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PEKKA PANKKA AND KAI RAJALA

ABSTRACT. We show that the only quasiregularly elliptic link complements are complements of the unknot and the Hopf link. The proof of non-existence of other link complements is obtained from a Varopoulos type theorem for open manifolds.

1. INTRODUCTION

This article is motivated by the following result.

Theorem 1.1. *There exist a smooth unknot S and a smooth Hopf link H in S^3, and Riemannian metrics g_S and g_H in $S^3 \setminus S$ and $S^3 \setminus H$, respectively, so that $(S^3 \setminus S, g_S)$ and $(S^3 \setminus H, g_H)$ are quasiregularly elliptic.*

A connected and oriented Riemannian n-manifold N is said to be *quasiregularly elliptic* if it receives a non-constant *quasiregular mapping* from $\mathbb{R}^n$. A continuous mapping f between oriented Riemannian n-manifolds M and N is *quasiregular* if it is a Sobolev mapping in $W^{1,n}_{\mathrm{loc}}(M,N)$ and satisfies the distortion inequality

$$|Df|^n \leq K J_f \quad \text{a.e. } M, \tag{1.1}$$

where $|Df|$ is the operator norm of the differential Df and J_f is the Jacobian determinant of Df.

Recall that a subset X of S^3 is a *knot* if it is homeomorphic to S^1, and a *link* if it is a disjoint union of finitely many knots. The unknot (flat circle) and the Hopf link (two flat circles linked once) in Theorem 1.1 are special cases among all links in S^3.

Theorem 1.2. *Let L be a link in S^3. If there exists a Riemannian metric g in $S^3 \setminus L$ so that $(S^3 \setminus L, g)$ is quasiregularly elliptic, then L is either an unknot or a Hopf link.*

We find it interesting that Theorem 1.2 can be viewed as an analog to the classical Picard theorem for analytic functions. In the case of Picard's theorem, the non-existence of analytic functions into twice punctured plane can be traced to the fundamental group $\pi_1(\mathbb{C} \setminus \{0,1\})$, which is a free group of two generators. The same topological obstruction is present in Theorem 1.2. In Appendix A, we give a direct proof for the classical topological fact

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that the fundamental group of $S^3 \setminus L$ contains a free group if L is any other link than the unknot or the Hopf link.

Theorem 1.2 can be viewed, in light of Theorem 1.1, as a non-Euclidean Picard type theorem for quasiregular mappings in dimension 3. In the context of Euclidean 3-spaces, Picard's theorem is due to Rickman. Rickman's fundamental result [10] states that quasiregular mappings from $\mathbb{R}^n$ to $\mathbb{S}^n$ can omit only finitely many values. In a celeberated construction [11] he also shows that this result is sharp in dimension 3, since for any finite set of points $\{q_1, \ldots, q_d\}$ in $\mathbb{S}^3$ there exists a quasiregular mapping from $\mathbb{R}^3$ into $\mathbb{S}^3$ omitting exactly those points.

Theorem 1.2 bears similarity to the classification of closed quasiregularly elliptic manifolds due to Jormakka [7]: *all closed quasiregularly elliptic* 3-*manifolds are quotients of* $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$, *or* $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. A proof of Theorem 1.2 can probably be obtained also along the lines of Jormakka's path-family argument.

Theorem 1.2, and parts of Jormakka's result, can also be viewed in light of a theorem due to Varopoulos [13, Theorem X.11]: *the order of growth of the fundamental group of a closed quasiregularly elliptic n-manifold is at most n.* Whereas Jormakka's result follows from Varopoulos's theorem and the *geometrization conjecture*, Theorem 1.2 can be deduced from the following Varopoulos type theorem for open manifolds.

