MAPPINGS WITH AN UPPER GRADIENT IN A LORENTZ SPACE

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ABSTRACT. Suppose that X is a metric measure space that is Ahlfors Q-regular on small scales and supports a Q-Poincaré inequality, $Q > 1$. We show that if a mapping has an upper gradient in the Lorentz space $L^{Q,1}(X)$, then it satisfies the Rado-Reichelderfer condition, and hence possesses a variety of desirable mapping properties, including continuity, Lusin's condition N, and the a.e. finiteness of the Lipschitz constant. These results are sharp.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^n$ and $m \in \mathbb{N}$. If $n = 1$, then each mapping in the Sobolev space $W^{1,1}(\Omega; \mathbb{R}^m)$ has a representative that is absolutely continuous and differentiable almost everywhere. However, when $n \geq 2$, mappings in the space $W^{1,n}(\Omega; \mathbb{R}^m)$ may fail to have these properties [Ser61, Section 9], [MM95, Section 5]. Hence, in this case, the Lebesgue space L^n is not an appropriate n-dimensional analogue of L^1. The correct generalization is given by the Lorentz spaces, which arise naturally in the theory of interpolation of operators and refine the Lebesgue spaces. The requirement that the weak gradient of mapping in $W^{1,n}(\mathbb{R}^n)$ lie in the Lorentz space $L^{n,1}(\mathbb{R}^n) \subseteq L^n(\Omega)$ is known to be a sharp condition guaranteeing the Rado-Reichelderfer condition, and consequently, a variety of desirable mapping properties, including continuity, Lusin's condition N, and differentiability almost everywhere [Ste81], [KKM99].

Recent developments in the study of Sobolev mappings between Heisenberg groups motivate the generalization of the above principle to the setting of abstract measure metric spaces [HT08], [WZ09]. Our main result, stated below, accomplishes this. Precise descriptions of the Rado-Reichelderfer condition and Lusin's condition N are given in Section 3.

Theorem 1.1. *Assume that (X, d, μ) is a complete and doubling metric measure space that supports a Q-Poincaré inequality, $Q > 1$, and is Ahlfors Q-regular at small scales. Let Y be a separable metric space, and suppose that $f \in L^1_{\mathrm{loc}}(X; Y)$ is continuous and has an upper gradient $g \in L^{Q,1}(X)$. Then f that satisfies the Q-Rado-Reichelderfer condition with a weight that depends only on the constants associated to the assumptions and g. Consequently, the mapping f satisfies Lusin's condition N and satisfies* $\operatorname{Lip} f(x) < \infty$ *for almost every $x \in X$.*

Theorem 1.1 has already been used to prove the non-existence of highly regular continuous surjections between certain metric spaces [WZ09, Theorem 1.4]. In turn, corresponding existence results show that the assumption of a Q-Poincaré inequality in Theorem 1.1 cannot be replaced by the assumption of a $(Q+\epsilon)$-Poincaré inequality for any $\epsilon > 0$ [WZ09, Theorem 1.5], and that the space $L^{Q,1}(X)$ cannot be replaced with a larger Lorentz space $L^{Q,q}(X)$ for any $q > 1$.

Results of Romanov establish the Q-absolute continuity of mappings as in Theorem 1.1, see [Rom08, Theorem 2]. A Sobolev-Lorentz embedding theorem has recently been achieved by Ranjbar-Motlagh in a similar setting [RM09]. We also note that a version of Theorem 1.1 concluding Lusin's condition N under a weaker Poincaré inequality assumption but stronger regularity assumption has recently been established by Marola and Ziemer [MZ08].

Our results nearly follow, with a small amount of extra work, from the proofs of the results of Romanov and Ranjbar-Motlagh mentioned above. A difference between this paper and the existing literature is that our interpretation of the Rado-Reichelderfer condition employs the "dyadic" cube

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structure of metric spaces. This simplifies some proofs and allows us to more easily conclude Lusin's condition N. The difficulty in dealing with cubes rather than balls is overcome by the technical Lemma 5.4, which allows us to replace the ball on the right-hand side of the Poincaré inequality with an arbitrarily small ball at the cost of passing to a maximal function.

A sketch of the proof of Theorem 1.1 is as follows. We first show that certain maximal function operators are bounded on the Lorentz space $L^{Q,1}(X)$. We note that in the presence of a suitable Poincaré inequality, the result of applying such a maximal function to an upper gradient of a mapping yields a so-called Hajłasz upper gradient, which satisfies a 1-Poincaré inequality with the original mapping. This implies a pointwise bound on the oscillation of the mapping by the Riesz potential of the Hajłasz upper gradient. The result then follows from imitating the proof in the Euclidean setting.

In Section 2 we discuss the general metric setting and set notation. In Section 3 we define the Rado-Reichelderfer condition and explore some of its consequences. Section 4 gives background information on Lorentz spaces and establishes results on the boundedness of maximal function operators. In Section 5 we improve the Q-Poincaré inequality to a 1-Poincaré inequality by passing to a maximal function, and discuss a key technical trick. Section 6 shows how the previous sections allow Euclidean techniques to be employed and completes the proof of Theorem 1.1.

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2. The metric setting

Given a metric space (X, d), we denote the ball of radius $r > 0$ centered at a point $x \in X$ by

$$B(x, r) = \{z \in X : d(x, z) < r\}.$$

Given an open ball $B = B(x, r)$ and a parameter $\lambda > 0$, we set $\lambda B = B(x, \lambda r)$.

A *metric measure space* is a triple (X, d, μ) where (X, d) is a metric space and μ is a measure on X. For our purposes, a *measure* is a nonnegative countably subadditive set function defined on all subsets of a measure space taking the value zero on the empty set. We further assume that measures are Borel inner and outer regular.

Let (X, d, μ) be a metric measure space and let Y be a separable metric space. We say that a mapping $f\colon X \to Y$ is in $L^1_{\mathrm{loc}}(X; Y)$ if there is a point $y_0 \in Y$ such that the function $x \mapsto d_Y(f(x), y_0)$ is locally integrable. A simple computation shows that if $f \in L^1_{\mathrm{loc}}(X; Y)$ and $T\colon Y \to \mathbb{R}$ is a Lipschitz function, then $T \circ f \in L^1_{\mathrm{loc}}(X)$.

The metric measure space (X, d, μ) is *doubling* if balls have finite and positive measure and there is a constant $C \geq 1$ such $\mu(2B) \leq C\mu(B)$ for any open ball B in X. It follows from the definitions that if (X, d, μ) is a doubling metric measure space, then the metric space (X, d) enjoys the following property, also called *doubling*: there is a number $n \in \mathbb{N}$ such that any ball in X of radius $r > 0$ can be covered by at most n balls of radius $r/2$. It is easy to see that a doubling metric space is complete if and only if it is proper, i.e., closed and bounded sets are compact.

Each metric space carries a structure analogous to that of dyadic cubes in Euclidean space [DS97, Section 5.5], [Chr90, Theorem 11].

