$$

and the choice of h assures that

$$\int\limits_{\{h<|u|<h+k\}} |\nabla u|^p \, dx \le C' \varepsilon$$

where C' depends on α, p and p'. □

Now we have all the tools we need for the main result in this chapter, the existence and uniqueness of the renormalized solution. The proof concentrates on the uniqueness part, since the existence part is obtained with the usual approximation technique.

Theorem 5.7. *Let $\mu \in \mathcal{M}_0(U)$ and $1<p\le N$. There exists a unique renormalized solution $u \in \widehat{W}^{1,p}(U)$ of the problem* (1.1).

Proof. The existence of the solution can be shown using the same methods as for the problem with homogeneous boundary values. Following the ideas from [1] (see also [5]), we first approximate the measure $\mu = F - \operatorname{div}(G)$ with the sequence of measures $\mu_n = F_n - \operatorname{div}(G_n) \in W^{-1,p'}(U)$, where $F \in L^1(U)$, $G, F_n, G_n \in L^{p'}(U)$ and $F_n \rightharpoonup F$ in $L^1(U)$, $G_n \to G$ in $L^{p'}(U)$. Classical results (see [14], page 177, Example 2.3.2) give us a sequence of solutions $u_n \in W^{1,p}(U)$ corresponding to each μ_n for Equation (1.1). Notice that u_n is also a renormalized solution for any n since it belongs to $W^{1,p}(U)$. Using similar techniques to those in [1], we then obtain, using the estimation (5.1) and Lemma 5.5 for solutions u_n that $u_n \to u \in \widehat{W}^{1,p}(U)$, $\nabla u_n \to \nabla u$ a.e. and $u_n \to u$ locally in measure.

We can, moreover, estimate the limit function u for fixed $k \ge 1$:

$$\begin{aligned} \big|\{|u|>k\}\big| &= \big|\{|u|>k\} \cap \{|u_n|>k-1\}\big| + \big|\{|u|>k\} \cap \{|u_n| \le k-1\}\big| \\ &\le C(k-1)^{-\frac{N(p-1)}{N-p}} + \big|\{|u_n-u| \ge 1\}\big| \overset{n\to\infty}{\longrightarrow} C(k-1)^{-\frac{N(p-1)}{N-p}}, \end{aligned}$$

from where it follows that u is finite almost everywhere in U.

For functions u_n it also holds

$$\begin{aligned}\int_U |\nabla u_n|^q\,dx &= \int_{\{|\nabla u|\le 1\}} |\nabla u_n|^q\,dx + \int_{\{|\nabla u|>1\}} |\nabla u_n|^q\,dx \\ &\le |U| + q\int_1^\infty t^{q-1}\big|\{|\nabla u|>t\}\big|\,dt \\ &\overset{(5.6)}{\le} |U| + C\int_1^\infty t^{(q-1)-\frac{N(p-1)}{N-1}}\,dt \le C' < \infty\end{aligned}$$

for every $q < \frac{N(p-1)}{N-1}$, and by using Fatou's lemma we find that

$$\int_U |\nabla u|^{r(p-1)}\,dx \le \liminf_{n\to\infty}\int_U |\nabla u_n|^{r(p-1)}\,dx \le C'$$

for every $r < \frac{N}{N-1}$. Thus $|\nabla u|^{p-1} \in L^r(U)$. Moreover, $u \in W^{1,q}(U)$ for every $q < \frac{N(p-1)}{N-1}$.

Finally, since for the test function w satisfying the conditions required in Definition 5.1, the term $a(x,\nabla u)\cdot\nabla w$ is integrable in U and since

$$a(x,\nabla u_n)\cdot\nabla w \to a(x,\nabla u)\cdot\nabla w \qquad \text{a.e. in } U\ ,$$

we have by the dominated convergence theorem

$$\begin{aligned}\int_U a(x,\nabla u)\cdot\nabla w\,dx &= \lim_{n\to\infty}\int_U a(x,\nabla u_n)\cdot\nabla w\,dx \\ &= \lim_{n\to\infty}\int_U F_n\,w\,dx + \lim_{n\to\infty}\int_U G_n\cdot\nabla w\,dx = \int_U w\,d\mu\,.\end{aligned}$$

This proves the existence of the renormalized solution.

Next we show the uniqueness of the renormalized solution. Let $\varepsilon > 0$, $k > 0$, and $u_1, u_2 \in \widehat{W}^{1,p}(U)$ be two renormalized solutions. For $h > 0$ (assume $h > \max\{k, \|g\|_{L^\infty}\}$) we define

$$\begin{aligned}A_0 &= \{x\in U : |u_1 - T_h(u_2)| < k,\ |u_2| < h,\ |u_1| < h\}\ ,\\ A_1 &= \{x\in U : |u_1 - T_h(u_2)| < k,\ |u_2| \ge h\} \qquad \text{and}\\ A_2 &= \{x\in U : |u_1 - T_h(u_2)| < k,\ |u_2| < h,\ |u_1| \ge h\}\end{aligned}$$

and

$$\begin{aligned}A_0' &= \{x\in U : |u_2 - T_h(u_1)| < k,\ |u_1| < h,\ |u_2| < h\}\ ,\\ A_1' &= \{x\in U : |u_2 - T_h(u_1)| < k,\ |u_1| \ge h\} \qquad \text{and}\\ A_2' &= \{x\in U : |u_2 - T_h(u_1)| < k,\ |u_1| < h,\ |u_2| \ge h\}\ .\end{aligned}$$

Notice that

$$
\begin{aligned}
&\{x \in U : |u_1 - T_h(u_2)| < k\} = A_0 \cup A_1 \cup A_2 \qquad \text{and}\\
&\{x \in U : |u_2 - T_h(u_1)| < k\} = A_0' \cup A_1' \cup A_2' \ .
\end{aligned}
$$

Now we choose $h_\varepsilon > 0$ (depending on k) such that for every $h \geq h_\varepsilon$

$$
\|\nabla u_i\|^{p-1}_{L^p(\{h \leq |u_i| \leq h+k\})} < \varepsilon \quad \text{and} \quad \|\nabla u_i\|_{L^p(\{h-k \leq |u_i| \leq h\})} < \varepsilon \quad \text{and} \tag{5.7}
$$

$$
k \int\limits_{\{|u_i|>h\}} F \, dx + \frac{1}{p'} \int\limits_{\{|u_i|>h\}} |G|^{p'} \, dx < \varepsilon \tag{5.8}
$$

for $i = 1, 2$. (Here we again use the decomposition $\mu = F - \operatorname{div}(G)$.) The estimates (5.7) follow from Lemma 5.6 (use the test function $T_k(u - T_{h-k}(u))$ to get the last estimate), and the estimate (5.8) follows from Lemma 5.5.

Next we fix $h \geq h_\varepsilon$ and notice that $T_k(u_1 - T_h(u_2))$ and $T_k(u_2 - T_h(u_1))$ are admissible test functions for renormalized solutions u_1 and u_2, respectively. Here $w^{+\infty} \equiv k$ when $u_i > k + h$ and $w^{-\infty} \equiv -k$ when $u_i < -k - h$. Using these test functions, we get

$$
\begin{aligned}
I :=& \int\limits_U a(x, \nabla u_1) \cdot \nabla T_k(u_1 - T_h(u_2)) \, dx + \int\limits_U a(x, \nabla u_2) \cdot \nabla T_k(u_2 - T_h(u_1)) \, dx\\
=& \int\limits_U T_k(u_1 - T_h(u_2)) \, d\mu + \int\limits_U T_k(u_2 - T_h(u_1)) \, d\mu =: J
\end{aligned}
\tag{5.9}
$$

Since $A_0 = A_0' = \{|u_1 - u_2| < k, \ |u_1| < h, \ |u_2| < h\}$, we have

$$
\begin{aligned}
I_0 :=& \int\limits_{A_0} a(x, \nabla u_1) \cdot (\nabla u_1 - \nabla \underbrace{T_h(u_2)}_{=u_2}) \, dx\\
&\qquad + \int\limits_{A_0'} a(x, \nabla u_2) \cdot (\nabla u_2 - \nabla \underbrace{T_h(u_1)}_{=u_1}) \, dx\\
=& \int\limits_{A_0} (a(x, \nabla u_1) - a(x, \nabla u_2)) \cdot (\nabla u_1 - \nabla u_2) \, dx \ .
\end{aligned}
\tag{5.10}
$$

