A JORDAN SOBOLEV EXTENSION DOMAIN

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A Jordan Sobolev Extension Domain

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Abstract Let $1 < q < 2$. In this paper, we construct a Jordan domain $G_q \subset \mathbb{R}^2$ such that $G_q \in \text{Ext}_p$ if and only if $1 \leq p < q$ and $\mathbb{R}^2 \setminus G_q \in \text{Ext}_s$ if and only if $q/(q-1) < s \leq \infty$.

1 Introduction

Let $D$ be a domain in $\mathbb{R}^2$, namely, $D$ is a connected open subset of $\mathbb{R}^2$. For $1 \leq p \leq \infty$, denote by $W^{1,p}(D)$ the set of all functions in $L^p(D)$ whose first distributional derivatives lie in $L^p(D)$. For any $u \in W^{1,p}(D)$, the norm of $u$ is given by $\|u\|_{W^{1,p}(D)} \equiv \|u\|_{L^p(D)} + \|\nabla u\|_{L^p(D)}$, where $\nabla u$ is the distributional gradient of $u$.

Definition 1.1. Let $1 \leq p \leq \infty$. A domain $D \subset \mathbb{R}^2$ is called a domain of class $\text{Ext}_p$ if there exists a bounded extension operator $\text{Ext} : W^{1,p}(D) \to W^{1,p}(\mathbb{R}^2)$, namely, for each $u \in W^{1,p}(D)$, there exists a function $\text{Ext}(u) \in W^{1,p}(\mathbb{R}^2)$ such that $\text{Ext}(u)(x) = u(x)$ for all $x \in D$ and $\|\text{Ext}(u)\|_{W^{1,p}(\mathbb{R}^2)} \leq C\|u\|_{W^{1,p}(D)}$, where $C$ is a positive constant independent of $u$.

For $p > 1$, one could in fact require above that $\text{Ext}$ is linear; see [1, Theorem 5].

In [5], Maz’ya constructed a planar Jordan domain $D$ such that $D \in \text{Ext}_p$ for all $1 \leq p < 2$ but $D \notin \text{Ext}_p$ for any $2 \leq p \leq \infty$. Furthermore the complementary domain $\mathbb{R}^2 \setminus D$ of $D$ satisfies $\mathbb{R}^2 \setminus D \in \text{Ext}_s$ exactly when $2 < s \leq \infty$. This shows that the possibility of $W^{1,p}(D)$-extensions depends not only on the structure of the domain $D$ but also on the exponent $p$. Motivated by this, for each $1 < q < 2$, Romanov [10] further constructed a planar domain $G_q$ such that $G_q \in \text{Ext}_p$ if and only if $1 \leq p < q$. In this paper, we establish the following results by generalizing the above two constructions in [5, 10].

Theorem 1.1. For each $1 < q < 2$, there exists a Jordan domain $G_q \subset \mathbb{R}^2$ such that $G_q \in \text{Ext}_p$ if and only if $1 \leq p < q$ and $\mathbb{R}^2 \setminus G_q \in \text{Ext}_s$ if and only if $q/(q-1) < s \leq \infty$.\[\text{Abstract}\]

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Our construction is an improvement on the one by Romanov [10] and it partially relies on his approach. We should point out that the boundary of $G_q$ of Romanov [10] contains a curve generated by a certain Cantor set. In order to deal with the complementary domain, we actually simplify the construction from [10] and apply a certain sufficient condition for extendability from [8].

Finally, we state some conventions. Throughout the whole paper, we denote by $C$ a positive constant which is independent of the main parameters, but which may vary from line to line. The symbol $A \lesssim B$ or $B \gtrsim A$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. For any measurable set of positive measure $E \subset \mathbb{R}^2$ and locally integrable function $f$, we set $\int_E f(x) \, dx \equiv \frac{1}{|E|} \int_E f(x) \, dx$.

2 Proof of Theorem 1.1

Theorem 1.1 follows from Lemmas 2.5, 2.7, 2.10 and 2.11 below. We begin with the construction of the domain $G_q$, which is inspired by [10] and [5].

Construction of the domain $G_q$. Assume $1 < q < 2$. Throughout the whole paper, let $a \equiv 2^{1/(q-2)}$ and $b \equiv 1 - 2a$. Then $0 < a < 1/2$ and $0 < b < 1$. Denote by $I$ the interval $[0, 1] \times \{0\}$.

First we generate a sequence of subintervals,

$$\tilde{I} \equiv \{ \tilde{I}^{i,k}_m : m \in \mathbb{N} \cup \{0\}; \, k = 0, \ldots, m + 1; \, i = 0, \ldots, 2^k - 1 \},$$

following the idea of the construction of a Cantor set. When $m = 0$, let $\tilde{I}^{0,0}_0$ be the closed middle interval of $I$ with length $b$ and $\tilde{I}^{i,1}_0$ with $i = 0, 1$ be the closure of the two intervals obtained by removing $\tilde{I}^{0,0}_0$ from $I$ and ordered from left to right. When $m = 1$, let $\tilde{I}^{0,0}_1$, $\tilde{I}^{1,0}_1$, $\tilde{I}^{0,1}_1$ be the closed middle interval of $\tilde{I}^{0,0}_0$ with length $ba$ for $i = 1, 2$, and $\tilde{I}^{i,2}_1$ with $i = 0, 1, 2, 3$ be the closure of the four intervals obtained by removing $\tilde{I}^{0,0}_1$, $\tilde{I}^{1,0}_1$, $\tilde{I}^{0,1}_1$ from $I$ and ordered from left to right. When $m \geq 2$, for $k \leq m - 1$ and $i = 0, \ldots, 2^k - 1$, let $\tilde{I}^{i,k}_m$ be the closed middle interval of $\tilde{I}^{i,k}_{m-1}$ with length $ba^m$; for $k = m$ and $i = 0, \ldots, 2^m - 1$, let $\tilde{I}^{i,m}_m$ be the closed middle interval of $\tilde{I}^{i,m}_{m-1}$ with length $ba^m$; for $k = m + 1$ and $i = 0, \ldots, 2^{m+1} - 1$, let $\tilde{I}^{i,m}_m$ be the closure of the $2^{m+1}$ intervals obtained by removing $\{ \tilde{I}^{i,m}_m : k = 0, \ldots, m; \, i = 0, \ldots, 2^k - 1 \}$ from $I$ and ordered from left to right.

