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On strictly convex subsets in negatively curved manifolds

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Abstract

In a complete simply connected Riemannian manifold X of pinched negative curvature, we give a sharp criterion for a subset C to be the ϵ -neighbourhood of some convex subset of X , in terms of the extrinsic curvatures of the boundary of C .¹

1 Introduction

Let X be a complete simply connected smooth Riemannian manifold of dimension $m \geq 2$ with pinched negative sectional curvature $-b^2 \leq K \leq -a^2 < 0$, with $a, b > 0$. Fix $\epsilon > 0$. A subset C of X will be called ϵ -strictly convex if there exists a nonempty convex subset C' in X such that C is the closed ϵ -neighbourhood of C' . One often encounters ϵ -strictly convex subsets in the literature, for instance when considering tubular neighbourhoods of geodesic lines or totally geodesic subspaces (see for instance [Gra]), or the ϵ -neighbourhood of the convex hull of the limit set of a nonelementary discrete subgroup of isometries of X (see for instance [MT]). In [PP], we studied penetration properties of geodesic lines in ϵ -strictly convex subsets of X (called ϵ -convex subsets in [PP]); given an appropriate family of them, we constructed geodesic lines having an exactly prescribed (big enough) penetration in exactly one of them, and otherwise avoiding (or not entering too much in) them.

In this paper, we give a criterion for a subset of X to be ϵ -strictly convex, in terms of the extrinsic curvature properties of its boundary. Note that an ϵ -strictly convex subset of X is a closed strictly convex subset of X with nonempty interior, whose boundary is $C^{1,1}$ -smooth (see for instance [Fed, Theo. 4.8 (9)] or [Wal, page 272]).

For every strictly convex $C^{1,1}$ -smooth hypersurface S in X , let $\Pi_S : TS \oplus TS \rightarrow \mathbb{R}$ be the (almost everywhere defined, scalar) second fundamental form of S associated to the inward normal unit vector field \vec{n} along S (see Section 3 for a definition). For

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every point x of S at which \mathbb{I}_S is defined, let $\overline{\mathbb{I}}_S(x)$ (resp. $\underline{\mathbb{I}}_S(x)$) be the least upper (resp. greatest lower) bound of $\mathbb{I}_S(v, v)$ for every unit tangent vector v to S at x .

Theorem 1 *Let C be a closed strictly convex subset of X with nonempty interior, whose boundary $S = \partial C$ is $C^{1,1}$ -smooth.*

- *If C is ϵ -strictly convex, then $a \tanh(a\epsilon) \leq \underline{\mathbb{I}}_S \leq \overline{\mathbb{I}}_S \leq b \coth(b\epsilon)$ almost everywhere.*
- *If $b \tanh(b\epsilon) \leq \underline{\mathbb{I}}_S \leq \overline{\mathbb{I}}_S \leq a \coth(a\epsilon)$ almost everywhere, then C is ϵ -strictly convex.*

The bounds that appear in the statement of the theorem are geometrically natural: $a \tanh(a\epsilon)$ is the value on unit tangent vectors of the second fundamental form of the ϵ -neighbourhood of a totally geodesic hyperplane in the real hyperbolic m -space of constant curvature $-a^2$, and $b \coth(b\epsilon)$ is the corresponding number for a sphere of radius ϵ in the real hyperbolic m -space with constant curvature $-b^2$. Furthermore, in the complex hyperbolic plane whose sectional curvatures are normalized to be between -4 and -1 , a geodesic line is contained in a copy of the real hyperbolic plane of curvature -1 and it is orthogonal to a copy of the real hyperbolic plane of curvature -4 . Thus, the ϵ -neighbourhood of a geodesic line has principal curvatures corresponding to both extremes $\tanh \epsilon$ and $2 \coth \epsilon$ in the first part of the statement of the theorem.

One should expect not only upper bounds on the extrinsic curvatures but also lower bounds. Indeed, if for instance C is a half-space in the real hyperbolic m -space, then C is closed, convex, with nonempty interior and smooth boundary, but is not the closed ϵ -neighbourhood of a convex subset (the natural subset of X of which it is the closed ϵ -neighbourhood is nonconvex, and even strictly concave).

A horoball in X is ϵ -strictly convex for every $\epsilon > 0$, hence we recover from the first assertion of Theorem 1 that the extrinsic curvatures of a horosphere in X belong to $[a, b]$.

Theorem 1 is sharp, given the pinching on the curvature, since it immediately implies the following result.

Corollary 2 *Let X be a complete simply connected smooth Riemannian manifold of dimension $m \geq 2$ with constant sectional curvature $-a^2 < 0$, let C be a closed strictly convex subset of X with nonempty interior, whose boundary $S = \partial C$ is $C^{1,1}$ -smooth, and let $\epsilon > 0$. Then C is ϵ -strictly convex if and only if $a \tanh(a\epsilon) \leq \underline{\mathbb{I}}_S \leq \overline{\mathbb{I}}_S \leq a \coth(a\epsilon)$ almost everywhere. \square*

Our notion of ϵ -strictly convexity is related to, but different from, the notion of λ -convexity studied for instance in [GR, BGR, BM]. Indeed, for every $\lambda \in [0, 1]$, there exist λ -convex (in the sense of these references) subsets of the real hyperbolic plane (with constant sectional curvature -1) that are not ϵ -strictly convex for any $\epsilon > 0$ (for instance the intersection of all λ -convex subsets containing two distinct

points). Furthermore, the lower bound in the first assertion of Theorem 1 says that any ϵ -strictly convex subset with C^2 -smooth boundary in X is $a \tanh(a\epsilon)$ -convex in the sense of [BGR, Def. 2.2].