Moreover,

$$\begin{aligned}
I_1 :=& \int\limits_{A_1} a(x, \nabla u_1) \cdot (\nabla u_1 - \nabla \underbrace{T_h(u_2)}_{=h})\, dx \\
&\qquad\qquad + \int\limits_{A_1'} a(x, \nabla u_2) \cdot (\nabla u_2 - \nabla \underbrace{T_h(u_1)}_{=h})\, dx \\
=& \int\limits_{A_1} a(x, \nabla u_1) \cdot \nabla u_1 \, dx + \int\limits_{A_1'} a(x, \nabla u_2) \cdot \nabla u_2 \, dx \\
\overset{(M1)}{\geq}& \alpha \Big(\int\limits_{A_1} |\nabla u_1|^p \, dx + \int\limits_{A_1'} |\nabla u_2|^p \, dx \Big) \geq 0
\end{aligned}$$

and

$$\begin{aligned}
I_2 :=& \int\limits_{A_2} a(x, \nabla u_1) \cdot (\nabla u_1 - \nabla \underbrace{T_h(u_2)}_{=u_2})\, dx \\
&\qquad\qquad + \int\limits_{A_2'} a(x, \nabla u_2) \cdot (\nabla u_2 - \nabla \underbrace{T_h(u_1)}_{=u_1})\, dx \\
\overset{(M1)}{\geq}& - \int\limits_{A_2} a(x, \nabla u_1) \cdot \nabla u_2 \, dx - \int\limits_{A_2'} a(x, \nabla u_2) \cdot \nabla u_1 \, dx \ .
\end{aligned}$$

Thus $I = I_0 + I_1 + I_2 \geq I_0 + I_2$.

Next we estimate the integral I_2. Using Hölder's inequality, we get

$$\begin{aligned}
&\int\limits_{A_2} a(x, \nabla u_1) \cdot \nabla u_2 \, dx \\
&\leq \left(\int\limits_{\{h \leq |u_1| \leq h+k\}} |a(x, \nabla u_1)|^{p'} \, dx \right)^{1/p'} \left(\int\limits_{\{h-k \leq |u_2| \leq h\}} |\nabla u_2|^p \, dx \right)^{1/p} \\
&\overset{(M2)}{\leq} \gamma \left(\int\limits_{\{h \leq |u_1| \leq h+k\}} |\nabla u_1|^{p'(p-1)} \, dx \right)^{1/p'} \left(\int\limits_{\{h-k \leq |u_2| \leq h\}} |\nabla u_2|^p \, dx \right)^{1/p} \\
&= \gamma \, \|\nabla u_1\|^{p-1}_{L^p(\{h \leq |u_1| \leq h+k\})} \, \|\nabla u_2\|_{L^p(\{h-k \leq |u_2| \leq h\})} \ ,
\end{aligned}$$

and

$$\int\limits_{A_2'} a(x, \nabla u_2) \cdot \nabla u_1 \, dx \leq \gamma \, \|\nabla u_2\|^{p-1}_{L^p(\{h \leq |u_2| \leq h+k\})} \, \|\nabla u_1\|_{L^p(\{h-k \leq |u_1| \leq h\})} \ .$$

Conditions (5.7) now assure that $I_2 > -C_1\varepsilon^2$, and thus

$$I_0 \le I - I_2 < I + C_1\varepsilon^2 \ . \tag{5.11}$$

Now we estimate the integral J in Equation (5.9). Since $A_0 = A_0'$, we first see that

$$\begin{aligned} J_0 &= \int\limits_{A_0} T_k(u_1 - T_h(u_2))\, d\mu + \int\limits_{A_0'} T_k(u_2 - T_h(u_1))\, d\mu \\ &= \int\limits_{A_0} T_k(u_1 - u_2)\, d\mu + \int\limits_{A_0} \underbrace{T_k(u_2 - u_1)}_{=-T_k(u_1-u_2)}\, d\mu = 0 \ . \end{aligned}$$

Using the decomposition $\mu = F - \operatorname{div}(G)$, we next calculate with Young's inequality

$$\begin{aligned} J_1 &:= \int\limits_{A_1} T_k(u_1 - T_h(u_2))\, d\mu \\ &= \int\limits_{A_1} F\, T_k(u_1 - T_h(u_2))\, dx + \int\limits_{A_1} G \cdot \nabla T_k(u_1 - T_h(u_2))\, dx \\ &\le k \int\limits_{A_1} F\, dx + \frac{1}{p'} \int\limits_{A_1} |G|^{p'}\, dx + \frac{1}{p} \int\limits_{A_1} |\nabla T_k(u_1 - T_h(u_2))|^p\, dx \ . \end{aligned} \tag{5.12}$$

In the set A_1 it holds that

$$\begin{cases} \nabla T_k(u_1 - T_h(u_2)) = \nabla(u_1 - h) = \nabla u_1 \\ h - k < |u_1| < h + k \ , \end{cases}$$

so that by conditions (5.7) and (5.8) we have $J_1 < C_2\,\varepsilon$.

A similar calculation for the set A_2, together with the facts that

$$\begin{cases} |\nabla T_k(u_1 - T_h(u_2))|^p = |\nabla(u_1 - u_2)|^p \le |\nabla u_1|^p + |\nabla u_2|^p \\ h < |u_1| < h + k \\ h - k \le |u_2| < h \end{cases}$$

in the set A_2, gives us

$$J_2 := \int\limits_{A_2} T_k(u_1 - T_h(u_2))\, d\mu < C_3\varepsilon \ .$$

Similarly,

$$J_1' := \int\limits_{A_1'} T_k(u_2 - T_h(u_1))\, d\mu < C_4\varepsilon$$

and

$$J_2' := \int\limits_{A_2'} T_k(u_2 - T_h(u_1))\, d\mu < C_5\varepsilon \ .$$

Combining these estimates, we have

$$\int_{\{|u_1-T_h(u_2)|<k\}} T_k(u_1-T_h(u_2))\, d\mu + \int_{\{|u_2-T_h(u_1)|<k\}} T_k(u_2-T_h(u_1))\, d\mu < C_6\varepsilon\ . \quad (5.13)$$

We still need to integrate over sets $U \cap \{|u_1 - T_h(u_2)| \geq k\}$ and $U \cap \{|u_2 - T_h(u_1)| \geq k\}$. For this we define

$$\begin{aligned}
B_0 &= \{x \in U : |u_1 - T_h(u_2)| \geq k, |u_1| < h,\ |u_2| < h\}\ , \\
B_1 &= \{x \in U : |u_1 - T_h(u_2)| \geq k, |u_1| \geq h\} \qquad \text{and} \\
B_2 &= \{x \in U : |u_1 - T_h(u_2)| \geq k, |u_2| \geq h\}
\end{aligned}$$

and

$$\begin{aligned}
B_0' &= \{x \in U : |u_2 - T_h(u_1)| \geq k, |u_1| < h,\ |u_2| < h\}\ , \\
B_1' &= \{x \in U : |u_2 - T_h(u_1)| \geq k, |u_1| \geq h\} \qquad \text{and} \\
B_2' &= \{x \in U : |u_2 - T_h(u_1)| \geq k, |u_2| \geq h\}\ .
\end{aligned}$$

First we see that $B_0' = B_0 = \{x \in U : |u_1 - u_2| \geq k, |u_1| < h,\ |u_2| < h\}$ so that

$$\begin{aligned}
J_4 &= \int_{B_0} T_k(u_1 - T_h(u_2))\, d\mu + \int_{B_0'} T_k(u_2 - T_h(u_1))\, d\mu \\
&= \int_{B_0} T_k(u_1 - u_2)\, d\mu + \int_{B_0'} \underbrace{T_k(u_2 - u_1)}_{=-T_k(u_1-u_2)}\, d\mu = 0\ .
\end{aligned}$$

