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Abstract Let $1 < q < 2$. In this paper, we construct a Jordan domain $G_q \subset \mathbb{R}^2$ such that $G_q \in \mathrm{Ext}_p$ if and only if $1 \le p < q$ and $\mathbb{R}^2 \setminus \overline{G_q} \in \mathrm{Ext}_s$ if and only if $q/(q-1) < s \le \infty$.

1 Introduction

Let D be a domain in $\mathbb{R}^2$, namely, D is a connected open subset of $\mathbb{R}^2$. For $1 \le p \le \infty$, denote by $W^{1,p}(D)$ the set of all functions in $L^p(D)$ whose first distributional derivatives lie in $L^p(D)$. For any $u \in W^{1,p}(D)$, the norm of u is given by $\|u\|_{W^{1,p}(D)} \equiv \|u\|_{L^p(D)} + \|\nabla u\|_{L^p(D)}$, where ∇u is the distributional gradient of u.

Definition 1.1. *Let $1 \le p \le \infty$. A domain $D \subset \mathbb{R}^2$ is called a domain of class Ext_p if there exists a bounded extension operator $\mathrm{Ext} : W^{1,p}(D) \to W^{1,p}(\mathbb{R}^2)$, namely, for each $u \in W^{1,p}(D)$, there exists a function $\mathrm{Ext}(u) \in W^{1,p}(\mathbb{R}^2)$ such that $\mathrm{Ext}(u)(x) = u(x)$ for all $x \in D$ and $\|\mathrm{Ext}(u)\|_{W^{1,p}(\mathbb{R}^2)} \le C\|u\|_{W^{1,p}(D)}$, where C is a positive constant independent of u.*

For $p > 1$, one could in fact require above that Ext is linear; see [1, Theorem 5].

In [5], Maz'ya constructed a planar Jordan domain D such that $D \in \mathrm{Ext}_p$ for all $1 \le p < 2$ but $D \notin \mathrm{Ext}_p$ for any $2 \le p \le \infty$. Furthermore the complementary domain $\mathbb{R}^2 \setminus \overline{D}$ of D satisfies $\mathbb{R}^2 \setminus \overline{D} \in \mathrm{Ext}_s$ exactly when $2 < s \le \infty$. This shows that the possibility of $W^{1,p}(D)$-extensions depends not only on the structure of the domain D but also on the exponent p. Motivated by this, for each $1 < q < 2$, Romanov [10] further constructed a planar domain G_q such that $G_q \in \mathrm{Ext}_p$ if and only if $1 \le p < q$. In this paper, we establish the following results by generalizing the above two constructions in [5, 10].

Theorem 1.1. *For each $1 < q < 2$, there exists a Jordan domain $G_q \subset \mathbb{R}^2$ such that $G_q \in \mathrm{Ext}_p$ if and only if $1 \le p < q$ and $\mathbb{R}^2 \setminus \overline{G_q} \in \mathrm{Ext}_s$ if and only if $q/(q-1) < s \le \infty$.*

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Our construction is an improvement on the one by Romanov [10] and it partially relies on his approach. We should point out that the boundary of G_q of Romanov [10] contains a curve generated by a certain Cantor set. In order to deal with the complementary domain, we actually simplify the construction from [10] and apply a certain sufficient condition for extendability from [8].

Finally, we state some conventions. Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but which may vary from line to line. The symbol $A \lesssim B$ or $B \gtrsim A$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. For any measurable set of positive measure $E \subset \mathbb{R}^2$ and locally integrable function f, we set $⨍_E f(x)\,dx \equiv \frac{1}{|E|}\int_E f(x)\,dx$.

2 Proof of Theorem 1.1

Theorem 1.1 follows from Lemmas 2.5, 2.7, 2.10 and 2.11 below. We begin with the construction of the domain G_q, which is inspired by [10] and [5].

Construction of the domain G_q. Assume $1 < q < 2$. Throughout the whole paper, let $a \equiv 2^{1/(q-2)}$ and $b \equiv 1 - 2a$. Then $0 < a < 1/2$ and $0 < b < 1$. Denote by I the interval $[0, 1] \times \{0\}$.

First we generate a sequence of subintervals,

$$\widetilde{\mathcal{I}} \equiv \{\widetilde{I}_m^{k,\,i}:\ m \in \mathbb{N} \cup \{0\};\ k = 0, \cdots, m+1;\ i = 0, \cdots, 2^k - 1\}, \tag{2.1}$$

following the idea of the construction of a Cantor set. When $m = 0$, let $\widetilde{I}_0^{0,\,0}$ be the closed middle interval of I with length b and $\widetilde{I}_0^{1,\,i}$ with $i = 0, 1$ be the closure of the two intervals obtained by removing $\widetilde{I}_0^{0,\,0}$ from I and ordered from left to right. When $m = 1$, let $\widetilde{I}_1^{0,\,0}$ be $\widetilde{I}_0^{0,\,0}$, $\widetilde{I}_1^{1,\,i}$ be the closed middle interval of $\widetilde{I}_0^{1,\,i}$ with length ba for $i = 1, 2$, and $\widetilde{I}_1^{2,\,i}$ with $i = 0, 1, 2, 3$ be the closure of the four intervals obtained by removing $\widetilde{I}_1^{0,\,0}$, $\widetilde{I}_1^{1,\,0}$ and $\widetilde{I}_1^{1,\,1}$ from I and ordered from left to right. When $m \ge 2$, for $k \le m-1$ and $i = 0, \cdots, 2^k - 1$, let $\widetilde{I}_m^{k,\,i}$ be $\widetilde{I}_{m-1}^{k,\,i}$; for $k = m$ and $i = 0, \cdots, 2^m - 1$, let $\widetilde{I}_m^{m,\,i}$ be the closed middle interval of $\widetilde{I}_{m-1}^{m,\,i}$ with length ba^m; for $k = m+1$ and $i = 0, \cdots, 2^{m+1} - 1$, let $\widetilde{I}_m^{m,\,i}$ be the closure of the 2^{m+1} intervals obtained by removing $\{\widetilde{I}_m^{k,\,i}:\ k = 0, \cdots, m;\ i = 0, \cdots, 2^k - 1\}$ from I and ordered from left to right.

Obviously, $\widetilde{\mathcal{I}}$ has the following properties:

(i) for each $m \in \mathbb{N} \cup \{0\}$, $I = \cup_{k=0}^{m+1} \cup_{i=0}^{2^k-1} \widetilde{I}_m^{k,\,i}$;

(ii) $|\widetilde{I}_m^{k,\,i}| = ba^k$ when $k \le m$ and $i = 0, \cdots, 2^k - 1$, and $|\widetilde{I}_m^{m+1,\,i}| = a^{m+1}$ when $i = 0, \cdots, 2^{m+1} - 1$.