Alexander and Bishop [AB] (see also [Lyt]), have introduced a natural notion of an “extrinsic curvature bounded from above” for subspaces of $\text{CAT}(\kappa)$ -spaces, extending the notion of having a bounded (absolute value of the) second fundamental form for submanifolds of Riemannian manifolds. Thus, this concept of [AB] is related to our notion of ϵ -strictly convex subsets (see in particular Proposition 6.1 in [AB]).

Comparison techniques in Riemannian geometry (as in [CE, Pet, Esc]) are at the heart of the proof of Theorem 1. We start by developing the (quite classical) vector space version of them in Section 2, in a symmetric way to treat upper and lower bounds. The motivations for such a vector space study will be given at the beginning of Section 3, along with the definitions of the Riemannian geometry tools that we will need. We then prove Theorem 1, using the Riemannian convolution smoothing process of Greene and Wu [GW1] to deal with regularity issues.

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2 Comparison results for the matrix Riccati equation

The arguments of the following proposition are standard (compare for instance with [Esc, §2], [Pet, §6.5]), but do not seem to appear in this precise form in the litterature. Recall that an endomorphism f of a (finite dimensional) Euclidean space E is said to be *nonnegative*, and we then write $f \geq 0$, if $\langle f(v) | v \rangle \geq 0$ for every $v \in E$, and *positive* if $\langle f(v) | v \rangle > 0$ for all $v \neq 0$. We denote by $\text{Sym}(E)$ the (finite dimensional) vector space of symmetric endomorphisms of E .

Proposition 3 *Let $\epsilon, a, b > 0$, let $R:] - \epsilon, +\epsilon[\rightarrow \text{Sym}(E)$ be a smooth map such that*

$$\forall t \in] - \epsilon, +\epsilon[, \quad a^2 \text{Id} \leq R(t) \leq b^2 \text{Id} , \quad (1)$$

and let $t \mapsto A(t)$ be a smooth map from a maximal neighbourhood of 0 in \mathbb{R} to $\text{Sym}(E)$ such that $-\dot{A}(t) + A(t)^2 + R(t) = 0$ for all t in the domain of A . Let $\lambda_-(t)$ and $\lambda_+(t)$ denote respectively the smallest and biggest eigenvalues of $A(t)$.

(i) *If*

$$b \tanh(b\epsilon) \leq \lambda_-(0) \leq \lambda_+(0) \leq a \coth(a\epsilon),$$

then the map $t \mapsto A(t)$ is defined at least on $[0, \epsilon[$, and $A(t)$ is positive for every t in $[0, \epsilon[$.

(ii) *Conversely, if $A(t)$ is defined and nonnegative on $[0, \epsilon[$, then*

$$a \tanh(a\epsilon) \leq \lambda_-(0) \leq \lambda_+(0) \leq b \coth(b\epsilon) .$$

Proof. Let us first prove that the functions λ_{\pm} are locally Lipschitz, hence are almost everywhere differentiable, and that they satisfy almost everywhere the inequalities

$$a^2 \leq -\dot{\lambda}_{\pm}(t) + \lambda_{\pm}^2(t) \leq b^2. \quad (2)$$

To prove the locally Lipschitz regularity of the path of biggest (resp. smallest) eigenvalues $t \mapsto \sigma(t)$ (resp. $t \mapsto \iota(t)$) of a smooth path of symmetric endomorphisms $t \mapsto \mathcal{A}(t)$ of a Euclidean space, up to adding a constant big multiple of the identity to $\mathcal{A}(t)$, we may assume that $\mathcal{A}(t)$ is positive. Then, using the standard operator norm, $\sigma(t) = \|\mathcal{A}(t)\|$ and $\iota(t) = \|\mathcal{A}(t)^{-1}\|^{-1}$. Hence σ and ι are indeed locally Lipschitz.