Moreover, since

$$\begin{aligned}
T_k(u_1 - T_h(u_2)) &= \pm k \quad \text{in} \quad B_1 \cup B_2\ , \\
T_k(u_2 - T_h(u_1)) &= \pm k \quad \text{in} \quad B_1' \cup B_2'\ ,
\end{aligned}$$

and hence

$$\begin{aligned}
\nabla T_k(u_1 - T_h(u_2)) &= 0 \quad \text{in} \quad B_1 \cup B_2 \quad \text{and} \\
\nabla T_k(u_2 - T_h(u_1)) &= 0 \quad \text{in} \quad B_1' \cup B_2'\ ,
\end{aligned}$$

thus the condition (5.8), together with calculations preceeding (5.12), shows that

$$\int_{\{|u_1-T_h(u_2)|\geq k\}} T_k(u_1-T_h(u_2))\, d\mu + \int_{\{|u_2-T_h(u_1)|\geq k\}} T_k(u_2-T_h(u_1))\, d\mu < C_7\varepsilon\ . \quad (5.14)$$

Combining (5.9) – (5.11), (5.13) and (5.14), we now have

$$\begin{aligned}
&\int_{A_0} \big(a(x, \nabla u_1) - a(x, \nabla u_2)\big) \cdot \big(\nabla u_1 - \nabla u_2\big)\, dx \\
&< I + C_1\varepsilon^2 = J + C_1\varepsilon^2 < C_6\varepsilon + C_7\varepsilon + C_1\varepsilon^2\ .
\end{aligned}$$

Since this limit does not depend on the choice of $h(\geq h_\varepsilon)$, and since A_0 converges to $\{|u_1-u_2|<k\}$ as $h\to\infty$, we finally have

$$\int\limits_{\{|u_1-u_2|<k\}} \big(a(x,\nabla u_1)-a(x,\nabla u_2)\big)\cdot\big(\nabla u_1-\nabla u_2\big)\,dx \leq 0$$

for any $k>0$. This implies that $\nabla u_1=\nabla u_2$ almost everywhere in U by (M3) . Considering the similar boundary conditions for both solutions, we have uniqueness of the solution in U. □

The last thing in this chapter is an important lemma for the future references. In chapter 6 we are dealing with renormalized solutions in an open set Ω, but we are mainly interested in their behaviour in a special quasi open subset of Ω. The uniqueness proof above holds only for open sets, and thus we have to replace it with a special version (see Lemma 6.4) in a way that also holds for quasi open subsets. For this purpose we need to know that the "renormality condition" 5.1 (2) also holds in quasi open subsets.

Sadly, the proof of the following lemma only holds for measures that are absolutely continuous with respect to the p-capacity. This is one of the main reasons why the results obtained in chapter 6 cannot be generalized straight away for any bounded Radon measures.

Lemma 5.8. *If $\mu\in\mathcal{M}_0(\Omega)$ and $u\in\widehat{W}^{1,p}(\Omega)$ is a renormalized solution in Ω, then it is a renormalized solution also in every quasi open subset $U\subset\Omega$.*

Proof. Let u be a renormalized solution in Ω and U a quasi open subset of Ω. It is known that u is a renormalized solution in every open subset $V\supset U$. Now we may calculate for every test function $w\in W_0^{1,p}(U)\cap L^\infty(U)$

$$\begin{aligned}
&\Big|\int\limits_U a(x,\nabla u)\cdot\nabla w\,dx-\int\limits_U w\,d\mu\Big|\\
&=\Big|\int\limits_V a(x,\nabla u)\cdot\nabla w\,dx-\int\limits_{V\setminus U} a(x,\nabla u)\cdot\nabla w\,dx-\int\limits_U w\,d\mu\Big|\\
&=\Big|\int\limits_V w\,d\mu-\int\limits_U w\,d\mu\Big|\\
&=\Big|\int\limits_U w\,d\mu+\int\limits_{V\setminus U} w\,d\mu-\int\limits_U w\,d\mu\Big|=0
\end{aligned}$$

since $w=0$ quasi everywhere on $V\setminus U$, and thus $\nabla w=0$ quasi everywhere on $V\setminus U$ and $\int_{V\setminus U} w\,d\mu=0$. □

6. Strong comparison principle

This chapter concentrates on representing a version of the comparison principle for renormalized solutions. Most of the (technical) work is done in Theorem 6.3 and in the following Lemma 6.4. The main result is, however, rephrased in a stronger form in Theorem 6.7 in order to cut out some technical details from the proof of 6.3. The end of the chapter concentrates on finding some a priori conditions to guarantee the favourable conditions for our comparison principle.

Throughout this chapter, we are considering the open and bounded set Ω, measures $\mu_1, \mu_1 \in \mathcal{M}_0(\Omega)$ such that $\mu_2 \geq \mu_1$, and respective renormalized solutions $u_1, u_2 \in \widehat{W}_0^{1,p}(\Omega)$. We also have to assume an additional hypothesis on the operator a, namely, the Hölder continuity with respect to ξ:

$$\begin{cases} |a(x,\xi) - a(x,\xi')| \leq \gamma(|\xi| + |\xi|)^{p-2}|\xi - \xi'|, & \text{if } p \geq 2, \\ |a(x,\xi) - a(x,\xi')| \leq \gamma|\xi - \xi'|^{p-1}, & \text{if } p < 2 \end{cases} \tag{6.1}$$

for almost every x in Ω and for every ξ, ξ' in $\mathbb{R}^N$, where $\gamma > 0$ is a constant.

The main theorem roughly states that the difference of two solutions is minimized in the neighbourhood of $\partial\Omega$. From that we may deduce that if we have two solutions that do not change order in some neighbourhood of $\partial\Omega$, then they keep the same order in the whole Ω. Unfortunately, the proof relies on the fact that the measures in question are absolutely continuous with respect to the p-capacity, and thus we cannot use these results straight away to show the uniqueness of renormalized solutions for any bounded Radon measure.

We start with a technical lemma that introduces some important sets and tools for the later calculations. Notice that the result 6.1 (2) is obvious in the whole Ω in the case of the N-dimensional measure. The main purpose for Lemma 6.1 is to show that this kind of estimates can be done also for $(N-1)$-dimensional measures in proper $(N-1)$-dimensional subsets of Ω.

Lemma 6.1. *Let $k > 0$, $u_1, u_2 \in \widehat{W}^{1,p}(\Omega)$ and define*

$$E_{tk} := \{x \in \Omega : \; T_k(u_2)(x) - T_k(u_1)(x) < t\} \ .$$

Then the following are true for almost every $t \in \mathbb{R}$:

1. *E_{tk} has finite perimeter*
2. $\big(a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))\big) \cdot \big(\nabla T_k(u_2) - \nabla T_k(u_1)\big) \geq 0$ *for $\mathcal{H}^{N-1}$-a.e. $x \in \partial_* E_{tk}$*
3. $\big(a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))\big) \cdot \nu_{E_{tk}} \geq 0$ *for $\mathcal{H}^{N-1}$-a.e. $x \in \partial_* E_{tk}$*

Proof. (1): Let $k > 0$. Since $T_k(u_i) \in W^{1,p}(\Omega)$ for $i = 1, 2$, Theorem 1 in section 5.5 from [7] shows that E_{tk} has finite perimeter for almost every $t \in \mathbb{R}$.