Obviously, $\tilde{I}$ has the following properties:

(i) for each $m \in \mathbb{N} \cup \{0\}$, $I = \bigcup_{k=0}^{m+1} \bigcup_{i=0}^{2^k-1} \tilde{I}^{i,k}_m$;

(ii) $|\tilde{I}^{i,k}_m| = ba^k$ when $k \leq m$ and $i = 0, \ldots, 2^k - 1$, and $|\tilde{I}^{m+1,i}_m| = a^{m+1}$ when $i = 0, \ldots, 2^{m+1} - 1$.

Then we translate and dilate these intervals in $\tilde{I}$ by setting

$$I^{k,i}_m = (1-a)a^m \tilde{I}^{k,i}_m + (a^{m+1}, 0)$$

for each $\tilde{I}^{k,i}_m \in \tilde{I}$. Then we write

$$I \equiv \{ I^{k,i}_m : m \in \mathbb{N} \cup \{0\}; \, k = 0, \ldots, m + 1; \, i = 0, \ldots, 2^k - 1 \}.$$
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Obviously from (i) and (ii), it is easy to see that

(iii) for each \( m \in \mathbb{N} \cup \{0\} \), \([a^{m+1}, a^m] \times \{0\} = \bigcup_{k=0}^{m+1} \bigcup_{i=0}^{2^k-1} I_{m}^{k,i} \);

(iv) \( |I_{m}^{k,i}| = b(1-a)a^{m+k} \) when \( k \leq m \) and \( i = 0, \cdots, 2^k-1 \), and \( |I_{m+1}^{n+1,i}| = (1-a)a^{2m+1} \) when \( i = 0, \cdots, 2^{m+1}-1 \).

For each \( I_{m}^{k,i} \in \mathcal{I} \), denote its upper hat by \( \Gamma_{m}^{k,i} \), namely,

\[
\bigcup_{k=0}^{m+1} \bigcup_{i=0}^{2^k-1} \Gamma_{m}^{k,i} = \bigcup_{m=0}^{\infty} \Gamma_{m}^{k,i} = \Gamma
\]

where and in what follows, for any set \( E \subset \mathbb{R}^2 \) and \( x \in \mathbb{R}^2 \), \( \text{dist}(x, E) = \inf\{|x-y| : y \in E\} \).

We also denote by \( T_{m}^{k,i} \) the closed triangle generated by \( \Gamma_{m}^{k,i} \) and \( I_{m}^{k,i} \). For \( m \in \mathbb{N} \cup \{0\} \), set

\[
\Gamma_{m} = \bigcup_{k=0}^{m+1} \bigcup_{i=0}^{2^k-1} \Gamma_{m}^{k,i}, \quad T_{m} = \bigcup_{k=0}^{m+1} \bigcup_{i=0}^{2^k-1} T_{m}^{k,i}
\]

and

\[
\Gamma = \{(0, 0)\} \bigcup \left( \bigcup_{m=0}^{\infty} \Gamma_{m} \right), \quad T = \{(0, 0)\} \bigcup \left( \bigcup_{m=0}^{\infty} T_{m} \right).
\]

Then we obtain a Lipschitz curve \( \Gamma \) joining \((0, 0)\) and \((1, 0)\). By abuse of notation, we always write \( \Gamma = \{(x_1, \Gamma(x_1)) : x_1 \in [0, 1]\} \).

The following figure shows the curve \( \Gamma_0 \cup \Gamma_1 \), when \( a = 1/4 \).

Let \( R \) be the rectangle \((-1, 1) \times (0, 1)\) and \( \varphi : R \to \mathbb{R}^2 \) such that \( \varphi(x_1, x_2) \equiv (x_1, x_2) \) if \( x_1 \leq 0 \) and \( \varphi(x_1, x_2) \equiv (x_1, x_2 + x_1^2) \) if \( x_1 > 0 \). Set \((G_q)_+ \equiv \varphi(R) \setminus \varphi(T)\) and let \((G_q)_-\) be the reflection of \((G_q)_+\) across the \( x_1 \)-axis. Then define

\[
G_q \equiv (G_q)_+ \bigcup (G_q)_- \bigcup (-1, 0) \times \{0\},
\]

which completes the construction of the domain \( G_q \).

Now we recall the following result, which was established when \( p > 1 \) in [11, Theorem 1] and when \( p = 1 \) by Lemma 4.9.1 of [12].
Lemma 2.1. Let \( 1 \leq p < \infty \) and \( w \) be a non-negative function on \( \mathbb{R}^2 \). If there exist constants \( s > 1 \) and \( C(w, s) > 0 \) such that for all \( r > 0 \) and \( x \in \mathbb{R}^2 \),

\[
\left( \int_{B(x,r)} |w(y)|^{ps} \, dy \right)^{1/(ps)} \leq C(w, s),
\]

then there exists positive constant \( C \) such that for all \( f \in W^{1,p}(\mathbb{R}^2) \), \( fw \in L^p(\mathbb{R}^2) \) and \( \|fw\|_{L^p(\mathbb{R}^2)} \leq C \|f\|_{W^{1,p}(\mathbb{R}^2)} \).

For any \( c > 0 \), set \( \Delta_c \equiv \{(x_1, x_2) : 0 < x_1 < c, 0 < x_2 < x_1\} \). For any real-valued function \( u \) on \( \mathbb{R}^2 \), define \( u_c(x_1, x_2) \equiv u(x_1, x_2)(x_2/x_1)^\chi_{\Delta_c} \). Then we have the following result; see [10, Lemma 2.4] and also [6, p. 75]

Lemma 2.2. Let \( 1 \leq p < 2 \). Then there exists constant \( C \) such that for all \( u \in W^{1,p}(\mathbb{R}^2) \) and \( 0 < c \leq 1 \),

\[
\|u_c\|_{W^{1,p}(\Delta_c)} \leq Cc^{1/p} \|u\|_{L^p(\Delta_c)} + C\|u\|_{W^{1,p}(\Delta_c)}.
\]

Similarly to Lemma 3 of [10], we have the following conclusion.