To prove the inequalities (2), let t be a time at which λ'_{\pm} is defined, and let v_t be a unit eigenvector of $A(t)$ corresponding to $\lambda_+(t)$. In particular, $\lambda_+(t) = \langle A(t)v_t | v_t \rangle$ and $\lambda_+^2(t) = \langle A^2(t)v_t | v_t \rangle$. For every $\eta > 0$ small enough, since by definition, $\lambda_+(t - \eta) = \max_{\|w\|=1} \langle A(t - \eta)w | w \rangle$, we have

$$\langle (A(t - \eta) - A(t))v_t | v_t \rangle \leq \lambda_+(t - \eta) - \lambda_+(t). \quad (3)$$

Dividing by $-\eta < 0$ and taking the limit as η goes to 0 gives $\langle A'(t)v_t | v_t \rangle \geq \lambda'_+(t)$. It follows that

$$-\dot{\lambda}_+(t) + \lambda_+^2(t) \geq -\langle A'(t)v_t | v_t \rangle + \langle A^2(t)v_t | v_t \rangle \geq a^2.$$

The inequality $-\dot{\lambda}_+(t) + \lambda_+^2(t) - b^2 \leq 0$ is proved similarly, by replacing η by $-\eta$ in the formula (3), and dividing by $\eta > 0$. The inequalities $a^2 \leq -\dot{\lambda}_-(t) + \lambda_-^2(t) \leq b^2$ are also proved similarly.

The proposition 3 will follow from the next two lemmatae.

Lemma 4 *Let $\epsilon > 0$. Let $s : t \mapsto s(t)$ be a real locally Lipschitz map defined on an open interval I of \mathbb{R} containing 0, such that $-s'(t) + s^2(t) - a^2 \geq 0$ almost everywhere and $s(0) \leq a \coth(a\epsilon)$. Then $s(t) \leq a \coth a(\epsilon - t)$ for every $t \in I \cap [0, \epsilon[$. Conversely, if s is a real locally Lipschitz map defined on $[0, \epsilon[$ such that $-s'(t) + s^2(t) - b^2 \leq 0$ almost everywhere, then $s(0) \leq b \coth(b\epsilon)$.*

Proof. For every $c > 0$, the maximal solution $s_{c,\epsilon}$ of the scalar differential equation $-x' + x^2 - c^2 = 0$, with value $c \coth(c\epsilon)$ at $t = 0$, is $s_{c,\epsilon} : t \mapsto c \coth c(\epsilon - t)$, defined on $] -\infty, \epsilon[$, which satisfies $\lim_{t \rightarrow \epsilon^-} s_{c,\epsilon}(t) = +\infty$. If

$$\varphi_{c,\epsilon}(t) = (s_{c,\epsilon}(t) - s(t))e^{-\int_0^t (s_{c,\epsilon}(u) + s(u)) du},$$

which is defined on the intersection of the intervals of definition of $s_{c,\epsilon}$ and s , then

$$\varphi'_{c,\epsilon}(t) = (s'_{c,\epsilon}(t) - s^2_{c,\epsilon}(t) - s'(t) + s^2(t))e^{-\int_0^t (s_{c,\epsilon}(u) + s(u)) du}$$

almost everywhere.

If $-s'(t) + s^2(t) - a^2 \geq 0$ almost everywhere, then $\varphi'_{a,\epsilon} \geq 0$ almost everywhere. Hence $\varphi_{a,\epsilon}$ is nondecreasing. If furthermore $s(0) \leq a \coth(a\epsilon)$, then $\varphi_{a,\epsilon}(0) = s_{a,\epsilon}(0) - s(0) \geq 0$, so that the map $\varphi_{a,\epsilon}$ is nonnegative at nonnegative times. Hence, for every $t \geq 0$, we have $s(t) \leq s_{a,\epsilon}(t)$ whenever both functions are defined. The first claim follows.

To prove the second one, if by absurd $s(0) > b \coth(b\epsilon)$, since the map $\epsilon \mapsto b \coth(b\epsilon)$ is continuous, there exists $\epsilon' < \epsilon$ such that $s(0) \geq b \coth(b\epsilon')$. If $-s'(t) + s^2(t) \leq b^2$ almost everywhere, then the map $\varphi_{b,\epsilon'}$ is nonincreasing. Since $\varphi_{b,\epsilon'}(0) \leq 0$, $\varphi_{b,\epsilon'}$ is nonpositive at nonnegative times. Therefore $s(t) \geq s_{b,\epsilon'}(t)$ for $t \in [0, \epsilon'[,$ so that $s(t)$ tends to $+\infty$ as $t \rightarrow \epsilon'^-$. But this contradicts the finiteness of $s(\epsilon')$, since s is continuous at ϵ' . \square

Lemma 5 *Let $\epsilon > 0$. Let $i : t \mapsto i(t)$ be a real locally Lipschitz map defined on $[0, \epsilon[$. If $-i'(t) + i^2(t) - b^2 \leq 0$ almost everywhere and $i(0) \geq b \tanh(b\epsilon)$, then i is positive on $[0, \epsilon[$. Conversely, if i is nonnegative on $[0, \epsilon[$ and $-i'(t) + i^2(t) - a^2 \geq 0$ almost everywhere, then $i(0) \geq a \tanh(a\epsilon)$.*

Proof. The proof is similar to that of the previous lemma. For every $c > 0$, the maximal solution $i_{c,\epsilon}$ of the scalar differential equation $-x' + x^2 - c^2 = 0$, with value $c \tanh(c\epsilon)$ at $t = 0$, is $i_{c,\epsilon} : t \mapsto c \tanh c(\epsilon - t)$, defined on \mathbb{R} . If

$$\psi_{c,\epsilon}(t) = (i(t) - i_{c,\epsilon}(t))e^{-\int_0^t (i(u) + i_{c,\epsilon}(u)) du},$$

then

$$\psi'_{c,\epsilon}(t) = (i'(t) - i^2(t) - i'_{c,\epsilon}(t) + i^2_{c,\epsilon}(t))e^{-\int_0^t (i(u) + i_{c,\epsilon}(u)) du}$$

almost everywhere.