Theorem 2.1. *Let $c, s > 0$ satisfy $c + s < 1/4$, and let (X, d, μ) be a doubling metric measure space. Then for each $i \in \mathbb{Z}$ there exists a countable collection $\mathcal{Q}_i$ of open subsets of X such that*

(i) *if $I \in \mathcal{Q}_i$ and $J \in \mathcal{Q}_j$ where $i \leq j$, then $J \subseteq I$ or $J \cap I = \emptyset$, and the latter possibility holds if $i = j$.*
(ii) *if $i < j$, then for any $J \in \mathcal{Q}_j$ there is a unique $I \in \mathcal{Q}_i$ with $J \subseteq I$.*
(iii) *for each $I \in \mathcal{Q}_i$, there is a point $z_I \in X$ such that*

$$B(z_I, cs^i) \subseteq I \subseteq B(z_I, 3s^i),$$

(iv) *for each $i \in \mathbb{Z}$,*

$$\mu(X \backslash \{x \in I : I \in \mathcal{Q}_i\}) = 0,$$

The collection $\mathcal{Q} = \bigcup_{i\in\mathbb{Z}} \mathcal{Q}_i$ is called a *cube structure* on X. An individual open set I in some $\mathcal{Q}_i$ is said to be a *cube*. As the constants c and s are independent of the metric space X, for the remainder of the paper we always assume that $c = 1/16 = s$.

Our main motivation for employing a cube structure is that it provides arbitrarily fine "nearly disjoint" covers.

Proposition 2.2. *Let (X, d, μ) be a doubling metric measure space, and let $\mathcal{Q}$ be a cube structure on X. Then for each $i \in \mathbb{Z}$,*

$$X = \{x \in \overline{I} : I \in \mathcal{Q}_i\}.$$

Proof. Fix $i \in \mathbb{N}$ and let $x \in X$. Consider the collection

$$\mathcal{C} = \{I \in \mathcal{Q}_i : I \cap B(x, 1) \neq \emptyset\}.$$

By Theorem 2.1 (i), cubes of the same generation do not intersect. Hence, Theorem 2.1 (iii) implies that the collection

$$\{B(z_I, cs^i) : I \in \mathcal{C}\}$$

is a collection of pairwise disjoint balls, each of which is contained in $B(x, 5)$. The doubling condition now shows that $\mathcal{C}$ must consist of finitely many cubes. However, by Theorem 2.1 (iv) and the fact that balls in X are assumed to have positive measure, for every integer $n \geq 1$, the ball $B(x, 1/n)$ intersects some cube $I_n \in \mathcal{Q}_i$. Hence $\{I_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}$, and so there is some cube $I \in \mathcal{C}$ which intersects arbitrarily small balls centered at x. It follows that $x \in \overline{I}$. □

Doubling metric spaces are precisely those that have finite Assouad dimension [Hei01, Chapter 10]. However, this notion of dimension is not uniform; a doubling metric space may have some parts or scales where the space appears to be of lower dimension than is actually the case. To prevent this, we employ a stronger notion of dimension. A metric measure space (X, d, μ) is said to be *Ahlfors Q-regular at scales below* $r_0 > 0$ if there is a constant $C \geq 1$ such that if $x \in X$ and $0 < r < r_0$, then

$$\frac{r^Q}{C} \leq \mu(B(x, r)) \leq Cr^Q.$$

If the threshold radius is unimportant, we say that (X, d, μ) is *Ahlfors Q-regular at small scales.*

Let $f\colon X \to Y$ be a mapping between metric spaces. An *upper gradient* of f is a Borel function $g\colon X \to [0, \infty]$ such that for each rectifiable path $\gamma\colon [0, 1] \to X$,

$$d_Y(f(\gamma(0)), f(\gamma(1))) \leq \int_\gamma g\, ds.$$

If X contains no rectifiable curves, then the constant function with value 0 is an upper gradient of any mapping. If f is locally Lipschitz, then the local Lipschitz constant of f, defined by

$$\mathrm{Lip}(f)(x) = \liminf_{r\to 0} \sup_{y\in B(x,r)} \frac{d_Y(f(x), f(y))}{r},$$

is an upper gradient of f [Che99, Proposition 1.11]. The concept of an upper gradient was introduced by Heinonen and Koskela [HK98], and substantial literature indicates that is a suitable analogue of the weak gradient of a mapping between open sets in Euclidean space [HKST01].

A key idea in theory of analysis on metric spaces is to measure the plentitude of curves in a given space. Fundamental work has resulted in an analytic condition which guarantees the presence of "many" rectifiable curves in a metric space [HK98]. Let $p \geq 1$, and let f and g be measurable functions on a metric measure space (X, d, μ). The pair (f, g) *satisfies a p-Poincaré inequality* with constant $C > 0$ and dilation factor $\sigma > 0$ if for each ball B in X,

$$\fint_B |f - f_B|\, d\mu \leq C(\operatorname{diam} B)\Big(\fint_{\sigma B} g^p\, d\mu\Big)^{\frac{1}{p}}. \tag{2.1}$$

For a set $E \subset X$ with $0 < \mu(E) < \infty$, we use the notation

$$f_E := \fint_E f\, d\mu := \frac{1}{\mu(E)} \int f\, d\mu.$$

The space (X, d, μ) *supports a p-Poincaré* inequality if there is a constant $C > 0$ and a dilation factor $\sigma > 0$ such that for each measurable function f on X and each upper gradient g of f, the pair (f, g) satisfies a p-Poincaré inequality with constant C and dilation factor σ.

A deep theorem of Keith and Zhong states that the Poincaré inequality is an open condition [KZ08, Theorem 1.0.1].

Theorem 2.3. *Let $p > 1$ and let (X, d, μ) be a complete and doubling metric measure space that supports a p-Poincaré inequality with constant C and dilation factor σ. Then there exists $1 \leq q < p$ such that (X, d, μ) supports a q-Poincaré inequality, with constant and dilation factor depending only on doubling constant and the constant C and dilation factor σ of the original Poincaré inequality.*

We note that a doubling metric measure space (X, d, μ) that supports a p-Poincaré inequality for some $p \geq 1$ is connected in a strong sense [Kor07].

For convenience, for the remainder of the paper we will consider only spaces with the following properties.

Standing Assumption 2.4. Let (X, d, μ) be a doubling metric measure space that is locally compact, connected, and equipped with a cube structure $\mathcal{Q}$.

3. The Rado-Reichelderfer condition and its consequences

Let Y be a separable metric space. As mentioned in the introduction, the condition that a mapping $f\colon X \to Y$ have an upper gradient in the Lorentz space $L^{Q,1}(X)$ implies that the mapping satisfies several desirable properties. Many of these are implied by the following Rado-Reichelderfer condition, which is analogous to the original version [RR55].

Definition 3.1. Let $Q > 1$. A mapping $f\colon X \to Y$ satisfies the *Q-Rado-Reichelderfer condition* with respect to $\mathcal{Q}$ if there is a non-negative function $\Theta \in L^1_{\mathrm{loc}}(X)$ and a scale $r_0 > 0$ such that for any cube $I \in \mathcal{Q}$ with compact closure in X and diameter less than r_0,

$$(\operatorname{diam} f(I))^Q \leq \int_I \Theta \, d\mu.$$

The reason for employing cubes instead of balls in the Rado-Reichelderfer condition, as was done in [KKM99], is to obtain Lusin's condition N in the general metric setting.

Definition 3.2. Let $Q > 0$. A mapping $f\colon X \to Y$ satisfies *Lusin's condition N_Q* if every set $E \subseteq X$ satisfying $\mu(E) = 0$ also satisfies $\mathcal{H}^Q(f(E)) = 0$.

Lusin's condition N is of great importance in analysis, as it implies the validity of a change of variables formula.