Then we translate and dilate these intervals in $\widetilde{\mathcal{I}}$ by setting

$$I_m^{k,\,i} = (1-a)a^m \widetilde{I}_m^{k,\,i} + (a^{m+1}, 0)$$

for each $\widetilde{I}_m^{k,\,i} \in \widetilde{\mathcal{I}}$. Then we write

$$\mathcal{I} \equiv \{I_m^{k,\,i}:\ m \in \mathbb{N} \cup \{0\};\ k = 0, \cdots, m+1;\ i = 0, \cdots, 2^k - 1\}. \tag{2.2}$$

Obviously from (i) and (ii), it is easy to see that

(iii) for each $m \in \mathbb{N} \cup \{0\}$, $[a^{m+1},\, a^m] \times \{0\} = \cup_{k=0}^{m+1} \cup_{i=0}^{2^k-1} I_m^{k,\,i}$;

(iv) $|I_m^{k,\,i}| = b(1-a)a^{m+k}$ when $k \le m$ and $i = 0, \cdots, 2^k-1$, and $|I_m^{m+1,\,i}| = (1-a)a^{2m+1}$ when $i = 0, \cdots, 2^{m+1}-1$.

For each $I_m^{k,\,i} \in \mathcal{I}$, denote its upper hat by $\Gamma_m^{k,\,i}$, namely,

$$\Gamma_m^{k,\,i} \equiv \{x = (x_1,\, x_2) : \ \operatorname{dist}((x_1,\, 0),\, \partial I_m^{k,\,i}) = x_2\},$$

where and in what follows, for any set $E \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, $\operatorname{dist}(x,\, E) = \inf\{|x-y| : \ y \in E\}$.

We also denote by $T_m^{k,\,i}$ the closed triangle generated by $\Gamma_m^{k,\,i}$ and $I_m^{k,\,i}$. For $m \in \mathbb{N} \cup \{0\}$, set

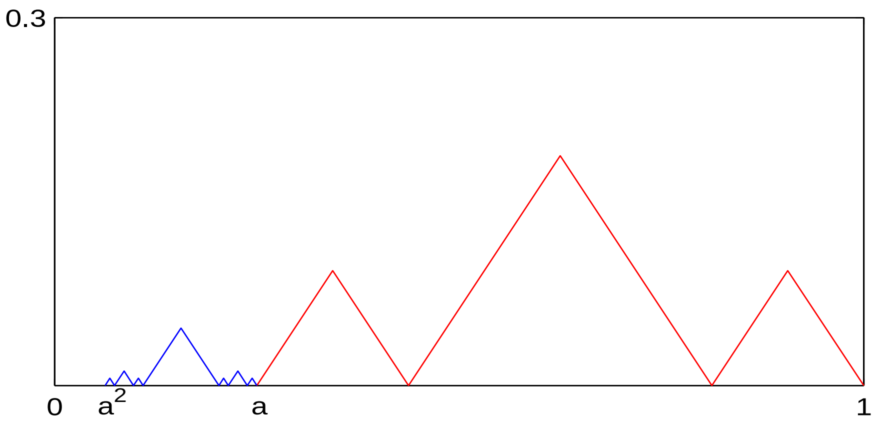
$$\Gamma_m \equiv \bigcup_{k=0}^{m+1} \bigcup_{i=0}^{2^k-1} \Gamma_m^{k,\,i}, \quad T_m \equiv \bigcup_{k=0}^{m+1} \bigcup_{i=0}^{2^k-1} T_m^{k,\,i}$$

and

$$\Gamma \equiv \{(0,\,0)\} \bigcup \left(\bigcup_{m=0}^{\infty} \Gamma_m \right), \quad T \equiv \{(0,\,0)\} \bigcup \left(\bigcup_{m=0}^{\infty} T_m \right).$$

Then we obtain a Lipschitz curve Γ joining $(0,\,0)$ and $(1,\,0)$. By abuse of notation, we always write $\Gamma \equiv \{(x_1,\, \Gamma(x_1)) : \ x_1 \in [0,\,1]\}$.

The following figure shows the curve $\Gamma_0 \cup \Gamma_1$, when $a = 1/4$.



Let R be the rectangle $(-1,\,1) \times (0,\,1)$ and $\varphi : R \to \mathbb{R}^2$ such that $\varphi(x_1,\, x_2) \equiv (x_1,\, x_2)$ if $x_1 \le 0$ and $\varphi(x_1,\, x_2) \equiv (x_1,\, x_2 + x_1^2)$ if $x_1 > 0$. Set $(G_q)_+ \equiv \varphi(R) \setminus \varphi(T)$ and let $(G_q)_-$ be the reflection of $(G_q)_+$ across the x_1- axis. Then define

$$G_q \equiv (G_q)_+ \bigcup (G_q)_- \bigcup (-1,\,0) \times \{0\},$$

which completes the construction of the domain G_q. $\square$

Now we recall the following result, which was established when $p > 1$ in [11, Theorem 1] and when $p = 1$ by Lemma 4.9.1 of [12].

Lemma 2.1. *Let $1 \le p < \infty$ and w be a non-negative function on $\mathbb{R}^2$. If there exist constants $s > 1$ and $C(w, s) > 0$ such that for all $r > 0$ and $x \in \mathbb{R}^2$,*

$$r \left(\fint_{B(x,r)} [w(y)]^{ps} \, dy \right)^{1/(ps)} \le C(w, s), \tag{2.3}$$

then there exists positive constant C such that for all $f \in W^{1,p}(\mathbb{R}^2)$, $fw \in L^p(\mathbb{R}^2)$ and $\|fw\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{W^{1,p}(\mathbb{R}^2)}$.

For any $c > 0$, set $\triangle_c \equiv \{(x_1, x_2) : \ 0 < x_1 < c, \ 0 < x_2 < x_1\}$. For any real-valued function u on $\mathbb{R}^2$, define $u_c(x_1, x_2) \equiv u(x_1, x_2)(x_2/x_1)\chi_{\triangle_c}$. Then we have the following result; see [10, Lemma 2] and also [6, p. 75]

Lemma 2.2. *Let $1 \le p < 2$. Then there exists constant C such that for all $u \in W^{1,p}(\mathbb{R}^2)$ and $0 < c \le 1$,*

$$\|u_c\|_{W^{1,p}(\triangle_c)} \le Cc^{-1}\|u\|_{L^p(\triangle_c)} + C\|u\|_{W^{1,p}(\triangle_c)}.$$

Similarly to Lemma 3 of [10], we have the following conclusion.