Theorem 1.3. *Let N be a connected and oriented Riemannian n-manifold without boundary and $f\colon \mathbb{R}^n \to N$ a quasiregular mapping. If $\pi_1(N)$ has order of growth at least $d > n$ then f is constant.*

Since the fundamental group of an open manifold need not be finitely generated, we use the following definition. A group Γ has *order of growth at least d* if there exists a finite set $S \subset \Gamma$ and a constant $C > 0$ so that any ball of radius r in the subgroup $\langle S \rangle$ generated by S has at least Cr^d elements for all $r \in \mathbb{Z}_+$ when $\langle S \rangle$ is endowed with the word metric determined by S; see [4, Section 5B] for the terminology and discussion on the growth of groups.

The constructions in Theorem 1.1 are based on the existence of covering maps from $\mathbb{R}^3$. Although there is no formal connection, our debt to the work of Semmes [12] is apparent regarding the construction of Riemannian metrics g_S and g_H.

The proof of Theorem 1.3 is a localized version of the original proof of Varopoulos' theorem. Since the universal cover $\tilde{N}$ of N need neither be roughly isometric to $\langle S \rangle$ nor have bounded local geometry, we construct a submanifold $\tilde{X}$ of $\tilde{N}$ satisfying these conditions. Using results of Coulhon and Saloff-Coste [3] and Kanai [8], we show that $\tilde{X}$ supports a Sobolev inequality

$$\|u\|_{\frac{d}{d-1}} \leq C \|\nabla u\|_1$$

for compactly supported Lipschitz functions u on $\tilde{X}$, where $C > 0$ does not depend on u. The Sobolev inequality then yields the n-hyperbolicity of $\tilde{N}$ since $d > n$.

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2. Proof of Theorem 1.1

We consider first the construction in the case of an unknot. Let $\hat{\mathbb{R}}^3$ be the one-point compactification of $\mathbb{R}^3$, and $\sigma\colon \hat{\mathbb{R}}^3 \to S^3$ the inverse of the stereographic projection. Let also $Z = \{(0,0)\} \times \mathbb{R} \subset \mathbb{R}^3$ and $S = \sigma(\bar{Z})$. We construct a Riemannian metric g in $\mathbb{R}^3 \setminus Z$ and a quasiregular mapping $f\colon \mathbb{R}^3 \to (\mathbb{R}^3 \setminus Z, g)$. Then $\sigma \circ f\colon \mathbb{R}^3 \to (S^3 \setminus S, (\sigma^{-1})_* g)$ is quasiregular.

The mapping we construct is, in complex notation, $(z,t) \mapsto (e^z, t)$. For simplicity, however, we use only real coordinates, and define $f\colon \mathbb{R}^3 \to \mathbb{R}^3 \setminus Z$, $f = (f_1, f_2, f_2)$, by $f(x,y,t) = (e^x \cos y, e^x \sin y, t)$.

To construct the metric g, we denote

$$A(r) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix}. \tag{2.1}$$

The metric g is now defined in the standard basis $(\partial_x, \partial_y, \partial_t)$ by the matrix field $G\colon \mathbb{R}^3 \setminus Z \to \mathbb{R}^{3\times 3}$, $G(x,y,t) = A(x^2+y^2)$. Thus, in this metric, $(\partial_x, \partial_y, (x^2+y^2)\partial_t)$ is an orthonormal basis in $T_{(x,y,t)}\mathbb{R}^3$.

Since $f_1(x,y,t)^2 + f_2(x,y,t)^2 = e^{2x}$, we have, for $p = (x,y,t)$, that

$$(Df)_p^t G_{f(p)} (Df)_p \;=\; e^{2x} I,$$

where I is the identity matrix.

Thus

$$\begin{aligned} g_{f(p)}((Df)_p(v), (Df)_p(w)) &= \langle G_{f(p)}(Df)_p(v), (Df)_p(w)\rangle \\ &= \langle (Df)_p^t G_{f(p)} (Df)_p(v), w\rangle = e^{2x}\langle v, w\rangle \end{aligned}$$

for $v, w \in T_p\mathbb{R}^3$ and $p \in \mathbb{R}^3$. Hence $f\colon \mathbb{R}^3 \to (\mathbb{R}^3 \setminus Z, g)$ is a conformal map. Especially, f is quasiregular.