Theorem 3.3. *If $f\colon X \to Y$ is a continuous mapping that satisfies the Q-Rado-Reichelderfer condition with respect to some cube structure $\mathcal{Q}$, then f satisfies Lusin's condition N_Q.*

Proof. Suppose that $E \subseteq X$ satisfies $\mu(E) = 0$. Since X is doubling, it is separable. This and the assumption of local compactness imply that we may find a countable open cover $\{U_n\}$ of E such that each set U_n has compact closure. By the countable subadditivity of $\mathcal{H}^Q$, we may assume that E itself is contained in an open set U with compact closure. Moreover, as μ is always assumed to be Borel outer regular, for any $\epsilon > 0$ we may find a smaller open set $U_\epsilon \subseteq U$ that satisfies $\mu(U_\epsilon) < \epsilon$. As f is continuous, it is uniformly continuous on U_ϵ. Hence we may find $\delta > 0$ such that if $I \subseteq U_\epsilon$ has diameter less than δ, then $f(I)$ has diameter less than ϵ. We may assume that $\delta < \min\{\epsilon, r_0\}$.

By Proposition 2.2 and Theorem 2.1, for each $x \in E$, we may find a cube $I_x \in \mathcal{Q}$ with $x \in \overline{I_x}$ that is so small that $\overline{I_x} \subseteq U_\epsilon$ and $\operatorname{diam}(\overline{I_x}) < \delta$. Using Theorem 2.1 (i) and the fact that the entire cube structure $\mathcal{Q}$ contains only countably many cubes, we may find a countable cover $\{\overline{I_n}\}_{n \in \mathbb{N}}$ of E by closures of cubes that are contained in U_ϵ and have diameter less than δ, and such that the collection $\{I_n\}_{n\in\mathbb{N}}$ is pairwise disjoint. Now, $\{f(\overline{I_n})\}_{n\in\mathbb{N}}$ is a cover of $f(E)$ by sets of diameter less than ϵ, and so by the continuity of f we see that

$$\mathcal{H}^{Q,\epsilon}(f(E)) \le \sum_{n\in\mathbb{N}} \left(\operatorname{diam} f(\overline{I_n})\right)^Q = \sum_{n\in\mathbb{N}} \left(\operatorname{diam} f(I_n)\right)^Q.$$

Applying the Rado-Reichelderfer condition and using the disjointness of the collection $\{I_n\}_{n\in\mathbb{N}}$, we see that

$$\mathcal{H}^{Q,\epsilon}(f(E)) \le \sum_n \int_{I_n} \Theta \, d\mu \le \int_{U_\epsilon} \Theta \, d\mu.$$

As $\Theta \in L^1(U)$, letting ϵ tend to 0 shows that $\mathcal{H}^Q(f(E)) = 0$. □

In appropriate circumstances, the Rado-Reichelderfer condition also implies that the mapping in question has finite Lipschitz constant almost everywhere. Combined with a Stepanov-type theorem, this has implications for differentiability [Che99], [BRZ04].

Proposition 3.4. *Assume that (X, d, μ) is Ahlfors Q-regular at small scales. If $f\colon X \to Y$ satisfies the Q-Rado-Reichelderfer condition, then* $\operatorname{Lip} f(x) < \infty$ *for almost every $x \in X$.*

Proof. By Theorem 2.1 (iv), we may find a set $N \subseteq X$ of measure zero such that if $x \in X\backslash N$, then for every $i \in \mathbb{N}$, there is a cube $I \in \mathcal{Q}_i$ containing x.

Let $r_0 > 0$ be the scale below which both the Ahlfors regularity condition and the Q-Rado-Reichelderfer condition hold. Let $x \in X\backslash N$ and $0 < r < csr_0/6$, and choose $i \in \mathbb{N}$ such that $cs^{i+1} \le r < cs^i$. Let $I \in \mathcal{Q}_i$ be the cube containing x. Then $\operatorname{diam} I \le 6s^i < r_0$, and so $I \subseteq B(x, 7s^i)$. Thus,

$$\begin{aligned}\frac{\operatorname{diam} f(B(x,r))}{r} \le \frac{\operatorname{diam} f(I)}{r} &\le \frac{1}{r}\left(\int_I \Theta \, d\mu\right)^{1/Q} \\ &\le \left((cs^{i+1})^{-Q} \int_{B(x,7s^i)} \Theta \, d\mu\right)^{1/Q} \\ &\le \frac{7C^{1/Q}}{cs}\left(\fint_{B(x,7s^i)} \Theta \, d\mu\right)^{1/Q},\end{aligned}$$

where C is the constant from the Ahlfors regularity condition. As $s < 1$ and i tends to infinity as r tends to 0, the Lebesgue Differentiation Theorem now yields that for almost every $x \in X\backslash N$

$$\operatorname{Lip} f(x) \le \frac{7C^{1/Q}}{cs}\Theta(x).$$

Since $\Theta \in L^1(X)$, it is finite almost everywhere. □

4. Maximal function operators on Lorentz spaces

In this section, we show that certain maximal function operators are bounded on appropriate Lorentz spaces. The main tool is the Marcinkiewicz Interpolation theorem. We begin with a review of the basics facts regarding Lorentz spaces.

4.1. **Lorentz spaces.** We denote by $\mathcal{M}$ the collection of all extended real-valued μ-measurable functions on X. Let $\mathcal{M}_0$ be the class of functions in $\mathcal{M}$ that are finite μ-almost everywhere.

Given $f \in \mathcal{M}_0(X)$, we define the *distribution function* $\omega_f\colon [0,\infty) \to [0,\infty]$ of f by

$$\omega_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\}).$$

The *non-increasing rearrangement* $f^*\colon [0,\infty) \to [0,\infty]$ of f is given by

$$f^*(t) = \inf\{\alpha \ge 0 : \omega_f(\alpha) \le t\}.$$

It is a useful exercise to show that given $0 < s < \mu(X)$ and $0 < t < \infty$,

$$s < f^*(t) \iff \omega(s) > t. \tag{4.1}$$

Let $1 \leq Q \leq \infty$ and $0 < q \leq \infty$. The (Q, q)-*Lorentz class* consists of those functions $f \in \mathcal{M}_0(X)$ such that the quantity

$$\|f\|_{Q,q} := \begin{cases} (\int_0^\infty (t^{1/Q} f^*(t))^q \, \frac{dt}{t})^{1/q}, & 0 < q < \infty, \\ \sup_{0<t<\infty}\{t^{1/Q} f^*(t)\}, & Q < \infty \text{ and } q = \infty, \\ f^*(0), & Q = \infty = q \end{cases}$$

is finite.

If $1 \leq q \leq Q$, then $||\cdot||_{Q,q}$ defines a semi-norm on the (Q, q)-Lorentz class, and the corresponding normed space $(L^{Q,q}(X), ||\cdot||_{Q,q})$ is a Banach space, which we refer to as the (Q, q)-Lorentz space [BS88, Theorem IV.4.3]. Without these restrictions on Q and q, the functional $f \to ||f||_{Q,q}$ is not always a semi-norm, though it is always equivalent to a semi-norm on the (Q, q)-Lorentz class that defines a Banach space [BS88, Theorem IV.4.6].

Remark 4.1. It follows from the definitions that if $f, g \in \mathcal{M}_0(X)$ satisfy $f \leq g$ almost everywhere, then $||f||_{Q,q} \leq ||g||_{Q,q}$ whenever the latter is finite.

The following statement collects the relationships between the various Lorentz spaces [BS88, Proposition IV.4.2] .