Lemma 2.3. *There exists a positive constant C and a sequence of functions, $\{v_m\}_{m=0}^\infty \subset W^{1,p}(\mathbb{R}^2)$, such that $v_m(x) = 1$ if $x \in \Gamma_m$ and $v_m(x) = 0$ if $x_1 \le 0$; moreover,*

$$\|v_m\|_{W^{1,p}(\mathbb{R}^2)} \le Ca^{(2/p-1)m}.$$

Proof. Let $v_0 \in W^{1,p}(\mathbb{R}^2)$ such that $v_0(x) = 0$ if $x_1 \le 0$ and $v_0(x) = 1$ if $0 \le x_2 \le x_1$ and $a \le |x| \le 1$. Set $v_m(x) \equiv v_0(a^{-m}x)$ for $m \in \mathbb{N}$. Then $v_m(x) = 1$ if $x \in \Gamma_m$ and $v_m(x) = 0$ if $x_1 = 0$ for $m \in \mathbb{N} \cup \{0\}$. Moreover,

$$\|v_m\|^p_{W^{1,p}(\mathbb{R}^2)} \lesssim a^{2m}\|v_0\|^p_{L^p(\mathbb{R}^2)} + a^{2m-mp}\|\nabla v_0\|^p_{L^p(\mathbb{R}^2)} \lesssim a^{2m-mp},$$

which completes the proof of Lemma 2.3. □

Let $R_{h,d} \equiv (0, h) \times [0, d]$ for $0 < d, h \le 1$. Let $E, F \subset \overline{R_{h,d}}$ be disjoint continua connecting the vertical sides of $R_{h,d}$. The following result has been proved in [10, Lemma 4].

Lemma 2.4. *Let $1 < p \le \infty$. Then for all $u \in W^{1p}(\mathbb{R}^2)$ with $u(x) = 1$ if $x \in E$ and $u(x) = 0$ if $x \in F$, $\|u\|_{W^{1p}(\mathbb{R}^2)} \ge h^{1/p}d^{1/p-1}$.*

Lemma 2.5. *If $\infty \ge p \ge q \equiv 2 + \log_a 2$, then $G_q \notin \mathrm{Ext}_p$.*

Proof. Assume that $G_q \in \mathrm{Ext}_p$. Notice that quasi-isometry keeps the space $W^{1,p}(\mathbb{R}^2)$ invariant under the change of the variable. By this and Lemma 2.3, there exists a sequence of functions, $\{v_m\}_{m\in\mathbb{N}\cup\{0\}} \subset W^{1,p}(\mathbb{R}^2)$, such that $v_m(x) = 1$ if $x \in \varphi(\Gamma_m)$ and $v_m(x) = 0$ if $x \in (G_q)_-$ or $x_1 \le 0$; moreover, $\|v_m\|^p_{W^{1,p}(G_q)} \lesssim a^{(2-p)m}$, where φ is as in the construction of the domain G_q. Let u_m be an extension of v_m. Then, $\|u_m\|^p_{W^{1,p}(\mathbb{R}^2)} \lesssim a^{(2-p)m}$.

On the other hand, since $u_m(x) = 1$ if $x \in \varphi(\Gamma_m^{k,\,i})$ and $u_m(x) = 0$ if $x_1 = 0$, by Lemma 2.4 and $2a^{2-p} \ge 1$,

$$\begin{aligned}\|u_m\|^p_{W^{1,\,p}(\mathbb{R}^2)} &\gtrsim \sum_{k=0}^{m+1}\sum_{i=0}^{2^k-1} |I_m^{k,\,i}||I_m^{k,\,i}|^{1-p}\\ &\gtrsim \sum_{k=0}^{m+1} 2^k a^{(2-p)(m+k)} \gtrsim a^{(2-p)m}\sum_{k=0}^{m+1} 2^k a^{(2-p)k} \gtrsim m a^{(2-p)m}.\end{aligned}$$

This is a contradiction, which completes the proof of Lemma 2.5. □

Lemma 2.6. *Let $1 \le p < q \equiv 2 + \log_a 2$. Set*

$$w \equiv \sum_{m=0}^{\infty}\sum_{k=0}^{m+1}\sum_{i=0}^{2^k-1} a^{-(k+m)}\chi_{T_m^{k,\,i}}.$$

If $1 < s < 2/p$ is such that $2a^{2-sp} < 1$, then w satisfies (2.3).

Proof. If $r \ge a/2$, then for all $x \in \mathbb{R}^2$, then by $2a^{2-sp} < 1$, we have

$$\begin{aligned}\int_{B(x,\,r)} [w(y)]^{sp}\,dy &\le \sum_{m=0}^{\infty}\sum_{k=0}^{m+1}\sum_{i=0}^{2^k-1} a^{-(k+m)sp}|T_m^{k,\,i}|\\ &\le \sum_{m=0}^{\infty}\sum_{k=0}^{m+1} 2^k a^{(k+m)(2-sp)} \lesssim \sum_{m=0}^{\infty} a^{m(2-sp)} \lesssim 1 \lesssim r^{2-sp}.\end{aligned}$$

Similarly, it is easy to see that

$$\int_{B(0,\,a^n)} [w(y)]^{sp}\,dy \le \sum_{m=n}^{\infty}\sum_{k=0}^{m+1}\sum_{i=0}^{2^k-1} a^{-(k+m)sp}|T_m^{k,\,i}| \lesssim a^{n(2-sp)}.$$

If $r < a/2$ and $w(y) \ne 0$ for all $y \in B(x,\,r)$, then $|y| \le 1$ and $|x| < 1 + r$. For $1 \le |x| < 1 + r$, observing that $w(y) \lesssim 1$ for all $y \in B(x,\,r)$, we then have (2.3). Assume now that $a^{n+1} \le |x| < a^n$ for certain $n \in \mathbb{N} \cup \{0\}$. If $r \ge (1-a)a^{n+1}/2$, similarly to above computations, then we have

$$\int_{B(x,\,r)} [w(y)]^{sp}\,dy \le \int_{B(0,\,r+2r/a(1-a))} [w(y)]^{sp}\,dy \lesssim a^{n(2-sp)} \lesssim r^{2-sp}.$$

If $r < (1-a)a^{2(n+1)}/2$, then for all $y \in B(x,\,r)$, $w(y) \lesssim a^{-2n}$, and hence,

$$\int_{B(x,\,r)} [w(y)]^{sp}\,dy \lesssim a^{-2nsp} r^2 \lesssim r^{2-sp}.$$

If $(1-a)a^{2(n+1)}/2 \le r < (1-a)a^{n+1}/2$, then $(1-a)a^{n+k_0+1}/2 \le r < (1-a)a^{n+k_0}/2$ for some $1 \le k_0 \le n+1$, and thus $B(x, r)$ contains at most 2^{k-k_0} many of the $T_m^{k,\,i}$ for each $m = n-1,\, n,\, n+1$ and $k \ge k_0$, which implies that

$$\int_{B(x,\,r)} [w(y)]^{sp}\,dy \le \sum_{k=k_0-1}^{n+2} 2^{k-k_0} a^{-(n+k)sp} a^{2(n+k)} \lesssim a^{(2-sp)(n+k_0)} \lesssim r^{2-sp}.$$

This finishes the proof of Lemma 2.6. □

Lemma 2.7. *If* $1 \le p < q \equiv 2 + \log_a 2$*, then* $G_q \in \operatorname{Ext}_p$.