If $-i'(t) + i^2(t) - b^2 \leq 0$ almost everywhere, then $\psi'_{b,\epsilon}(t) \geq 0$ almost everywhere. Hence $\psi_{b,\epsilon}$ is nondecreasing. If furthermore $i(0) \geq b \tanh(b\epsilon)$, then $\psi_{b,\epsilon}(0) = i(0) - i_{b,\epsilon}(0) \geq 0$, so that the map $\psi_{b,\epsilon}$ is nonnegative at nonnegative times. Hence, if $t \geq 0$, we have $i(t) \geq i_{b,\epsilon}(t)$ whenever defined. Since $i_{b,\epsilon}(t) > 0$ if and only if $t < \epsilon$, the first assertion follows.

To prove the second one, assume that $-i'(t) + i^2(t) - a^2 \geq 0$ almost everywhere. Then $\psi_{a,\epsilon}$ is nonincreasing. If furthermore $i(t) \geq 0$ for $t \in [0, \epsilon[$, then $\liminf_{t \rightarrow \epsilon^-} \psi_{a,\epsilon}(t) \geq 0$ since $i_{a,\epsilon}(\epsilon) = 0$, so that $\psi_{a,\epsilon}(0) \geq 0$, which implies that $i(0) \geq a \tanh(a\epsilon)$. \square

Let us now prove Assertion (i) of Proposition 3. The first claim of Lemma 4 applied (using Equation (2)) to the biggest eigenvalue $s(t) = \lambda_+(t)$ of $A(t)$ shows that the matrix $A(t)$ remains bounded as long as it is defined and $t \leq \epsilon$. By the assumption of maximality on the domain of definition of A , and by the non explosion theorem at a finite bound for ordinary differential equations, this proves that $A(t)$ is defined for every t in the interval $[0, \epsilon[$. The fact that $A(t)$ is positive there follows from the first claim of Lemma 5 applied (using Equation (2)) to $i = \lambda_-$. This proves Assertion (i).

Similarly, Assertion (ii) of Proposition 3 follows from the second claims of Lemmata 4 and 5. This concludes the proof of Proposition 3. \square

3 Proof of the main result

Let X be as in the introduction, let C be a closed strictly convex subset of X with nonempty interior, whose boundary $S = \partial C$ is $C^{1,1}$ -smooth, and let \vec{n} be the inward normal unit vector field along S . Note that \vec{n} is well-defined (since S is a C^1 strictly convex hypersurface). It is locally Lipschitz, hence is differentiable at almost every point of S (for the (well defined) Lebesgue measure on S).

Let $\Pi_S : TS \oplus TS \rightarrow \mathbb{R}$ be the (almost-everywhere defined) scalar second fundamental form of S associated to the inward normal unit vector field \vec{n} along S , that is, with $\langle \cdot, \cdot \rangle$ the first fundamental form,

$$\Pi_S(V, W) = \langle \nabla_V W, \vec{n} \rangle = -\langle \nabla_V \vec{n}, W \rangle, \quad (4)$$

where V, W are tangent vectors to S at the same point, extended to Lipschitz vector fields on a neighbourhood of this point, which are tangent to S at every point of S , and are differentiable at every twice differentiable point of S . The definition of Π_S depends on the choice between \vec{n} and $-\vec{n}$, and the various references differ on that point; we have chosen the inward pointing vector field in order for the symmetric bilinear form Π_S to be nonnegative, by convexity of C .

As a motivation, here is a short proof that if C is ϵ -strictly convex, then $\overline{\Pi}_S \leq b \coth(b\epsilon)$ almost everywhere. For every x in S , let $y = \exp_x(\epsilon \vec{n}(x))$. Note that the sphere $S_X(y, \epsilon)$ of center y and radius ϵ in X is contained in C , as C is ϵ -strictly convex. Locally over the tangent space $T_x S = T_x(S_X(y, \epsilon)) \subset T_x X$, the graph of $S_X(y, \epsilon)$ is above the graph of S (when \vec{n} points upwards). Hence, for almost every x in S , for every v in $T_x S$, we have $\Pi_S(v, v) \leq \Pi_{S_X(y, \epsilon)}(v, v)$. As the sectional curvature of X is at least $-b^2$, we have by comparison $\overline{\Pi}_{S_X(y, \epsilon)} \leq b \coth(b\epsilon)$ (see for instance [Pet, page 175]).

We now explain the curvature equation that will allow us to apply the results of Section 2.