Proposition 4.2. *Let $1 \leq Q \leq \infty$ and $0 < q \leq r \leq \infty$. Then there is a quantity $C > 0$ depending only on Q, q, and r, such that for all $f \in \mathcal{M}_0(X)$,*

$$\|f\|_{Q,r} \leq C \, \|f\|_{Q,q} \,.$$

That is, the (Q, q)-Lorentz class is a subset of the (Q, r)-Lorentz class, quantitatively.

Combined with Proposition 4.2, the following result shows how the Lorentz classes refine the Lebesgue classes. It follows easily from [BS88, Proposition II.1.8].

Proposition 4.3. *Let $1 \leq Q \leq \infty$ and let $f \in \mathcal{M}_0(X)$. Then $f \in L^{Q,Q}(X)$ if and only if $f \in L^Q(X)$, and*

$$\|f\|_{Q,Q} = \|f\|_Q \,.$$

Given $1 \leq Q < \infty$ and $0 < r \leq Q$, a function $f \in \mathcal{M}_0(X)$ is in $L^Q(X)$ if and only if $|f|^r$ is in $L^{Q/r}(X)$. An analogous result holds for Lorentz classes.

Proposition 4.4. *Let $1 \leq Q \leq \infty$, $0 < q \leq \infty$, and $0 < r \leq Q$. If $f \in \mathcal{M}_0(X)$, then $f \in L^{Q,q}(X)$ if and only if $|f|^r \in L^{\frac{Q}{r},\frac{q}{r}}(X)$, and*

$$\||f|^r\|_{\frac{Q}{r},\frac{q}{r}} = \|f\|^r_{Q,q} \,.$$

Proposition 4.4 follows quickly from the following lemma, which states that we may interchange exponents and the non-increasing rearrangement.

Lemma 4.5. *Let $f \in \mathcal{M}_0(X)$, and let $0 < r < \infty$. Then for all $t \in [0, \infty)$,*

$$(|f|^r)^*(t) = (f^*)^r(t).$$

Proof. We calculate

$$\begin{aligned} (|f|^r)^*(t) &= \inf\{\alpha \geq 0 : \, \mu_{|f|^r}(\alpha) \leq t\} = \inf\{\alpha \geq 0 : \, \mu\{x \in X : \, |f(x)|^r > \alpha\} \leq t\} \\ &= \inf\{\alpha \geq 0 : \, \mu\{x \in X : |f(x)| > \alpha^{1/r}\} \leq t\} = \inf\{\beta^r : \, \mu\{x \in X : \, |f(x)| > \beta\} \leq t\} \\ &= (\inf\{\beta : \, \mu\{x \in X : \, |f(x)| > \beta\} \leq t\})^r = (f^*(t))^r, \end{aligned}$$

as desired. $\square$

4.2. **Maximal function theorems for Lorentz spaces on metric spaces.** We establish the boundedness of the Hardy-Littlewood maximal function on Lorentz spaces on X. Our proof of this result basically follows the proof of the classical maximal function theorems in Euclidean space given in [BS88, Section III.3].

Recall that the Hardy-Littlewood maximal function M is defined for all locally integrable functions f with values in the extended-reals by

$$M(f)(x) := \sup_{r>0} \fint_{B(x,r)} |f| \, d\mu.$$

Similarly, the R-restricted Hardy-Littlewood maximal function M_R is defined by

$$M_R(f)(x) := \sup_{R>r>0} \fint_{B(x,r)} |f| \, d\mu$$

The standard maximal function theorems remain valid in our setting [Hei01, Theorem 2.2].

Theorem 4.6. *There is a quantity $C_1 \geq 1$ depending only on the doubling constant of μ such that for all $f \in L^1(X)$ and $t > 0$*

$$\mu(\{x \in X : |M(f)(x)| > t\}) \leq \frac{C_1 ||f||_{L^1}}{t}. \tag{4.2}$$

Moreover, for each $1 < p \leq \infty$, there is a constant $C_p \geq 1$ depending only on the doubling constant of μ such that for all $f \in L^p(X)$

$$||Mf||_{L^p} \leq C_p ||f||_{L^p}. \tag{4.3}$$

The constant C_∞ may be chosen to be 1.

The weak estimate (4.2) easily passes to the non-increasing rearrangement of the maximal function.

Lemma 4.7. *Let $f \in L^1(X)$ and $t > 0$. Then*

$$(M(f))^*(t) \leq \frac{C_1 \, \|f\|_{L^1}}{t}.$$

Proof. By (4.2),

$$\begin{aligned}(M(f))^*(t) &= \inf\{\alpha \geq 0 : \ \mu(\{x \in X : |M(f)(x)| > \alpha\}) \leq t\} \\ &\leq \inf\{\alpha > 0 : \ \frac{C_1 ||f||_{L^1}}{\alpha} \leq t\} = \inf\{\alpha > 0 : \ \frac{C_1 ||f||_{L^1}}{t} \leq \alpha\} \\ &= \frac{C_1 ||f||_{L^1}}{t},\end{aligned}$$

as desired. □

Let $f \in L^1_{\text{loc}}(X)$, and let $t > 0$. Set

$$f^{**}(t) := \fint_0^t f^*(s) \, ds.$$

From the fact that f^* is non-decreasing, we see that f^{**} is the restricted maximal function of the non-increasing rearrangement of f. The following result establishes the relationship of f^{**} to the non-increasing rearrangement of the maximal function of f. The proof given in [BS88, Theorem III.3.8] remains valid in this setting.

Lemma 4.8. *There is a constant $c > 0$ such that for all $f \in L^1_{\text{loc}}(X)$ and $t > 0$,*

$$c(Mf)^*(t) \leq f^{**}(t).$$

The following inequality, due to Hardy, plays a central role in many interpolation theorems [BS88, Lemma III.3.9]. Stated in this fashion, its relevance to the matters at hand is clear.

Lemma 4.9 (G. H. Hardy)**.** *Let ψ be a nonnegative measurable function on $(0,\infty)$, and let $Q>1$ and $1\le q\le Q$. Then there is a quantity $C>0$ depending only on q and Q such that*

$$\int_0^\infty \left(t^{1/Q}\fint_0^t \psi(s)\,ds\right)^q \frac{dt}{t} \le C\int_0^\infty (t^{1/Q}\psi(t))^q\frac{dt}{t}.$$

Proposition 4.10. *Let $Q>1$ and $1\le q\le Q$. Then there is a constant $C>0$ depending only on q and Q such that for all $f\in L^{Q,q}(X)$,*

$$||Mf||_{L^{Q,q}} \le C||f||_{L^{Q,q}}.$$

Proof. By Lemma 4.8,

$$||Mf||^q_{L^{Q,q}} = \int_0^\infty (t^{1/Q}(M(f))^*(t))^q\,\frac{dt}{t} \le c^{-q}\int_0^\infty (t^{1/Q}f^{**}(t))^q\,\frac{dt}{t}.$$

Recalling the definition of f^{**} and applying Lemma 4.9, we see that

$$\begin{aligned}||Mf||^q_{L^{Q,q}} &\le c^{-q}\int_0^\infty \left(t^{1/Q}\fint_0^t f^*(s)\,ds\right)^q\frac{dt}{t}\\ &\le Cc^{-q}\int_0^\infty (t^{1/Q}f^*(t))^q\frac{dt}{t} = Cc^{-q}||f||^q_{L^{Q,q}},\end{aligned}$$

as desired. □

4.3. **Perturbed maximal operators on Lorentz spaces.** For a number $p\ge 1$, consider the operator T_p defined on $L^p_{\mathrm{loc}}(X)$ by

$$T_p(g) = (M(g^p))^{\frac{1}{p}}. \tag{4.4}$$

Fix $Q>1$ and choose $0<\epsilon<1$ such that $Q>1+\epsilon$. For the remainder of this section, we set $T=T_{Q-\epsilon}$. Our aim is to show that T maps $L^{Q,1}(X)$ into itself. To do so, we will employ the Marcinkiewicz interpolation theorem, which applies to quasilinear operators of weak type. We refer to [BS88, Sections III.5 and IV.4] for more information.