Lemma 2.3. There exists a positive constant \( C \) and a sequence of functions, \( \{v_m\}_{m=0}^{\infty} \subset W^{1,p}(\mathbb{R}^2) \), such that \( v_m(x) = 1 \) if \( x \in \Gamma_m \) and \( v_m(x) = 0 \) if \( x_1 \leq 0 \); moreover,

\[
\|v_m\|_{W^{1,p}(\mathbb{R}^2)} \leq Ca^{(2/p-1)m}.
\]

Proof. Let \( v_0 \in W^{1,p}(\mathbb{R}^2) \) such that \( v_0(x) = 0 \) if \( x_1 \leq 0 \) and \( v_0(x) = 1 \) if \( 0 \leq x_2 \leq x_1 \) and \( a \leq |x| \leq 1 \). Set \( v_m(x) \equiv v_0(a^{-m}x) \) for \( m \in \mathbb{N} \). Then \( v_m(x) = 1 \) if \( x \in \Gamma_m \) and \( v_m(x) = 0 \) if \( x_1 = 0 \) for \( m \in \mathbb{N} \cup \{0\} \). Moreover,

\[
\|v_m\|_{W^{1,p}(\mathbb{R}^2)} \lesssim a^{2m-mp}\|v_0\|_{L^p(\mathbb{R}^2)} + a^{2m-mp}\|
abla v_0\|_{L^p(\mathbb{R}^2)} \lesssim a^{2m-mp},
\]

which completes the proof of Lemma 2.3.

Let \( R_{h,d} \equiv (0, h) \times [0, d] \) for \( 0 < d, h \leq 1 \). Let \( E, F \subset \overline{R_{h,d}} \) be disjoint continua connecting the vertical sides of \( R_{h,d} \). The following result has been proved in [10, Lemma 4].

Lemma 2.4. Let \( 1 < p \leq \infty \). Then for all \( u \in W^{1,p}(\mathbb{R}^2) \) with \( u(x) = 1 \) if \( x \in E \) and \( u(x) = 0 \) if \( x \in F \), \( \|u\|_{W^{1,p}(\mathbb{R}^2)} \geq h^{1/p}d^{1/p-1} \).

Lemma 2.5. If \( \infty \geq p \geq q \equiv 2 + \log_a 2 \), then \( G_q \notin \text{Ext}_p \).

Proof. Assume that \( G_q \notin \text{Ext}_p \). Notice that quasi-isometry keeps the space \( W^{1,p}(\mathbb{R}^2) \) invariant under the change of the variable. By this and Lemma 2.3, there exists a sequence of functions, \( \{v_m\}_{m \in \mathbb{N} \cup \{0\}} \subset W^{1,p}(\mathbb{R}^2) \), such that \( v_m(x) = 1 \) if \( x \in \varphi(\Gamma_m) \) and \( v_m(x) = 0 \) if \( x \in (G_q)_- \) or \( x_1 \leq 0 \); moreover, \( \|v_m\|_{W^{1,p}(G_q)} \lesssim a^{(2-p)m} \), where \( \varphi \) is as in the construction of the domain \( G_q \). Let \( u_m \) be an extension of \( v_m \). Then, \( \|u_m\|_{W^{1,p}(\mathbb{R}^2)} \lesssim a^{(2-p)m} \).
On the other hand, since \( u_m(x) = 1 \) if \( x \in \varphi(\Gamma_m^k) \) and \( u_m(x) = 0 \) if \( x_1 = 0 \), by Lemma 2.4 and \( 2a^{2-p} \geq 1 \),

\[
\|u_m\|_{W^1, r(\mathbb{R}^2)}^p \gtrsim \sum_{k=0}^{m+1} 2^{k-1} \sum_{i=0}^{2^{k-1}-1} \left| I_m^{k, i} \right| \left| I_m^{k, i} \right|^{1-p} \\
\gtrsim \sum_{k=0}^{m+1} 2^k a^{(2-p)(m+k)} \gtrsim a^{(2-p)m} \sum_{k=0}^{m+1} 2^k a^{(2-p)k} \gtrsim ma^{(2-p)m}.
\]

This is a contradiction, which completes the proof of Lemma 2.5. \( \Box \)

**Lemma 2.6.** Let \( 1 \leq p < q \equiv 2 + \log_a 2 \). Set

\[
w \equiv \sum_{m=0}^{\infty} \sum_{k=0}^{m+1} 2^{k-1} \sum_{i=0}^{2^{k-1}-1} a^{-(k+m)} \chi_{I_m^{k, i}}.
\]

If \( 1 < s < 2/p \) is such that \( 2a^{2-sp} < 1 \), then \( w \) satisfies (2.3).

**Proof.** If \( r \geq a/2 \), then for all \( x \in \mathbb{R}^2 \), then by \( 2a^{2-sp} < 1 \), we have

\[
\int_{B(x, r)} [w(y)]^{sp} dy \leq \sum_{m=0}^{\infty} \sum_{k=0}^{m+1} 2^{k-1} \sum_{i=0}^{2^{k-1}-1} a^{-(k+m)sp} |T_m^{k, i}|
\]

\[
\leq \sum_{m=0}^{\infty} \sum_{k=0}^{m+1} 2^k a^{(k+m)(2-sp)} \lesssim \sum_{m=0}^{\infty} a^{m(2-sp)} \lesssim 1 \lesssim r^{2-sp}.
\]

Similarly, it is easy to see that

\[
\int_{B(0, a^n)} [w(y)]^{sp} dy \leq \sum_{m=n}^{\infty} \sum_{k=0}^{m+1} 2^{k-1} \sum_{i=0}^{2^{k-1}-1} a^{-(k+m)sp} |T_m^{k, i}| \lesssim a^{n(2-sp)}.
\]

If \( r < a/2 \) and \( w(y) \neq 0 \) for all \( y \in B(x, r) \), then \( |y| \leq 1 \) and \( |x| < 1 + r \). For \( 1 \leq |x| < 1 + r \), observing that \( w(y) \lesssim 1 \) for all \( y \in B(x, r) \), we then have (2.3). Assume now that \( a^{n+1} \leq |x| < a^n \) for certain \( n \in \mathbb{N} \cup \{0\} \). If \( r \geq (1 - a)a^{n+1}/2 \), similarly to above computations, then we have

\[
\int_{B(x, r)} [w(y)]^{sp} dy \leq \int_{B(0, r+2r/a(1-a))} [w(y)]^{sp} dy \lesssim a^{n(2-sp)} \lesssim r^{2-sp}.
\]