For every $t \in \mathbb{R}$ and x in S , let $x_t = \exp_x(t \vec{n}(x))$, and let N be the vector field defined by $N(x_t) = \dot{x}_t$ on an open neighbourhood U of S (containing $X - C$ and x_s for $s \in [0, t]$ if it contains x_t). Note that N is locally Lipschitz on U , is differentiable at each $x_t \in U$ such that S is twice differentiable at x , and is smooth along each geodesic $t \mapsto x_t$ while it stays in U . Identify the tangent space $T_{x_t} X$ with $T_x X$ by the parallel transport $\parallel_{x_t}^x : T_{x_t} X \rightarrow T_x X$ along the geodesic $s \mapsto x_s$ whenever it stays in U . Let $A(t)$ (which, for every twice differentiable point x in S , is defined and smooth at every t such that $x_t \in U$) be the symmetric endomorphism of $T_x S$ defined by $v \mapsto -\parallel_{x_t}^x(\nabla_{\parallel_{x_t}^x(v)} N)$. Let $R(t)$ (which is defined for every $x \in S$ and $t \in \mathbb{R}$, is locally Lipschitz in (x, t) , and is smooth in t) be the symmetric endomorphism of $T_x S$ defined by $v \mapsto \parallel_{x_t}^x R(\parallel_{x_t}^x(v), \dot{x}_t) \dot{x}_t$, where $R(\cdot, \cdot) \cdot$ is the curvature tensor of X .

For every twice differentiable point x in S , the map $t \mapsto A(t)$, defined and smooth on a neighbourhood of 0, satisfies on it the following matrix Riccati equation (see for instance [Pet, page 44])

$$-\dot{A}(t) + A(t)^2 + R(t) = 0. \quad (5)$$

For every twice differentiable point $x \in S$ and every unit tangent vector $v \in T_x^1 S$, note that $\langle R(t)v, v \rangle$, being the sectional curvature of the plane generated by the orthonormal tangent vectors $\|_{x_t}^{x_t} v$ and \dot{x}_t at x_t , is at most $-a^2$, and at least $-b^2$. Hence, the map $t \mapsto A(t)$ satisfies the two inequalities (1).

Proof of the first assertion of Theorem 1. If C is ϵ -strictly convex, then there exists a nonempty convex subset C' of X such that $C = \mathcal{N}_\epsilon C'$. For every $t \in [0, \epsilon[$, the map $x \mapsto x_t$ is a $C^{1,1}$ -diffeomorphism from S to $S_t = \partial \mathcal{N}_{\epsilon-t} C'$. For every twice differentiable point x of S , the point x_t is a twice differentiable point of S_t , and $A(t)$ is well-defined on $[0, \epsilon[$ (see [Wal, §3] for these two facts). By the definition of the endomorphism $A(t)$, and since the parallel transport preserves the first fundamental form, we have for every v in $T_{x_t}^1 S_t$,

$$\text{II}_{S_t}(v, v) = -\langle \nabla_v N, v \rangle = \langle A(t) \|_{x_t}^{x_t} v, \|_{x_t}^{x_t} v \rangle . \quad (6)$$

Since S_t is locally convex, its second fundamental form is nonnegative at each twice differentiable point, hence the endomorphism $A(t)$ is nonnegative for $t \in [0, \epsilon[$, for almost every $x \in S$. The first assertion of Theorem 1 now follows from the second assertion of Proposition 3.

Proof of the second assertion of Theorem 1. First assume that S is smooth.

Recall (see [BC, page 222]) that the Σ -Jacobi fields for a C^1 -smooth submanifold Σ are the variations of its normal geodesics. More precisely, for every $x \in S$, a map $J: \mathbb{R} \rightarrow TX$ is a S -Jacobi field along the normal geodesic $\tau: t \mapsto x_t$ to S at x if there exists a C^1 -smooth map $f: \mathbb{R}^2 \rightarrow X$ such that for every $s, t \in \mathbb{R}$, we have $f(t, 0) = \tau(t)$, the map $t \mapsto f(t, s)$ is a geodesic line in X which starts at time $t = 0$ perpendicularly to S , and $J(t) = \frac{\partial f}{\partial s}(t, 0)$. Note that by Schwarz' theorem, the vector field J commutes with the vector field $N = \frac{\partial f}{\partial t}(t, 0)$ along τ . Since the Riemannian connection is torsion-free, by the definition of $A(t)$ and denoting again by $J(t)$ the parallel transport of $J(t)$ from $T_{x_t} X$ to $T_x X$, we have

$$\dot{J}(t) = -A(t)J(t) . \quad (7)$$

Recall that $t_0 \geq 0$ is a *focal time* for $x_0 \in S$ if the differential of $x \mapsto x_{t_0}$ is not injective at x_0 . Note that t_0 is a focal time for $x \in S$ if and only if there is a nonzero S -Jacobi field J along $t \mapsto x_t$ which vanishes at x_{t_0} , and that for every $\epsilon' > 0$, by the triviality of the normal bundle to S and the convexity of C , the map $(x, t) \mapsto x_t$ from $S \times]-\infty, \epsilon'[$ to X is a proper immersion if there is no focal time in $[0, \epsilon']$, see [BC, §11.3].