Definition 4.11 (quasilinearity)**.** Let S be an operator that is defined on a linear subspace of $\mathcal{M}_0(X)$, and that has range contained in the set of measurable functions on X. Then S is *quasilinear* if there is a constant $k\ge 1$ such that for all functions f and g in the domain of S, and all $\lambda\in\mathbb{R}$, the following relations hold μ-a.e. on X:

$$\begin{aligned}|S(f+g)| &\le k(|Sf|+|Sg|),\\ |S(\lambda f)| &= |\lambda|\,|Sf|\,.\end{aligned}$$

We recall for future use the elementary estimate

$$(a+b)^p \le 2^p(a^p+b^p), \tag{4.5}$$

which is valid for all $p>0$ and all non-negative real numbers a and b. We also recall that a locally integrable function is finite almost everywhere, i.e., $L^1_{\mathrm{loc}}(X)\subseteq\mathcal{M}_0(X)$.

Lemma 4.12. *The operator T is quasilinear on the domain $L^{Q-\varepsilon/2}(X)+L^{Q+\varepsilon/2}(X)$. Moreover, the range of T on this domain is contained in the set of measurable functions that are finite almost everywhere.*

Proof. We begin by showing the statement regarding the range of T. Let $g=g_1+g_2$, where $g_1\in L^{Q-\varepsilon/2}(X)$ and $g_2\in L^{Q+\varepsilon/2}(X)$. We can write this as

$$|g_1|^{Q-\varepsilon}\in L^{\frac{Q-\varepsilon/2}{Q-\varepsilon}}(X)\subseteq L^1_{\mathrm{loc}}(X), \quad\text{and}\quad |g_2|^{Q-\varepsilon}\in L^{\frac{Q+\varepsilon/2}{Q-\varepsilon}}(X)\subseteq L^1_{\mathrm{loc}}(X). \tag{4.6}$$

By (4.5), the inclusions (4.6) imply that $|g|^{Q-\epsilon}$ is locally integrable. Hence $T(g)$ is sensibly defined.

By Theorem 4.6, the inclusions (4.6) also imply that

$$M(|g_1|^{Q-\varepsilon})\in L^{\frac{Q-\varepsilon/2}{Q-\varepsilon}}(X)\subseteq L^1_{\mathrm{loc}}(X), \quad\text{and}\quad M(|g_2|^{Q-\varepsilon})\in L^{\frac{Q+\varepsilon/2}{Q-\varepsilon}}(X)\subseteq L^1_{\mathrm{loc}}(X). \tag{4.7}$$

Again using (4.5), we now see that for each $x \in X$

$$\begin{aligned} T(g)(x) &= \left(\sup_{R>0} \fint_{B(x,R)} |(g_1+g_2)(y)|^{Q-\varepsilon} \, d\mu(y) \right)^{\frac{1}{Q-\epsilon}} \\ &\leq 2 \left(\sup_{R>0} \left(\fint_{B(x,R)} |g_1(y)|^{Q-\varepsilon} \, d\mu(y) + \fint_{B(x,R)} |g_2(y)|^{Q-\varepsilon} \, d\mu(y) \right) \right)^{\frac{1}{Q-\epsilon}} \\ &\leq 2 \left(M(|g_1|^{Q-\varepsilon})(x) + M(|g_2|^{Q-\varepsilon})(x) \right)^{\frac{1}{Q-\epsilon}} \end{aligned}$$

Thus the inclusions (4.7) in fact show that $T(g)$ is locally integrable, and in particular it is in $\mathcal{M}_0(X)$.

Moreover, if g and h are arbitrary elements of the domain of T, the same argument as above and another application of (4.5) show that for each $x \in X$,

$$\begin{aligned} T(g+h)(x) &\leq 2 \left(M(|g|^{Q-\varepsilon})(x) + M(|h|^{Q-\varepsilon})(x) \right)^{\frac{1}{Q-\epsilon}} \\ &\leq 2 \cdot 2^{\frac{1}{Q-\epsilon}} \left(T(g)(x) + T(h)(x) \right). \end{aligned}$$

Finally, for $\alpha \in \mathbb{R}$, we get

$$|T(\alpha g)(x)| = \left(\sup_{R>0} \fint_{B(x,R)} |\alpha g(y)|^{Q-\varepsilon} \, d\mu(y) \right)^{\frac{1}{Q-\varepsilon}} = |\alpha| \, |T(g)(x)| \, ,$$

establishing the quasilinearity of T. □

Proposition 4.2 and Proposition 4.3 show that the above lemma implies the following statement.

Lemma 4.13. *If $Q > 1 + \epsilon/2$, then the operator T is quasilinear on the domain $L^{Q-\varepsilon/2,1}(X) + L^{Q+\varepsilon/2,1}(X)$. Moreover, the range of T on this domain is contained in the set of measurable functions that are finite almost everywhere.*

Definition 4.14 (weak type)**.** Let (X, μ) and (Y, ν) be two σ-finite measure spaces and suppose that $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Let S be an operator defined on $L^{p,1}(X)$ and taking values in $\mathcal{M}_0(Y)$. Then S is said to be of *weak type* (p, q) if it is a bounded operator from $L^{p,1}(X)$ into $L^{q,\infty}(Y)$, that is, if there is a constant $C > 0$ such that

$$\|Sf\|_{q,\infty} \leq C \, \|f\|_{p,1} \, .$$

Lemma 4.15. *If $Q > 1 + \epsilon$, then the operator T is of weak types $(Q - \epsilon/2, Q - \epsilon/2)$ and $(Q + \epsilon/2, Q + \epsilon/2)$.*

Proof. We show that T has weak type $(Q - \epsilon/2, Q - \epsilon/2)$. The proof that T has weak type $(Q + \epsilon/2, Q + \epsilon/2)$ is identical. For the remainder of the proof, we denote by C a positive number, possibly varying in each occurrence, that depends only Q and ϵ.

Let $g \in L^{Q-\epsilon/2,1}(X)$. By Proposition 4.4 and Proposition 4.2,

$$\|T(g)\|_{Q-\varepsilon/2,\infty} = \left\| (M(|g|^{Q-\varepsilon})) \right\|^{\frac{1}{Q-\varepsilon}}_{\frac{Q-\varepsilon/2}{Q-\varepsilon},\infty} \leq C \left\| M(|g|^{Q-\varepsilon}) \right\|^{\frac{1}{Q-\varepsilon}}_{\frac{Q-\varepsilon/2}{Q-\varepsilon},1} . \tag{4.8}$$

Similarly,

$$\left\| |g|^{Q-\varepsilon} \right\|_{\frac{Q-\varepsilon/2}{Q-\varepsilon},1} \leq C \left\| |g|^{Q-\varepsilon} \right\|_{\frac{Q-\varepsilon/2}{Q-\varepsilon},\frac{1}{Q-\varepsilon}} = C \, \|g\|^{Q-\varepsilon}_{Q-\varepsilon/2,1} .$$

Thus, $|g|^{Q-\varepsilon} \in L^{\frac{Q-\varepsilon/2}{Q-\varepsilon},1}(X)$. Proposition 4.10 states that the maximal function is bounded on Lorentz spaces. Hence, by applying (4.8), followed by Lemma 4.5 and Proposition 4.2, we obtain

$$\|T(g)\|_{Q-\varepsilon/2,\infty} \leq C \left\| |g|^{Q-\varepsilon} \right\|^{\frac{1}{Q-\varepsilon}}_{\frac{Q-\varepsilon/2}{Q-\varepsilon},1} = C \, \|g\|_{Q-\varepsilon/2,Q-\varepsilon} \leq C \, \|g\|_{Q-\varepsilon/2,1} \, ,$$

as desired. □

For the reader's convenience, we record the Marcinkiewicz Interpolation Theorem, as stated in [BS88, Theorem IV.4.13].