If \( r < (1 - a)a^{2(n+1)}/2 \), then for all \( y \in B(x, r) \), \( w(y) \lesssim a^{-2n} \), and hence,

\[
\int_{B(x, r)} [w(y)]^{sp} dy \lesssim a^{-2nsp}r^2 \lesssim r^{2-sp}.
\]
If \((1-a)a^{2(n+1)/2} \leq r < (1-a)a^{n+1}/2\), then \((1-a)a^{n+k_0+1}/2 \leq r < (1-a)a^{n+k_0}/2\) for some \(1 \leq k_0 \leq n+1\), and thus \(B(x, r)\) contains at most \(2^{k-k_0}\) many of the \(T_{m}^{k,i}\) for each \(m = n-1, n, n+1\) and \(k \geq k_0\), which implies that

\[
\int_{B(x, r)} |u(y)|^{sp} dy \leq \sum_{k=k_0-1}^{n+2} 2^{k-k_0} a^{-(n+k)sp} a^{2(n+k)} \lesssim a^{(2-sp)(n+k_0)} \lesssim r^{2-sp}.
\]

This finishes the proof of Lemma 2.6. \(\square\)

**Lemma 2.7.** If \(1 \leq p < q \equiv 2 + \log_{a} 2\), then \(G_{q} \in \text{Ext}_{p}\).

**Proof.** Notice that quasi-isometries keep the space \(W^{1,p}(\mathbb{R}^2)\) invariant under the change of the variable. By this and the symmetry of \(G_{q}\) with respect to \(x_1\)-axis, we only need to prove that for any \(u \in W^{1,p}(R \setminus T)\), there exists a function \(\overline{u} \in W^{1,p}(R)\) such that \(\overline{u}(x) = 0\) for almost all \(x \in (0,1) \times \{0\}\) and \(\overline{u}(x) = u(x)\) for almost all \(x \in R \setminus T\). Since the boundary of \(R \setminus T\) is Lipschitz, there exists a bounded extension operator \(\text{Ext} : W^{1,p}(R \setminus T) \to W^{1,p}(\mathbb{R}^2)\); see [10, p. 725]. Let \(v \equiv \text{Ext}(u)\) and \(T_{m}^{k,i}\) be the interior of \(T_{m}^{k,i}\). Now we will obtain \(\overline{u}\) by redefining \(v\) on \(T\). In fact, applying Lemma 2.2 to \(v\) on each \(T_{m}^{k,i}\), we obtain a function \(u_{m}^{k,i}\) such that \(u_{m}^{k,i}(x) = v(x)\) for all \(x \in T_{m}^{k,i}\) and \(u_{m}^{k,i}(x) = 0\) for all \(x \notin T_{m}^{k,i}\), and moreover,

\[
\|u_{m}^{k,i}\|_{W^{1,p}(T_{m}^{k,i})} \lesssim a^{-(k+m)p}\|v\|_{L^{p}(T_{m}^{k,i})}^{p} + \|v\|_{W^{1,p}(T_{m}^{k,i})}^{p}.
\]

Set

\[
\overline{u} \equiv u\chi_{R\setminus T} + \sum_{m=0}^{\infty} \sum_{k=0}^{2^{k}-1} \sum_{i=0}^{2^{k}-1} u_{m}^{k,i} \chi_{T_{m}^{k,i}}.
\]

Then

\[
\|\overline{u}\|_{W^{1,p}(R)} \lesssim \|u\|_{W^{1,p}(R\setminus T)} + \sum_{m=0}^{\infty} \sum_{k=0}^{2^{k}-1} \sum_{i=0}^{2^{k}-1} \|u_{m}^{k,i}\|_{W^{1,p}(T_{m}^{k,i})}^{p} + \|w\|_{L^{p}(\mathbb{R}^2)}^{p} + \|u\|_{W^{1,p}(R\setminus T)}^{p},
\]

where

\[
w \equiv \sum_{m=0}^{\infty} \sum_{k=0}^{2^{k}-1} \sum_{i=0}^{2^{k}-1} a^{-(k+m)p} \chi_{T_{m}^{k,i}}.
\]

Since \(2a^{2-p} < 1\), we can find \(1 < s < 2/p\) such that \(2a^{2-sp} < 1\). By Lemma 2.6, we know that \(w\) satisfies (2.3). Then, by Lemma 2.1, we have that

\[
\|vw\|_{L^{p}(\mathbb{R}^2)} \lesssim \|v\|_{W^{1,p}(\mathbb{R}^2)} \lesssim \|u\|_{W^{1,p}(R\setminus T)},
\]

which further implies that \(\|\overline{u}\|_{W^{1,p}(R)} \lesssim \|u\|_{W^{1,p}(R\setminus T)}\), This finishes the proof of Lemma 2.7. \(\square\)
So far, for \(1 < q < 2\), we have already proved that \(G_q \in \text{Ext}_p\) if and only if \(1 \leq p < q\). To prove the extendability properties for the domain \(\mathbb{R}^2 \setminus G_q\), we need the following two auxiliary conclusions.

**Lemma 2.8.** The mapping \(\varphi\) from \(\{x \in \mathbb{R}^2 : 0 \leq x_1 \leq a\}\) to itself given by \(\varphi(x_1, x_2) \equiv (x_1, x_2 + x_1^2)\) is bi-Lipschitz.

**Proof.** In fact, since \(0 \leq |x_1^2 - y_1^2| \leq 2a|x_1 - y_1|\) for all \(0 \leq x_1, y_1 \leq a\), we have

\[
(1 - 2a)|x - y| \leq |x - y| - |x_1^2 - y_1^2|
\]

\[
\leq |\varphi(x) - \varphi(y)| \leq |x - y| + |x_1^2 - y_1^2| \leq (1 + 2a)|x - y|,
\]

which completes the proof of Lemma 2.8. \(\square\)

We always write \(\gamma(x, y) \subset D\) for a rectifiable curve joining \(x\) and \(y\) in a domain \(D \subset \mathbb{R}^2\). By abuse of notation, we also sometimes use \(\gamma\) to denote \(\gamma(x, y)\). Denote by \(\ell(\gamma)\) the arc length of \(\gamma\), \(\gamma(s)\) the arc length representation of \(\gamma\), \(\gamma(0) = x\) and \(\gamma(\ell(\gamma)) = y\). If \(g\) is a real-valued function in \(D\), we let

\[
\int_{\gamma(x, y)} g(z) |dz| \equiv \int_0^{\ell(\gamma)} g(\gamma(s)) \, ds
\]

be the line integral of \(g\) along \(\gamma\) whenever the integral exists.

A domain \(D\) is called a Lip\(_a\)-extension domain if for any pair of points \(x, y \in D\), there exists a curve \(\gamma(x, y) \subset D\) such that

\[
(2.4) \quad \int_{\gamma(x, y)} [\text{dist}(z, \partial D)]^{a-1} |dz| \leq C|x - y|^a,
\]

where \(C\) is a positive constant independent of \(x\) and \(y\); see [4].