The construction of a quasiregular mapping $\mathbb{R}^3 \to S^3 \setminus H$, where H is a Hopf link, is based on observation that $\mathbb{R}^3$ is the universal cover of $S^3 \setminus H$ and that S^3 is a union of two solid tori. The construction of the Riemannian metric and the mapping could be done similarly as in the case of the unknot, but to avoid technicalities, we proceed more abstractly.

Let $S_0 = \{(x,y,0) \in \mathbb{R}^3 \colon x^2 + y^2 = 4\}$ and $S_1 = Z$. Then $H = \sigma(S_0 \cup \bar{Z})$ is a Hopf link in S^3. We denote $L = S_0 \cup S_1$.

Let $T = \{p \in \mathbb{R}^3 \colon \mathrm{dist}\,(p, S_0) = 1/2\}$. Then T is a 2-torus and $S^3 \setminus \sigma(T)$ consists of two open solid tori in S^3. We fix an orientation preserving diffeomorphism $\Theta\colon T \times \mathbb{R} \to \mathbb{R}^3 \setminus L$ so that $\Theta(p, 0) = p$ for $p \in T$. We define g_L to be the push-forward metric in $\mathbb{R}^3 \setminus L$ by

$$(g_L)_p(v, w) = \left\langle (D\Theta)_p^{-1} v, (D\Theta)_p^{-1} w \right\rangle,$$

where $v, w \in T_p\mathbb{R}^3$ and $p \in T \times \mathbb{R}$.

We fix a diffeomorphism $\varphi\colon S^1 \times S^1 \to T$ and define $\tilde{\varphi}\colon \mathbb{R}^2 \to T$ to be the composition of φ with the covering map $\mathbb{R}^2 \to S^1 \times S^1$, $(t,s) \mapsto (e^{i2\pi t}, e^{i2\pi s})$. We define $\hat{\varphi}\colon \mathbb{R}^3 \to T \times \mathbb{R}$ by $\hat{\varphi}(x,y,t) = (\tilde{\varphi}(x,y), t)$. Since φ is a diffeomorphism and $S^1 \times S^1$ is compact, φ is bilipschitz. Thus $\hat{\varphi}$ is quasiregular. Since $\Theta\colon T\times\mathbb{R}\to(\mathbb{R}^3\setminus L,g)$ is conformal by construction, we have that $f\colon\mathbb{R}^3\to(S^3\setminus H,g_H)$, $f=\sigma\circ\Theta\circ\hat{\varphi}$, is quasiregular, where g_H is the push-forward metric of g_L under σ.

Remark 2.1. Semmes [12] has constructed topologically interesting metric spaces whose geometry is controlled in the sense that they are Ahlfors regular and satisfy Poincaré inequalities. Concerning the examples above, the metric constructed in $S^3\setminus H$ can be modified to have controlled geometry. In contrast, we do not know if $S^3\setminus S$ has a metric with controlled geometry such that the resulting space is quasiregularly elliptic. Similarly, we do not know if the Whitehead manifold, an example also considered by Semmes, admits a metric as above which makes it quasiregularly elliptic.

3. Proof of Theorem 1.3

Since the quasiregularity of the mapping depends only on the conformal class of the Riemannian metric on N, we may assume that N is complete, see e.g. [14]. We denote by g a fixed complete Riemannian metric on N.

3.1. **Construction of the submanifold.** Since the fundamental group of N has a order of growth at least $d>n$, we can fix a finite set $S\subset\pi_1(N)$ so that the subgroup $\langle S\rangle$ generated by S has order of growth at least d.

Let $x_0\in N$. We fix loops $\gamma_s\colon S^1\to N$, $s\in S$, so that $S=\{[\gamma_s]\colon s\in S\}$ and $\gamma_s(1)=x_0$ for every $s\in S$.

