Interpolation and approximation in L_p

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Abstract

When discretizing certain stochastic integrals along equidistant time nets, the approximation error converges to zero in the pth mean, $p > 2$, with the optimal rate $\frac{1}{\sqrt{n}}$ if the function we start with belongs to Malliavin Sobolev space $\mathbb{D}_{1,p}$. For other L_p functions, the L_p convergence rate of the approximation error depends only on their fractional smoothness, and vice versa.

Keywords: Besov spaces, Malliavin Sobolev spaces, stochastic integrals, approximation, L_p convergence

1 Introduction

The main tasks of this paper are to estimate the discretization error of a stochastic integral, i.e.

$$\int_0^1 \phi(s, W_s)dW_s - \sum_{i=1}^{n} \phi(t_{i-1}, W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})$$

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in the L_p norm with $p > 2$, where $\int_0^1 \phi(s, W_s) dW_s = f(W_1) - \mathbb{E}\,(\,f(W_1)\,)$, and to connect the convergence properties of the error to the smoothness properties (in the Malliavin sense) of the function f.

We extend some convergence results that are known in the L_2 case to functions in L_p, $p > 2$. In applications such as stochastic finance, this kind of improvement in integrability leads to better tail estimates, and thus to more accurate estimates of risk. Mathematically, this step out of orthogonality is not trivial, and we employ new techniques. For an introduction to the literature concerning similar results in L_2, see e.g. [7].

For simplicity, this paper is restricted to stochastic integrals with respect to the Brownian motion. Applications in option pricing require a positive price process, such as geometric Brownian motion. This work serves as the first step towards similar results for suitable price processes. In the current form, these results can be applied to simulations of stochastic differential equations retaining the martingale property (for more details, see e.g. [7], p. 2).

We begin by establishing notation and discussing the results of the paper.

1.1 Notation

Let $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,1]}\right)$ be a stochastic basis, and let $W = (W_t)_{t\in[0,1]}$ be a standard Brownian motion, with continuous paths and $W_0 = 0$ for all $\omega \in \Omega$. Assume that $(\mathcal{F}_t)_{t\in[0,1]}$ is the augmentation of the filtration generated by W and that $\mathcal{F} = \mathcal{F}_1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function satisfying $f(W_1) \in L_2$ and define the function $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ by setting

$$F(t, x) := \mathbb{E}\,(\,f(W_1)\,\mid W_t = x) = \mathbb{E}f(x + W_{1-t})$$

Then $F \in C^\infty([0,1[\, \times \mathbb{R})$ (see e.g. [9, Lemma A.2] or [8, p. 4]) and satisfies

$$\begin{cases} \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} & = 0, \qquad 0 \le t < 1, x \in \mathbb{R} \\ F(1, x) & = f(x), \quad x \in \mathbb{R} \end{cases} \tag{1}$$

and by Itô's formula, $f(W_1) = F(1, W_1) = \mathbb{E}f(W_1) + \int_0^1 \frac{\partial F}{\partial x}(s, W_s) dW_s$ a.s.

We discretize the integral on the interval $[0, t]$ with $t \le 1$ using a deterministic time net $\tau^{(n)} := (t_i^n)_{i=0}^n$ with $0 = t_0^n < t_1^n < \ldots < t_n^n = 1$, and get the approximation error process

$$C_t(f, \tau^{(n)}) := \int_0^t \frac{\partial F}{\partial x}(s, W_s) dW_s - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}^n, W_{t_{i-1}^n}) \left(W_{t_i^n \wedge t} - W_{t_{i-1}^n \wedge t}\right).$$

Moreover, we denote the size of the time net $\tau^{(n)}$ by

$$\left|\left|\tau^{(n)}\right|\right|_\infty := \max_{1\le i\le n} |t_i^n - t_{i-1}^n|.$$

For convenience, we denote by τ_n the equidistant time net of $n+1$ time points, i.e. $\tau_n = \left(\frac{i}{n}\right)_{i=0}^n$ and $||\tau_n||_\infty = \frac{1}{n}$.

Throughout the paper, γ denotes the standard Gaussian measure on the real line.

Furthermore, we use the notation $A \sim_c B$ with $c > 0$ for the two-sided inequality $c^{-1}A \le B \le cA$.

1.2 Results

The first result, Theorem 1.1, claims that if f is smooth enough, i.e. $L_p(\gamma)$ integrable for some $p > 2$ and differentiable in the Malliavin sense with derivative in $L_p(\gamma)$, then the L_p norm of the final approximation error $C_1(f,\tau)$ converges to zero as the timenets get tighter, with the rate $||\tau||_\infty^{\frac{1}{2}}$. For equidistant time nets with n time steps this rate equals $\frac{1}{\sqrt{n}}$, which is optimal as long as there are no constants $a, b \in \mathbb{R}$ such that $f(W_1) = a + bW_1$ a.s. (which would lead to zero error with any time net); see [8] and the references therein for geometric Brownian motion - the same applies to the Brownian motion.

Theorem 1.1. *Let* $2 < p < \infty$. *If* $f \in \mathbb{D}_{1,p}(\gamma)$, *then there exists a constant* $c_{(1.1)} > 0$ *depending only on* p *and* f *such that*

$$||C_1(f,\tau)||_{L_p} \le c_{(1.1)} ||\tau||_\infty^{\frac{1}{2}}$$

for any time net $\tau = (t_i)_{i=0}^n$.

The case $p = 2$ is included in [7] and [11] and will be used in the proof of Theorem 1.1.

The second result, Theorem 1.2, reveals a close connection between smoothness of f and the convergence rate of the L_p norm of the final error: the rate $\left(\frac{1}{\sqrt{n}}\right)^\theta$, where $0 < \theta < 1$, is achieved if and only if the function f has its fractional smoothness index equal to θ. Fractional smoothness is measured by interpolating between "smooth" (Malliavin differentiable $\mathbb{D}_{1,p}$) functions (index 1) and all L_p functions (index 0), see Definitions 2.4 and 2.11.

Theorem 1.2. *Let* $2 \leq p < \infty$ *and* $0 < \theta < 1$. *Then*

$$\|f\|_{\theta,\infty} \sim_{c_{(1.2)}} \sup_n \left\{ n^{\frac{\theta}{2}} \|C_1(f,\tau_n)\|_{L_p} \right\} + \|f\|_{L_p(\gamma)}$$

for some constant $c_{(1.2)} > 0$ *depending only on* p *and* θ, *where* $\|f\|_{\theta,\infty}$ *denotes the norm of* f *in the interpolation space* $(L_p(\gamma), \mathbb{D}_{1,p}(\gamma))_{\theta,\infty}$.

The proof of Theorem 1.2 is independent of Theorem 1.1. The case $p = 2$ in Theorem 1.2 has already been studied in [7] and [11] but with a different proof. Here it is treated along with the general case.

Theorem 1.2 shows that Theorem 1.1 is nearly sharp:

Corollary 1.3. *Let* $2 \leq p < \infty$. *If*

$$\|C_1(f,\tau_n)\|_{L_p} \leq c \left(\frac{1}{n}\right)^{\frac{1}{2}}$$

for all $n = 1, 2, \ldots$ *and for some* $c > 0$ *not depending on* n, *then*

$$f \in \bigcap_{0<\theta<1} (L_p(\gamma), \mathbb{D}_{1,p}(\gamma))_{\theta,\infty}.$$

For the proof of the corollary, notice that the condition $C_1(f, \tau_1) \in L_p$ implies that $f \in L_p(\gamma)$.

One more result, Theorem 5.3 is presented in Section 5, showing how the convergence rate $\frac{1}{\sqrt{n}}$ is possible also when f is not smooth, provided we take special non-equidistant time nets instead of equidistant ones. This direction is open for more development (see Section 6).

Section 2 contains definitions and basic results that will be needed later on. Some of these results are proven in the appendix.

Section 3 is dedicated to the smooth case: Theorem 1.1 is proved by interpolation using earlier results concerning L_2-approximation.

The core of the paper is Section 4, where we discuss the proof of Theorem 1.2. Fractional smoothness is first connected to the growth rate of the first and second derivatives before approaching the original question of approximation rates.

In Section 5 we apply the results to some usual examples, and make an observation about how we can improve the convergence rate by using non-equidistant time nets.

Section 6 concludes the paper with remarks on further extensions and ideas.

2 Preliminaries

Recall that
$$\left\|\left(\int_a^b X_s^2 ds\right)^{\frac{1}{2}}\right\|_{L_p} \le \left(\int_a^b \|X_s\|_{L_p}^2 \, ds\right)^{\frac{1}{2}}$$
for any $2 \le p < \infty$, $0 \le a < b \le 1$ whenever $X = (X_t)_{t\in[0,1]}$ is a progressively measurable process. We will use this inequality without reference.