To prove the second assertion of Theorem 1, let us assume that

$$b \tanh(b\epsilon) \leq \underline{\text{II}}_S \leq \overline{\text{II}}_S \leq a \coth(a\epsilon).$$

By the definition of the endomorphism $A(0)$, we have $\text{II}_S(v, v) = -\langle \nabla_v \vec{n}, v \rangle = \langle A(0)v, v \rangle$ for every v in $T_x^1 S$. Hence by the first assertion of Proposition 3, the endomorphism $A(t)$ is defined and positive for all $t \in [0, \epsilon[$.

We claim that no nonzero S -Jacobi field along a normal geodesic $t \mapsto x_t$ to S vanishes in $[0, \epsilon[$. Indeed, since $A(t)$ is defined for every $t \in [0, \epsilon[$, a S -Jacobi field, which satisfies the first order linear equation (7), vanishes at one point of $[0, \epsilon[$ if and only if it vanishes at all points of $[0, \epsilon[$. Since the biggest eigenvalue of $A(0)$ is at most $a \coth(a\epsilon)$, the claim also follows from the comparison theorem for S -Jacobi fields [Esc, Thm. 3.4] applied to S and the sphere of radius ϵ in the real hyperbolic space of constant curvature $-a^2$ (and keeping in mind that the sign convention for A in [Esc] is different from ours).

Hence $x \mapsto x_t$ is a proper immersion of S , whose image we denote by S_t , for every $t \in [0, \epsilon[$. By the definition of the endomorphism $A(t)$, Equation (6) is still valid, and hence S_t is, for every $t \in [0, \epsilon[$, a strictly convex immersed hypersurface whose smallest eigenvalue of the second fundamental form has a positive lower bound. Let $t_* \geq 0$ be the upper bound of all $t \in [0, \epsilon[$ such that the map $(x, u) \mapsto x_u$ from $S \times [0, t]$ to X is an embedding.

If t_* was strictly less than ϵ , then there would exist a sequence $(t_n)_{n \in \mathbb{N}}$ in $[0, t_*[$ converging to t_* and two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in S_{t_n} such that $d(x_n, y_n)$ tends to 0 and $N(x_n) + \frac{x_n}{y_n} N(y_n)$ converges to 0. Hence as $n \rightarrow +\infty$, the tangent planes at x_n and y_n are closer and closer, and the two germs of the hypersurface S_{t_n} at x_n and y_n , being contained between them, are more and more flat. But this contradicts the fact that the smallest eigenvalue of the second fundamental form of S_{t_*} has a positive lower bound.

If $t_* = \epsilon$, then for every $s \in [0, \epsilon[$, the subset $C_s = C - \bigcup_{t \in [0, s[} S_t$ is closed, and convex since its boundary is locally convex, by the Schmidt theorem (see for instance [Hei, Appendix] as explained in [Ale, page 323]). Define $C = \bigcap_{s \in [0, \epsilon[} C_s$, which is a closed convex subset. We have $C = \mathcal{N}_\epsilon C'$ by construction, so that C is ϵ -strictly convex. This ends the proof of the second assertion of Theorem 1 when S is smooth.

The following approximation result will allow us to extend the result from the smooth case to the general case.

Proposition 6 *Let $a, b, \alpha, \beta > 0$. Let M be a complete simply connected smooth Riemannian manifold of dimension $m \geq 2$ with pinched sectional curvature $-b^2 \leq K \leq -a^2 < 0$. Let C be a closed strictly convex subset of M with nonempty interior, whose boundary $S = \partial C$ is $C^{1,1}$ -smooth and satisfies the inequalities $\alpha \leq \underline{\Pi}_S \leq \overline{\Pi}_S \leq \beta$ almost everywhere. Then for every $\eta > 0$, $\alpha' \in]0, \alpha[$ and $\beta' \in]\beta, +\infty[$, there exists a closed convex subset C' in X containing C , with smooth boundary $S' = \partial C'$ such that $C' \subset \mathcal{N}_\eta C$ and $\alpha' \leq \underline{\Pi}_{S'} \leq \overline{\Pi}_{S'} \leq \beta'$.*

Proof. We denote by $\exp : TM \rightarrow M$ the Riemannian exponential map, by $\|_x^y : T_x M \rightarrow T_y M$ the parallel transport along the geodesic from x to y in M , by $\nabla f : M \rightarrow TM$ the Riemannian gradient of a C^1 map $f : M \rightarrow \mathbb{R}$, by $\pi : TM \rightarrow M$ the canonical projection, by $TTM = V \oplus H$ the orthogonal decomposition into the vertical and horizontal subbundles of $TTM \rightarrow TM$ defined by the Riemannian metric of M (with $V_v = T_{\pi(v)} M$ and $T\pi|_{H_v} : H_v \rightarrow T_{\pi(v)} M$ a linear isomorphism for

every $v \in TM$), and by $\pi_V : TTM \rightarrow V$ the bundle projection to the vertical factor parallel onto the horizontal one. Recall that the covariant derivative of a C^1 vector field $Y : M \rightarrow TM$ is defined by $\nabla Y = \pi_V \circ TY$. Also recall that if $f : M \rightarrow \mathbb{R}$ and $g : M' \rightarrow M$ are C^1 maps, then