Theorem 4.16 (Marcinkiewicz Interpolation Theorem)**.** *Suppose* $1 \leq p_0 < p_1 < \infty$ *and* $1 \leq q_0, q_1 \leq \infty$ *with* $q_0 \neq q_1$*. Let* $0 < \theta < 1$ *and define* p *and* q *by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$
$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let T *be a quasilinear operator defined on* $(L^{p_0,1} + L^{p_1,1})(X,\mu)$ *and taking values in* $\mathcal{M}_0(Y,\nu)$*, where* (X,μ) *and* (Y,ν) *are* σ*-finite measure spaces. Suppose* T *is of weak types* (p_0, q_0) *and* (p_1, q_1)*. If* $1 \leq r \leq \infty$*, then* $T\colon L^{p,r}(X) \to L^{q,r}(Y)$ *is a bounded operator, with constant depending only on* p_0*,* p_1*,* q_0*,* q_1*,* θ*,* r*, and the constants associated to the weak type conditions.*

Proposition 4.17. *The operator* T *is a bounded operator from* $L^{Q,1}(X)$ *to itself, with constant depending only on* Q *and* ϵ*.*

Proof. Let $p_0 = Q - \epsilon/2 = q_0$ and $p_1 = Q + \epsilon/2 = q_1$. Set $\theta = \frac{Q+\varepsilon/2}{2Q}$. Then $0 < \theta < 1$ and

$$\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{Q-\varepsilon/2}\left(\frac{2Q - Q - \varepsilon/2}{2Q}\right) + \frac{1}{Q+\varepsilon/2}\left(\frac{Q+\varepsilon/2}{2Q}\right) = \frac{1}{Q}.$$

This also implies that

$$\frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1}{Q}.$$

By Lemma 4.13 and Lemma 4.15, T satisfies the hypotheses of Marcinkiewicz's Interpolation Theorem 4.16 for these parameters, yielding the desired result. □

5. Improving the Poincaré inequality

If the metric space X contains no rectifiable curves, the upper gradient condition is vacuous. Hajłasz introduced the following notion of an upper gradient, which does not rely on curves [Haj03].

Definition 5.1 (Hajłasz upper gradient)**.** Let $f \in \mathcal{M}_0(X)$. A *Hajłasz upper gradient* of f is a measurable function $g\colon X \to [0,\infty]$ such that for almost every $x, y \in X$,

$$|f(x) - f(y)| \leq d(x,y)(g(x) + g(y)). \tag{5.1}$$

A function and Hajłasz upper gradient pair always satisfies a p-Poincaré inequality, for any $p \geq 1$ [Haj03, Section 9].

Theorem 5.2 (Hajłasz)**.** *Let* $p \geq 1$*. If* $f \in \mathcal{M}_0(X)$ *has a Hajłasz upper gradient* g*, then the pair* (f,g) *satisfies a* p*-Poincaré inequality with constant depending only on the doubling constant, and with dilation factor* 1*.*

Theorem 5.2 has a converse of sorts [Haj03, Theorem 9.4]. Recall the definition of the operator T_p, $p \geq 1$, from (4.4).

Theorem 5.3 (Hajłasz)**.** *Let* $p \geq 1$*. If the pair* (f,g) *satisfies a* p*-Poincaré inequality with constant* C *and any dilation factor, then there is a quantity* $\widetilde{C} > 0$*, depending only on* C *and the doubling constant, such that* $\widetilde{C}T_p(g)$ *is a Hajłasz upper gradient of* f*.*

In the standard Poincaré inequality, the ball on the right-hand side may be larger than that on the left. The following technical lemma essentially states that we may avoid this, provided we pass to a maximal function.

Lemma 5.4. *Fix* $0 < \lambda \leq \sigma < \infty$*. Suppose that* $f, g \in L^1_{\mathrm{loc}}(X)$*, the function* g *is non-negative almost everywhere, and the pair* (f,g) *satisfies a* 1*-Poincaré inequality with constant* $C > 1$ *and dilation* σ*. Then the pair* $(f, M(g))$ *satisfies a* 1*-Poincaré inequality with dilation factor* λ*, and with constant depending only* C *and the doubling constant. In particular, the constant does not depend on* λ*.*

Proof. Fix a ball $B = B(z, r)$ in X. Note that if $x \in \lambda B$, then

$$\sigma B \subseteq B(x, (\sigma + \lambda)r) \subseteq (\sigma + 2\lambda)B.$$

Using this and the assumptions, we estimate

$$\begin{aligned}
\fint_{\lambda B} M(g)(x)\, d\mu(x) &\geq \fint_{\lambda B} \left(\fint_{B(x,(\sigma+\lambda)r)} g(y)\, d\mu(y) \right) d\mu(x) \\
&\geq \fint_{\lambda B} \frac{1}{\mu(B(x, (\sigma+\lambda)r))} \left(\int_{\sigma B} g(y)\, d\mu(y) \right) d\mu(x) \\
&\geq \fint_{\lambda B} \frac{1}{\mu((\sigma+2\lambda)B)} \left(\int_{\sigma B} g(y)\, d\mu(y) \right) d\mu(x) \\
&= \frac{1}{\mu((\sigma+2\lambda)B)} \int_{\sigma B} g(y)\, d\mu(y)
\end{aligned}$$

By the Lebesgue differentiation theorem, at almost every point $y \in \lambda B$, the function g satisfies

$$g(y) = \lim_{R \to 0} \fint_{B(y,R)} g(x)\, d\mu(x) \leq M(g)(y).$$

Thus, renaming variables and using the fact that $\lambda \leq \sigma$, we see that

$$\fint_{\lambda B} M(g)(x)\, d\mu(x) \geq \frac{\mu(\sigma B)}{\mu(3\sigma B)} \fint_{\sigma B} M(g)(x)\, d\mu(x).$$

The desired result now follows from the doubling property and the assumption that the pair (f, g) satisfies a 1-Poincaré inequality with dilation factor σ. $\square$

We may combine these statements with the results of Section 4.

Theorem 5.5. *Let $Q > 1$ and $\lambda \leq 1$. Assume that (X, d, μ) is complete and supports a Q-Poincaré inequality with constant $C > 1$ and any dilation factor. If $f \in \mathcal{M}_0(X)$ has an upper gradient $g \in L^{Q,1}(X)$, then there is a Hajłasz upper gradient $\rho \in L^{Q,1}(X)$ of f such that the pair (f, ρ) satisfies a 1-Poincaré inequality with dilation factor λ and with constant depending only on C and the doubling constant.*

Proof. For this proof only, we refer to C and the doubling constant as the *data*. By Theorem 2.3, the pair (f, g) satisfies a $(Q - \epsilon)$-Poincaré inequality for some $\epsilon > 0$, with constant depending only on the data, and with some unspecified dilation factor. By Hölder's inequality, we may assume that $\epsilon < Q - 1$.