When $f \in L_p(\gamma)$ for some $2 \le p < \infty$, it is known that $\left\|\frac{\partial F}{\partial x}(t, W_t)\right\|_{L_p} < \infty$ and $\left\|\frac{\partial^2 F}{\partial x^2}(t, W_t)\right\|_{L_p} < \infty$ for all $0 \le t < 1$ (see e.g. [12, Lemma 3.1]). Thus Itô's formula and (1) imply that

$$\left(\frac{\partial F}{\partial x}(t, W_t)\right)_{0\le t<1} \text{ and } \left(\frac{\partial^2 F}{\partial x^2}(t, W_t)\right)_{0\le t<1} \text{ are } L_p \text{ integrable martingales.} \tag{2}$$

2.1 The derivative

Definition 2.1 (Hermite polynomials). *The family of (normalized) Hermite polynomials* $h_k : \mathbb{R} \to \mathbb{R}$, $k = 0, 1, 2, \ldots$ *is defined by*

$$h_k(x) := \frac{1}{\sqrt{k!}}(-1)^k e^{\frac{x^2}{2}} D^k e^{-\frac{x^2}{2}}, k = 0, 1, 2, \ldots$$

where $0! := 1$ *and* D^k *denotes the* k*th derivative.*

The Hermite polynomials form a complete orthonormal system in $L_2(\gamma)$, so that for any $f \in L_2(\gamma)$ there is a unique expansion

$$f = \sum_{k=0}^{\infty} \alpha_k h_k,$$

where the limit is considered in $L_2(\gamma)$ and $\alpha_k \in \mathbb{R}$, $k = 0, 1, 2, \ldots$ The norm satisfies

$$\|f\|_{L_2} = \left(\sum_{k=0}^{\infty} \alpha_k^2\right)^{\frac{1}{2}}.$$

The Malliavin Sobolev space $\mathbb{D}_{1,2}$, which we also denote by $\mathbb{D}_{1,2}(\gamma)$ to emphasize the Gaussian weight on the real line, can be defined using this expansion:

Definition 2.2 (Sobolev space $\mathbb{D}_{1,2}(\gamma)$). *The Sobolev space $\mathbb{D}_{1,2}(\gamma)$ is the space of those functions*

$$f = \sum_{k=0}^{\infty} \alpha_k h_k \in L_2(\gamma)$$

for which the norm

$$\|f\|_{\mathbb{D}_{1,2}} = \left(\sum_{k=0}^{\infty} (k+1)\alpha_k^2 \right)^{\frac{1}{2}}$$

is finite.

For all $f \in \mathbb{D}_{1,2}(\gamma)$ we define the (weak) derivative by

$$f' := \sum_{k=1}^{\infty} \sqrt{k}\alpha_k h_{k-1}, \tag{3}$$

where the limit is considered in $L_2(\gamma)$. Notice that the classical derivative of h_k is $D^1 h_k = \sqrt{k} h_{k-1}$, and that $\mathbb{D}_{1,2}$ is a Banach space.

Throughout this paper, the term "derivative" and the notation f' refer to the above formulation. When considering the limit of difference quotients, we speak about the "classical derivative" and use the notation D^k as in Definition 2.1. Under some regularity conditions, these two concepts coincide. We will formulate this as a remark for future use.

Remark 2.3. *Let $f \in L_2(\gamma)$ be continuous. Assume that there exist $x_1, \ldots, x_n \in \mathbb{R}$ such that f is continuously differentiable on $]x_i, x_{i+1}[$ for each $i = 1, \ldots, n-1$, on $]-\infty, x_1[$ and on $]x_n, \infty[$. Define the function f'_{cl} by setting*

$$f'_{cl}(x) := \begin{cases} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, & x \in \mathbb{R} \setminus \{x_1, \ldots, x_n\} \\ 0, & x \in \{x_1, \ldots, x_n\}. \end{cases}$$

If $f'_{cl} \in L_2(\gamma)$, then

(i) $f \in \mathbb{D}_{1,2}(\gamma)$, *and*

(ii) $f' = f'_{cl}$ *a.s.*

Proof. Let $m < x_1 \leq x_n < M$ and notice that

$$[D^1 h_k](x) - x h_k(x) = -\sqrt{k+1} h_{k+1}(x).$$

Integration by parts on each interval yields, for any $k = 0, 1, 2, \ldots$,

$$\int_m^M f'_{cl}(x) h_k(x) d\gamma(x)$$

$$= \int_m^M f(x)h_k(x)e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}} + \int_m^M f(x)\sqrt{k+1}h_{k+1}(x)d\gamma(x). \quad (4)$$

Since each h_k, f and f'_{cl} are in $L_2(\gamma)$, we know by Hölder's inequality that

$$\int_{\mathbb{R}} |g(x)h_k(x)|\, d\gamma(x) < \infty \quad (5)$$

for all $k = 0, 1, 2, \dots$ and $g = f, f'_{cl}$. Therefore, both integrals in (4) converge as $M \to \infty$, and the limit $\lim_{x\to\infty} f(x)h_k(x)e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}$ has to be finite. A limit $c \neq 0$ would require that for some $x_c \in \mathbb{R}$, $|f(x)h_k(x)| > \frac{c}{2}e^{\frac{x^2}{2}}$ for all $x > x_c$, which leads to a contradiction with (5). Similar observation on m ensures that

$$\int_{\mathbb{R}} f'_{cl}(x)h_k(x)d\gamma(x) = \sqrt{k+1}\int_{\mathbb{R}} f(x)h_{k+1}(x)d\gamma(x).$$

Considering $L_2(\gamma)$ a Hilbert space with $\langle g_1, g_2\rangle = \int_{\mathbb{R}} g_1(x)g_2(x)d\gamma(x)$, we see that

$$\langle f'_{cl}, h_k\rangle = \sqrt{k+1}\langle f, h_{k+1}\rangle = \sqrt{k+1}\alpha_{k+1}$$

for all $k = 0, 1, 2, \dots$, where $\alpha_k = \langle f, h_k\rangle$ as in Definition 2.1. Since $f'_{cl} \in L_2(\gamma)$, this means that

$$\sum_{k=0}^{\infty}(k+1)\alpha_{k+1}^2 < \infty,$$

which proves (i). Also, it implies that $f'_{cl} = f'$ in $L_2(\gamma)$ and thus almost surely (wrt. both γ and the Lebesgue measure). $\square$

Definition 2.4 (Sobolev space $\mathbb{D}_{1,p}(\gamma)$)**.** *Let $2 < p < \infty$. The Sobolev space $\mathbb{D}_{1,p}(\gamma)$ is the space of those $f \in \mathbb{D}_{1,2}(\gamma)$ for which both f and its derivative f' are in $L_p(\gamma)$, that is, the norm*

$$\|f\|_{\mathbb{D}_{1,p}} := \left(\|f\|_{L_p}^p + \|f'\|_{L_p}^p\right)^{\frac{1}{p}}$$

is finite.

We assume a priori that a function in $\mathbb{D}_{1,p}(\gamma)$ is in $\mathbb{D}_{1,2}(\gamma)$ and hence has a well defined derivative. We do not consider the L_p convergence of the infinite sum in (3); the sum does not necessarily converge, since Hermite polynomials do not form a basis in $L_p(\gamma)$ when $p > 2$. The space $\mathbb{D}_{1,p}(\gamma)$ is a Banach space when $p > 2$ as well:

Proposition 2.5. *Let $2 < p < \infty$. Then* *(i)* $\mathbb{D}_{1,p}(\gamma)$ *is a Banach space, and*

(ii) if $f \in \mathbb{D}_{1,2}(\gamma)$ *and* $f' \in L_p(\gamma)$*, then* $f \in \mathbb{D}_{1,p}(\gamma)$*.*

Proof. Notice that by (3), $\|f\|_{\mathbb{D}_{1,2}} = \left(\|f\|_{L_2}^2 + \|f'\|_{L_2}^2\right)^{\frac{1}{2}}$ for all $f \in \mathbb{D}_{1,2}(\gamma)$.

To prove (i), assume that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{D}_{1,p}(\gamma)$. This means that the sequence is Cauchy also in $\mathbb{D}_{1,2}(\gamma)$, and we find a function $f \in \mathbb{D}_{1,2}(\gamma)$ such that $f_n \underset{L_2}{\longrightarrow} f$ and $f'_n \underset{L_2}{\longrightarrow} f'$ as $n \to \infty$. Since $(f_n)_{n=1}^{\infty}$ and $(f'_n)_{n=1}^{\infty}$ are Cauchy sequences in $L_p(\gamma)$, there exist functions $g \in L_p(\gamma)$ and $h \in L_p(\gamma)$ such that $f_n \underset{L_p}{\longrightarrow} g$ and $f'_n \underset{L_p}{\longrightarrow} h$. But then $f_n \underset{L_2}{\longrightarrow} g$ and $f'_n \underset{L_2}{\longrightarrow} h$ so that $g = f$ and $h = f'$ in $L_2(\gamma)$ and thus almost surely.

For (ii), we need to prove that $f \in L_p(\gamma)$. In preparation, we establish some formulae that are valid for all functions in $\mathbb{D}_{1,2}(\gamma)$, and the condition $f' \in L_p(\gamma)$ will be only used at the very end of the proof.

By Itô's formula, (1), and the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
\|f\|_{L_p(\gamma)} &= \|f(W_1)\|_{L_p} \\
&= \left\| \mathbb{E}\,(\,f(W_1)\,) + \int_0^1 \frac{\partial F}{\partial x}(t, W_t)\,dW_t \right\|_{L_p} \\
&\leq |\mathbb{E}\,(\,f(W_1)\,)| + c_p \left\| \left(\int_0^1 \left[\frac{\partial F}{\partial x}(t, W_t)\right]^2 dt \right)^{\frac{1}{2}} \right\|_{L_p}.
\end{aligned}$$

The first term is finite. To estimate the second one, we need to prove that

$$\frac{\partial F}{\partial x}(t, W_t) = \mathbb{E}\,(\,f'(W_1) \mid \mathcal{F}_t) \text{ a.s.} \tag{6}$$

for any $f \in \mathbb{D}_{1,2}(\gamma)$ and $t \in \,]0,1[$. It follows from [16], Section 4.3 (see also [10]) that, for all $f \in L_2(\gamma)$,

$$\frac{\partial F}{\partial x}(t, x) = \mathbb{E}\left(f(x + W_{1-t}) \frac{W_{1-t}}{1-t} \right) \tag{7}$$

for all $x \in \mathbb{R}$ and $t \in \,]0,1[$, and it remains to show that

$$\mathbb{E}\left(f(x + W_{1-t}) \frac{W_{1-t}}{1-t} \right) = \mathbb{E}\,(\,f'(x + W_{1-t})\,) \tag{8}$$

for all $x \in \mathbb{R}$ and $t \in \,]0,1[$ whenever $f \in \mathbb{D}_{1,2}(\gamma)$.