Since N is complete, we may fix, by Sard's theorem, a closed ball X in N so that ∂X is a smooth manifold and loops $\gamma_s(S^1)$ are contained in the interior of X.

Let $\tilde{N}$ be the universal cover of N and $\pi\colon\tilde{N}\to N$ a covering map. We fix a component $\tilde{X}$ of $\pi^{-1}(X)$. Then $\tilde{X}$ is a submanifold of $\tilde{N}$ with smooth boundary $\partial\tilde{X}=\pi^{-1}(\partial X)$. We show first that $\tilde{X}$ is *roughly isometric* to the subgroup $\langle S\rangle$ with a word metric determined by S. A mapping $\varphi\colon Y\to Z$ between metric spaces (Y,d_Y) and (Z,d_Z) is said to be a *rough isometry* if there exist constants $a\ge1$, $b>0$, and $\varepsilon>0$ so that

$$\frac{1}{a}d_Y(y,y')-b\le d_Z(\varphi(y),\varphi(y'))\le ad_Y(y,y')+b$$

for all $y,y'\in Y$ and

$$\operatorname{dist}_Z(z,\varphi(Y))<\varepsilon$$

for all $z\in Z$. Spaces Y and Z is said to be roughly isometric if there exists a rough isometry $Y\to Z$; see e.g. [8] for more details on rough isometries.

To see that $\langle S\rangle$ and $\tilde{X}$ are roughly isometric, we observe that, by compactness of X, the submanifold $\tilde{X}$ and the net $\tilde{P}=\pi^{-1}(x_0)\cap\tilde{X}$ are roughly isometric. Since $\tilde{P}$ and $\langle S\rangle$ are bilipschitz equivalent, also $\tilde{X}$ and $\langle S\rangle$ are roughly isometric.

3.2. **Proof of Theorem 1.3.** The proof of Theorem 1.3 is based on the following Sobolev inequality. For the statement, we say that an ε-net P on a manifold admits a *d-dimensional isoperimetric inequality* if there exists a constant $C>0$ so that

$$\#Q\le C\left(\#\partial_P Q\right)^{d/(d-1)}$$

for all finite subset $Q \subset P$, where $\partial_P Q = \{p \in P \setminus Q \colon \mathrm{dist}\,(p,Q) \le 2\varepsilon\}$; see [8] for more details.

For the statement of the Sobolev inequality, we denote by $\mathrm{Lip}_0(\tilde{M})$ the space of compactly supported Lipschitz functions on $\tilde{M}$.

Proposition 3.1. *Let M be a compact submanifold with boundary of a Riemannian manifold N, $\tilde{N}$ the universal cover of N, and $\tilde{M}$ a component of the lift of M in a covering map $\tilde{N} \to N$. If $\tilde{M}$ contains an ε-net P admitting a d-dimensional isoperimetric inequality for $d \ge n$, then there exists $C > 0$ so that*

$$\|u\|_{\frac{d}{d-1}} \le C\|\nabla u\|_1 \tag{3.1}$$

for all $u \in \mathrm{Lip}_0(\tilde{M})$.

We postpone the proof of this proposition to the next section and consider first the proof of Theorem 1.3.

Proof of Theorem 1.3. Since $\langle S\rangle$ has an order of growth at least $d > n$, it supports a d-dimensional isoperimetric inequality; see [3, Théorème 1] for a precise statement. Since $P = \pi^{-1}(x_0) \cap \tilde{X}$ is bilipschitz equivalent to $\langle S\rangle$, we have that P supports a d-dimensional isoperimetric inequality by [8, Lemma 4.2]. Thus $\tilde{X}$ admits d-dimensional Sobolev inequality (3.1) by Proposition 3.1.