Proof. Notice that quasi-isometries keep the space $W^{1,\,p}(\mathbb{R}^2)$ invariant under the change of the variable. By this and the symmetry of G_q with respect to x_1-axis, we only need to prove that for any $u \in W^{1,\,p}(R \setminus T)$, there exists a function $\overline{u} \in W^{1,\,p}(R)$ such that $\overline{u}(x) = 0$ for almost all $x \in (0,\,1) \times \{0\}$ and $\overline{u}(x) = u(x)$ for almost all $x \in R \setminus T$. Since the boundary of $R \setminus T$ is Lipschitz, there exists a bounded extension operator $\operatorname{Ext}:$ $W^{1,\,p}(R \setminus T) \to W^{1,\,p}(\mathbb{R}^2)$; see [10, p. 725]. Let $v \equiv \operatorname{Ext}(u)$ and $\mathring{T}_m^{k,\,i}$ be the interior of $T_m^{k,\,i}$. Now we will obtain $\overline{u}$ by redefining v on T. In fact, applying Lemma 2.2 to v on each $T_m^{k,\,i}$, we obtain a function $u_m^{k,\,i}$ such that $u_m^{k,\,i}(x) = v(x)$ for all $x \in \Gamma_m^{k,\,i}$ and $u_m^{k,\,i}(x) = 0$ for all $x \in I_m^{k,\,i}$, and moreover,

$$\|u_m^{k,\,i}\|^p_{W^{1,\,p}(\mathring{T}_m^{k,\,i})} \lesssim a^{-(k+m)p} \|v\|^p_{L^p(\mathring{T}_m^{k,\,i})} + \|v\|^p_{W^{1,\,p}(\mathring{T}_m^{k,\,i})}.$$

Set

$$\overline{u} \equiv u\chi_{R\setminus T} + \sum_{m=0}^{\infty} \sum_{k=0}^{m+1} \sum_{i=0}^{2^k-1} u_m^{k,\,i} \chi_{T_m^{k,\,i}}.$$

Then

$$\begin{aligned}
\|\overline{u}\|^p_{W^{1,\,p}(R)} &\lesssim \|u\|^p_{W^{1,\,p}(R\setminus T)} + \sum_{m=0}^{\infty} \sum_{k=0}^{m+1} \sum_{i=0}^{2^k-1} \|u_m^{k,\,i}\|^p_{W^{1,\,p}(\mathring{T}_m^{k,\,i})} \\
&\lesssim \|u\|^p_{W^{1,\,p}(R\setminus T)} + \sum_{m=0}^{\infty} \sum_{k=0}^{m+1} \sum_{i=0}^{2^k-1} a^{-(k+m)p} \|v\|^p_{L^p(\mathring{T}_m^{k,\,i})} + \|v\|^p_{W^{1,\,p}(\mathbb{R}^2)} \\
&\lesssim \|vw\|^p_{L^p(\mathbb{R}^2)} + \|u\|^p_{W^{1,\,p}(R\setminus T)},
\end{aligned}$$

where

$$w \equiv \sum_{m=0}^{\infty} \sum_{k=0}^{m+1} \sum_{i=0}^{2^k-1} a^{-(k+m)} \chi_{T_m^{k,\,i}}.$$

Since $2a^{2-p} < 1$, we can find $1 < s < 2/p$ such that $2a^{2-sp} < 1$. By Lemma 2.6, we know that w satisfies (2.3). Then, by Lemma 2.1, we have that

$$\|vw\|^p_{L^p(\mathbb{R}^2)} \lesssim \|v\|^p_{W^{1,\,p}(\mathbb{R}^2)} \lesssim \|u\|^p_{W^{1,\,p}(R\setminus T)},$$

which further implies that $\|\overline{u}\|_{W^{1,\,p}(R)} \lesssim \|u\|_{W^{1,\,p}(R\setminus T)}$. This finishes the proof of Lemma 2.7. □

So far, for $1 < q < 2$, we have already proved that $G_q \in \operatorname{Ext}_p$ if and only if $1 \le p < q$. To prove the extendability properties for the domain $\mathbb{R}^2 \setminus \overline{G_q}$, we need the following two auxiliary conclusions.

Lemma 2.8. *The mapping φ from $\{x \in \mathbb{R}^2 : \ 0 \le x_1 \le a\}$ to itself given by $\varphi(x_1, x_2) \equiv (x_1, x_2 + x_1^2)$ is bi-Lipschitz.*

Proof. In fact, since $0 \le |x_1^2 - y_1^2| \le 2a|x_1 - y_1|$ for all $0 \le x_1, y_1 \le a$, we have

$$\begin{aligned}(1-2a)|x-y| &\le |x-y| - |x_1^2 - y_1^2| \\ &\le |\varphi(x) - \varphi(y)| \le |x-y| + |x_1^2 - y_1^2| \le (1+2a)|x-y|,\end{aligned}$$

which completes the proof of Lemma 2.8. □

We always write $\gamma(x, y) \subset D$ for a rectifiable curve joining x and y in a domain $D \subset \mathbb{R}^2$. By abuse of notation, we also sometimes use γ to denote $\gamma(x, y)$. Denote by $\ell(\gamma)$ the arc length of γ, $\gamma(s)$ the arc length representation of γ, $\gamma(0) = x$ and $\gamma(\ell(\gamma)) = y$. If g is a real-valued function in D, we let

$$\int_{\gamma(x,\,y)} g(z)\,|dz| \equiv \int_0^{\ell(\gamma)} g(\gamma(s))\,ds$$

be the line integral of g along γ whenever the integral exists.

A domain D is called a $\operatorname{Lip}_\alpha$-extension domain if for any pair of points $x, y \in D$, there exists a curve $\gamma(x, y) \subset D$ such that

$$\int_{\gamma(x,\,y)} [\operatorname{dist}(z, \partial D)]^{\alpha-1}\,|dz| \le C|x-y|^\alpha, \tag{2.4}$$

where C is a positive constant independent of x and y; see [4].

Then by [8, Theorem A and Corollary 4.1] and [3, Theorem 5.2], we have the following conclusion.