We begin by observing that, for any $x \in \mathbb{R}$ and any $t \in \,]0,1[$,

$$0 < e^{-\frac{(y-x)^2}{2(1-t)} + \frac{y^2}{2}} \leq e^{\frac{x^2}{2t}} \text{ for all } y \in \mathbb{R}. \tag{9}$$

Integration by parts thus implies, for any polynomial h, that

$$\begin{aligned}
\mathbb{E}\left(h(x+W_{1-t})\frac{W_{1-t}}{1-t} \right) &= \int_{\mathbb{R}} h(x+z)\frac{z}{1-t}e^{-\frac{z^2}{2(1-t)}}\frac{1}{\sqrt{2\pi(1-t)}}dz \\
&= \int_{\mathbb{R}} [D^1 h](x+z)e^{-\frac{z^2}{2(1-t)}}\frac{1}{\sqrt{2\pi(1-t)}}dz \\
&\quad -\Big/_{-\infty}^{\infty} h(y)e^{-\frac{(y-x)^2}{2(1-t)}}\frac{1}{\sqrt{2\pi(1-t)}} \\
&= \mathbb{E}\left([D^1 h](x+W_{1-t}) \right), \qquad (10)
\end{aligned}$$

where D^1 denotes the classical derivative. Applying (9) again we see that

$$\begin{aligned}
\mathbb{E}\left(f(x+W_{1-t}) \right)^2 &= \int_{\mathbb{R}} f(x+z)^2 e^{-\frac{z^2}{2(1-t)}}\frac{1}{\sqrt{2\pi(1-t)}}dz \\
&= \int_{\mathbb{R}} f(y)^2 e^{-\frac{y^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-x)^2}{2(1-t)}+\frac{y^2}{2}}\frac{1}{\sqrt{1-t}}dy \\
&\leq \|f\|^2_{L_2(\gamma)}\, e^{\frac{x^2}{2t}}\frac{1}{\sqrt{1-t}} < \infty \qquad (11)
\end{aligned}$$

for any $x \in \mathbb{R}$ and any $0 < t < 1$ since $f \in L_2(\gamma)$. The same is true for the derivative, so that

$$\mathbb{E}\,|f'(x+W_{1-t})| < \infty$$

and, by Hölder's inequality,

$$\mathbb{E}\left| f(x+W_{1-t})\frac{W_{1-t}}{1-t} \right| \leq \left(\mathbb{E}\left(f(x+W_{1-t}) \right)^2 \mathbb{E}\left(\frac{W_{1-t}}{1-t} \right)^2 \right)^{\frac{1}{2}} < \infty.$$

We continue by recalling that since $f \in \mathbb{D}_{1,2}(\gamma)$,

$$f_N := \sum_{k=0}^{N} \alpha_k h_k \underset{L_2}{\longrightarrow} \sum_{k=0}^{\infty} \alpha_k h_k \underset{L_2}{=} f$$

and

$$D^1 f_N = \sum_{k=0}^{N} \alpha_k D^1 h_k \underset{L_2}{\longrightarrow} \sum_{k=1}^{\infty} \alpha_k D^1 h_k \underset{L_2}{=} f'.$$

For these polynomials, (10) yields

$$\mathbb{E}\left(f(x+W_{1-t})\frac{W_{1-t}}{1-t} \right) = \mathbb{E}\left([f(x+W_{1-t}) - f_N(x+W_{1-t})]\frac{W_{1-t}}{1-t} \right)$$

$$
\begin{aligned}
&\quad + \mathbb{E}\left(f_N(x + W_{1-t}) \frac{W_{1-t}}{1-t} \right) \\
&= \mathbb{E}\left([f(x + W_{1-t}) - f_N(x + W_{1-t})] \frac{W_{1-t}}{1-t} \right) \\
&\quad + \mathbb{E}\left([D^1 f_N](x + W_{1-t}) \right) \\
&= \mathbb{E}\left([f(x + W_{1-t}) - f_N(x + W_{1-t})] \frac{W_{1-t}}{1-t} \right) \\
&\quad + \mathbb{E}\left(\left[[D^1 f_N](x + W_{1-t}) - f'(x + W_{1-t}) \right] \right) \quad (12) \\
&\quad + \mathbb{E}\left(f'(x + W_{1-t}) \right).
\end{aligned}
$$

As before, we obtain

$$
\begin{aligned}
&\mathbb{E}\left| [f(x + W_{1-t}) - f_N(x + W_{1-t})] \frac{W_{1-t}}{1-t} \right| \\
&\leq \left(\mathbb{E}\left(f(x + W_{1-t}) - f_N(x + W_{1-t}) \right)^2 \mathbb{E}\left(\frac{W_{1-t}}{1-t} \right)^2 \right)^{\frac{1}{2}} \\
&\leq \|f - f_N\|_{L_2} \left(e^{\frac{x^2}{2t}} \frac{1}{\sqrt{1-t}} \right)^{\frac{1}{2}} \frac{1}{1-t} \|W_{1-t}\|_{L_2},
\end{aligned}
$$

which converges to zero as $N \to \infty$, as well as the second term (12). This proves (8).

Now we employ the condition $f' \in L_p(\gamma)$ and complete the proof of (ii) with the estimate

$$
\begin{aligned}
\left\| \left(\int_0^1 \left[\frac{\partial F}{\partial x}(t, W_t) \right]^2 dt \right)^{\frac{1}{2}} \right\|_{L_p} &\leq \left(\int_0^1 \left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_p}^2 dt \right)^{\frac{1}{2}} \\
&= \left(\int_{]0,1[} \|\mathbb{E}\left(f'(W_1) \mid \mathcal{F}_t \right)\|_{L_p}^2 \, dt \right)^{\frac{1}{2}} \\
&\leq \left(\int_{]0,1[} \|f'(W_1)\|_{L_p}^2 \, dt \right)^{\frac{1}{2}} \\
&= \|f'(W_1)\|_{L_p},
\end{aligned}
$$

where we have used the condition $2 \leq p < \infty$ and equation (6). □

2.2 BMO spaces

Definition 2.6 (BMO). *Let* $M = (M_t)_{0 \leq t \leq 1}$ *be a continuous square-integrable martingale on* $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,1]}\right)$ *with* $M_0 = 0$ *a.s. We say that* M *is of* *bounded mean oscillation*, $M \in \mathrm{BMO}$, *if*

$$\|M\|_{\mathrm{BMO}} := \sup_{0 \le t \le 1} \left\| \mathbb{E}\left((M_1 - M_t)^2 \mid \mathcal{F}_t \right) \right\|_{L_\infty}^{\frac{1}{2}} < \infty.$$

We identify the martingale with its last element, at time 1, and say that the random variable X has bounded mean oscillation if the martingale $(\mathbb{E}(X \mid \mathcal{F}_t))_{t \in [0,1]}$ has that property.

Definition 2.6 differs from the usual definition of the BMO_2 spaces found in the literature. However, we see in Remark 2.7 that in our setting, it leads to the same concept as defined in [10] (with weight $\phi \equiv 1$), as well as, for instance, in [4] or in [6] for discrete time. Notice that in our setting (see Section 1.1), all martingales have a continuous modification.

Remark 2.7. *Let* $M = (M_t)_{0 \le t \le 1}$ *be a continuous square-integrable martingale on* $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,1]}\right)$ *with* $M_0 = 0$ *a.s. Then*

$$\|M\|_{\mathrm{BMO}} = \sup_{\sigma} \left\| \mathbb{E}\left((M_1 - M_{\sigma-})^2 \mid \mathcal{F}_\sigma \right) \right\|_{L_\infty}^{\frac{1}{2}},$$

where the supremum extends over all stopping times $\sigma : \Omega \to [0,1]$ *and* $M_{\sigma-} := \lim_{n \to \infty} M_{\left(\sigma - \frac{1}{n}\right)^+}$.