Since $d > n$, we have $d/(d-1) < n/(n-1)$. Let $\gamma > 1$ be so that

$$\frac{\gamma}{\gamma-1}\frac{d}{d-1} = \frac{n}{n-1}.$$

We fix a closed ball $B \subset \tilde{X}$. Let $v \in C_0^\infty(\tilde{N})$ be so that $v|B \ge 1$. We define $u = v|\tilde{X}$. Then, by Proposition 3.1 and Hölder's inequality,

$$\begin{aligned}\left(\int_{\tilde{X}} |u|^{\gamma d/(d-1)}\right)^{(d-1)/d} &\le C\int_{\tilde{X}} |\nabla |u|^\gamma| \le C\int_{\tilde{X}} \gamma|u|^{\gamma-1}|\nabla u| \\ &\le C\gamma\left(\int_{\tilde{X}} |u|^{(\gamma-1)n/(n-1)}\right)^{(n-1)/n}\left(\int_{\tilde{X}} |\nabla u|^n\right)^{1/n},\end{aligned}$$

where $C > 0$ us as in (3.1).

Since $\gamma d/(d-1) = (\gamma-1)n/(n-1)$ and $(d-1)/d > (n-1)/n$, we have

$$\begin{aligned}|B|^{\frac{d-1}{d}-\frac{n-1}{n}} &\le \left(\int_B |u|^{\gamma d/(d-1)}\right)^{\frac{d-1}{d}-\frac{n-1}{n}} \le C\gamma\left(\int_{\tilde{X}} |\nabla u|^n\right)^{1/n} \\ &\le C\gamma\left(\int_{\tilde{N}} |\nabla v|^n\right)^{1/n}\end{aligned}$$

Thus the ball B has positive n-capacity with respect to $\tilde{N}$ and hence $\tilde{N}$ is n-hyperbolic by definition; see e.g. [6].

Since every quasiregular mapping from $\mathbb{R}^n$ to N has a lift to $\tilde{N}$ and the lifted mapping is constant by n-hyperbolicity of $\tilde{N}$ (recall that n-hyperbolic spaces do not receive non-constant quasiregular mappings from $\mathbb{R}^n$, see [13]), there are no non-constant quasiregular mappings into N. This completes the proof. □

4. A Sobolev inequality

In this section we prove Proposition 3.1. We obtain the Sobolev inequality by constructing a *double of* $\tilde{M}$. Proposition 3.1 follows almost directly from this construction. The *bounded local geometry* in the following statement refers to the standard assumptions that the Ricci curvature is bounded from below and the manifold has positive injectivity radius; see e.g. condition (*) in [8, p.394], or [2].

Lemma 4.1. *Let M, N, $\tilde{M}$, and $\tilde{N}$ be as in the statement of Proposition 3.1. Then there exists $L \geq 1$, a connected and complete Riemannian manifold $\hat{M}$ with bounded local geometry, and mappings $\iota\colon \tilde{M} \to \hat{M}$ and $\hat{\pi}\colon \hat{M} \to \hat{M}$ so that*

- (i) *the mapping ι is an L-bilipschitz embedding, and*
- (ii) *$\hat{\pi}$ is a 2-Lipschitz mapping so that $\hat{\pi}|\iota(\tilde{M}) = \mathrm{id}$ and $\hat{\pi}|(\hat{M} \setminus \iota(\tilde{M}))$ is a local isometry.*

Proof. We construct a *double* M_D of M by $M_D = (M \times \{0,1\})/\sim$, where $\sim$ is the equivalence relation $(x,0) \sim (x,1)$ if $x \in \partial M$. Then M_D is an n-manifold without boundary and there exists an open neighborhood Ω of M in N and a smooth embedding $\psi_M\colon \Omega \to M_D$, so that $\psi_M(x) = [(x,0)]$ for $x \in M$; see e.g. [9, Chapter IV §5.]. We denote by $\pi_D\colon M_D \to M_D$ the projection $[(x,k)] \mapsto [(x,0)]$ and by $\sigma_D\colon M_D \to M_D$ the reflection $[(x,k)] \mapsto [(x,1-k)]$. The double $\tilde{M}_D$ of $\tilde{M}$ is constructed similarly.