(2): Let $k > 0$ and define

$$A_t := \{x \in \partial_* E_{tk} : \big(a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))\big) \cdot (\nabla T_k(u_2) - \nabla T_k(u_1)) < 0\}$$

We claim that $|A_t|_{n-1} = 0$ for almost every $t \in \mathbb{R}$. Assume by contradiction that

$$|T|_1 := \big|\{t \in \mathbb{R} : |A_t|_{n-1} > 0\}\big|_1 = a > 0 \ ,$$

and define for $i = 1, 2, 3, ...$

$$T_i := \{t \in \mathbb{R} : |A_t|_{n-1} > \frac{1}{i}\} \ .$$

Now

$$\bigcup_{i=1}^{\infty} T_i = T \qquad \text{and} \qquad T_1 \subset T_2 \subset T_3 \subset ... \ ,$$

so that

$$\lim_{i \to \infty} |T_i| = |T| = a > 0 \ .$$

This means that there exists $i_0 \in \mathbb{N}$ such that $|T_{i_0}| \geq a/2$ and $|A_t|_{n-1} > 1/i_0$ for every $t \in T_{i_0}$. From this we get

$$\Big|\{x \in U : \big(a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))\big) \cdot (\nabla T_k(u_2) - \nabla T_k(u_1))\} < 0\Big|_n > \frac{a}{2i_0} > 0$$

which is a contradiction because of the assumption (M3).

(3): Let $k > 0$ and define

$$B_t := \{x \in \partial_* E_{tk} : \big(a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))\big) \cdot \nu_{E_{tk}} < 0\} \ .$$

We claim that $|B_t|_{n-1} = 0$ for almost every $t \in \mathbb{R}$. Assuming by contradiction that

$$|T|_1 := \big|\{t \in \mathbb{R} : |B_t|_{n-1} > 0\}\big|_1 = b > 0$$

leads us to the same conclusion as before, that is,

$$\Big|\bigcup_{t \in \mathbb{R}} B_t\Big|_n > c > 0 \ .$$

Since $T_k(u_2) - T_k(u_1) \in W^{1,p}(\Omega)$, we know that

$$\nu_{E_{tk}} = \frac{\nabla T_k(u_2) - \nabla T_k(u_1)}{|\nabla T_k(u_2) - \nabla T_k(u_1)|} \qquad \mathcal{L}^N\text{-a.e.}$$

(see [7], section 5.1, Example 1). From this it follows that we can find a set $D \subset \cup_{t \in \mathbb{R}} B_t$ such that $|D|_n > 0$ and

$$\begin{aligned} &\big(a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))\big) \cdot \nu_{E_{tk}} \\ &= \big(a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))\big) \cdot \frac{\nabla T_k(u_2) - \nabla T_k(u_1)}{|\nabla T_k(u_2) - \nabla T_k(u_1)|} > 0 \end{aligned}$$

in D by the assumption (M3). This is, however, a contraction because of the definition of the sets B_t. $\square$

Definition 6.2. For $\delta > 0$ define a δ-neighbourhood $\mathcal{U}^{\delta}_{\partial\Omega}$ of $\partial\Omega$ as

$$\mathcal{U}^{\delta}_{\partial\Omega} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \delta\} \ .$$

Theorem 6.3. *Let Ω be an open and bounded set, $1 < p \le N$ and $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ such that $\mu_2 \ge \mu_1$. If the condition* (6.1) *holds and u_1, u_2 are respective renormalized solutions such that $u_2 \ge u_1$ quasi everywhere in Ω, then*

$$\operatorname*{\mathit{ess\ inf}}_{\Omega} (u_2 - u_1) \ge \operatorname*{\mathit{ess\ inf}}_{\mathcal{U}^{\delta}_{\partial\Omega}} (u_2 - u_1) \qquad \textit{for a.e. } \delta > 0 \tag{6.2}$$

where essential infimums are considered with respect to the p-capacity.

Proof. Let $\tau = \operatorname*{ess\,inf}_{\mathcal{U}^{\delta}_{\partial\Omega}} (u_2 - u_1)$ with respect to p-capacity. We may assume that $\tau > 0$, otherwise the claim is trivial. Assume by contradiction that there exists a Borel set $B \subset \Omega$ of positive p-capacity such that $u_2 - u_1 < \tau$ on B. Choose $x_0 \in B$ and $t > 0$ such that $(u_2 - u_1)(x_0) < t < \tau$.

First, define

$$E_t = \{u_2 - u_1 < t\} \ ,$$
$$E_{tk} = \{T_k(u_2) - T_k(u_1) < t\} \ .$$

Since u_1 and u_2 are quasi continuous functions by Remark 3.8, we know from Remark 3.2 that the set E_t is quasi open. Moreover,

$$E_{tk} \subset E_{t_l} \text{ for } l < k \ , \qquad E_t \subset E_{tk} \text{ for any } k > 0 \ .$$

Also notice that

$$E_{tk} \backslash E_t \subset \{x \in \Omega : |u_1| > k \text{ or } |u_2| > k\}$$

and since $\operatorname{cap}_p\big((\mathcal{U}^{\delta}_{\partial\Omega})^C \cap \{|u_i| > k\}\big) \to 0$ as $k \to \infty$ by Lemma 3.9, we deduce that for every $\epsilon > 0$ there is $k_\epsilon > 0$ such that

$$\operatorname{cap}_p((\mathcal{U}^{\delta}_{\partial\Omega})^C \cap [E_{tk} \backslash E_t]) < \epsilon \tag{6.3}$$

for every $k \ge k_\epsilon$.

Now let $\varepsilon > 0$ and choose k_0 such that

$$\int\limits_{(\mathcal{U}^{\delta}_{\partial\Omega})^C \cap \partial_* E_{tk} \cap \ [\{|u_1| > k_0\} \cup \{|u_2| > k_0\}]} \gamma\left(|\nabla u_1|^{p-1} + |\nabla u_2|^{p-1}\right) d\mathcal{H}^{N-1} < \varepsilon \tag{6.4}$$

where $k \ge k_0$ and $\gamma > 0$ is the constant from the assumption (M2). This inequality is true for some $\mathcal{H}^1$-positive measured set of real numbers t from the fixed interval $T := \big((u_2 - u_1)(x_0), \tau\big)$ (which is enough for our purposes, since all the other following requirements are true for almost every t, and also the integrand is an L^1-function for almost every t).

If not, then the integral above would be greater or equal than ε for almost every $t \in T$. Since

$$\bigcup_{t\in T} \left(\partial_* E_{tk} \cap [\{|u_1| > k_0\} \cup \{|u_2| > k_0\}]\right) \subset \left(\{|u_1| > k_0\} \cup \{|u_2| > k_0\}\right)$$

and the sets $\partial_* E_{tk}$ are disjoint with respect to t, we would then have

$$\int\limits_{(\mathcal{U}^\delta_{\partial\Omega})^C \cap\ [\{|u_1|>k_0\}\cup\{|u_2|>k_0\}]} \gamma \left(|\nabla u_1|^{p-1} + |\nabla u_2|^{p-1}\right) d\mathcal{H}^N \geq |T|\, \varepsilon$$

by Fubini's theorem. This would be a contradiction for big enough k_0 by Lemma 5.5 and the absolute continuity of the integral with respect to the measure. Notice that the choice of k_0 does not depend on k.

The second assumption for the choice of k_0 is that

$$\int\limits_{\{E_{tk_0}\setminus E_t\}\cap\partial_*(\mathcal{U}^\delta_{\partial\Omega})^C} \gamma \left(|\nabla u_1|^{p-1} + |\nabla u_2|^{p-1}\right) d\mathcal{H}^{N-1} < \varepsilon\ . \tag{6.5}$$

This is possible because of (6.3) and the proper choice of $\mathcal{U}^\delta_{\partial\Omega}$.

Now define

$$V_{k_0} := E_{tk_0} \cap (\mathcal{U}^\delta_{\partial\Omega})^C \subset\subset \Omega$$

and notice that V_{k_0} has finite perimeter. The standard mollification of the characteristic function of V_{k_0} is denoted by $\chi^\xi_{V_{k_0}} \in C^\infty_0(\Omega)$. Also define

$$\mu_0 := \mu_2 - \mu_1 \geq 0\ .$$

Now we choose ξ (depending on k_0) such that

$$\int\limits_{V_{k_0}} \chi_{V_{k_0}} d\mu_0 \leq \int\limits_{V_{k_0}} \chi^\xi_{V_{k_0}} d\mu_0 + \epsilon \tag{6.6}$$

and

$$\left| \int\limits_{\partial_* V_{k_0}} \left([a(x,\nabla u_2) - a(x,\nabla u_1)]^\xi - [a(x,\nabla u_2) - a(x,\nabla u_1)]\right) \nu_{V_{k_0}} d\mathcal{H}^{N-1} \right| < \varepsilon\ . \tag{6.7}$$

The first inequality is true since quasi every point from V_{k_0} is a Lebesgue point, and thus the mollification of characteristic function converges almost everywhere with respect to the absolutely continuous measure μ_0. The second inequality follows from the L^1-convergence of mollification with the proper choice of t (see Lemma 6.1).