Proof. By continuity, $M_{\sigma-} = M_\sigma$ a.s. and one inequality is clear. For the other one, assume that $\|M\|_{\mathrm{BMO}} \le c$ for some $c > 0$. This implies that, for any $t \in [0,1]$,

$$\mathbb{E}\left(M_1^2 \mid \mathcal{F}_t \right) - M_t^2 \le c^2 \text{ a.s.}$$

Continuity of the process implies that

$$\mathbb{E}\left(M_1^2 \mid \mathcal{F}_t \right) - M_t^2 \le c^2$$

for all $t \in [0,1]$ and all $\omega \in \Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$. This ensures that the same is true for any stopping time and any $\omega \in \Omega_0$, and the optional stopping theorem thus implies that

$$\begin{aligned} c^2 &\ge \mathbb{E}\left(M_1^2 \mid \mathcal{F}_\sigma \right) - M_\sigma^2 \\ &= \mathbb{E}\left(M_1^2 \mid \mathcal{F}_\sigma \right) - 2M_\sigma \mathbb{E}(M_1 \mid \mathcal{F}_\sigma) + M_\sigma^2 \\ &= \mathbb{E}\left((M_1 - M_\sigma)^2 \mid \mathcal{F}_\sigma \right), \end{aligned}$$

almost surely, and the proof is complete. □

Theorem 2.8. *There exists a constant* $c_{(2.8)} > 0$ *such that, for all* $f \in \mathbb{D}_{1,2}(\gamma)$,

$$\|C_1(f,\tau)\|_{BMO} \leq c_{(2.8)} \|\tau\|_\infty^{\frac{1}{2}} \sup_{\substack{0<t<1 \\ x\in R}} \left| \frac{\partial F}{\partial x}(t,x) \right|$$

for any time net τ.

The above result for Lipschitz functions f, geometric Brownian motion, and weighted BMO space is a part of Theorem 7 of [10]. For the convenience of the reader, the proof of Theorem 2.8 is included in the Appendix.

2.3 Interpolation

We now take a brief look at interpolation spaces; for more information on interpolation, see e.g. [2] or [3].

Definition 2.9 (Compatible couple). *A pair* (X_0, X_1) *of Banach spaces* X_0 *and* X_1 *is called a compatible couple if there is a Hausdorff topological vector space in which each of* X_0 *and* X_1 *is continuously embedded.*

Definition 2.10 (K-functional). *The K-functional of the compatible couple* (X_0, X_1) *for an element* $f \in X_0 + X_1$ *at* $t > 0$ *is defined by*

$$K(f,t;X_0,X_1) := \inf_{f=f_0+f_1} \{\|f_0\|_{X_0} + t\,\|f_1\|_{X_1}\},$$

where the infimum extends over all representations $f = f_0 + f_1$ *of* f *with* $f_0 \in X_0$ *and* $f_1 \in X_1$.

When there is no danger of misunderstanding, the spaces are omitted in the notation: $K(f,t;X_0,X_1) = K(f,t)$. The interpolation method using the K-functional is called real interpolation:

Definition 2.11 (Intermediate spaces). *Let* (X_0, X_1) *be a compatible couple and suppose* $0 < \theta < 1$ *and* $1 \leq q \leq \infty$. *The space* $(X_0, X_1)_{\theta,q}$ *consists of all functions* $f \in X_0 + X_1$ *for which the functional*

$$\|f\|_{\theta,q} = \begin{cases} \left[\int_0^\infty \left(t^{-\theta}K(f,t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}}, & 1 \leq q < \infty \\ \sup_{t>0} t^{-\theta}K(f,t), & q = \infty \end{cases}$$

is finite.

Notice that Definition 2.11 is symmetric with respect to θ:

$$(X_0, X_1)_{\theta,q} = (X_1, X_0)_{1-\theta,q}$$

for all $0 < \theta < 1$ and $1 \le q \le \infty$. Also, there is some monotonicity: if $Y_i \subset X_i$ with $\|y\|_{X_i} \le c_i \|y\|_{Y_i}$ for some $c_i > 0$ and for all $y \in Y_i$, $i = 0,1$, then

$$(Y_0, Y_1)_{\theta,q} \subset (X_0, X_1)_{\theta,q} \tag{13}$$

for all $0 < \theta < 1$ and $1 \le q \le \infty$, and the norms satisfy

$$\|f\|_{(X_0,X_1)_{\theta,q}} \le \max\{c_0, c_1\} \|f\|_{(Y_0,Y_1)_{\theta,q}} .$$

Furthermore, if $X_1 \subset X_0$ with $\|x\|_{X_0} \le c \|x\|_{X_1}$ for some $c > 0$ and for all $x \in X_1$, then

$$(X_0, X_1)_{\theta_2,q_2} \subset (X_0, X_1)_{\theta_1,q_1} \tag{14}$$

for all $0 < \theta_1 < \theta_2 < 1$ and $1 \le q_1, q_2 \le \infty$, and the norms satisfy

$$\|f\|_{(X_0,X_1)_{\theta_1,q_1}} \le c' \|f\|_{(X_0,X_1)_{\theta_2,q_2}}$$

for some $c' > 0$ depending at most on θ_1, θ_2, q_1, q_2, and c. In particular,

$$\|f\|_{(X_0,X_1)_{\theta,q}} \le c_{\theta,q} \|f\|_{X_1}$$

for all $0 < \theta < 1$, $1 \le q \le \infty$ and some $c_{\theta,q} > 0$ depending only on θ and q, and

$$c^{-1} \|f\|_{X_0} \le \|f\|_{(X_0,X_1)_{\theta,\infty}} \le c \|f\|_{X_1} \tag{15}$$

for all $0 < \theta < 1$.

In this paper, we will mostly interpolate between $L_p(\gamma)$ and $\mathbb{D}_{1,p}(\gamma)$. These interpolation spaces are called Besov spaces in the literature; see e.g. [2, Corollary V.4.13] for the real line.

When proving Theorem 1.1, we will also employ some other interpolation results. For $2 < p < \infty$ and $\theta = 1 - \frac{2}{p}$, we need the identity

$$(L_2(\gamma), L_\infty(\gamma))_{\theta,p} = L_p(\gamma), \tag{16}$$

which follows from Theorems V.1.9 and V.2.4 of [2]. The norms of the spaces are equivalent up to a multiplicative constant depending on p.

We will need to interpolate between L_2 and BMO as well:

Theorem 2.12. *Let $0 < \theta < 1$ and $p = \frac{2}{1-\theta}$. Then*

$$\left(L_2^0(\mathbb{P}), \mathrm{BMO}\right)_{\theta,p} = L_p^0(\mathbb{P}),$$

where $L_q^0 \subset L_q$, $2 \le q < \infty$, is the subspace of mean zero random variables, and the norms are equivalent up to a multiplicative constant depending only on p.

This kind of result was included e.g. in [19] (see also [1], [14], and [15] as well as [5], [6], and [20]); nevertheless, the proof of this theorem is included in the Appendix.

Recall also the following interpolation theorem:

Theorem 2.13 (Theorem V.1.12 of [2])**.** *Let* (X_0, X_1) *and* (Y_0, Y_1) *be compatible couples, and let* $0 < \theta < 1$, $1 \leq q \leq \infty$. *Let* T *be an admissible linear operator with respect to* (X_0, X_1) *and* (Y_0, Y_1), *i.e.* $T : X_0 + X_1 \to Y_0 + Y_1$ *is linear and the restriction of* T *to* X_i *is a bounded operator from* X_i *to* Y_i *with the norm* M_i, $i = 0, 1$.

Then $T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$ *and*

$$||Tf||_{\theta,q} \leq M_0^{1-\theta} M_1^{\theta} ||f||_{\theta,q}$$

for all f *in* $(X_0, X_1)_{\theta,q}$.

3 Approximation and smooth functions - the "$\mathbb{D}_{1,p}$ case"

In this section we prove Theorem 1.1 using an interpolation argument on the level of the derivative.

Proof of Theorem 1.1. The approximation error $C_1(f, \tau)$ does not, in fact, depend on f but on the derivative, as follows. Recall the definition

$$F(t, x) := \mathbb{E}\left(f(W_1) \mid W_t = x \right) = \mathbb{E}\left(f(x + W_{1-t}) \right)$$

from Section 1.1. Formula (6) for the derivative implies that

$$\frac{\partial F}{\partial x}(t, x) = \mathbb{E}\left(f'(x + W_{1-t}) \right)$$

for all $x \in \mathbb{R}$ and $t \in]0, 1[$ whenever $f \in \mathbb{D}_{1,2}(\gamma)$. Defining $G(t, x) := \mathbb{E}\left(g(x + W_{1-t}) \right)$ for $x \in \mathbb{R}$, $t \in [0, 1]$, and a function $g \in L_2(\gamma)$, and

$$\tilde{C}_t(g, \tau) := \int_0^t \left[G(s, W_s) - \sum_{i=1}^{n} \chi_{]t_{i-1}, t_i]}(s)\, G(t_{i-1}, W_{t_{i-1}}) \right] dW_s$$

for all $0 \leq t \leq 1$, we see that

$$\tilde{C}_1(f', \tau) = C_1(f, \tau)$$

for all $f \in \mathbb{D}_{1,2}(\gamma)$ and all $\omega \in \Omega$.

It follows from [11, Theorem 3.2] (see also [7, Theorem 2.6]) that there exists a constant $c_1' > 0$ such that if $f \in \mathbb{D}_{1,2}(\gamma)$, then

$$\|C_1(f,\tau)\|_{L_2} \leq c_1' \|\tau\|_\infty^{\frac{1}{2}} \|f\|_{\mathbb{D}_{1,2}(\gamma)}$$

for any time net τ. Using the expansion by Hermite polynomials

$$f \underset{L_2}{=} \sum_{k=0}^{\infty} \alpha_k h_k,$$

we observe that

$$\|f\|_{\mathbb{D}_{1,2}(\gamma)} = \left(\sum_{k=0}^{\infty} (k+1)\alpha_k^2 \right)^{\frac{1}{2}} \leq \left(\alpha_0^2 + 2\sum_{k=1}^{\infty} k\alpha_k^2 \right)^{\frac{1}{2}} \leq |\alpha_0| + \sqrt{2}\,\|f'\|_{L_2(\gamma)}.$$

Since α_0 has no effect on $\tilde{C}_1(f',\tau)$, we obtain

$$\left\|\tilde{C}_1(f',\tau)\right\|_{L_2} \leq c_1 \|\tau\|_\infty^{\frac{1}{2}} \|f'\|_{L_2(\gamma)}$$

with $c_1 = \sqrt{2}c_1'$, for any time net τ, and the only condition on f' being that $\|f'\|_{L_2} < \infty$. This means that

$$\left\|\tilde{C}_1(g,\tau)\right\|_{L_2} \leq c_1 \|\tau\|_\infty^{\frac{1}{2}} \|g\|_{L_2(\gamma)} \tag{17}$$

for any $g \in L_2(\gamma)$.