Lemma 2.9. *Let $p > 2$.*

(i) If D is a simply connected planar domain and $D \in \operatorname{Ext}_p$, then D is $\operatorname{Lip}_{(p-2)/(p-1)}$-extension domain.

(ii) If D is $\operatorname{Lip}_{(p-2)/(p-1)}$-extension domain, then $D \in \operatorname{Ext}_s$ for all $s > p$.

Lemma 2.10. *If $1 < q < 2$ and $1 \le p \le q/(q-1)$, then $\mathbb{R}^2 \setminus \overline{G_q} \notin \operatorname{Ext}_p$.*

Proof. By Theorem 6.4 of [2] for $1 < p \le 2$, if D is a $W^{1,p}$ extension domain, then D has the property LLC(2), namely, there exists a constant $c \ge 1$ such that for all $z \in \mathbb{R}^2$ and $r > 0$, any pair of points $x, y \in D \setminus B(z, r)$ can be joined in $D \setminus B(z, r/c)$. For $p = 1$, we claim that $\mathbb{R}^2 \setminus \overline{G_q} \in \operatorname{Ext}_1$ implies that G_q is a quasiconvex domain. Assume this for the moment. Then by [7], G_q is a bounded turning domain, which together with [7, Theorem 4.5] further implies that $\mathbb{R}^2 \setminus \overline{G_q}$ has the LLC(2) property.

So for $1 \le p \le 2$, the proof of Lemma 2.10 is reduced to proving that $\mathbb{R}^2 \setminus \overline{G_q}$ does not have the property LLC(2). To see this, obviously, for any fixed positive constant c, we always find m large enough such that $a^{2m} \le (1-a)a^m/(cN)$, where N is a fixed positive constant such that $\varphi(T_m^{m+1,0}) \subset B((a^m, 0), Na^{2m})$. Thus the pair of points $(a^{m+1}, 0)$ and $(a^{m-1}, 0)$, which lie in $\mathbb{R}^2 \setminus \overline{G_q}$ but not in $B((a^m, 0), (1-a)a^{m+1})$, cannot be joined in $(\mathbb{R}^2 \setminus \overline{G_q}) \setminus B((a^m, 0), Na^{2m})$ and thus not in $(\mathbb{R}^2 \setminus \overline{G_q}) \setminus B((a^m, 0), (1-a)a^{m+1}/c)$. This implies that $\mathbb{R}^2 \setminus \overline{G_q}$ does not have the property LLC(2) and thus $\mathbb{R}^2 \setminus \overline{G_q} \notin \mathrm{Ext}_p$ for any $1 \le p \le 2$.

Now we turn to prove the above claim that $\mathbb{R}^2 \setminus \overline{G_q} \in \mathrm{Ext}_1$ implies that G_q is a quasiconvex domain. To this end, we first observe that for any $1 \le p < \infty$, $\mathbb{R}^2 \setminus \overline{G_q} \in \mathrm{Ext}_p$ implies that $E_{-1} \equiv (\{(x_1, x_2) : x_1 > -1\} \setminus \overline{G_q}) \in \mathrm{Ext}_p$. In fact, let η be a smooth function such that $0 \le \eta(x) \le 1$ and $|\nabla \eta(x)| \le 4$ for all $x \in \mathbb{R}^2$, and $\eta(x) = 0$ for $x_1 \le -1$ and $\eta(x) = 1$ for $x_1 \ge 0$. Let $S \equiv \{(x_1, x_2) : x_1 > -1\} \setminus ([-1, 0] \times [-1, 1])\}$. If $u \in W^{1,p}(E_{-1})$, then $u\eta \in W^{1,p}(\mathbb{R}^2 \setminus \overline{G_q})$, $u(1-\eta) \in W^{1,p}(S)$ and

$$\|u\eta\|_{W^{1,p}(\mathbb{R}^2 \setminus \overline{G_q})} + \|u(1-\eta)\|_{W^{1,p}(S)} \lesssim \|u\|_{W^{1,p}(E_{-1})}.$$

By the assumption $\mathbb{R}^2 \setminus \overline{G_q} \in \mathrm{Ext}_p$, we have that $\mathrm{Ext}\,(u\eta) \in W^{1,p}(\mathbb{R}^2)$ and

$$\|\mathrm{Ext}\,(u\eta)\|_{W^{1,p}(\mathbb{R}^2)} \lesssim \|u\eta\|_{W^{1,p}(\mathbb{R}^2 \setminus \overline{G_q})} \lesssim \|u\|_{W^{1,p}(E_{-1})}.$$

Since S is a uniform domain, $u(1-\eta)$ can be extended to the entire $\mathbb{R}^2$ (see [2, p. 9]). The extension, $\mathrm{Ext}\,(u(1-\eta))$, satisfies $\|\mathrm{Ext}\,(u(1-\eta))\|_{W^{1,p}(\mathbb{R}^2)} \lesssim \|u(1-\eta)\|_{W^{1,p}(S)} \lesssim \|u\|_{W^{1,p}(E_{-1})}$. Obviously $\mathrm{Ext}\,(u(1-\eta)) + \mathrm{Ext}\,(u\eta)$ coincides with u on E_{-1}, which implies $E_{-1} \in \mathrm{Ext}_p$. Then an argument similar to but easier than the above shows that $E_{-1} \cap B(0, 10) \in \mathrm{Ext}_p$. Observe that $E_{-1} \cap B(0, 10)$ is a bounded, simply connected $W^{1,1}$-extension domain. Applying [9, Corollary 1.2], we know that the complement domain of $E_{-1} \cap B(0, 10)$ is quasiconvex, which further implies that $(\overline{E_{-1}})^{\complement}$, and thus G_q, is a quasiconvex domain. This proves the above claim.

For $2 < p \le q/(q-1)$, since $\mathbb{R}^2 \setminus \overline{G_q} \in \mathrm{Ext}_p$ implies $E_{-1} \in \mathrm{Ext}_p$ as above, to prove Lemma 2.10, by Lemma 2.9, it suffices to prove that E_{-1} is not a Lip_α-extension domain for any $0 < \alpha \le 2-q$.