Another standard mollification of a function is denoted by $[\cdot]^\eta$. Next we choose η (depending on ξ and thus from k_0) such that

$$\left| \int\limits_\Omega \left([a(x,\nabla u_2) - a(x,\nabla u_1)]^\eta - [a(x,\nabla u_2) - a(x,\nabla u_1)]\right) \nabla \chi^\xi_{V_{k_0}}\, dx \right| < \varepsilon, \tag{6.8}$$

$$\left| \int\limits_{\partial_* V_{k_0}} \Big[\big([a(x,\nabla u_2) - a(x,\nabla u_1)]^\eta \big)^\xi - [a(x,\nabla u_2) - a(x,\nabla u_1)]^\xi \Big] \nu_{V_{k_0}} d\mathcal{H}^{N-1} \right| < \varepsilon. \tag{6.9}$$

The first one follows from the L^1-convergence of mollification. Since the η-sequence $\big([a(x,\nabla u_2) - a(x,\nabla u_1)]^\eta\big)^\xi$ is uniformly bounded and equicontinuous for any fixed (small) ξ, we have uniform convergence by Arzela-Ascoli's theorem, and thus the second assumption is possible.

Now we start our calculation noticing that since $u_2 \geq u_1 + \tau > u_1 + t$ q.e. in $\mathcal{U}^\delta_{\partial\Omega}$, we have $\mathrm{cap}_p(E_t \cap \mathcal{U}^\delta_{\partial\Omega}) = 0$, and thus

$$\mu_0(E_t) = \mu_0(E_t \cap (\mathcal{U}^\delta_{\partial\Omega})^C) \leq \mu_0(E_{t_{k_0}} \cap (\mathcal{U}^\delta_{\partial\Omega})^C) = \int\limits_{V_{k_0}} \chi_{V_{k_0}} \, d\mu_0 \tag{6.10}$$

because of the fact that $E_t \subset E_{t_{k_0}}$. Using (6.6), the definition of the renormalized solution and the non-negativity of the measure and the mollification, we may now calculate

$$\int\limits_{V_{k_0}} \chi_{V_{k_0}} \, d\mu_0 \leq \int\limits_\Omega \chi^\xi_{V_{k_0}} \, d\mu_0 + \varepsilon = \int\limits_\Omega [a(x,\nabla u_2) - a(x,\nabla u_1)] \cdot \nabla \chi^\xi_{V_{k_0}} \, dx + \varepsilon \,. \tag{6.11}$$

Then we continue using (6.8)

$$\begin{aligned} &\int\limits_\Omega [a(x,\nabla u_2) - a(x,\nabla u_1)] \cdot \nabla \chi^\xi_{V_{k_0}} \, dx + \varepsilon \\ &< \int\limits_\Omega [a(x,\nabla u_2) - a(x,\nabla u_1)]^\eta \cdot \nabla \chi^\xi_{V_{k_0}} \, dx + 2\varepsilon \\ &= -\int\limits_\Omega \mathrm{div}\,\big([a(x,\nabla u_2) - a(x,\nabla u_1)]^\eta\big)\, \chi^\xi_{V_{k_0}} \, dx + 2\varepsilon \;, \end{aligned} \tag{6.12}$$

since by Theorem 4.9 (we may assume that Ω has finite perimeter by considering a slightly smaller set) and the fact that $\chi^\xi_{V_{k_0}} = 0$ on $\partial\Omega$, we have

$$\begin{aligned} &\int\limits_\Omega \mathrm{div}\,\big([a(x,\nabla u_2) - a(x,\nabla u_1)]^\eta \; \chi^\xi_{V_{k_0}}\big) \, dx \\ &= \int\limits_{\partial\Omega} [a(x,\nabla u_2) - a(x,\nabla u_1)]^\eta \, \chi^\xi_{V_{k_0}} \, \nu_\Omega \, d\mathcal{H}^{N-1} = 0 \;. \end{aligned}$$

Using Fubini's theorem with the fact that $\operatorname{dist}(V_{k_0},\Omega) \geq \delta > 0$, we may, furthermore, calculate (assuming $\xi < \delta$)

$$\begin{aligned}
&-\int_{\Omega} \operatorname{div}\left([a(x,\nabla u_2)-a(x,\nabla u_1)]^{\eta}\right)\chi^{\xi}_{V_{k_0}}\ dx + 2\varepsilon \\
&= -\int_{V_{k_0}} \operatorname{div}\left(\left([a(x,\nabla u_2)-a(x,\nabla u_1)]^{\eta}\right)^{\xi}\right)\ dx + 2\varepsilon \\
&= -\int_{\partial_* V_{k_0}} \left([a(x,\nabla u_2)-a(x,\nabla u_1)]^{\eta}\right)^{\xi}\nu_{V_{k_0}}\ d\mathcal{H}^{N-1} + 2\varepsilon\ .
\end{aligned} \tag{6.13}$$

by Theorem 4.9.

Now we use (6.9) to obtain

$$\begin{aligned}
&-\int_{\partial_* V_{k_0}} \left([a(x,\nabla u_2)-a(x,\nabla u_1)]^{\eta}\right)^{\xi}\nu_{V_{k_0}}\ d\mathcal{H}^{N-1} + 2\varepsilon \\
&= -\int_{\partial_* V_{k_0}} \Big[\left([a(x,\nabla u_2)-a(x,\nabla u_1)]^{\eta}\right)^{\xi} \\
&\qquad\qquad - [a(x,\nabla u_2)-a(x,\nabla u_1)]^{\xi}\Big]\nu_{V_{k_0}}\ d\mathcal{H}^{N-1} \\
&- \int_{\partial_* V_{k_0}} [a(x,\nabla u_2)-a(x,\nabla u_1)]^{\xi}\,\nu_{V_{k_0}}\ d\mathcal{H}^{N-1} + 2\varepsilon \\
&< -\int_{\partial_* V_{k_0}} [a(x,\nabla u_2)-a(x,\nabla u_1)]^{\xi}\,\nu_{V_{k_0}}\ d\mathcal{H}^{N-1} + 3\varepsilon\ .
\end{aligned} \tag{6.14}$$

Moreover, using (6.7), we have

$$\begin{aligned}
&-\int_{\partial_* V_{k_0}} [a(x,\nabla u_2)-a(x,\nabla u_1)]^{\xi}\,\nu_{V_{k_0}}\ d\mathcal{H}^{N-1} + 3\varepsilon \\
&< -\int_{\partial_* V_{k_0}} [a(x,\nabla u_2)-a(x,\nabla u_1)]\,\nu_{V_{k_0}}\ d\mathcal{H}^{N-1} + 4\varepsilon\ .
\end{aligned} \tag{6.15}$$

Finally, since $\nabla T_k(u_i) \to \nabla u_i$ by definition ($\mathcal{H}^{N-1}$-a.e. for a.e. t from which the integration set $\partial_* V_{k_0}$ depends) and $a(x, \nabla T_k(u))$ is continuous with respect to $\nabla T_k(u)$, we can choose $k \geq k_0$ such that

$$
\begin{aligned}
&- \int_{\partial_* V_{k_0}} [a(x, \nabla u_2) - a(x, \nabla u_1)]\, \nu_{V_{k_0}}\ d\mathcal{H}^{N-1} + 4\varepsilon \\
&< - \int_{\partial_* V_{k_0}} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_{k_0}}\ d\mathcal{H}^{N-1} + 5\varepsilon\ .
\end{aligned}
\tag{6.16}
$$