If $\|f'\|_{L_\infty(\gamma)} < \infty$, then formula (6) implies that

$$\begin{aligned} \sup_{\substack{0<t<1 \\ x\in R}} \left| \frac{\partial F}{\partial x}(t,x) \right| &= \sup_{0<t<1} \left\| \frac{\partial F}{\partial x}(t,W_t) \right\|_{L_\infty} \\ &\leq \sup_{0\leq t\leq 1} \|\mathbb{E}\left(f'(W_1) \mid \mathcal{F}_t \right)\|_{L_\infty} \\ &\leq \|f'(W_1)\|_{L_\infty} = \|f'\|_{L_\infty(\gamma)}. \end{aligned}$$

By Theorem 2.8, there exists a constant $c_{(2.8)} > 0$ such that

$$\left\|\tilde{C}_1(f',\tau)\right\|_{\mathrm{BMO}} = \|C_1(f,\tau)\|_{\mathrm{BMO}} \leq c_{(2.8)} \|\tau\|_\infty^{\frac{1}{2}} \|f'\|_{L_\infty(\gamma)}$$

and thus

$$\left\|\tilde{C}_1(g,\tau)\right\|_{\mathrm{BMO}} \leq c_{(2.8)} \|\tau\|_\infty^{\frac{1}{2}} \|g\|_{L_\infty(\gamma)} \tag{18}$$

for all $g \in L_\infty(\gamma)$.

Now we employ Theorem 2.13 together with (16) and Theorem 2.12. Notice that $\mathbb{E}\left(\tilde{C}_1(g,\tau)\right) = 0$ for any $g \in L_2(\gamma)$ and any time net τ so that we can use Theorem 2.12 for interpolation. As a result, applying (17) and (18) at the endpoints,

$$\left|\left|\tilde{C}_1(g,\tau)\right|\right|_{L_p} \leq C_p c_1^{\frac{2}{p}} c_{(2.8)}^{1-\frac{2}{p}} ||\tau||_\infty^{\frac{1}{2}} ||g||_{L_p(\gamma)},$$

whenever $g \in L_p(\gamma)$. The new constant $C_p > 0$ comes from the interpolations in (16) and Theorem 2.12, and depends only on p. Thus

$$||C_1(f,\tau)||_{L_p} \leq c_{(1.1)} ||\tau||_\infty^{\frac{1}{2}},$$

where $c_{(1.1)} = C_p c_1^{\frac{2}{p}} c_{(2.8)}^{1-\frac{2}{p}} ||f'||_{L_p(\gamma)}$. □

4 Approximation and fractional smoothness

This section contains the proof of Theorem 1.2. The proof is divided into three parts as follows:

We start by developing an auxiliary but critical result, Lemma 4.6, which is a decoupling argument for estimating a norm of a martingale using stochastic integrals. In the second subsection, we connect the first and second derivatives of the function F of formula (1) with the fractional smoothness of f. Linking the approximation rate to the second derivative of F in the third subsection completes the proof - as we see in the last subsection.

4.1 Some important lemmas

Our aim in this section is Lemma 4.6, which is a kind of "forward mean value theorem" for martingales: we estimate from above the L_p norm of a martingale at time $a \geq 0$ by the norm of its stochastic integral from a to b divided by $\sqrt{b-a}$, or, more importantly, by the norm of its double stochastic integral from a to b divided by $b-a$, allowing in both cases a multiplicative constant.

We begin by defining progressive measurability on a closed interval.

Definition 4.1. *A real-valued stochastic process* $L = (L_t)_{t\in[0,1]}$ *is progressively measurable if, for any* $t \in [0,1]$ *and any* A *in the Borel* σ*-algebra of* $\mathbb{R}$*, the set* $\{(s,\omega) : 0 \leq s \leq t, \omega \in \Omega, L_s(\omega) \in A\}$ *belongs to the product* σ*-algebra* $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$*, where* $\mathcal{B}([0,t])$ *is the Borel* σ*-algebra of* $[0,t]$.

To define the stochastic integral of a Hilbert space valued process, we need to extend the concept of progressive measurability.

Let H be a separable Hilbert space. For simplicity, let $(e_n)_{n=1}^{\infty}$ be a fixed orthonormal basis in H. The following definitions do not, in effect, depend on the choice of the basis, but we will pass the details.

Definition 4.2. *A process $(G_t)_{t\in[0,1]}$, where $G_t : \Omega \to H$ is measurable for each $t \in [0,1]$, is progressively measurable if for all $n = 1, 2, \ldots$, the coordinate process $(\langle G_t, e_n\rangle_H)_{t\in[0,1]}$ is a real-valued progressively measurable process.*

Now, the stochastic integral $\int_0^t G_s dW_s$, where W is a real-valued Brownian motion, can be defined coordinate-wise:

Definition 4.3. *Let $(G_t)_{t\in[0,1]}$ be a progressively measurable process with values in H satisfying $\mathbb{E}\left(\int_0^1 \|G_s\|_H^2 ds\right) < \infty$, and let $0 < t \le 1$. We define the stochastic integral $\int_0^t G_s dW_s$ by*

$$\langle \int_0^t G_s dW_s, e_n\rangle_H = \int_0^t \langle G_s, e_n\rangle_H dW_s, \; n = 1, 2, \ldots,$$

where $(W_t)_{t\in[0,1]}$ is a standard Brownian motion.

The following is a special case of the Burkholder-Davis-Gundy inequalities for continuous local martingales with values in Hilbert spaces, see e.g. [18, E.2 on p. 212] and the Doob inequality:

Theorem 4.4. *Let $2 \le p < \infty$ and let $(G_t)_{t\in[0,1]}$ be a progressively measurable process with values in a separable Hilbert space H satisfying $\mathbb{E}\left(\int_0^1 \|G_s\|_H^2 ds\right) < \infty$. Then there exists a constant $c_{(4.4)} > 0$ depending only on p such that, for any $0 < t \le 1$,*

$$\mathbb{E}\left\|\int_0^t G_s dW_s\right\|_H^p \sim_{c_{(4.4)}} \mathbb{E}\left|\int_0^t \|G_s\|_H^2 ds\right|^{\frac{p}{2}},$$

where $(W_t)_{t\in[0,1]}$ is a standard Brownian motion.

For convenience, we formulate this for a double stochastic integral:

Lemma 4.5. *Let $2 \le p < \infty$ and let $L = (L_t)_{t\in[0,1]}$ be a progressively measurable process such that $\mathbb{E}\int_0^1 L_s^2 ds < \infty$. Then, for any $0 \le a < b \le 1$,*

$$\left\|\int_a^b \int_a^u L_s dW_s dW_u\right\|_{L_p} \sim_{c_{(4.5)}} \left\|\left(\int_a^b \int_a^u L_s^2 ds du\right)^{\frac{1}{2}}\right\|_{L_p}$$

for some constant $c_{(4.5)} > 0$ depending only on p.

Proof of Lemma 4.5. Let $H := L_2([0,b])$ with $\langle h_1, h_2\rangle_H = \int_0^b h_1(s)h_2(s)ds$. Theorem 4.4 applied for the process

$$G_t(u) := \begin{cases} 0, & 0 \le t \le a \\ L_t\chi_{[t,b]}(u) & a < t \le b \end{cases}$$

together with the Burkholder-Davis-Gundy inequalities and Fubini's theorem for stochastic processes (see e.g. [17, Theorem 5.15]) yield the desired equivalence. $\square$

Lemma 4.6. *Let* $2 \le p < \infty$. *If* $M = (M_t)_{0\le t<1}$ *is a* p*-integrable martingale, then, for any* $0 \le a < b < 1$,

(i)

$$\|M_a\|_{L_p} \le c_{(4.6)}\,(b-a)^{-\frac{1}{2}} \left\| \int_a^b M_u dW_u \right\|_{L_p}$$

and

(ii)

$$\|M_a\|_{L_p} \le c_{(4.6)}\,(b-a)^{-1} \left\| \int_a^b \int_a^u M_s dW_s dW_u \right\|_{L_p}$$

for some constant $c_{(4.6)} > 0$ *depending only on* p.

Proof. Since we have a Brownian filtration, we can assume that all paths of M and $\left(\int_a^u M_s dW_s\right)_{u\in]a,1]}$ are continuous.