$$\nabla(f \circ g) = (Tg)^* \circ (\nabla f) \circ g , \quad (8)$$

where $h^* : TM \rightarrow TM'$ is the adjoint bundle morphism of a bundle morphism $h : TM' \rightarrow TM$ defined by the Riemannian metrics. Here is a proof by lack of reference. For every $x \in M'$ and $Z \in T_x M'$, we have

$$\begin{aligned} \langle (\nabla(f \circ g))(x), Z \rangle_x &= d(f \circ g)_x(Z) = df_{g(x)}(Tg(Z)) = \langle \nabla f(g(x)), Tg(Z) \rangle_{g(x)} \\ &= \langle (Tg)^*(\nabla f(g(x))), Z \rangle_x . \end{aligned}$$

The main tool we use to prove Proposition 6 is the *Riemannian convolution smoothing* process of Greene and Wu. Introduced in [GW1, page 646], it has already been used for instance in [GW2, Theo. 2] to approximate continuous strictly convex functions on Riemannian manifolds by smooth ones. A new property of this process introduced in this paper is a good control of its second order derivatives. The smoothing is defined as follows. Let $\psi : \mathbb{R} \rightarrow [0, +\infty[$ be a smooth map with compact support contained in $[-1, 1]$, constant on a neighbourhood of 0, such that $\int_{v \in \mathbb{R}^m} \psi(\|v\|) d\lambda(v) = 1$, where $d\lambda$ is the Lebesgue measure on the standard Euclidean space \mathbb{R}^m . For every $\kappa > 0$, let $\psi_\kappa : t \mapsto \frac{1}{\kappa^m} \psi(\frac{t}{\kappa})$, whose support is contained in $[-\kappa, \kappa]$. For every continuous map $f : M \rightarrow \mathbb{R}$, define a map $f_\kappa : M \rightarrow \mathbb{R}$ by

$$f_\kappa : x \mapsto \int_{v \in T_x M} \psi_\kappa(\|v\|) f(\exp_x v) d\lambda_x(v) ,$$

where $d\lambda_x$ is the Lebesgue measure on the Euclidean space $T_x M$. Note that f_κ is nonnegative if f is nonnegative.

Given η, α', β' as in the statement of Proposition 6, let us prove that if f is the distance function to C , for some $\kappa, t > 0$ small enough, then $C' = f_\kappa^{-1}([0, t])$ satisfies the conclusions of Proposition 6.

Let $j_x = \frac{d(\exp_x)^* \lambda_x}{d\text{vol}} : M \rightarrow [0, +\infty]$ be the jacobian of the map \exp_x from $T_x M$ to M , with respect to the Lebesgue measure $d\lambda_x$ on $T_x M$ and the Riemannian measure $d\text{vol}$ on M . Since M is complete, simply connected and negatively curved, for every x in M , the map \exp_x is a smooth diffeomorphism, whose inverse and whose jacobian are hence well-defined, and depend smoothly on (x, y) where y is the variable point in M . For every continuous map $f : M \rightarrow \mathbb{R}$, by an easy change of variables, we have $f_\kappa : x \mapsto \int_{y \in M} \psi_\kappa(\|\exp_x^{-1}(y)\|) f(y) j_x(y) d\text{vol}(y)$. Since ψ is constant near 0 and by a standard argument of differentiation under the integral sign, the map f_κ is smooth.

If $f : M \rightarrow \mathbb{R}$ is a 1-Lipschitz map, then since $\int_{v \in T_x M} \psi_\kappa(\|v\|) d\lambda_x(v) = 1$, since $d(x, \exp_x v) = \|v\|$ and since $v \mapsto \psi_\kappa(\|v\|)$ vanishes outside $\{v \in T_x M : \|v\| \leq \kappa\}$, we have, for every $x \in M$,

$$|f_\kappa(x) - f(x)| \leq \kappa . \quad (9)$$

For every $x_0 \in M$ and $v \in T_{x_0}M$, let $g_v : M \rightarrow M$ be the map $x \mapsto \exp_x(\|x_{x_0} v)$. Note that g_0 is the identity map, and that $(v, x) \mapsto g_v(x)$ is a smooth map. Since the parallel transport is an isometry, for every continuous map $f : M \rightarrow \mathbb{R}$, we have as in [GW1]

$$f_\kappa : x \mapsto \int_{v \in T_{x_0}M} \psi_\kappa(\|v\|) f \circ g_v(x) d\lambda_{x_0}(v) .$$

By Equation (8), and by differentiation under the integral sign, if f is C^1 on $B(x_0, 2\kappa)$ and $x \in B(x_0, \kappa)$, we have

$$\nabla f_\kappa(x) = \int_{v \in T_{x_0}M} \psi_\kappa(\|v\|) (Tg_v)^* \circ (\nabla f) \circ g_v(x) d\lambda_{x_0}(v) .$$