We fix a Riemannian metric g_D on M_D so that σ_D is a local isometry, that is, $\sigma_{D*}g_D = g_D$.

Since M is compact, there exists a constant $L \geq 1$ so that

$$\frac{1}{L^2} g \leq g_D \leq L^2 g$$

as tensors on $M \subset M_D$. Here g is the Riemannian metric fixed in the beginning of this section. Since σ_D is a local isometry, we have that the standard embedding $\iota_D\colon M \to M_D$ is L-bilipschitz. By the choice of g_D, the projection π_D is a local isometry on $M_D \setminus M$.

By construction of the double, $\tilde{M}_D$ is a covering space of M_D and $\tilde{M}_D \to M_D$, $[(\tilde{x},k)] \mapsto [(\pi(\tilde{x}),k)]$, is a covering map. We denote by $\tilde{g}$ and $\tilde{g}_D$ the lifts of Riemannian metrics g and g_D on $\tilde{M}$ and $\tilde{M}_D$, respectively. We denote also by $\tilde{\iota}_D\colon \tilde{M} \to \tilde{M}_D$ and by $\tilde{\pi}_D\colon \tilde{M}_D \to \tilde{M}_D$ the lifts of ι_D and π_D, respectively. Then $\tilde{\iota}_D$ is an L-bilipschitz embedding and $\tilde{\pi}_D$ is an local isometry in $\tilde{M}_D \setminus \tilde{M}$. Thus we may take $\hat{M} = \tilde{M}_D$, $\iota = \tilde{\iota}_D$ and $\hat{\pi} = \tilde{\pi}_D$. This concludes the proof. □

Proof of Proposition 3.1. Let $\hat{M}$ be a manifold as in Lemma 4.1. We denote $M_0 = \iota(M)$ and $M_1 = \hat{M} \setminus M_0$. We show first that $\hat{M}$ and M_0 are roughly isometric.

Let $\hat{x} \in \hat{M}$. Since M is compact there exists $y \in \partial\tilde{M}$ so that $d(\hat{\pi}(\hat{x}), \iota(y)) \leq 2\,\mathrm{diam}\, M$. Let $\hat{y} = \iota(y)$. Since $\hat{y} \in \partial M_1$, we have that $d(\hat{x},\hat{y}) \leq Ld(\hat{\pi}(x),\hat{y}) \leq 2L\,\mathrm{diam}\, M$. Since also $d(\hat{\pi}(\hat{x}),\hat{x}) \leq d(\hat{\pi}(\hat{x}),\hat{y}) + d(\hat{y},\hat{x}) \leq 4L\,\mathrm{diam}\, X$, we have that $\hat{\pi}$ is a rough isometry. Thus $\hat{M}$ and M_0 are roughly isometric.

Let P be an ε-net on $\tilde{M}$ admitting a d-dimensional isoperimetric inequality. Then $\iota(P)$ is bilipschitz equivalent to P and roughly equivalent to $\hat{M}$.

Since $\iota(P)$ supports a d-dimensional isoperimetric inequality by [8, Lemma 4.2] and $\hat{M}$ has locally bounded geometry, we have by [8, Lemma 4.5] that $\hat{M}$ supports d-dimensional Sobolev inequality, that is, there exists $C > 0$ so that

$$\|v\|_{d/(d-1)} \leq C\|\nabla v\|_1 \tag{4.1}$$

for all $v \in C_0^\infty(\hat{M})$.