To see this, choose $N \in \mathbb{N}$, and $x = (a^m, 0)$ and $y = (a^{m-1}, 0)$. Then for any $\gamma(x, y) \subset E_{-1}$, take $\widetilde{\gamma}$ to be the component of $\gamma \cap \{(z_1, z_2) : 0 \le z_1 \le a\}$ containing x. Obviously, $[a^m, a^{m-1}] \subset \{z_1 : (z_1, z_2) \in \widetilde{\gamma}\}$, and without loss of generality, we may assume that $z_2 \ge 0$ for all $(z_1, z_2) \in \widetilde{\gamma}$. Moreover, for all $z \in \widetilde{\gamma}$, by Lemma 2.8,

$$\mathrm{dist}\,(z, \partial G_q) = \mathrm{dist}\,(z, \varphi(\Gamma \setminus \Gamma_0)) \sim \mathrm{dist}\,(\varphi^{-1}(z), \Gamma \setminus \Gamma_0).$$

Assume that $(z_1, 0) \in T_m^{k,i}$. Then

$$\mathrm{dist}\,(\varphi^{-1}(z), \Gamma \setminus \Gamma_0) = \mathrm{dist}\,(\varphi^{-1}(z), \Gamma_m^{k,i}) \le \mathrm{dist}\,((z_1, -z_1^2), \Gamma_m^{k,i}) \le a^{2(m-1)} + \Gamma(z_1),$$

where $(z_1, \Gamma(z_1)) \in \Gamma$. Since $2a^\alpha \ge 2a^{2-q} = 1$, we have

$$\int_{\gamma(x,y)} [\,\mathrm{dist}\,(z, \partial G_q)]^{\alpha-1}\,|dz| \gtrsim \int_{a^m}^{a^{m-1}} [a^{2(m-1)} + \Gamma(z_1)]^{\alpha-1}\,dz_1$$

$$\begin{aligned}
&\gtrsim \sum_{k=0}^{m+1}\sum_{i=0}^{2^k-1}\int_0^{|I_m^{k,\,i}|/2}[a^{2m}+t]^{\alpha-1}\,dt\\
&\gtrsim \sum_{k=0}^{m+1}2^k([a^{2m}+a^{k+m}]^{\alpha}-a^{2m\alpha})\\
&\gtrsim \sum_{k=0}^{m}2^k a^{m\alpha}a^{k\alpha}\gtrsim ma^{m\alpha}\gtrsim m|x-y|^{\alpha},
\end{aligned}$$

which implies that E_{-1} is not a Lip_{α}-extension domain. This finishes the proof of Lemma 2.10. □

Lemma 2.11. *If $q/(q-1)<s\le\infty$, then $\mathbb{R}^2\setminus\overline{G_q}\in\mathrm{Ext}_s$.*

Proof. By Lemma 2.9, it suffices to prove that $\mathbb{R}^2\setminus\overline{G_q}$ is Lip_{α}-extension domain for all $\alpha>2-q$. Let $\varphi_-(T\setminus T_0)$ be the reflection of $\varphi(T\setminus T_0)$ with respect to x_1-axis, namely, $\varphi_-(T\setminus T_0)\equiv\{(x_1,\,-x_2):(x_1,\,x_2)\in\varphi(T\setminus T_0)\}$ and for $m\in\mathbb{N}\cup\{0\}$

$$E_m\equiv\varphi(\mathring{T}\setminus\cup_{n=0}^m T_n)\cup\varphi_-(\mathring{T}\setminus\cup_{n=0}^m T_n)\cup\{(x_1,\,x_2):\ |x_2|\le x_1^2,\ 0\le x_1<a^m\}.$$

Then $G_q\cup E_m$ is a Jordan domain with Lipschitz boundary since $\Gamma(z_1)$ is Lipschitz function. Obviously, $\mathbb{R}^2\setminus\overline{G_q}=E_1\cup(\mathbb{R}^2\setminus\overline{G_q\cup E_2})$. Then the proof of Lemma 2.11 is reduced to proving that for any $x,\,y\in E_1$, there exists a curve $\gamma(x,\,y)\subset E_1$ such that

$$\int_{\gamma(x,\,y)}[\,\mathrm{dist}\,(z,\,\varphi(\Gamma)\cup\varphi_-(\Gamma))]^{\alpha-1}\,|dz|\lesssim|x-y|^{\alpha}. \tag{2.5}$$

Assuming that (2.5) holds for the moment, we now establish Lemma 2.11. Since $\mathrm{dist}\,(z,\,\varphi(\Gamma)\cup\varphi_-(\Gamma))=\mathrm{dist}\,(z,\partial G_q)$ for all $z\in E_1$, then for any $x,\,y\in E_1$, there exists a curve $\gamma(x,\,y)\subset E_1$ such that

$$\int_{\gamma(x,\,y)}[\,\mathrm{dist}\,(z,\,\partial G_q)]^{\alpha-1}\,|dz|\lesssim|x-y|^{\alpha}. \tag{2.6}$$

Obviously, $\mathbb{R}^2\setminus\overline{G_q\cup E_2}$ is a uniform domain and thus Lip_{α}-extension domain for all $0<\alpha\le1$; see [4]. Thus for any $x,\,y\in\mathbb{R}^2\setminus\overline{G_q\cup E_2}$, there exists a curve $\gamma(x,\,y)\subset\mathbb{R}^2\setminus\overline{G_q\cup E_2}$ satisfying (2.6) with $\mathrm{dist}\,(x,\,\partial G_q)$ replaced by $\mathrm{dist}\,(x,\,\partial(G_q\cup E_2))$. Observe that for all $x\in\mathbb{R}^2\setminus(G_q\cup E_2)$, $\mathrm{dist}\,(x,\,\partial(G_q\cup E_2))\le\mathrm{dist}\,(x,\,\partial G_q)$, which implies that $\gamma(x,\,y)\subset\mathbb{R}^2\setminus\overline{G_q}$ satisfies (2.6). For any $x\in\overline{E_1}$ and $y\in\mathbb{R}^2\setminus(G_q\cup E_1)$, assume that there exists a point $w\in J\equiv\{(a,\,y_2):(a,\,y_2)\in\partial E_1\}$ such that

$$|x-y|\sim|x-w|+|w-y|. \tag{2.7}$$

Then there exist curves $\gamma_1(x,w)\subset E_0$ and $\gamma_2(w,\,y)\subset\mathbb{R}^2\setminus\overline{G_q\cup E_1}$ satisfying (2.6). Let $\gamma\equiv\gamma_1\cup\gamma_2$. Therefore, $\gamma(x,\,y)\subset\mathbb{R}^2\setminus\overline{G_q}$ satisfies (2.6). To see (2.7), if $y_1\ge1$, since $|x_2|\le a^2+ba$, then $|x_1-y_1|>1-a$ and

$$|y-(a,\,0)|+|(a,\,0)-x|\le|y-x|+2|(a,\,0)-x|\le|x-y|+4a\lesssim|x-y|,$$

which implies (2.7) with $w \equiv (a, 0)$. Set

$$f(x, y) \equiv |x-y|^{-1} \inf_{w \in J} \{|x-w| + |w-y|\}.$$

Obviously, $f(x, y) \geq 1$ whenever defined. Moreover, f is continuous on the bounded closed set

$$\{(x, y) \in \mathbb{R}^4 : \ x \in \overline{E}_1, \ y \in \overline{E_0 \setminus E_1}, \ |x-y| \geq a^4/4\},$$

which implies f is bounded on this set and thus (2.7) holds for (x, y) in this set. Finally, if $x \in \overline{E}_1$, $y \in \overline{E_0 \setminus E_1}$ and $|x-y| < a^4/4$, then it is easy to see that (2.7) holds. Thus, so far, we proved that the claim (2.7) is true, and therefore, except (2.5), we have finished this proof of Lemma 2.11.