Let now $f : M \rightarrow [0, +\infty[$ be the distance map to C , which is 1-Lipschitz on M and is a $C^{1,1}$ Riemannian submersion outside C . Recall that (adapting [Pet, §2.4.1] to the $C^{1,1}$ regularity), the Hessian $Hf = \nabla^2 f : TM \rightarrow TM$ of f is almost everywhere defined. For every $t > 0$, the level hypersurface $S_t = f^{-1}(t)$ is C^1 with inward normal unit vector field equal to $-\nabla f$ along S_t , and is twice differentiable at almost every point. By Equation (4), the second fundamental form of S_t at a twice differentiable point x satisfies, for every $Z \in T_x S_t$,

$$\Pi_{S_t}(Z, Z) = \langle Hf(Z), Z \rangle .$$

Let $\alpha'' \in]\alpha', \alpha[$ and $\beta'' \in]\beta, \beta'[$. Let $\eta' \in]0, \eta]$ be small enough so that $\alpha'' \leq \underline{\Pi}_{S_s} \leq \overline{\Pi}_{S_s} \leq \beta''$ almost everywhere on S_s for every $s \in [0, \eta']$. Let $t \in [\frac{\eta'}{3}, \frac{2\eta'}{3}]$. Then by Equation (9), for every $\kappa \in]0, \frac{\eta'}{12}]$, we have

$$C \subset \mathcal{N}_{3\kappa}C \subset f_\kappa^{-1}([0, t]) \subset \mathcal{N}_{\eta'}C \subset \mathcal{N}_\eta C .$$

By the linearity of π_V and again by differentiation almost everywhere of Lipschitz maps under the integral sign, if $d(x_0, C) > 2\kappa$ and $Z \in T_{x_0}M$, we have

$$Hf_\kappa(Z) = \nabla^2 f_\kappa(Z) = \int_{v \in T_{x_0}M} \psi_\kappa(\|v\|) \pi_V \circ T((Tg_v)^*) \circ T(\nabla f) \circ (Tg_v)(Z) d\lambda_{x_0}(v) .$$

The maps $T((Tg_v)^*)$ and Tg_v are the identity maps of respectively TTM and TM when $v = 0$, and they depend continuously of v for the uniform convergence of maps, since M has pinched curvature. If κ is small enough, since π_V is 1-Lipschitz, for every $x_0 \in M$ such that $\frac{\eta'}{3} \leq d(x_0, C) \leq \frac{2\eta'}{3}$, we hence have

$$\alpha' \leq \min_{Z \in T_{x_0}^1 M} \langle Hf_\kappa(Z), Z \rangle \leq \max_{Z \in T_{x_0}^1 M} \langle Hf_\kappa(Z), Z \rangle \leq \beta' .$$

Using Sard's theorem, let $t \in [\frac{\eta'}{3}, \frac{2\eta'}{3}]$ be such that $f_\kappa^{-1}(t)$ is smooth, and we have

$$\alpha' \leq \underline{\Pi}_{f_\kappa^{-1}(t)} \leq \overline{\Pi}_{f_\kappa^{-1}(t)} \leq \beta' .$$

In particular, the smooth hypersurface $f_\kappa^{-1}(t)$ is strictly convex, since $\alpha' > 0$.

Using again the Schmidt theorem, it is then easy to check that $C' = f_\kappa^{-1}([0, t])$ satisfies the conclusion of Proposition 6. \square

Now, if C satisfies the assumption of the second assertion of Theorem 1, for every $n \in \mathbb{N}$ bigger than some $N \in \mathbb{N}$, there exists, by Proposition 6, a closed convex subset C_n with smooth boundary containing C such that $C_n \subset \mathcal{N}_{\frac{1}{n}}C$ and $b \tanh(b(\epsilon - \frac{1}{n})) \leq \underline{\Pi}_S \leq \overline{\Pi}_S \leq a \coth(a(\epsilon - \frac{1}{n}))$. By the already proven smooth case of the second assertion of Theorem 1, C_n is hence $(\epsilon - \frac{1}{n})$ -strictly convex. Let C'_n be a closed convex subset such that $C_n = \mathcal{N}_{\epsilon - \frac{1}{n}}(C'_n)$.

Then C is the closed ϵ -neighbourhood of the intersection of the C'_n 's. Indeed, since $\mathcal{N}_{\epsilon - \frac{1}{n}}(C'_n)$ is contained in $\mathcal{N}_\epsilon(C'_n)$ for every $n \geq N$, the set $C = \bigcap_{n \geq N} C_n$ is contained in $\mathcal{N}_\epsilon(\bigcap_{n \geq N} C'_n)$. Conversely, let $x \in \mathcal{N}_\epsilon(\bigcap_{n \geq N} C'_n)$. Since $\mathcal{N}_\epsilon(C'_n) = \mathcal{N}_{\frac{1}{n}}C_n \subset \mathcal{N}_{\frac{2}{n}}C$, for every $n \geq N$, there exists $x_n \in C$ such that $d(x, x_n) \leq \frac{2}{n}$. By a compactness argument and since C is closed, we hence have $x \in C$.

This proves that C is ϵ -strictly convex, and concludes the proof of Theorem 1.

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