Let $u \in \mathrm{Lip}_0(\tilde{M})$. Then $v = u \circ \iota^{-1} \circ \hat{\pi}$ is a compactly supported Lipschitz function on $\hat{M}$. By the density of smooth functions, we have that

$$\|v\|_{d/(d-1)} \leq C\|\nabla v\|_1,$$

where C is the constant in (4.1). Since ι^{-1} and $\hat{\pi}$ are L-Lipschitz and $\hat{\pi} = \mathrm{id}$ on $\iota(\tilde{M})$, we have that

$$\begin{aligned}\left(\int_{\tilde{M}} |u|^{d/(d-1)}\, \mathrm{d}\mathcal{H}^n\right)^{(d-1)/d} &\leq \left(L^n \int_{\hat{M}} |v|^{d/(d-1)}\, \mathrm{d}\mathcal{H}^n\right)^{(d-1)/d} \\ &\leq C' \int_{\hat{M}} |\nabla v|\, \mathrm{d}\mathcal{H}^n \leq C' \int_{\tilde{M}} |\nabla u|\, \mathrm{d}\mathcal{H}^n,\end{aligned}$$

where $C' = C'(C, L, n)$. This concludes the proof. □

Appendix A

The purpose of this appendix is to give a direct proof for the well-known fact that $\pi_1(S^3 \setminus L)$ contains a free group if L is a link that is neither the unknot nor the Hopf link. In the following proof, our considerations are in the PL category.

Lemma A.1. *Let L be a link in S^3. Then either L is an unknot, a Hopf link, or $\pi_1(S^3 \setminus L)$ contains a free group.*

Proof. Suppose first that L is a knot. By the structure theorem [1, Theorem 4.6], $[\pi_1(S^3 \setminus L), \pi_1(S^3 \setminus L)]$ contains a free group if L is not an unknot. Thus if L is a knot then it is either an unknot or $\pi_1(S^3 \setminus L)$ contains a free group.

Suppose now that L is not a knot. We may assume that one of the circles, say S, in L is an unknot. Indeed, since $\pi_1(S^3 \setminus L) \to \pi_1(S^3 \setminus S)$ is surjective, we may apply the structure theorem for commutator subgroups again.

Let T be a solid torus that is a regular neighborhood of S so that $L \cap T = S$. Since S is an unknot, $T' = S^3 \setminus \mathrm{int}T$ is a solid torus. Let $L' = L \setminus S$ and let V' be a union of disjoint tori that are regular neighborhoods of circles in L'. Then $\pi_1(S^3 \setminus L) = \pi_1(T' \setminus L') = \pi_1(T' \setminus V')$. It suffices to show that either L is a Hopf link or $\pi_1(T' \setminus L')$ contains a free group.

Let $C = \partial B^2 \times \{1\} \subset \partial(B^2 \times S^1) = \partial T'$ and let D be a PL-disk in T' so that $\partial D = C$ and that the number of intersections $D \cap L'$ is minimal. Let $P' = D \cap L'$. By minimality assumption and [5, 6.1], we have that

$$\ker(\pi_1(D \setminus P') \to \pi_1(T' \setminus L')) = 1.$$

Indeed, suppose that the kernel is non-trivial, then $\ker(\pi_1(D \setminus V') \to \pi_1(T' \setminus V')) \neq 1$ and there exists an embedded disk $D' \subset T' \setminus V'$ so that $\partial D' = D' \cap D$ and $\partial D'$ is not contractible in D. Then there exists a disk $D'' \subset D \cup D'$

so that D'' has a smaller number of intersections with L' than D. This contradicts the minimality of D.

It suffices now to consider $\pi_1(D \setminus P')$. If $D \cap P'$ contains at least two points, then $\pi_1(D \setminus P')$ contains a free group. Suppose that $D \cap P'$ is a point. Then either L' is a circle or L' contains a circle, say S', that does not meet D. In the latter case, $S \cup S'$ is an unlink in S^3. In this case, $\pi_1(S^3 \setminus L)$ contains a free group.