Now we turn to proving the above claim (2.5). Set $D \equiv (T \setminus T_0) \cup \{(x_1, x_2) : \ 0 \leq x_1 \leq a, \ -x_1^2 \leq x_2 \leq 0\}$. Observe that the union of $\varphi(D)$ and $\varphi_-(D)$, the reflection of $\varphi(D)$ with respect to x_1-axis, is just the set E_1. Then the claim (2.5) is reduced to proving that for any $x, y \in D \setminus \Gamma$, there exists a curve $\gamma(x, y) \subset D$ such that

$$\int_{\gamma(x,y)} [\operatorname{dist}(z, \Gamma)]^{\alpha-1} |dz| \lesssim |x-y|^{\alpha}. \tag{2.8}$$

In fact, assume that (2.8) holds for the moment. Then for any $x, y \in \varphi(D \setminus \Gamma)$, there exists a curve $\gamma(\varphi^{-1}(x), \varphi^{-1}(y)) \subset D$ satisfying (2.8). Since φ is bi-Lipschitz, the curve $\varphi(\gamma)(x, y) \subset \varphi(D \setminus \Gamma)$ also satisfies (2.5). A similar argument applies to any $x, y \in \varphi_-(D \setminus \Gamma)$. For any $x \in \varphi(D \setminus \Gamma)$ and $y \in \varphi_-(D \setminus \Gamma)$, letting w be the intersection of the x_1-axis and the line joining x and y, we have that $w \in \overline{E_1}$ and $|x-y| \sim |x-w| + |w-y|$. Applying similar arguments to x, w and w, y, we obtain the curves $\gamma_1(x, w) \subset E_0$ and $\gamma_2(w, y) \subset E_0$ satisfying (2.5). Taking $\gamma \equiv \gamma_1 \cup \gamma_2$ gives the desired result.

To prove (2.8), we consider three cases.

Case 1. $x, y \in D_m^{k,i} \equiv T_m^{k,i} \cup \{(x_1, x_2) : \ (x_1, 0) \in T_m^{k,i}, \ -x_1^2 \leq x_2 \leq 0\}$ for $m \in \mathbb{N}$.

It suffices to verify that if $x_1 = y_1$ or $x_2 = y_2$, then there exists $\gamma(x, y) \subset D$ satisfying (2.8). In fact, assume this for the moment. In general, we assume that $x_i \neq y_i$ for $i = 1, 2$, and we may further assume that $x_1 < y_1$ without loss of generality. If $x_2 < y_2$, then let $z \equiv (y_1, x_2)$, and if $x_2 > y_2 \geq -x_1^2$, then let $z \equiv (x_1, y_2)$. Obviously, $z \in D_m^{k,i}$ and $|x-y| \sim |x-z| + |z-y|$. Moreover, by the choices of z and the assumptions, there exist curves γ_1 and γ_2 such that $\gamma_1(x, z) \subset D$ and $\gamma_2(z, y) \subset D$ satisfy (2.8), respectively. Taking $\gamma \equiv \gamma_1 \cup \gamma_2$, we know that $\gamma(x, y) \subset D$ satisfies (2.8). If $y_2 < -x_1^2$, then let $z \equiv (x_1, -x_1^2)$ and $u \equiv (y_1, -x_1^2)$. Obviously, $z, u \in D_m^{k,i}$ and $|x-y| \sim |x-z| + |z-u| + |u-y|$. Moreover, by the choices of z and u and the assumptions, there exist curves γ_1, γ_2 and γ_3 such that $\gamma_1(x, z) \subset D$, $\gamma_2(z, u) \subset D$ and $\gamma_3(u, y) \subset D$ satisfy (2.8), respectively. Taking $\gamma \equiv \gamma_1 \cup \gamma_2 \cup \gamma_3$, we have that $\gamma(x, y) \subset D$ satisfies (2.8).

Now assume that $x_1 = y_1$ or $x_2 = y_2$. If $x_1 = y_1$, taking γ to be the line segment joining x and y, we have that $\gamma(x, y) \subset D_m^{k,i}$. Since

$$\operatorname{dist}(z, \Gamma) = \operatorname{dist}(z, \Gamma_m^{k,i}) \sim \Gamma(z_1) - z_2 \tag{2.9}$$

for all z with $(z_1, 0) \in T_m^{k,i}$ and $z_2 < \Gamma(z_1) = \Gamma(x_1)$, we have

$$\int_\gamma [\,\mathrm{dist}\,(z, \Gamma)]^{\alpha-1}\,|dz| \sim \int_\gamma [\Gamma(x_1) - z_2]^{\alpha-1}\,|dz|$$
$$\lesssim \int_0^{|x_2-y_2|} t^{\alpha-1}\,dt \lesssim |x_2 - y_2|^\alpha \sim |x-y|^\alpha.$$

If $x_2 = y_2$, taking γ to be the line segment joining x and y, we have that $\gamma(x, y) \subset D_m^{k,i}$. Moreover, we have

$$\int_\gamma [\,\mathrm{dist}\,(z, \Gamma)]^{\alpha-1}\,|dz| \sim \int_\gamma [\Gamma(z_1) - x_2]^{\alpha-1}\,|dz|$$
$$\lesssim \int_0^{|x_1-y_1|} t^{\alpha-1}\,dt \lesssim |x_1 - y_1|^\alpha \sim |x-y|^\alpha.$$

Case 2. $x \in D_m^{k,i}$ and $y \in D_n^{\ell,j}$, where $T_n^{\ell,j}$ is adjacent to $T_m^{k,i}$. Let $\{(w_1, 0)\} = T_m^{k,i} \cap T_n^{\ell,j}$ and assume that $x_1 < y_1$.

If $x_2 < 0$, letting $u \equiv (y_1, x_2)$ and $w \equiv (w_1, x_2)$, then $|x-y| \sim |x-w| + |w-u| + |u-y|$. By Case 1, there exists curve $\gamma_1(x, w) \subset D$, $\gamma_2(w, u) \subset D$ and $\gamma_3(u, y) \subset D$ satisfying (2.8), respectively. Taking $\gamma \equiv \gamma_1 \cup \gamma_2 \cup \gamma_3$, we know that $\gamma(x, y) \subset D$ satisfies (2.8).