By the Burkholder-Davis-Gundy inequalities, there exists a constant $c_p > 0$ depending only on p such that

$$\begin{aligned} \left\| \int_a^b M_u dW_u \right\|_{L_p} &\le c_p \left\| \left(\int_a^b M_u^2 du \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\le c_p \left(\int_a^b \|M_u\|_{L_p}^2 \, du \right)^{\frac{1}{2}} \\ &\le c_p (b-a)^{\frac{1}{2}} \|M_b\|_{L_p} < \infty. \end{aligned}$$

Similarly, Lemma 4.5 implies that

$$\left\| \int_a^b \int_a^u M_s dW_s dW_u \right\|_{L_p} \le c_{(4.5)} \left\| \left(\int_a^b \int_a^u M_s^2 ds du \right)^{\frac{1}{2}} \right\|_{L_p}$$

$$\leq c_{(4.5)} \left(\int_a^b \int_a^u \|M_s\|_{L_p}^2 \, ds du \right)^{\frac{1}{2}}$$
$$\leq c_{(4.5)} \frac{(b-a)}{\sqrt{2}} \|M_b\|_{L_p} < \infty.$$

We first show (i) for a piecewise constant approximation of M. Let $n \in \{1, 2, \ldots\}$ and define

$$M_u^{(n)} := M_{a+\frac{b-a}{n}(i-1)} \quad \text{for} \quad u \in \left[a + \tfrac{b-a}{n}(i-1), a + \tfrac{b-a}{n} i\right[, \; i = 1, \ldots, n, \text{ and}$$
$$M_b^{(n)} := M_b.$$

Using the Burkholder-Davis-Gundy inequalities we obtain

$$\left\| \int_a^b M_u^{(n)} dW_u \right\|_{L_p} \geq \frac{1}{c_p} \left\| \left(\int_a^b \left(M_u^{(n)}\right)^2 du \right)^{\frac{1}{2}} \right\|_{L_p}$$
$$= \frac{1}{c_p} \left(\frac{b-a}{n} \right)^{\frac{1}{2}} \left\| \left(\sum_{i=1}^n \left(M_{a+\frac{b-a}{n}(i-1)} \right)^2 \right)^{\frac{1}{2}} \right\|_{L_p}. \quad (19)$$

Assume another probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$ and independent random variables $r_i : \tilde{\Omega} \to \{-1, 1\}$ with $\tilde{\mathbb{P}}(r_i = 1) = \tilde{\mathbb{P}}(r_i = -1) = \frac{1}{2}$, $i = 1, 2, \ldots, n$. Khintchine inequalities imply that, for some constant $d_p > 0$ depending only on p,

$$\left\| \left(\sum_{i=1}^n \left(M_{a+\frac{b-a}{n}(i-1)} \right)^2 \right)^{\frac{1}{2}} \right\|_{L_p}$$
$$\geq \frac{1}{d_p} \left\| \left\| \sum_{i=1}^n r_i M_{a+\frac{b-a}{n}(i-1)} \right\|_{L_p(\tilde{\Omega})} \right\|_{L_p(\Omega)}$$
$$= \frac{1}{d_p} \left\| \left\| \mathbb{E} \left(\sum_{i=1}^n r_i M_{a+\frac{b-a}{n}(i-1)} \mid \mathcal{F}_a \right) \right\|_{L_p(\Omega)} \right\|_{L_p(\tilde{\Omega})}$$
$$= \frac{1}{d_p} \|M_a\|_{L_p(\Omega)} \left\| \sum_{i=1}^n r_i \right\|_{L_p(\tilde{\Omega})}$$
$$\geq \frac{1}{d_p^2} \|M_a\|_{L_p(\Omega)} n^{\frac{1}{2}}.$$

Together with (19) this proves that

$$\left|\left|\int_a^b M_u^{(n)} dW_u\right|\right|_{L_p} \geq \frac{1}{c_p d_p^2} \|M_a\|_{L_p} (b-a)^{\frac{1}{2}}.$$

To complete the proof of (i), we need to show that

$$\lim_{n\to\infty} \left|\left|\int_a^b M_u^{(n)} dW_u\right|\right|_{L_p} = \left|\left|\int_a^b M_u dW_u\right|\right|_{L_p}. \tag{20}$$

By the Burkholder-Davis-Gundy inequalities,

$$\begin{aligned} \left|\left|\int_a^b M_u^{(n)} dW_u - \int_a^b M_u dW_u\right|\right|_{L_p} &\leq c_p \left|\left|\left(\int_a^b \left(M_u^{(n)} - M_u\right)^2 du\right)^{\frac{1}{2}}\right|\right|_{L_p} \\ &\leq c_p \left(\int_a^b \left|\left|M_u^{(n)} - M_u\right|\right|_{L_p}^2 du\right)^{\frac{1}{2}}. \end{aligned}$$

Doob's maximal inequality yields, for any $n = 1, 2, \ldots$ and any $u \in [a, b]$,

$$\left|\left|M_u^{(n)} - M_u\right|\right|_{L_p} \leq 2 \left|\left|\sup_{a\leq s\leq b} |M_s|\right|\right|_{L_p} \leq \frac{2p}{p-1} \|M_b\|_{L_p} < \infty.$$

Dominated convergence and continuity of M ensure that

$$\lim_{n\to\infty} \left|\left|M_u^{(n)} - M_u\right|\right|_{L_p} = \left|\left|\lim_{n\to\infty} M_u^{(n)} - M_u\right|\right|_{L_p} = 0$$

for all $u \in [a, b]$, and (20) is proven.

To prove (ii), notice first that $\int_a^b \int_a^u g(s) ds du = \int_a^b (b-u) g(u) du$ whenever both sides are well defined. Employing Lemma 4.5 and the Burkholder-Davis-Gundy inequalities we see that

$$\begin{aligned} \left|\left|\int_a^b \int_a^u M_s dW_s dW_u\right|\right|_{L_p} &\geq \frac{1}{c_{(4.5)}} \left|\left|\left[\int_a^b \int_a^u M_s^2 ds du\right]^{\frac{1}{2}}\right|\right|_{L_p} \\ &= \frac{1}{c_{(4.5)}} \left|\left|\left[\int_a^b (b-u) M_u^2 du\right]^{\frac{1}{2}}\right|\right|_{L_p} \\ &\geq \frac{1}{c_{(4.5)}\, c_p} \left|\left|\int_a^b (b-u)^{\frac{1}{2}} M_u dW_u\right|\right|_{L_p}. \end{aligned}$$

Then we use the same approach as in the proof of (i) but for the process $\left((b-u)^{\frac{1}{2}}M_u\right)_{u\in[a,b]}$. Define

$$N_u^{(n)} := \left(\tfrac{b-a}{n}(n-i+1)\right)^{\frac{1}{2}} M_{a+\frac{b-a}{n}(i-1)}, \quad u\in\left[a+\tfrac{b-a}{n}(i-1), a+\tfrac{b-a}{n}i\right[, \quad i=1,\dots,n, \text{ and}$$
$$N_b^{(n)} := M_b,$$

so that similarly to (20),

$$\lim_{n\to\infty}\left\|\int_a^b N_u^{(n)} dW_u\right\|_{L_p} = \left\|\int_a^b (b-u)^{\frac{1}{2}} M_u dW_u\right\|_{L_p}.$$

As above,

$$\begin{aligned}
&\left\|\int_a^b N_u^{(n)} dW_u\right\|_{L_p} \\
&\geq \frac{1}{c_p d_p}\left(\frac{b-a}{n}\right)^{\frac{1}{2}} \|M_a\|_{L_p(\Omega)} \left\|\sum_{i=1}^n r_i \left(\frac{b-a}{n}(n-i+1)\right)^{\frac{1}{2}}\right\|_{L_p(\tilde\Omega)} \\
&\geq \frac{1}{c_p d_p^2}\left(\frac{b-a}{n}\right)^{\frac{1}{2}} \|M_a\|_{L_p(\Omega)} \left(\sum_{i=1}^n \frac{b-a}{n}(n-i+1)\right)^{\frac{1}{2}}
\end{aligned}$$

which converges to $\tilde c_p(b-a)\|M_a\|_{L_p}$ as $n\to\infty$, where $\tilde c_p = (\sqrt{2}c_p d_p^2)^{-1}$, and (ii) is proven. □

4.2 Interpolation

The following result is the principal tool for the proof of Theorem 1.2. It connects the fractional smoothness of f and the growth rate of the first and second state derivatives of F.

Lemma 4.7. *Let* $2\le p<\infty$, $f\in L_p(\gamma)$ *and* $0<\theta<1$. *Then there are constants* $c>0$ *and* $c'>0$ *depending only on* p *and* θ *such that*

$$\begin{aligned}
\|f\|_{\theta,\infty} &\sim_c \|f\|_{L_p} + \sup_{0<t<1}(1-t)^{\frac{1-\theta}{2}}\left\|\frac{\partial F}{\partial x}(t,W_t)\right\|_{L_p} && (21)\\
&\sim_{c'} \|f\|_{L_p} + \sup_{0<t<1}(1-t)^{\frac{2-\theta}{2}}\left\|\frac{\partial^2 F}{\partial x^2}(t,W_t)\right\|_{L_p}, && (22)
\end{aligned}$$

where $\|f\|_{\theta,\infty}$ *denotes the norm of* f *in the interpolation space* $(L_p(\gamma),\mathbb{D}_{1,p}(\gamma))_{\theta,\infty}$.