Then by [8, Theorem A and Corollary 4.1] and [3, Theorem 5.2], we have the following conclusion.

**Lemma 2.9.** Let \(p > 2\).

(i) If \(D\) is a simply connected planar domain and \(D \in \text{Ext}_p\), then \(D\) is Lip\(_{(p-2)/(p-1)}\)-extension domain.

(ii) If \(D\) is Lip\(_{(p-2)/(p-1)}\)-extension domain, then \(D \in \text{Ext}_s\) for all \(s > p\).

**Lemma 2.10.** If \(1 < q < 2\) and \(1 \leq p \leq q/(q - 1)\), then \(\mathbb{R}^2 \setminus G_q \notin \text{Ext}_p\).

**Proof.** By Theorem 6.4 of [2] for \(1 < p \leq 2\), if \(D\) is a \(W^{1,p}\) extension domain, then \(D\) has the property LLC(2), namely, there exists a constant \(c \geq 1\) such that for all \(z \in \mathbb{R}^2\) and \(r > 0\), any pair of points \(x, y \in D \setminus B(z, r)\) can be joined in \(D \setminus B(z, r/c)\). For \(p = 1\), we claim that \(\mathbb{R}^2 \setminus G_q \in \text{Ext}_1\) implies that \(G_q\) is a quasiconvex domain. Assume this for the moment. Then by [7], \(G_q\) is a bounded turning domain, which together with [7, Theorem 4.5] further implies that \(\mathbb{R}^2 \setminus G_q\) has the LLC(2) property.

So for \(1 \leq p \leq 2\), the proof of Lemma 2.10 is reduced to proving that \(\mathbb{R}^2 \setminus G_q\) does not have the property LLC(2). To see this, obviously, for any fixed positive constant \(c\), we
always find \( m \) large enough such that \( a^{2m} \leq (1 - a)a^m/(cN) \), where \( N \) is a fixed positive constant such that \( \varphi(T_m^{m+1,0}) \subset B((a^m, 0), Na^{2m}) \). Thus the pair of points \((a^{m+1}, 0)\) and \((a^{m-1}, 0)\), which lie in \( \mathbb{R}^2 \setminus G_q \) but not in \( B((a^m, 0), (1 - a)a^{m+1}) \), cannot be joined in \( (\mathbb{R}^2 \setminus G_q) \setminus B((a^m, 0), Na^{2m}) \) and thus not in \( (\mathbb{R}^2 \setminus G_q) \setminus B((a^m, 0), (1 - a)a^{m+1}/c) \). This implies that \( \mathbb{R}^2 \setminus G_q \) does not have the property LLC(2) and thus \( \mathbb{R}^2 \setminus G_q \notin \text{Ext}_p \) for any \( 1 \leq p \leq 2 \).

Now we turn to prove the above claim that \( \mathbb{R}^2 \setminus G_q \in \text{Ext}_1 \) implies that \( G_q \) is a quasiconvex domain. To this end, we first observe that for any \( \alpha \leq \eta \leq \rho(1 - \alpha) \), by Lemma 2.9, it suffices to prove that \( E_{\eta, \rho} = \varphi(T_m^{m+1,0}) \). By the assumption \( \alpha \leq \eta \leq \rho(1 - \alpha) \), we know that the complement domain \( E \) is a uniform domain, applying [9, Corollary 1.2], we know that the complement domain \( \mathbb{R}^2 \setminus \{ \mathbb{R}^2 \setminus G_q \} \), \( u(1 - \eta) \in W^{1,p}(E_{\eta, \rho}) \), then \( u(1 - \eta) \in W^{1,p}(E_{\eta, \rho}) \) and

\[
\|u(1 - \eta)\|_W^{1,p}(\mathbb{R}^2 \setminus G_q) \leq \|u\|_W^{1,p}(E_{\eta, \rho}).
\]

By the assumption \( \mathbb{R}^2 \setminus G_q \in \text{Ext}_p \), we have that \( \text{Ext}(u(1 - \eta)) \in W^{1,p}(\mathbb{R}^2) \) and

\[
\|\text{Ext}(u(1 - \eta))\|_W^{1,p}(\mathbb{R}^2) \leq \|u(1 - \eta)\|_W^{1,p}(\mathbb{R}^2 \setminus G_q) \leq \|u\|_W^{1,p}(E_{\eta, \rho}).
\]

Since \( S \) is a uniform domain, \( u(1 - \eta) \) can be extended to the entire \( \mathbb{R}^2 \) (see [2, p. 9]). The extension \( \text{Ext}(u(1 - \eta)) \), satisfies \( \|\text{Ext}(u(1 - \eta))\|_W^{1,p}(\mathbb{R}^2) \leq \|u(1 - \eta)\|_W^{1,p}(\mathbb{R}^2) \leq \|u\|_W^{1,p}(E_{\eta, \rho}) \). Obviously \( \text{Ext}(u(1 - \eta)) = \text{Ext}(u) \) coincides with \( u \) on \( E_{\eta, \rho} \), which implies \( E_{\eta, \rho} \in \text{Ext}_p \). Then an argument similar to but easier than the above shows that \( E_{\eta, \rho} \) is a quasiconvex domain, further implies that \( (\mathbb{R}^2 \setminus G_q) \) is quasiconvex, and thus \( G_q \) is a quasiconvex domain. This proves the above claim.

For \( 2 < p \leq q/(q - 1) \), since \( \mathbb{R}^2 \setminus G_q \in \text{Ext}_p \) implies \( E_{\eta, \rho} \in \text{Ext}_p \) as above, to prove Lemma 2.10, by Lemma 2.9, it suffices to prove that \( E_{\eta, \rho} \) is not a \( \text{Lip}_\alpha \)-extension domain for any \( 0 < \alpha < 2 - q \).