If $x_2 \geq 0$ and $y_2 \geq 0$, noticing that $x_2 \leq w_1 - x_1$ and $y_2 \leq y_1 - w_1$, and letting $u \equiv (x_1, -c)$, $w = (w_1, -c)$ and $z \equiv (y_1, -c)$, where $0 < c \leq \min\{x_2, y_2, x_1^2\}$, we conclude that

$$|x_2 + c| + |y_2 + c| \leq 2|y_1 - w_1| + 2|x_1 - w_1| \sim |x_1 - y_1|,$$

which implies that

$$|x-y| \sim |x-u| + |u-w| + |w-z| + |z-y| \sim |x_1 - y_1|.$$

Thus, there exist curves $\gamma_1(x, u) \subset D$, $\gamma_2(u, w) \subset D$, $\gamma_3(w, z) \subset D$ and $\gamma_4(z, y) \subset D$ satisfying (2.8), respectively. Taking $\gamma \equiv \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, we know that $\gamma(x, y) \subset D$ satisfies (2.8).

If $x_2 \geq 0$ and $-x_1^2 \leq y_2 < 0$, then similarly to the proof for the case $x_1 < 0$, we obtain a curve $\gamma(x, y) \subset D$ that satisfies (2.8).

If $x_2 \geq 0$ and $-x_1^2 > y_2 \geq -y_1^2$, letting $u \equiv (y_1, -x_1^2)$, and $z \equiv (x_1, -x_1^{-2})$, we have that $|x-y| = |x-z| + |z-u| + |u-y|$. Since there exist curves $\gamma_1(x, z) \subset D$, $\gamma_1(z, u) \subset D$ and $\gamma_3(u, y) \subset D$ satisfying (2.8), respectively, taking $\gamma \equiv \gamma_1 \cup \gamma_2 \cup \gamma_3$ gives the desired curve.

Case 3. $x \in D_m^{k,i}$ and $y \in D_n^{\ell,j}$, where $T_n^{\ell,j}$ is not the one adjacent to $T_m^{k,i}$. We may assume that $x_1 < y_1$ without loss of generality.

Let $(u_1, 0)$ be the right endpoint of $I_m^{k,i}$ and $(z_1, 0)$ be the left endpoint of $I_n^{\ell,j}$. Then $-x_1^2 \leq x_2 \leq u_1 - x_1$ and $-y_1^2 \leq y_2 \leq y_1 - z_1$. Since $x_1^2 + y_1^2 \lesssim a^{2m} + a^{2n} \lesssim |u_1 - z_1| \lesssim |x_1 - y_1|$, we know that $|x_2 - y_2| \lesssim |x_1 - y_1|$, which implies that $|x-y| \sim |x_1 - y_1|$ and $|x - (u_1, 0)| + |(u_1, 0) - (z_1, 0)| + |(z_1, 0) - y| \lesssim |x-y|$.

Let γ_2 be the line segment joining the pair of points $u=(u_1,\,-x_1^2)$ and $z=(z_1,\,-x_1^2)$. Then we claim that

$$\int_{\gamma_2}[\,\mathrm{dist}\,(v,\,\Gamma)]^{\alpha-1}\,dv\lesssim|u_1-z_1|^\alpha. \tag{2.10}$$

Assume that (2.10) holds for the moment. Then by Case 1, there exist curves $\gamma_1(x,\,u)\subset D$ and $\gamma_3(z,\,y)\subset D$ satisfying (2.8), respectively. Thus the curve $\gamma\equiv\gamma_1\cup\gamma_2\cup\gamma_3$ is as desired.

Now we show (2.10). If $m=n$, let $|I_m^{k_0,\,i_0}|$ be the largest for the subintervals from our construction contained in $\widetilde{\gamma}_2\equiv\gamma_2+(0,\,x_1^2)$. Then $|u_1-z_1|\sim a^{m+k_0}$ and $\widetilde{\gamma}_2\subset\cup_{k=k_0}^{m+1}\cup_{j=1}^{2^{k-k_0}}I_m^{k,\,i_j}$. Since $\mathrm{dist}\,(v,\Gamma)\sim\Gamma(v_1)$ for all $v\in I_m^{k,\,i}$, by Case 1, we have

$$\int_{I_m^{k,\,i}}[\,\mathrm{dist}\,(v,\,\Gamma)]^{\alpha-1}\,dv\lesssim a^{\alpha(m+k)}.$$

Thus, by $2a^\alpha<1$, i. e. $\alpha>-\log_a 2=2-q$, we have

$$\begin{aligned}\int_{\gamma_2}[\,\mathrm{dist}\,(v,\,\Gamma)]^{\alpha-1}\,dv&\le\int_{\gamma_2}[\,\mathrm{dist}\,(v,\,\Gamma)]^{\alpha-1}\,dv\\&\lesssim\sum_{k=k_0}^{m+1}2^{k-k_0}a^{\alpha(m+k)}\\&\lesssim a^{\alpha(m+k_0)}\sum_{k=0}^{m-k_0+1}2^k a^{\alpha k}\lesssim a^{\alpha(m+k_0)}\lesssim|u_1-z_1|^\alpha.\end{aligned}$$

If $m=n+1$, then $|u_1-z_1|=|a^m-u_1|+|a^m-z_1|$, and thus, by the above estimate in the case $m=n$,

$$\begin{aligned}\int_{\gamma_2}[\,\mathrm{dist}\,(v,\,\Gamma)]^{\alpha-1}\,dv&\le\int_{\widetilde{\gamma}_2}[\,\mathrm{dist}\,(v,\,\Gamma)]^{\alpha-1}\,dv\\&\lesssim\left(\int_{\widetilde{\gamma}_2\cap(a^n,\,a^n]}+\int_{\widetilde{\gamma}_2\cap[a^n,\,a^{n-1}]}\right)[\,\mathrm{dist}\,(v,\,\Gamma)]^{\alpha-1}\,dv\\&\lesssim|a^m-u_1|^\alpha+|z_1-a^m|^\alpha\lesssim|u_1-z_1|^\alpha.\end{aligned}$$

Similarly, if $m\ge n+2$, then $|u_1-z_1|\sim a^n$, and thus

$$\begin{aligned}\int_{\gamma_2}[\,\mathrm{dist}\,(v,\,\Gamma)]^{\alpha-1}\,dv&\lesssim\sum_{m'=n}^{m}\int_{\widetilde{\gamma}_2\cap[a^{m'},\,a^{m'-1}]}[\,\mathrm{dist}\,(v,\,\Gamma)]^{\alpha-1}\,dv\\&\lesssim\sum_{m'=n}^{m}a^{m'\alpha}\lesssim a^{n\alpha}\lesssim|u_1-z_1|^\alpha.\end{aligned}$$

This shows (2.10) and finishes the proof of Lemma 2.11. □

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