Though not needed in the proofs, it may be of interest to observe that for $\theta = 1$ we have a slightly different situation:

Lemma 4.8. *Let* $2 \le p < \infty$ *and* $f \in L_p(\gamma)$. *Then there exist constants* $c, c_1, c_2 > 0$ *depending only on* p *such that*

$$\|f\|_{\mathbb{D}_{1,p}(\gamma)} \sim_c \|f\|_{L_p} + \sup_{0<t<1} \left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_p} \tag{23}$$

and

$$\begin{aligned} \sup_{0<t<1} \frac{1}{\left(\log \frac{1}{(1-t)}\right)^{\frac{1}{2}} + 1} \left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_p} \\ &\le c_1 \left(\|f\|_{L_p} + \sup_{0<t<1} (1-t)^{\frac{1}{2}} \left\| \frac{\partial^2 F}{\partial x^2}(t, W_t) \right\|_{L_p} \right) && (24) \\ &\le c_2 \left(\|f\|_{L_p} + \sup_{0<t<1} \left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_p} \right). && (25) \end{aligned}$$

Proof of Lemma 4.7. Throughout the proof we use the notation $\tilde{W}$ to indicate a Brownian motion independent of W, and $\tilde{\mathbb{E}}$ for the corresponding expectation.

Assuming first that $\|f\|_{\theta,\infty} < \infty$, for any $\epsilon > 0$ and any $s > 0$ we find functions $f_1^{s,\epsilon} \in L_p(\gamma)$ and $f_2^{s,\epsilon} \in \mathbb{D}_{1,p}(\gamma)$ such that $f = f_1^{s,\epsilon} + f_2^{s,\epsilon}$ and

$$\|f_1^{s,\epsilon}\|_{L_p} + s \|f_2^{s,\epsilon}\|_{\mathbb{D}_{1,p}} \le s^\theta \|f\|_{\theta,\infty} + \epsilon. \tag{26}$$

For the solutions $F_1^{s,\epsilon}$ and $F_2^{s,\epsilon}$ of (1) for terminal conditions $f_1^{s,\epsilon}$ and $f_2^{s,\epsilon}$, respectively, we see that $F(t, W_t) = F_1^{s,\epsilon}(t, W_t) + F_2^{s,\epsilon}(t, W_t)$ for all $t \in [0, 1]$. For any $t \in [0, 1[$ and any $t < b < 1$, Lemma 4.6 and (2) allow the estimate

$$\left\| \frac{\partial F_1^{s,\epsilon}}{\partial x}(t, W_t) \right\|_{L_p} \le c_{(4.6)} (b-t)^{-\frac{1}{2}} \|F_1^{s,\epsilon}(b, W_b) - F_1^{s,\epsilon}(t, W_t)\|_{L_p}.$$

Letting $b \to 1$ and combining this with (26) leads to

$$\left\| \frac{\partial F_1^{s,\epsilon}}{\partial x}(t, W_t) \right\|_{L_p} \le 2c_{(4.6)} (1-t)^{-\frac{1}{2}} \left(s^\theta \|f\|_{\theta,\infty} + \epsilon \right).$$

Furthermore, $f_2^{s,\epsilon} \in \mathbb{D}_{1,p}(\gamma)$ and (6) imply

$$\left\| \frac{\partial F_2^{s,\epsilon}}{\partial x}(t, W_t) \right\|_{L_p} = \|\mathbb{E}\left((f_2^{s,\epsilon})'(W_1) \mid \mathcal{F}_t \right)\|_{L_p} \le \|(f_2^{s,\epsilon})'(W_1)\|_{L_p}$$

for all $0 < t < 1$. Therefore, for any $0 < t < 1$,

$$\begin{aligned}
\left\|\frac{\partial F}{\partial x}(t, W_t)\right\|_{L_p} &\le \left\|\frac{\partial F_1^{s,\epsilon}}{\partial x}(t, W_t)\right\|_{L_p} + \left\|\frac{\partial F_2^{s,\epsilon}}{\partial x}(t, W_t)\right\|_{L_p} \\
&\le 2c_{(4.6)}(1-t)^{-\frac{1}{2}}\left(s^\theta \|f\|_{\theta,\infty} + \epsilon\right) + s^{\theta-1}\|f\|_{\theta,\infty} + s^{-1}\epsilon
\end{aligned}$$

for all $\epsilon > 0$ and all $s > 0$. Then, fix $t \in]0, 1[$. Choosing $s = \sqrt{1-t}$ we achieve

$$\begin{aligned}
\left\|\frac{\partial F}{\partial x}(t, W_t)\right\|_{L_p} &\le 2c_{(4.6)}(1-t)^{-\frac{1}{2}}(1-t)^{\frac{\theta}{2}}\|f\|_{\theta,\infty} + 2c_{(4.6)}(1-t)^{-\frac{1}{2}}\epsilon \\
&\quad +(1-t)^{\frac{\theta-1}{2}}\|f\|_{\theta,\infty} + (1-t)^{-\frac{1}{2}}\epsilon \\
&= c(1-t)^{\frac{\theta-1}{2}}\|f\|_{\theta,\infty} + c(1-t)^{-\frac{1}{2}}\epsilon
\end{aligned}$$

with $c := 2c_{(4.6)} + 1$. Letting $\epsilon \to 0$ leads to

$$\sup_{0<t<1}(1-t)^{\frac{1-\theta}{2}}\left\|\frac{\partial F}{\partial x}(t, W_t)\right\|_{L_p} \le c\|f\|_{\theta,\infty}.$$

Observing that $\|f\|_{L_p} \le \|f\|_{\theta,\infty}$ (see e.g. (15)), we obtain the first part of (21).

Let us then assume that

$$\|f\|_{L_p} + \sup_{0<t<1}(1-t)^{\frac{1-\theta}{2}}\left\|\frac{\partial F}{\partial x}(t, W_t)\right\|_{L_p} \le C < \infty.$$

We will show that the K-functional of $L_p(\gamma)$ and $\mathbb{D}_{1,p}(\gamma)$ for f at $s > 0$ can be bounded from above by s^θ with a constant proportional to C, that is, there is a constant $c_\theta > 0$ depending only on p and θ such that

$$K(f, s; L_p, \mathbb{D}_{1,p}) \le c_\theta C s^\theta \text{ for all } 0 < s < \infty, \tag{27}$$

which implies that

$$\|f\|_{\theta,\infty} = \sup_{s>0} s^{-\theta} K(f, s; L_p, \mathbb{D}_{1,p}) \le c_\theta C.$$

To show (27), define for all $t \in [0, 1]$ the functions g_t and h_t by setting

$$\begin{aligned}
g_t(x) &:= F(t, \sqrt{t}x), \\
h_t(x) &:= f(x) - F(t, \sqrt{t}x)
\end{aligned}$$

and note that $f(x) = g_t(x) + h_t(x)$ for any $t \in [0, 1]$. By definition of the K-functional,

$$K(f, s; L_p, \mathbb{D}_{1,p}) \le \|h_t\|_{L_p} + s\|g_t\|_{\mathbb{D}_{1,p}}$$

for any $t \in [0,1[$. Since $f \in L_p(\gamma)$, we see that, for $s \geq 1$,

$$K(f,s;L_p,\mathbb{D}_{1,p}) \leq \|f\|_{L_p} \leq \|f\|_{L_p} s^\theta \leq Cs^\theta.$$

Thus we may assume that $s < 1$. Our aim is to find, for a given $s \in]0,1[$, a number $t \in]0,1[$ such that

$$\|h_t\|_{L_p} + s\,\|g_t\|_{\mathbb{D}_{1,p}} \leq dCs^\theta,$$

where the constant d may depend on p and θ, but not on s or t.

Using Remark 2.3 and the fact that $(F(t,W_t))_{t\in[0,1]}$ is a martingale (see (2)), we compute for the smooth part g_t that, for any $t \in [0,1[$,

$$\begin{aligned}
\|g_t\|^p_{\mathbb{D}_{1,p}} &= \left\|F(t,\sqrt{t}x)\right\|^p_{L_p} + \left\|\frac{\partial F}{\partial x}(t,\sqrt{t}x)\sqrt{t}\right\|^p_{L_p} \\
&= \|F(t,W_t)\|^p_{L_p} + \left\|\frac{\partial F}{\partial x}(t,W_t)\sqrt{t}\right\|^p_{L_p} \\
&\leq \|F(1,W_1)\|^p_{L_p} + \left\|\frac{\partial F}{\partial x}(t,W_t)\right\|^p_{L_p} \\
&\leq \|f\|^p_{L_p} + C^p(1-t)^{-\frac{1-\theta}{2}p},
\end{aligned}$$

which means that

$$\|g_t\|_{\mathbb{D}_{1,p}} \leq \|f\|_{L_p} + C(1-t)^{-\frac{1-\theta}{2}}. \tag{28}$$