To see this, choose \( N \in \mathbb{N} \), and \( x = (a^m, 0) \) and \( y = (a^{m-1}, 0) \). Then for any \( \gamma(x, y) \subset E_{\eta, \rho} \), take \( \hat{\gamma} \) to be the component of \( \gamma \cap \{z_1, z_2\} : 0 \leq z_1 \leq a \} \) containing \( x \). Obviously, \( \{a^m, a^{m-1}\} \subset \{z_1 : (z_1, z_2) \in \hat{\gamma}\} \), and without loss of generality, we may assume that \( z_2 \geq 0 \) for all \( z_1, z_2 \in \hat{\gamma} \). Moreover, for all \( z \in \hat{\gamma} \), by Lemma 2.8,

\[
\text{dist}(z, \partial G_q) = \text{dist}(z, \varphi(\Gamma \setminus \Gamma_0)) \sim \text{dist}(\varphi^{-1}(z), \Gamma \setminus \Gamma_0).
\]

Assume that \( (z_1, 0) \in T_m^{k, i} \). Then

\[
\text{dist}(\varphi^{-1}(z), \Gamma \setminus \Gamma_0) = \text{dist}(\varphi^{-1}(z), \Gamma_m^{k, i}) \leq \text{dist}((z_1, -z_2^2), \Gamma_m^{k, i}) \leq a^{2(m-1)} + \Gamma(z_1),
\]

where \( (z_1, \Gamma(z_1)) \in \Gamma \). Since \( 2a^\alpha \geq 2a^{2-q} = 1 \), we have

\[
\int_{\gamma(x, y)} [\text{dist}(z, \partial G_q)]^{-\alpha}|dz| \gtrsim \int_{a^m} [a^{2(m-1)} + \Gamma(z_1)]^{-\alpha} dz_1
\]
If \( \text{Lemma 2.9, it suffices to prove that} \)
\( \phi_z \text{dist (} R \cap E G \text{)} \)

Then there exist curves \( R < \alpha \equiv \gamma \text{ | (2.7)} \)

\( \gamma (2.5) \)

\( \phi \text{Lemma 2.11.} \)

2.10.\)

\( \text{Proof.} \) By Lemma 2.9, it suffices to prove that \( \mathbb{R}^2 \setminus \overline{G_q} \in \text{Ext}_s. \)

\( \text{Proof.} \) By Lemma 2.9, it suffices to prove that \( \mathbb{R}^2 \setminus \overline{G_q} \) is Lip\( _\alpha \)-extension domain for all \( \alpha > 2 - q \). Let \( \varphi_-(T \setminus T_0) \) be the reflection of \( \varphi_-(T \setminus T_0) \) with respect to \( x_1 \)-axis, namely, \( \varphi_-(T \setminus T_0) \equiv \{(x_1, -x_2) : (x_1, x_2) \in \varphi(T \setminus T_0)\} \) and for \( m \in \mathbb{N} \cup \{0\} \)

\( E_m \equiv \varphi(T \setminus \bigcup_{n=0}^m T_n) \cup \varphi_-(T \setminus \bigcup_{n=0}^m T_n) \cup \{(x_1, x_2) : |x_2| \leq x_1^2, 0 \leq x_1 < a^m\}. \)

Then \( G_q \cup E_m \) is a Jordan domain with Lipschitz boundary since \( \Gamma(z_1) \) is Lipschitz function. Obviously, \( \mathbb{R}^2 \setminus \overline{G_q} = E_1 \cup (\mathbb{R}^2 \setminus \overline{G_q \cup E_2}). \) Then the proof of Lemma 2.11 is reduced to proving that for any \( x, y \in E_1 \), there exists a curve \( \gamma(x, y) \subset E_1 \) such that

\[
\int_{\gamma(x,y)} \text{dist (} z, \varphi(T) \cup \varphi_-(T)\text{)} |dz| \lesssim |x - y|^\alpha.
\]

Assuming that (2.5) holds for the moment, we now establish Lemma 2.11. Since \( \text{dist (} z, \varphi(T) \cup \varphi_-(T)\text{)} = \text{dist (} z, \partial G_q \text{)} \) for all \( z \in E_1 \), then for any \( x, y \in E_1 \), there exists a curve \( \gamma(x, y) \subset E_1 \) such that

\[
\int_{\gamma(x,y)} \text{dist (} z, \partial G_q\text{)} |dz| \lesssim |x - y|^\alpha.
\]

Obviously, \( \mathbb{R}^2 \setminus \overline{G_q \cup E_2} \) is a uniform domain and thus Lip\( _\alpha \)-extension domain for all \( 0 < \alpha \leq 1 \); see [4]. Thus for any \( x, y \in \mathbb{R}^2 \setminus \overline{G_q \cup E_2} \), there exists a curve \( \gamma(x, y) \subset \mathbb{R}^2 \setminus \overline{G_q \cup E_2} \) satisfying (2.6) with \( \text{dist (} x, \partial G_q\text{)} \) replaced by \( \text{dist (} x, \partial (G_q \cup E_2)\text{)} \). Observe that for all \( x \in \mathbb{R}^2 \setminus (G_q \cup E_2) \), \( \text{dist (} x, \partial (G_q \cup E_2)\text{)} \) \( \leq \text{dist (} x, \partial G_q\text{)}, \) which implies that \( \gamma(x, y) \subset \mathbb{R}^2 \setminus \overline{G_q} \) satisfies (2.6). For any \( x \in E_1 \) and \( y \in \mathbb{R}^2 \setminus (G_q \cup E_1) \), assume that there exists a point \( w \in J \equiv \{(a, y_2) : (a, y_2) \in \partial E_1\} \) such that

\[
|x - y| \sim |x - w| + |w - y|.
\]

Then there exist curves \( \gamma_1(x, w) \subset E_0 \) and \( \gamma_2(w, y) \subset \mathbb{R}^2 \setminus \overline{G_q \cup E_1} \) satisfying (2.6). Let \( \gamma \equiv \gamma_1 \cup \gamma_2 \). Therefore, \( \gamma(x, y) \subset \mathbb{R}^2 \setminus \overline{G_q} \) satisfies (2.6). To see (2.7), if \( y_1 \geq 1 \), since \( |x_2| \leq a^2 + ba \), then \( |x_1 - y_1| > 1 - a \) and

\[
|y - (a, 0)| + |(a, 0) - x| \leq |y - x| + 2|a, 0 - x| \leq |x - y| + 4a \lesssim |x - y|,
\]
which implies (2.7) with \( w \equiv (a, 0) \). Set
\[
f(x, y) \equiv |x - y|^{-1} \inf_{w \in f} \{|x - w| + |w - y|\}.
\]

Obviously, \( f(x, y) \geq 1 \) whenever defined. Moreover, \( f \) is continuous on the bounded closed set
\[
\{(x, y) \in \mathbb{R}^2 : x \in \overline{E}_1, y \in \overline{E}_0 \setminus \overline{E}_1, |x - y| \geq a^4/4\},
\]
which implies \( f \) is bounded on this set and thus (2.7) holds for \( (x, y) \) in this set. Finally, if \( x \in \overline{E}_1, y \in \overline{E}_0 \setminus \overline{E}_1 \) and \( |x - y| < a^4/4 \), then it is easy to see that (2.7) holds. Thus, so far, we proved that the claim (2.7) is true, and therefore, except (2.5), we have finished (2.9) \( \text{dist} (z, \Gamma) = \text{dist} (z, \Gamma_{m}) \sim \Gamma(z_1) - z_2 \).