By Theorem 5.3, there is a constant $\widetilde{C} > 0$, depending only on the data, such that the function $\widetilde{g} := \widetilde{C} T_{Q-\epsilon}(g)$ is a Hajłasz upper gradient of f. Theorem 5.2 shows that the pair $(f, \widetilde{g})$ satisfies a 1-Poincaré inequality with constant depending only on the data and dilation factor 1, and Proposition 4.17 implies that $\widetilde{g} \in L^{Q,1}(X)$.

By Lemma 5.4, the pair $(f, M(\widetilde{g}))$ satisfies a 1-Poincaré inequality with constant depending only on the data, and with smaller dilation factor λ. By the Lebesgue differentiation theorem, $M(\widetilde{g}) \geq \widetilde{g}$ almost everywhere, and hence $M(\widetilde{g})$ is also a Hajłasz upper gradient of f. Finally, Proposition 4.10 implies that $M(\widetilde{g}) \in L^{Q,1}(X)$. $\square$

Remark 5.6. A Rado-Reichelderfer condition using balls now follows from Theorem 5.5 and the proof of [RM09, Theorem 3.2].

6. Proof of the main result

6.1. **The Poincaré inequality and the Riesz potential.** We first show that if a pair (f, g) satisfies a 1-Poincaré inequality, then the Riesz potential of g provides a pointwise bound on f. In the Euclidean setting, such a result may be found in [MZ97]. Since this bound will eventually lead to the Rado-Reichelderfer condition, we employ cubes rather than the usual balls. We recall that $c = 1/16$ is the fixed universal constant used to specify the "size" of a cube; see Theorem 2.1 (iii).

Proposition 6.1. *Assume that X is Ahlfors Q-regular, $Q > 1$, on scales below r_0 with constant C_Q. Suppose that f and g are in the space $L^1_{\mathrm{loc}}(X)$, and that the pair (f, g) satisfies a 1-Poincaré inequality with constant C_P and dilation factor $0 < \sigma < c/3$. Then there exists a quantity C depending only on C_P, σ, C_Q, and Q, such that for almost every point $z \in X$, if I is a cube of diameter less than $cr_0/6$ that contains z, then*

$$|f(z) - f_I| \leq C \int_I \frac{d(z,x)}{\mu(B(z, d(z,x)))} g(x)\, d\mu(x). \tag{6.1}$$

Proof. Throughout this proof we denote by C a quantity, possibly changing at each occurrence, that depends only on C_P, σ, C_Q, and Q.

By the Lebesgue density theorem and Theorem 2.1 (iv), there is a set N of measure zero such that if $z \notin N$, then z is a Lebesgue point if f and for all $k \in \mathbb{Z}$ there is a cube $I_k \in \mathcal{Q}_k$ that contains z. Recalling Theorem 2.1 (iii), for each $k \in \mathbb{Z}$ we write

$$A_k := B(z_{I_k}, cs^k) \subseteq I_k \subseteq B(z_{I_k}, 3s^k) =: B_k. \tag{6.2}$$

Fix $z \notin N$ and let I be a cube containing z. By Theorem 2.1 (i) and (ii), we may assume that $I = I_{k_0}$ for some $k_0 \in \mathbb{Z}$. The assumption that I have diameter less than $cr_0/6$ implies by (6.2) that we may apply the Ahlfors regularity condition to all balls of radius $3s^{k_0}$ and smaller.

As z is a Lebesgue point of f, the triangle inequality yields

$$|f(z) - f_I| \leq \sum_{k=k_0}^{\infty} |f_{I_{k+1}} - f_{I_k}| \leq \sum_{k=k_0}^{\infty} \left(\fint_{I_{k+1}} |f(y) - f_{B_k}|\, d\mu(y) + |f_{I_k} - f_{B_k}| \right). \tag{6.3}$$

Fix $k \geq k_0$. Using the Ahlfors regularity condition and the Poincaré inequality, we estimate

$$\begin{aligned}
\fint_{I_{k+1}} |f(y) - f_{B_k}|\, d\mu(y) + |f_{I_k} - f_{B_k}| &\leq \left(\frac{\mu(B_k)}{\mu(I_{k+1})} + \frac{\mu(B_k)}{\mu(I_k)} \right) \fint_{B_k} |f(y) - f_{B_k}|\, d\mu(y) \\
&\leq C \fint_{B_k} |f(y) - f_{B_k}|\, d\mu(y) \leq C \frac{\operatorname{diam}(B_k)}{\mu(B_k)} \int_{\sigma B_k} g\, d\mu.
\end{aligned}$$

As $\sigma < c/3$, we see that $\sigma B_k \subseteq A_k \subseteq I_k$. Using this fact and again applying the Ahlfors regularity condition, it follows that

$$\begin{aligned}
|f(z) - f_I| &\leq \sum_{k=k_0}^{\infty} C \int_{I_k} (s^k)^{1-Q} g\, d\mu \\
&\leq C \sum_{k=k_0}^{\infty} \sum_{m=0}^{\infty} (s^{Q-1})^m \int_{I_{k+m} \setminus I_{k+m+1}} (s^{k+m})^{1-Q} g\, d\mu.
\end{aligned}$$

By assumption, for each integer $m \geq 0$ the cube I_{k+m} contains the point z, and so if $x \in I_{k+m}$, then $d(z,x) \leq Cs^{k+m}$. Thus,

$$\begin{aligned}
|f(z) - f_I| &\leq C \sum_{k=k_0}^{\infty} \sum_{m=0}^{\infty} (s^{Q-1})^m \int_{I_{k+m} \setminus I_{k+m+1}} d(z,x)^{1-Q} g(x)\, d\mu(x) \\
&\leq C \sum_{m=0}^{\infty} (s^{Q-1})^m \sum_{k=k_0}^{\infty} \int_{I_{k+m} \setminus I_{k+m+1}} d(z,x)^{1-Q} g(x)\, d\mu(x) \\
&\leq C \sum_{m=0}^{\infty} (s^{Q-1})^m \sum_{k=k_0}^{\infty} \int_{I_k \setminus I_{k+1}} d(z,x)^{1-Q} g(x)\, d\mu(x) \\
&\leq C \sum_{m=0}^{\infty} (s^{Q-1})^m \int_{I_{k_0}} d(z,x)^{1-Q} g(x)\, d\mu(x) \\
&\leq C \int_I d(z,x)^{1-Q} g(x)\, d\mu(x).
\end{aligned}$$

A final application of the Ahlfors regularity condition shows that if $z \in I$, then

$$d(z,x)^{1-Q} \le C \frac{d(z,x)}{\mu(B(z,d(z,x)))}.$$

This now yields the desired result. □

6.2. **The Riesz potential and Lorentz spaces.** We now discuss the characterization of the Lorentz space $L^{Q,1}(X)$ given in [KKM99], and use it to connect the Riesz potential to Lorentz spaces. We say that a *gauge* is a non-increasing function $\varphi\colon (0,\infty) \to [0,\infty)$. Given a gauge φ, we define $F^Q_\varphi\colon [0,\infty) \to [0,\infty)$ by

$$F^Q_\varphi(r) = \begin{cases} r\varphi^{(1-Q)/Q}(r) & r > 0, \\ 0 & r = 0. \end{cases}$$

The following theorem states that the Lorentz spaces are determined by a family of Orlicz conditions, defined in terms of the admissible gauges [KKM99, Corollary 2.4]. The main advantage of this representation is that it does not require the calculation of g^* in order to determine the membership of a function g in a given Lorentz space. Instead, the Lorentz norm of g is estimated by the integral of a function of g itself.