Next, we want to modify our integration set from $\partial_* V_{k_0}$ to $\partial_* V_k := \partial_*(E_{tk} \cap (\mathcal{U}^\delta_{\partial\Omega})^C)$. To this end we first write

$$
\begin{aligned}
\partial_* V_{k_0} &= \left[(\mathcal{U}^\delta_{\partial\Omega})^C \cap \partial_* E_{tk_0}\right] \cup \left[E_{tk_0} \cap \partial_*(\mathcal{U}^\delta_{\partial\Omega})^C\right] \\
\partial_* V_k &= \left[(\mathcal{U}^\delta_{\partial\Omega})^C \cap \partial_* E_{tk}\right] \cup \left[E_{tk} \cap \partial_*(\mathcal{U}^\delta_{\partial\Omega})^C\right]\ .
\end{aligned}
$$

First notice that since $k \geq k_0$, we have $E_t \subset E_{tk} \subset E_{tk_0}$, and thus

$$
\begin{aligned}
&\Bigg| \int_{E_{tk_0} \cap \partial_*(\mathcal{U}^\delta_{\partial\Omega})^C} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_{k_0}}\ d\mathcal{H}^{N-1} \\
&\qquad - \int_{E_{tk} \cap \partial_*(\mathcal{U}^\delta_{\partial\Omega})^C} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_{k_0}}\ d\mathcal{H}^{N-1} \Bigg| \\
&= \Bigg| \int_{(E_{tk_0} \setminus E_{tk}) \cap \partial_*(\mathcal{U}^\delta_{\partial\Omega})^C} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_{k_0}}\ d\mathcal{H}^{N-1} \Bigg| \\
&\leq \int_{(E_{tk_0} \setminus E_t) \cap \partial_*(\mathcal{U}^\delta_{\partial\Omega})^C} \gamma \left(|\nabla T_k(u_2)|^{p-1} + |\nabla T_k(u_1)|^{p-1} \right) d\mathcal{H}^{N-1} < \varepsilon\ .
\end{aligned}
\tag{I}
$$

by the assumption (M2) and (6.5).

Next we see that

$$
\begin{aligned}
&(\mathcal{U}^\delta_{\partial\Omega})^C \cap \partial_* E_{tk_0} \cap \{|u_1| \leq k_0, |u_2| \leq k_0\} \\
&= (\mathcal{U}^\delta_{\partial\Omega})^C \cap \partial_* E_{tk} \cap \{|u_1| \leq k_0, |u_2| \leq k_0\}
\end{aligned}
$$

so that we only need to compare the difference in the set $\{|u_1| > k_0\} \cup \{|u_2| > k_0\}$. The requirement (6.4) now gives us

$$
\begin{aligned}
&\Bigg| \int\limits_{(\mathcal{U}^\delta_{\partial\Omega})^C \cap \partial_* E_{tk_0}} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_{k_0}}\, d\mathcal{H}^{N-1} \\
&\qquad - \int\limits_{(\mathcal{U}^\delta_{\partial\Omega})^C \cap \partial_* E_{tk}} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_k}\, d\mathcal{H}^{N-1} \Bigg| \\
&\le \Bigg| \int\limits_{(\mathcal{U}^\delta_{\partial\Omega})^C \cap \partial_* E_{tk_0} \cap\, [\{|u_1|>k_0\} \cup \{|u_2|>k_0\}]} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_{k_0}}\, d\mathcal{H}^{N-1} \Bigg| \\
&\qquad + \Bigg| \int\limits_{(\mathcal{U}^\delta_{\partial\Omega})^C \cap \partial_* E_{tk} \cap\, [\{|u_1|>k_0\} \cup \{|u_2|>k_0\}]} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_k}\, d\mathcal{H}^{N-1} \Bigg| \\
&\le \int\limits_{\partial_* E_{tk_0} \cap\, [\{|u_1|>k_0\} \cup \{|u_2|>k_0\}]} \gamma\, (|\nabla u_1|^{p-1} + |\nabla u_2|^{p-1}) d\mathcal{H}^{N-1} \\
&\qquad + \int\limits_{\partial_* E_{tk} \cap\, [\{|u_1|>k_0\} \cup \{|u_2|>k_0\}]} \gamma\, (|\nabla u_1|^{p-1} + |\nabla u_2|^{p-1}) d\mathcal{H}^{N-1} \\
&< 2\varepsilon\ .
\end{aligned}
\tag{II}
$$

Using the estimates (I) and (II), we can now continue from (6.16) to get

$$
\begin{aligned}
&- \int\limits_{\partial_* V_{k_0}} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_{k_0}}\, d\mathcal{H}^{N-1} + 5\varepsilon \\
&< - \int\limits_{E_{tk} \cap \partial_* (\mathcal{U}^\delta_{\partial\Omega})^C} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_{k_0}}\, d\mathcal{H}^{N-1} \\
&\qquad - \int\limits_{(\mathcal{U}^\delta_{\partial\Omega})^C \cap \partial_* E_{tk}} [a(x, \nabla T_k(u_2)) - a(x, \nabla T_k(u_1))]\, \nu_{V_k}\, d\mathcal{H}^{N-1} + 8\varepsilon\ .
\end{aligned}
\tag{6.17}
$$

Now we want to show that

$$
\mathcal{H}^{N-1}\big(E_{tk} \cap \partial_* (\mathcal{U}^\delta_{\partial\Omega})^C\big) \longrightarrow 0 \qquad \text{as } k \to \infty\ . \tag{*}
$$

To obtain this, we first calculate

$$
\begin{aligned}
&\mathcal{H}^{N-1}\big(E_{tk} \cap \partial_* (\mathcal{U}^\delta_{\partial\Omega})^C\big) \\
&= \mathcal{H}^{N-1}\big(E_t \cap \partial_* (\mathcal{U}^\delta_{\partial\Omega})^C\big) && \text{(A)} \\
&\quad + \mathcal{H}^{N-1}\big((E_{tk} \backslash E_t) \cap \partial_* (\mathcal{U}^\delta_{\partial\Omega})^C\big)\ . && \text{(B)}
\end{aligned}
$$

Since $\mathrm{cap}_p(\mathcal{U}_{\partial\Omega}^{\delta/2})^C \cap E_{tk}\setminus E_t) \to 0$ as $k \to \infty$ and $\mathcal{H}^{N-1}\big(\partial_*(\mathcal{U}_{\partial\Omega}^{\delta})^C\big) < \infty$ (δ-neighbourhood has finite perimeter), we instantly get (B) $\to 0$ from the absolute continuity of the $\mathcal{H}^{N-1}$-measure with respect to the p-capacity.
Now we only have to show that (A) $= 0$, and this follows by using a smaller δ-neighbourhood if necessary. The claim is actually true for almost every $0 < \delta' < \delta$, since if there is a set $E = \{\delta' : \delta' < \delta,\ (A) > 0\}$ with positive $\mathcal{H}^1$-measure, we find by using arguments similar to those in Lemma 6.1 that $|E_t \cap \mathcal{U}_{\partial\Omega}^{\delta}|_n > 0$. This is, however, a contradiction with the original assumption $u_2 > u_1$ almost everywhere in $\mathcal{U}_{\partial\Omega}^{\delta}$.

Finally, by the assumption (M2), we may estimate for every k

$$\begin{aligned}\Big|[a(x,\nabla T_k(u_2))-a(x,\nabla T_k(u_1))]\,\nu_{V_{k_0}}\Big| &\le \big|a(x,\nabla T_k(u_2))\big|+\big|a(x,\nabla T_k(u_1))\big|\\ &\le \gamma\big(|\nabla T_k(u_2)|^{p-1}+|\nabla T_k(u_1)|^{p-1}\big)\\ &\le \gamma\big(|\nabla u_2|^{p-1}+|\nabla u_1|^{p-1}\big)\end{aligned}$$

$\mathcal{H}^N$-a.e. in Ω. Again, by the arguments of Lemma 6.1 (see also arguments used to show Equation (A)), by choosing a smaller δ-neighbourhood if necessary, we may assume that this estimate is valid $\mathcal{H}^{N-1}$-a.e. in the set $E_{tk} \cap \partial_*(\mathcal{U}_{\partial\Omega}^{\delta})^C$. This dominating function belongs to $L^1(\Omega)$ (with respect to $\mathcal{H}^{N-1}$ for a.e. δ) by the definition of renormalized functions, so that by the absolute continuity of integrals, (*) and Lemma 6.1 we have

$$\begin{aligned}&-\int\limits_{E_{tk}\cap\partial_*(\mathcal{U}_{\partial\Omega}^{\delta})^C}[a(x,\nabla T_k(u_2))-a(x,\nabla T_k(u_1))]\,\nu_{V_{k_0}}\,d\mathcal{H}^{N-1}\\ &\qquad-\int\limits_{(\mathcal{U}_{\partial\Omega}^{\delta})^C\cap\partial_*E_{tk}}[a(x,\nabla T_k(u_2))-a(x,\nabla T_k(u_1))]\,\nu_{V_k}\,d\mathcal{H}^{N-1}+8\varepsilon\\ &< 9\varepsilon\end{aligned} \tag{6.18}$$

for every k large enough.