Moreover, by the choices of this proof of Lemma 2.11. In fact, assume this for the moment. In general, we assume that \( \phi \) exists a curve \( \gamma \equiv \gamma(z, u) \subset \Gamma \). Obviously, \( \phi \) is bi-Lipschitz, the curve \( \phi(\gamma)(x, y) \subset \phi(\gamma(D \setminus \Gamma)) \) also satisfies (2.5). A similar argument applies to any \( x, y \in \phi(\gamma(D \setminus \Gamma)) \) and \( y \in \phi(\gamma(D \setminus \Gamma)) \), letting \( w \) be the intersection of the \( x_1 \)-axis and the line joining \( x \) and \( y \), we have that \( w \in \overline{E}_1 \) and \( |x - y| \sim |x - w| + |w - y| \).

Applying similar arguments to \( x, w \) and \( w, y \), we obtain the curves \( \gamma_1(x, w) \subset E_0 \) and \( \gamma_2(w, y) \subset E_0 \) satisfying (2.5). Taking \( \gamma \equiv \gamma_1 \cup \gamma_2 \) gives the desired result.

To prove (2.8), we consider three cases.

Case 1. \( x, y \in D_{m}^{k,i} \equiv T_{m}^{k,i} \cup \{(x_1, x_2) : (x_1, 0) \in T_{m}^{k,i}, -x_2^2 \leq x_2 \leq 0\} \) for \( m \in \mathbb{N} \).

It suffices to verify that if \( x_1 = y_1 \) or \( x_2 = y_2 \), then there exists \( \gamma(x, y) \subset D \) satisfying (2.8). In fact, assume this for the moment. In general, we assume that \( x_i \neq y_i \) for \( i = 1, 2 \), and we may further assume that \( x_1 < y_1 \) without loss of generality. If \( x_2 < y_2 \), then let \( z \equiv (y_1, x_2) \), and if \( x_2 > y_2 \geq -x_1^2 \), then let \( z \equiv (x_1, y_2) \). Obviously, \( z \in D_{m}^{k,i} \) and \( |x - y| \sim |x - z| + |z - y| \).

Moreover, by the choices of \( z \) and the assumptions, there exist curves \( \gamma_1 \) and \( \gamma_2 \) such that \( \gamma_1(x, z) \subset D \) and \( \gamma_2(z, y) \subset D \) satisfy (2.8), respectively. Taking \( \gamma \equiv \gamma_1 \cup \gamma_2 \), we know that \( \gamma(x, y) \subset D \) satisfies (2.8). If \( y_2 < -x_1^2 \), then let \( z \equiv (x_1, -x_2^2) \) and \( u \equiv (y_1, -x_2^2) \). Obviously, \( z, u \in D_{m}^{k,i} \) and \( |x - y| \sim |x - z| + |z - u| + |u - y| \).

Moreover, by the choices of \( z \) and \( u \) and the assumptions, there exist curves \( \gamma_1 \) and \( \gamma_2 \) such that \( \gamma_1(x, z) \subset D \), \( \gamma_2(z, u) \subset D \), and \( \gamma_3(u, y) \subset D \) satisfy (2.8), respectively. Taking \( \gamma \equiv \gamma_1 \cup \gamma_2 \cup \gamma_3 \), we have that \( \gamma(x, y) \subset D \) satisfies (2.8).

Now assume that \( x_1 = y_1 \) or \( x_2 = y_2 \). If \( x_1 = y_1 \), taking \( \gamma \) to be the line segment joining \( x \) and \( y \), we have that \( \gamma(x, y) \subset D_{m}^{k,i} \). Since
\[
\text{dist} (z, \Gamma) = \text{dist} (z, \Gamma_{m}) \sim \Gamma(z_1) - z_2
\]
for all $z$ with $(z_1, 0) \in T_{m}^{k,i}$ and $z_2 < \Gamma(z_1) = \Gamma(x_1)$, we have

$$
\int_{\gamma} \|\text{dist}(z, \Gamma)\|^{\alpha-1} |dz| \sim \int_{\gamma} |\Gamma(x_1) - z_2|^{\alpha-1} |dz|
\lesssim \int_{0}^{\|x_2 - y_2\|} t^{\alpha-1} dt \lesssim \|x_2 - y_2\|^{\alpha} \sim |x - y|^{\alpha}.
$$

If $x_2 = y_2$, taking $\gamma$ to be the line segment joining $x$ and $y$, we have that $\gamma(x, y) \subset D_{m}^{k,i}$. Moreover, we have

$$
\int_{\gamma} \|\text{dist}(z, \Gamma)\|^{\alpha-1} |dz| \sim \int_{\gamma} |\Gamma(z_1) - x_2|^{\alpha-1} |dz|
\lesssim \int_{0}^{\|x_1 - y_1\|} t^{\alpha-1} dt \lesssim \|x_1 - y_1\|^{\alpha} \sim |x - y|^{\alpha}.
$$

**Case 2.** $x \in D_{m}^{k,i}$ and $y \in D_{m}^{k,j}$, where $T_{n}^{k,j}$ is adjacent to $T_{m}^{k,i}$. Let $\{(w_1, 0)\} = T_{m}^{k,i} \cap T_{n}^{k,j}$ and assume that $x_1 < y_1$.

If $x_2 < 0$, letting $u \equiv (y_1, x_2)$ and $w \equiv (w_1, x_2)$, then $|x - y| \sim |x - w| + |w - u| + |u - y|$. By Case 1, there exists curve $\gamma_1(x, u) \subset D$, $\gamma_2(w, u) \subset D$ and $\gamma_3(u, y) \subset D$ satisfying (2.8), respectively. Taking $\gamma \equiv \gamma_1 \cup \gamma_2 \cup \gamma_3$, we know that $\gamma(x, y) \subset D$ satisfies (2.8).