Theorem 6.2 (Kauhanen-Koskela-Malý)**.** *Let $Q > 1$. A non-negative function $g \in \mathcal{M}_0(X)$ is in the space $L^{Q,1}(X)$ if and only if there is a gauge $\varphi \in L^{1/Q}((0,\infty))$ such that $\varphi(g(x)) > 0$ for almost every $x \in X$ with $g(x) > 0$, and*

$$\int_X F^Q_\varphi(g(x))\, d\mu(x) < \infty.$$

In addition, there is a constant C depending only on φ and Q such that

$$||g||^Q_{Q,1} \le C \int_X F^Q_\varphi(g(x))\, d\mu(x). \tag{6.4}$$

We next adapt Theorem 3.1 in [KKM99], which connects the Riesz potential to the Lorentz norm, to our more general setting. The proof given in [KKM99] remains valid in our setting with only minor modifications, and so we omit it.

Theorem 6.3. *Suppose that (X,d,μ) is Ahlfors Q-regular, $Q > 1$, on scales below $r_0 > 0$, with constant C_Q. Then there is a number $C \ge 1$ depending only C_Q and Q with the following property. If g is a nonnegative measurable function on X, φ is a gauge, $E \subseteq X$ is a measurable set with* $\operatorname{diam} E < r_0$, *and $z \in E$, then*

$$\left(\int_E \frac{d(z,x)}{\mu(B(z,d(z,x)))} g(x)\, d\mu(x)\right)^Q \le C \left(\int_0^\infty \varphi^{\frac{1}{Q}}(t)\, dt\right)^{Q-1} \int_E F^Q_\varphi(g(x))\, dx,$$

whenever the right-hand side is finite.

6.3. **The final steps.** We now show how our previous results combine to yield Theorem 1.1.

Proposition 6.4. *Suppose that (X,d,μ) is Ahlfors Q-regular, $Q > 1$, on scales below $r_0 > 0$, with constant C_Q. Let $f \in L^1_{\mathrm{loc}}(X)$ and $g \in L^{Q,1}(X)$, and assume that there is a constant $C \ge 1$ such that for almost every $z \in X$ and every cube I of diameter less than r_0 in some cube structure $\mathcal{Q}$ on X,*

$$|f(z) - f_I| \le C_0 \int_I \frac{d(z,x)}{\mu(B(z,d(z,x)))} g(x)\, d\mu(x). \tag{6.5}$$

Then there is a continuous function $\widetilde{f} \in L^1_{\mathrm{loc}}(X)$ that agrees with f almost everywhere, and satisfies the Q-Rado-Reichelderfer condition with a weight depending only on C_Q, Q, C_0, and g.

Proof. Through out this proof, we denote by C a number, possibly varying at each instance, that depends only on C_Q, Q, and C_0.

By Theorem 6.2, there exists a nonnegative function $\varphi \in L^{1/Q}((0,\infty))$ such that $\varphi(g(x)) > 0$ for almost every x with $g(x) > 0$, and such that $F^Q_\varphi \circ g \in L^1(X)$. Let N be a set of measure zero in X with the property that (6.5) holds at each point $z \in X \backslash N$.

Let I be a cube with diameter at most r_0. We first consider $\operatorname{diam} f(I\backslash N)$. We may find a point $z \in I\backslash N$ such that

$$\operatorname{diam} f(I\backslash N) \leq 4|f(z) - f_I| \leq 4C_0 \int_I \frac{d(z,x)}{\mu(B(z,d(z,x)))}\rho(x)\, d\mu(x).$$

By Theorem 6.3,

$$(\operatorname{diam} f(I\backslash N))^Q \leq C \left(\int_0^\infty \varphi^{\frac{1}{Q}}(t)\, dt\right)^{Q-1} \int_I F^Q_\varphi(g(x))\, d\mu(x). \tag{6.6}$$

Define $\Theta\colon X \to \mathbb{R}$ by

$$\Theta(x) := C \left(\int_0^\infty \varphi^{\frac{1}{Q}}(t)\, dt\right)^{Q-1} F^Q_\varphi(g(x)).$$

Then $\Theta \in L^1(X)$, and it depends only on C_0, C_Q, Q, and g. In particular, it does not depend on I. These facts, along with (6.6) implies that f is uniformly continuous on $X\backslash N$. As N has measure zero, it has empty interior, and so we may extend $f|_{X\backslash N}$ to a continuous function $\widetilde{f}$ on X. By the continuity of $\widetilde{f}$,

$$(\operatorname{diam} \widetilde{f}(I))^Q = (\operatorname{diam} \widetilde{f}(I\backslash N))^Q = (\operatorname{diam} f(I\backslash N))^Q \leq \int_I \Theta\, d\mu,$$

as desired. □

Remark 6.5. The technique used in the above proof also shows how a Rado-Reichelderfer condition using balls would follow from [Rom08, Lemma 5].

Corollary 6.6. *Assume that (X, d, μ) is complete, Ahlfors Q-regular on small scales, and supports a Q-Poincaré inequality. If $f \in L^1_{\mathrm{loc}}(X)$ has an upper gradient $g \in L^{Q,1}(X)$, then there is a continuous representative $\widetilde{f}$ of f that satisfies the Q-Rado-Reichelderfer condition with a weight that depends only on the constants associated to the assumptions and g.*

Proof. Theorem 5.5 implies that there is a Hajłasz upper gradient ρ of f, depending only on the data and g, such that the pair (f, ρ) satisfies a 1-Poincaré inequality with dilation factor $c/3$, and constant depending only on the data. Proposition 6.1 and Proposition 6.4 now provide the desired property. □

Corollary 6.6 only applies to real-valued mappings f. However, it easily extends to the general case.

Proof of Theorem 1.1. Recall that as Y is separable, there is an isometric embedding $\iota\colon Y \hookrightarrow l^\infty$ [Hei01, Exercise 12.6]. For each $k \in \mathbb{N}$, let $T_k\colon l^\infty \to \mathbb{R}$ denote the 1-Lipschitz projection defined by

$$T_k(\{a_n\}_{n\in\mathbb{N}}) = a_k.$$

Then g is again an upper gradient of the real-valued mapping $T_k \circ \iota \circ f \in L^1_{\mathrm{loc}}(X)$. Hence, by Corollary 6.6 each mapping $T_k \circ \iota \circ f$ satisfies the Q-Rado-Reichelderfer with the same weight Θ, which depends only on the constants associated with the space and g. By the definition of the metric on l^∞,

$$\begin{aligned}(\operatorname{diam} f(I))^Q = (\operatorname{diam} \iota \circ f(I))^Q &= \left(\sup_{x,y\in I}\sup_{k\in\mathbb{N}} |T_k \circ \iota \circ f(x) - T_k \circ \iota \circ f(y)|\right)^Q \\ &= \sup_{k\in\mathbb{N}} \left(\sup_{x,y\in I} |T_k \circ \iota \circ f(x) - T_k \circ \iota \circ f(y)|\right)^Q \\ &\leq \int_I \Theta\, d\mu,\end{aligned}$$

yielding the desired result. □

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