From Equations (6.10) to (6.18) we now conclude $\mu_0(E_t) < 9\varepsilon$ for any $\varepsilon > 0$, and for the positive measure this means $\mu_0(E_t) = 0$, that is $\mu_1 = \mu_2$ in E_t. From the construction of E_t we also know that $(u_2 - u_1)(x) = t$ on $\partial_* E_t$. From the following Lemma 6.4 (see also Remark 6.5) we now obtain that $u_2 = u_1 + t$ in E_t, and this is a contradiction with the fact that $x_0 \in E_t$. □

Lemma 6.4. *Let $1<p\le N$, $U \subset \Omega$ be a quasi open set, $\mu_2, \mu_1 \in \mathcal{M}_0(\Omega)$ such that $\mu_2 \ge \mu_1$ in Ω and $\mu_2 = \mu_1$ in U. If the condition* (6.1) *holds and $u_1, u_2 \in \widehat{W}^{1,p}(\Omega)$ are two renormalized solutions for Equation* (1.1) *respective to measures μ_1 and μ_2 such that $u_2 = u_1 + C$ quasi everywhere in ∂U, then $u_2 = u_1 + C$ a.e. in U.*

Remark 6.5. The difference between Theorem 5.7 and Lemma 6.4 is that here we do not need to assume anything special about the regularity of the functions on the boundary of U. We only need to know that the solutions agree quasi everywhere on ∂U. In Theorem 5.7 we need to know that the boundary value g is bounded, and thus we cannot apply the uniqueness proof directly for the set U even though u_2 and u_1 are solutions also in the set U by Lemma 5.8. In Lemma 6.4 we, however, need to assume the Hölder continuity (6.1) for the operator a. This is not needed in Theorem 5.7.

Proof of lemma 6.4: Define

$$h_n(s) := \begin{cases} 0, & \text{if } |s| > 2n \\ 1, & \text{if } |s| \le n \end{cases}$$

continuously such that $|h_n'(s)| = 1/n$ when $n < |s| \le 2n$. Now let $k > 0$ and consider a function

$$w(x) := h_n(u_1)h_n(u_2)T_k(u_2 - u_1 - C) \ .$$

Obviously, $w(x) \in L^\infty(U)$ and $w(x) = 0$ when $|u_i| > 2n$, $i = 1, 2$. We also have $T_k(u_2 - u_1) \in W^{1,p}(U)$ (see Remarks 10.5 and 10.7 from [5] and Lemma 5.4) so that $T_k(u_2 - u_1 - C) \in W_0^{1,p}(U)$, and thus $w(x)$ is an admissible test function for renormalized solutions.

We now use $w(x)$ as a test function in Definition 5.1 for both solutions u_1, u_2 and subtract the equations to obtain

$$\begin{aligned}
&I_1 + I_2 + I_3 \\
&:= \int_U \big(a(x, \nabla u_2) - a(x, \nabla u_1)\big) \cdot \nabla T_k(u_2 - u_1 - C)\, h_n(u_2)\, h_n(u_1)\, dx \\
&\quad + \int_U \big(a(x, \nabla u_2) - a(x, \nabla u_1)\big) \cdot \nabla u_2\, h_n'(u_2)\, h_n(u_1) T_k(u_2 - u_1 - C)\, dx \\
&\quad + \int_U \big(a(x, \nabla u_2) - a(x, \nabla u_1)\big) \cdot \nabla u_1\, h_n(u_2)\, h_n'(u_1) T_k(u_2 - u_1 - C)\, dx \\
&= \int_U w(x)\, d(\mu_2 - \mu_1) = 0 \ .
\end{aligned}$$

Since $\big(a(x, \nabla u_2) - a(x, \nabla u_1)\big) \cdot \nabla T_k(u_2 - u_1 - C) \ge 0$ by (M3) and $0 \le h_n(u_2)h_n(u_1) \to 1$ almost everywhere as $n \to \infty$, we have

$$\int_U \big(a(x, \nabla u_2) - a(x, \nabla u_1)\big) \cdot \nabla T_k(u_2 - u_1 - C)\, dx \le |I_2| + |I_3|$$

by Fatou's lemma. If we now show that $|I_2|+|I_3|\to 0$ as $n\to\infty$, we then have

$$\int\limits_{U\cap\{|u_2-u_1-C|<k\}} \big(a(x,\nabla u_2)-a(x,\nabla u_1)\big)\cdot(\nabla u_2-\nabla u_1)\,dx=0$$

for every $k>0$. It follows from (M3) that $\nabla u_2=\nabla u_1$ almost everywhere in U, and the boundary condition then implies the result.

Because of symmetry we only need to show that $\lim_{n\to\infty}|I_2|=0$. To show this we use the condition (6.1). For $p\ge 2$, we may calculate

$$\begin{aligned}
|I_2| &\le \frac{\gamma\,k}{n}\int\limits_{\left\{\substack{n\le|u_2|\le 2n\\ |u_1|\le 2n}\right\}} (|\nabla u_2|+|\nabla u_1|)^{p-2}|\nabla u_2-\nabla u_1||\nabla u_1|\,dx\\
&\le \frac{\gamma\,k}{n}\int\limits_{\left\{\substack{n\le|u_2|\le 2n\\ |u_1|\le 2n}\right\}} (|\nabla u_2|+|\nabla u_1|)^{p-1}|\nabla u_2-\nabla u_1|\,dx\\
&\le \gamma k\Big(\frac{1}{n}\int\limits_{\left\{\substack{n\le|u_2|\le 2n\\ |u_1|\le 2n}\right\}}(|\nabla u_2|+|\nabla u_1|)^p\,dx\Big)^{\frac{1}{p'}}\Big(\frac{1}{n}\int\limits_{\left\{\substack{n\le|u_2|\le 2n\\ |u_1|\le 2n}\right\}}|\nabla u_2-\nabla u_1|^p\,dx\Big)^{\frac{1}{p}}\\
&\le \gamma k\Big(\frac{1}{n}\int\limits_{\left\{\substack{n\le|u_2|\le 2n\\ |u_1|\le 2n}\right\}}2^p(|\nabla u_2|^p+|\nabla u_1|^p)\,dx\Big)^{\frac{1}{p'}}\Big(\frac{1}{n}\int\limits_{\left\{\substack{n\le|u_2|\le 2n\\ |u_1|\le 2n}\right\}}|\nabla u_2-\nabla u_1|^p\,dx\Big)^{\frac{1}{p}}
\end{aligned}$$

by using Hölder's inequality in the third inequality.

From the proof of Corollary 5.4 (using a test function $\frac{1}{n}T_{2n}(u-T_hu)$), we get

$$\lim_{n\to\infty}\frac{1}{n}\int\limits_{\left\{\substack{|u_2|\le 2n\\ |u_1|\le 2n}\right\}}|\nabla u_i|^p\,dx=0\ ,\qquad i=1,2.$$

From this we also get

$$\frac{1}{n}\int\limits_{\left\{\substack{n\le|u_2|\le 2n\\ |u_1|\le 2n}\right\}}|\nabla u_2-\nabla u_1|^p\,dx\le\frac{1}{n}\int\limits_{\{|u_1|\le 2n\}}|\nabla u_1|^p\,dx+\frac{1}{n}\int\limits_{\{|u_2|\le 2n\}}|\nabla u_2|^p\,dx\to 0$$

as $n\to\infty$. Thus we have $\lim_{n\to\infty}|I_2|=0$.