If $x_2 \geq 0$ and $y_2 \geq 0$, noticing that $x_2 \leq w_1 - x_1$ and $y_2 \leq y_1 - w_1$, and letting $u \equiv (x_1, -c)$, $w = (w_1, -c)$ and $z \equiv (y_1, -c)$, where $0 < c \leq \min\{x_2, y_2, x_1^2\}$, we conclude that

$$
|x_2 + c| + |y_2 + c| \leq 2|y_1 - w_1| + 2|x_1 - w_1| \sim |x_1 - y_1|,
$$

which implies that

$$
|x - y| \sim |x - u| + |u - w| + |w - z| + |z - y| \sim |x_1 - y_1|.
$$

Thus, there exist curves $\gamma_1(x, u) \subset D$, $\gamma_2(u, w) \subset D$, $\gamma_3(w, z) \subset D$ and $\gamma_4(z, y) \subset D$ satisfying (2.8), respectively. Taking $\gamma \equiv \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, we know that $\gamma(x, y) \subset D$ satisfies (2.8).

If $x_2 \geq 0$ and $-x_1^2 \leq y_2 < 0$, then similarly to the proof for the case $x_1 < 0$, we obtain a curve $\gamma(x, y) \subset D$ that satisfies (2.8).

If $x_2 \geq 0$ and $-x_1^2 > y_2 \geq -y_1^2$, letting $u \equiv (y_1, -x_1^2)$, and $z \equiv (x_1, -x_1^{-2})$, we have that $|x - y| = |x - z| + |z - u| + |u - y|$. Since there exist curves $\gamma_1(x, z) \subset D$, $\gamma_1(z, u) \subset D$ and $\gamma_3(u, y) \subset D$ satisfying (2.8), respectively, taking $\gamma \equiv \gamma_1 \cup \gamma_2 \cup \gamma_3$ gives the desired curve.

**Case 3.** $x \in D_{m}^{k,i}$ and $y \in D_{m}^{k,j}$, where $T_{n}^{k,j}$ is not the one adjacent to $T_{m}^{k,i}$. We may assume that $x_1 < y_1$ without loss of generality.

Let $(u_1, 0)$ be the right endpoint of $I_{m}^{k,i}$ and $(z_1, 0)$ be the left endpoint of $I_{n}^{k,j}$. Then $-x_1^2 \leq x_2 \leq u_1 - x_1$ and $-y_1^2 \leq y_2 \leq y_1 - z_1$. Since $x_2^2 + y_2^2 \lesssim a^{2m} + a^{2n} \lesssim |u_1 - z_1| \lesssim |x_1 - y_1|$, we know that $|x_2 - y_2| \lesssim |x_1 - y_1|$, which implies that $|x - y| \sim |x_1 - y_1|$ and $|x - (u_1, 0)| + |(u_1, 0) - (z_1, 0)| + |(z_1, 0) - y| \lesssim |x - y|$. 


Let $\gamma_2$ be the line segment joining the pair of points $u = (u_1, -x_1^2)$ and $z = (z_1, -x_1^2)$. Then we claim that

$$\int_{\gamma_2} [\text{dist} (v, \Gamma)]^{\alpha-1} dv \lesssim |u_1 - z_1|^{\alpha}.$$  

Assume that (2.10) holds for the moment. Then by Case 1, there exist curves $\gamma_1(x, u) \subset D$ and $\gamma_3(z, y) \subset D$ satisfying (2.8), respectively. Thus the curve $\gamma \equiv \gamma_1 \cup \gamma_2 \cup \gamma_3$ is as desired.

Now we show (2.10). If $m = n$, let $|I^{k_0, i_0}_m|$ be the largest for the subintervals from our construction contained in $\tilde{\gamma}_2 \equiv \gamma_2 + (0, x_1^2)$. Then $|u_1 - z_1| \sim a^{m+k_0}$ and $\tilde{\gamma}_2 \subset \cup_{k=k_0}^{m+1} \cup_{j=1}^{k-k_0} I^{k, i_j}_m$. Since $\text{dist} (v, \Gamma) \sim \Gamma(v_1)$ for all $v \in I^{k, i}_m$, by Case 1, we have

$$\int_{I^{k, i}_m} [\text{dist} (v, \Gamma)]^{\alpha-1} dv \lesssim a^{(m+k)}.$$  

Thus, by $2a^\alpha < 1$, i.e. $\alpha > -\log_2 2 = 2 - q$, we have

$$\int_{\tilde{\gamma}_2} [\text{dist} (v, \Gamma)]^{\alpha-1} dv \leq \int_{\gamma_2} [\text{dist} (v, \Gamma)]^{\alpha-1} dv \lesssim \sum_{k=k_0}^{m+1} 2^{k-k_0} a^{\alpha(m+k)} \lesssim a^{\alpha(m+k_0)} \sum_{k=0}^{m-k_0+1} 2^{k} a^{\alpha(k)} \lesssim a^{\alpha(m+k_0)} \lesssim |u_1 - z_1|^{\alpha}.$$  

If $m = n + 1$, then $|u_1 - z_1| = |a^m - u_1| + |a^n - z_1|$, and thus, by the above estimate in the case $m = n$,

$$\int_{\tilde{\gamma}_2} [\text{dist} (v, \Gamma)]^{\alpha-1} dv \leq \int_{\gamma_2} [\text{dist} (v, \Gamma)]^{\alpha-1} dv \lesssim \left( \int_{\tilde{\gamma}_2 \cap [a^n, a^n]} + \int_{\tilde{\gamma}_2 \cap [a^n, a^{n-1}]} \right) [\text{dist} (v, \Gamma)]^{\alpha-1} dv \lesssim |a^m - u_1|^{\alpha} + |z_1 - a^n|^{\alpha} \lesssim |u_1 - z_1|^{\alpha}.$$  

Similarly, if $m \geq n + 2$, then $|u_1 - z_1| \sim a^n$, and thus

$$\int_{\tilde{\gamma}_2} [\text{dist} (v, \Gamma)]^{\alpha-1} dv \lesssim \sum_{m'=n}^{m} \int_{\tilde{\gamma}_2 \cap [a^{m'}, a^{m'-1}]} [\text{dist} (v, \Gamma)]^{\alpha-1} dv \lesssim \sum_{m'=n}^{m} a^{m'\alpha} \lesssim a^{n\alpha} \lesssim |u_1 - z_1|^{\alpha}.$$  

This shows (2.10) and finishes the proof of Lemma 2.11.

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