We have now reduced our considerations to the case that L consists of two unknotted circles S and S' so that S' is contained in a torus T' that is a complement of the regular neighborhood of S and that there exists a disk D_S so that $\partial D_S = S$ and $D_S \cap S'$ is a point. The disk D_S is obtained by attaching an annulus to D. We show that, under these assumptions, L is a Hopf link. More precisely, let $\gamma = \partial B^2 \times \{1\} = \partial T$. We show that L and $S \cup \gamma$ are isotopic by finding an embedded annulus $A \subset S^3 \setminus S$ so that $\partial A = \gamma \cup S'$. Then S' and γ are isotopic in a regular neighborhood of A and L is isotopic to a Hopf link $S \cup \gamma$.

Since $D_S \cap L'$ is a point, we may fix a regular neighborhood, i.e. a 3-cell, B of D_S so that $B \cap L'$ is a closed unknotted arc in B and $\partial B \cap L'$ consists of two points. Moreover, we may choose B so that $T \subset B$ and $T \cap \partial B$ is an annulus. By an unknotted arc, we mean that $(B, B \cap L')$ is homeomorphic to $(B^3, \{0\} \times [-1,1])$. We denote by γ' the arc $\gamma \cap B$.

We now fix a circle S'' in S^3 such that $\gamma \subset S''$, $S'' \setminus B = L' \setminus B$, $S'' \cap B \subset \partial B$, and S'' is isotopic to L' in S^3. Since S'' and L' are isotopic and L' is unknotted in S^3, there exists a disk ω so that $\partial\omega = S''$ and $\omega \cap \operatorname{int} B = \emptyset$. Let $A \subset S^3 \setminus \operatorname{int} T$ be an annulus so that $\omega \subset A$, $A \cap B \subset B \setminus \operatorname{int} T$, and ∂A consists of L' and γ. Then L' is isotopic to γ in $S^3 \setminus S$. Hence L is isotopic to a Hopf link $S \cup \gamma$. □

References

[1] G. Burde and H. Zieschang. *Knots*, volume 5 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985.

[2] I. Chavel. *Isoperimetric inequalities*, volume 145 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001. Differential geometric and analytic perspectives.

[3] T. Coulhon and L. Saloff-Coste. Isopérimétrie pour les groupes et les variétés. *Rev. Mat. Iberoamericana*, 9(2):293–314, 1993.

[4] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, english edition, 2007. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[5] J. Hempel. 3-*Manifolds*. Princeton University Press, Princeton, N. J., 1976. Ann. of Math. Studies, No. 86.

[6] I. Holopainen. Quasiregular mappings and the p-Laplace operator. In *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, volume 338 of *Contemp. Math.*, pages 219–239. Amer. Math. Soc., Providence, RI, 2003.

[7] J. Jormakka. The existence of quasiregular mappings from $\mathbf{R}^3$ to closed orientable 3-manifolds. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (69):44, 1988.

[8] M. Kanai. Rough isometries, and combinatorial approximations of geometries of non-compact Riemannian manifolds. *J. Math. Soc. Japan*, 37(3):391–413, 1985.

[9] S. Lang. *Fundamentals of differential geometry*, volume 191 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[10] S. Rickman. On the number of omitted values of entire quasiregular mappings. *J. Analyse Math.*, 37:100–117, 1980.

[11] S. Rickman. The analogue of Picard's theorem for quasiregular mappings in dimension three. *Acta Math.*, 154(3-4):195–242, 1985.

[12] S. Semmes. Good metric spaces without good parameterizations. *Rev. Mat. Iberoamericana*, 12(1):187–275, 1996.

[13] N. T. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992.

[14] V. A. Zorich and V. M. Kesel'man. On the conformal type of a Riemannian manifold. *Funktsional. Anal. i Prilozhen.*, 30(2):40–55, 96, 1996.

P.P. University of Helsinki, Department of Mathematics and Statistics (P.O. Box 68), FI-00014 University of Helsinki, Finland
e-mail: pekka.pankka@helsinki.fi

K.R. University of Jyväskylä, Department of Mathematics and Statistics (P.O. Box 35), FI-40014 University of Jyväskylä, Finland
e-mail: kai.i.rajala@jyu.fi