In the case $p<2$ we use the second inequality from (6.1) to calculate

$$\begin{aligned}
|I_2| &\le \frac{\gamma\,k}{n}\int\limits_{\left\{\substack{n\le|u_2|\le 2n\\ |u_1|\le 2n}\right\}} (|\nabla u_2|-|\nabla u_1|)^{p-1}|\nabla u_1|\,dx\\
&\le \gamma k\ \Big(\frac{1}{n}\int\limits_{\left\{\substack{n\le|u_2|\le 2n\\ |u_1|\le 2n}\right\}}|\nabla u_2-\nabla u_1|^p\,dx\Big)^{\frac{1}{p'}}\Big(\frac{1}{n}\int\limits_{\left\{\substack{n\le|u_2|\le 2n\\ |u_1|\le 2n}\right\}}|\nabla u_1|^p\,dx\Big)^{\frac{1}{p}}\ ,
\end{aligned}$$

and the result follows as before. □

Remark 6.6. In Theorem 6.3 we assume that $u_2 \geq u_1$ in Ω. This assumption can be relaxed. The only place where this assumption was used was at the beginning of the proof, where we must have $\tau \geq 0$. To obtain this we only need to assume that $u_2 \geq u_1$ in $\mathcal{U}^\delta_{\partial\Omega}$ for some $\delta > 0$. Then we have $u_2 \geq u_1$ in Ω by Theorem 6.3.

Notice that here the case $\tau = 0$ is not trivial as it was in Theorem 6.3. If $\tau = 0$, we have to consider the function $u_2 + \varepsilon > u_1$ to obtain $\tau > 0$, and the result follows when $\varepsilon \to 0$.

On the other hand, if

$$\operatorname*{ess\,inf}_{\mathcal{U}^\delta_{\partial\Omega}} (u_2 - u_1) =: C < 0 \ ,$$

we may then consider functions $\widetilde{u}_2 := u_2 + C$ and u_1 for which the essential infimum in the boundary is non-negative, and the claim again follows.

With these observations we may rephrase Theorem 6.3 in the following form:

Theorem 6.7. *Let Ω be an open and bounded set, $1 < p \leq N$ and $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ such that $\mu_2 \geq \mu_1$. If the condition* (6.1) *holds and $u_1,\ u_2$ are respective renormalized solutions, then*

$$\operatorname*{\mathit{ess\,inf}}_{\Omega} (u_2 - u_1) \geq \operatorname*{\mathit{ess\,inf}}_{\mathcal{U}^\delta_{\partial\Omega}} (u_2 - u_1) \qquad \text{for a.e.} \quad \delta > 0 \ ,$$

where essential infimums are considered with respect to the p-capacity.

Next we want to find some a priori conditions which would be sufficient to satisfy the assumptions of Theorem 6.3.

Lemma 6.8. *Let Ω be an open and bounded set, $1<p\leq N$ and $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ such that $\mu_2 \geq \mu_1$ in Ω and define $\mu_0 = \mu_2 - \mu_1$. Let $u_1,\ u_2$ be renormalized solutions corresponding to measures μ_1 and μ_2. If there exists a set $U \subset \Omega$ such that dist$(U, \partial\Omega) \geq \delta$ for some $\delta > 0$ and $\mu_2(E) > \mu_1(E)$ for every $E \subset U$ with $\operatorname{cap}_p(E) > 0$, then $u_2 > u_1$ μ_0-everywhere in U.*

Proof. Define for $k, t > 0$

$$\begin{aligned} Z_k &= \{x \in U : T_k(u_2) = T_k(u_1)\} \\ Z &= \{x \in U : u_2 = u_1\} \\ E_{tk} &= \{x \in U : |T_k(u_2) - T_k(u_1)| < t\} \end{aligned}$$

and fix any open ball $B \subset U$. Following the ideas from the proof of Theorem 6.3 we define

$$\mu_0 := \mu_2 - \mu_1 \geq 0 \quad \text{and} \qquad V_{k_0} := E_{tk_0} \cap B \subset\subset \Omega \ .$$

As in Theorem 6.3, we use $\chi^{\xi}_{V_{k_0}} \in C_0^\infty(\Omega)$ as a test function to get

$$\int_\Omega [a(x,\nabla T_k(u_2)) - a(x,\nabla T_k(u_1))]\; \nabla \chi^{\xi}_{V}\; dx = \int_\Omega \chi^{\xi}_{V}\; d\mu_0 \;.$$

Since $Z \subset Z_{k_0} \subset E_{tk_0}$, we again have

$$\int_\Omega \chi^{\xi}_{V_{k_0}}\; d\mu_0 > \mu_0(E_{tk_0} \cap B) - \varepsilon \geq \mu_0(Z \cap B) - \varepsilon$$

for any $\varepsilon > 0$. Following the proof of Theorem 6.3 (replacing $(\mathcal{U}^\delta_{\partial\Omega})^C$ with B), we eventually get

$$\mu_0(Z \cap B) < -\int_{E_{tk}\cap\partial B} [a(x,\nabla T_k(u_2)) - a(x,\nabla T_k(u_1))]\, \nu_{_B}\; d\mathcal{H}^{N-1} + 8\varepsilon \;.$$

Since $E_{tk} \supset E_{sk} \supset Z_k$ for $s < t$ and $\cap_{t>0} E_{tk} = Z_k$, we have

$$-\int_{E_{tk}\cap\partial B} [a(x,\nabla T_k(u_2)) - a(x,\nabla T_k(u_1))]\, \nu_{_B}\; d\mathcal{H}^{N-1}$$
$$\longrightarrow -\int_{Z_k\cap\partial B} [a(x,\nabla T_k(u_2)) - a(x,\nabla T_k(u_1))]\, \nu_{_B}\; d\mathcal{H}^{N-1}$$

as $t \to 0$. Using Corollary 3.7 from [13], we find out that for appropriately chosen balls $B \subset \Omega$ it holds

$$\nabla T_k(u_2) = \nabla T_k(u_1) \qquad \mathcal{H}^{N-1}\text{-a.e. on the set} \quad Z_k \cap \partial B$$

(otherwise, we would get a contradiction, as in Lemma 6.1). From this it follows that the last integral is zero so that $\mu_0(Z \cap B) < 9\varepsilon$ for any $B \subset \Omega$. Since μ_0 is a positive measure and the result holds for arbitrary $\varepsilon > 0$, we may conclude that $\mu_0(Z) = 0$. □

As a corollary of Lemma 6.8 we now get a version of the strong comparison principle:

Corollary 6.9. *Let Ω be an open and bounded set, $1 < p \leq N$, and $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ such that $\mu_2 \geq \mu_1$ in Ω. Let u_1, u_2 be renormalized solutions corresponding to measures μ_1 and μ_2. If there exists a δ-neighbourhood $\mathcal{U}^\delta_{\partial\Omega}$ of $\partial\Omega$ such that $\mu_2(E) > \mu_1(E)$ for every $E \subset \mathcal{U}^\delta_{\partial\Omega}$ such that $\mathrm{cap}_p(E) > 0$, then $u_2 \geq u_1$ quasi everywhere in Ω.*

Proof. The result follows from Theorem 6.7 and Lemma 6.8 by choosing $U = \mathcal{U}^\delta_{\partial\Omega} \backslash \mathcal{U}^\epsilon_{\partial\Omega}$ where $\epsilon < \delta$, and then letting $\epsilon \to 0$. From Lemma 6.8 we now obtain that $u_2 \geq u_1$ everywhere in $\mathcal{U}^\delta_{\partial\Omega}$ with the exception of the set where $\mu_2 = \mu_1$. This set, however, has zero capacity according to our assumption, and the claim follows from Theorem 6.7 (see also Remark 6.6). □

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