For h_t, we use the fact that both $\sqrt{t}W_1 + \sqrt{1-t}\tilde{W}_1$ and $W_{\sqrt{t}} + \tilde{W}_{1-\sqrt{t}}$ have standard normal distribution, and covariance $\sqrt{t}$ with W_1. Applying Jensen's inequality, triangle inequality, and Itô's formula yields

$$\begin{aligned}
\|h_t\|_{L_p(\gamma)} &= \left[\mathbb{E}\left|f(W_1) - \tilde{\mathbb{E}}f(\sqrt{t}W_1 + \sqrt{1-t}\tilde{W}_1)\right|^p\right]^{\frac{1}{p}} \\
&\leq \left[\mathbb{E}\tilde{\mathbb{E}}\left|\left(f(W_1) - f(\sqrt{t}W_1 + \sqrt{1-t}\tilde{W}_1)\right)\right|^p\right]^{\frac{1}{p}} \\
&= \left[\mathbb{E}\tilde{\mathbb{E}}\left|\left(f(W_1) - f(W_{\sqrt{t}} + \tilde{W}_{1-\sqrt{t}})\right)\right|^p\right]^{\frac{1}{p}} \\
&\leq \left\|f(W_1) - F(\sqrt{t},W_{\sqrt{t}})\right\|_{L_p} \\
&\quad + \left\|F(\sqrt{t},W_{\sqrt{t}}) - f(W_{\sqrt{t}} + \tilde{W}_{1-\sqrt{t}})\right\|_{L_p} \\
&= 2\left\|f(W_1) - F(\sqrt{t},W_{\sqrt{t}})\right\|_{L_p}
\end{aligned}$$

$$= 2\left\|\int_{\sqrt{t}}^{1}\frac{\partial F}{\partial x}(u,W_u)dW_u\right\|_{L_p},$$

where the norm $\|\cdot\|_{L_p}$ denotes the L_p-norm in the product space when necessary. By the Burkholder-Davis-Gundy inequalities we thus obtain the estimate

$$\begin{aligned}
\|h_t\|_{L_p(\gamma)} &\le 2\left\|\int_{\sqrt{t}}^{1}\frac{\partial F}{\partial x}(u,W_u)dW_u\right\|_{L_p} \\
&\le 2c_p\left\|\left(\int_t^1\left[\frac{\partial F}{\partial x}(u,W_u)\right]^2du\right)^{\frac{1}{2}}\right\|_{L_p} \\
&\le 2c_p\left(\int_t^1\left\|\left[\frac{\partial F}{\partial x}(u,W_u)\right]\right\|_{L_p}^2du\right)^{\frac{1}{2}} \\
&\le 2c_p\left(\int_t^1\left[C(1-u)^{-\frac{1-\theta}{2}}\right]^2du\right)^{\frac{1}{2}} \\
&\le c_{p,\theta}\,C(1-t)^{\frac{\theta}{2}}, \qquad (29)
\end{aligned}$$

where c_p is the constant from the Burkholder-Davis-Gundy inequalities and $c_{p,\theta} = 2c_p\theta^{-\frac{1}{2}}$.

Recall that $0 < s < 1$ and define $t_1 := 1 - s^2$. Then $t_1 \in\,]0,1[$ and $(1-t_1)^{-\frac{1-\theta}{2}} = s^{\theta-1}$. The claim (27) follows by combining (28) and (29):

$$\begin{aligned}
\|h_{t_1}\|_{L_p} + s\,\|g_{t_1}\|_{\mathbb{D}_{1,p}} &\le c_{p,\theta}\,C\,(1-t_1)^{\frac{\theta}{2}} + s\left(\|f\|_{L_p} + C(1-t_1)^{-\frac{1-\theta}{2}}\right) \\
&= c_{p,\theta}\,Cs^{\theta} + s\,\|f\|_{L_p} + Cs^{\theta} \le C(c_{p,\theta}+2)s^{\theta},
\end{aligned}$$

because $s \le s^{\theta}$. The proof of (21) is now complete.

For (22), assume first that

$$\|f\|_{L_p} + \sup_{0<t<1}(1-t)^{\frac{1-\theta}{2}}\left\|\frac{\partial F}{\partial x}(t,W_t)\right\|_{L_p} \le C < \infty.$$

Then, the Burkholder-Davis-Gundy inequalities imply that

$$\left\|\int_t^1\frac{\partial F}{\partial x}(u,W_u)dW_u\right\|_{L_p} \le c_p\left\|\left(\int_t^1\frac{\partial F}{\partial x}(u,W_u)^2du\right)^{\frac{1}{2}}\right\|_{L_p}$$

$$
\begin{aligned}
&\le c_p \left(\int_t^1 \left\| \frac{\partial F}{\partial x}(u, W_u) \right\|_{L_p}^2 du \right)^{\frac{1}{2}} \\
&\le c_p C \left(\int_t^1 (1-u)^{\theta-1} du \right)^{\frac{1}{2}} \\
&\le c_p C \frac{1}{\sqrt{\theta}} (1-t)^{\frac{\theta}{2}},
\end{aligned}
$$

where c_p is the constant from the Burkholder-Davis-Gundy inequalities. Since $f \in L_2(\gamma)$ implies that $\frac{\partial^2 F}{\partial x^2}(t, \cdot) \in L_2(\gamma)$ for $0 \le t < 1$ (see (2)), we can use Itô's formula to obtain $\frac{\partial F}{\partial x}(u, W_u) = \frac{\partial F}{\partial x}(t, W_t) + \int_t^u \frac{\partial^2 F}{\partial x^2}(s, W_s) dW_s$ a.s. for $0 \le t < u < 1$. Therefore,

$$
\begin{aligned}
\left\| \int_t^1 \int_t^u \frac{\partial^2 F}{\partial x^2}(s, W_s) dW_s dW_u \right\|_{L_p} &\le \left\| \int_t^1 \frac{\partial F}{\partial x}(u, W_u) dW_u \right\|_{L_p} \\
&\quad + \left\| \frac{\partial F}{\partial x}(t, W_t)(W_1 - W_t) \right\|_{L_p} \\
&\le C \left(c_p \frac{1}{\sqrt{\theta}} + \hat{c}_p \right) (1-t)^{\frac{\theta}{2}},
\end{aligned}
$$

where $\hat{c}_p$ is the pth moment of the standard normal distribution. Lemma 4.6 and (2) yield

$$
\left\| \frac{\partial^2 F}{\partial x^2}(t, W_t) \right\|_{L_p} \le c_{(4.6)} (1-t)^{-1} \left\| \int_t^1 \int_t^u \frac{\partial^2 F}{\partial x^2}(s, W_s) dW_s dW_u \right\|_{L_p},
$$

where we first consider the outer integral over the interval $]t, b]$ for $t < b < 1$ and then let $b \to 1$. Thus

$$
\sup_{0<t<1} (1-t)^{1-\frac{\theta}{2}} \left\| \frac{\partial^2 F}{\partial x^2}(t, W_t) \right\|_{L_p} \le C c_{(4.6)} \left(c_p \frac{1}{\sqrt{\theta}} + \hat{c}_p \right),
$$

and the first part of (22) is proven.

Finally, assume that

$$
\|f\|_{L_p} + \sup_{0<t<1} (1-t)^{\frac{2-\theta}{2}} \left\| \frac{\partial^2 F}{\partial x^2}(t, W_t) \right\|_{L_p} \le C < \infty.
$$

As above, Lemma 4.6 with (2) yields

$$
\left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_p} \le c_{(4.6)} (1-t)^{-\frac{1}{2}} \left\| \int_t^1 \frac{\partial F}{\partial x}(s, W_s) dW_s \right\|_{L_p},
$$

where we first consider the integral over the interval $]t, b]$ for $t < b < 1$ and then let $b \to 1$. For $t = 0$, this means that

$$\left\| \frac{\partial F}{\partial x}(0, W_0) \right\|_{L_p} \leq c_{(4.6)} \|f(W_1) - \mathbb{E}\,(f(W_1))\|_{L_p} \leq 2c_{(4.6)} \|f(W_1)\|_{L_p}$$

(the same conclusion could be drawn from the inequality $\left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_p} \leq \hat{c}_p(1-t)^{-\frac{1}{2}} \|f(W_1) - \mathbb{E}\,(f(W_1) \mid \mathcal{F}_t)\|_{L_p}$ in [12, Proof of Proposition 3.5] with a different constant). Using the Burkholder-Davis-Gundy inequalities we achieve

$$\begin{aligned}
\left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_p} &= \left\| \int_0^t \frac{\partial^2 F}{\partial x^2}(s, W_s) dW_s + \frac{\partial F}{\partial x}(0, W_0) \right\|_{L_p} \\
&\leq \left\| \int_0^t \frac{\partial^2 F}{\partial x^2}(s, W_s) dW_s \right\|_{L_p} + \left\| \frac{\partial F}{\partial x}(0, W_0) \right\|_{L_p} \\
&\leq c_p \left\| \left(\int_0^t \left[\frac{\partial^2 F}{\partial x^2}(s, W_s) \right]^2 ds \right)^{\frac{1}{2}} \right\|_{L_p} + 2c_{(4.6)} C \\
&\leq c_p \left(\int_0^t \left\| \frac{\partial^2 F}{\partial x^2}(s, W_s) \right\|_{L_p}^2 ds \right)^{\frac{1}{2}} + 2c_{(4.6)} C \\
&\leq c_p C \left(\int_0^t (1-s)^{\theta-2} ds \right)^{\frac{1}{2}} + 2c_{(4.6)} C \\
&= C \left(\frac{c_p}{\sqrt{1-\theta}} \left((1-t)^{\theta-1} - 1 \right)^{\frac{1}{2}} + 2c_{(4.6)} \right) \\
&\leq C \frac{\left(c_p + 2c_{(4.6)} \right)}{\sqrt{1-\theta}} (1-t)^{\frac{\theta-1}{2}}
\end{aligned}$$

for any $0 < t < 1$. This completes the proof. □

Proof of Lemma 4.8. Let us first consider (23) for $p = 2$. Similarly to [11, Lemma 3.7], it can be shown that

$$\mathbb{E} \left(\frac{\partial F}{\partial x}(t, W_t) \right)^2 = \sum_{k=1}^{\infty} \alpha_k^2 k t^{k-1}$$

for all $0 \leq t < 1$. Thus

$$\sum_{k=1}^{\infty} k \alpha_k^2 = \lim_{t \to 1} \sum_{k=1}^{\infty} \alpha_k^2 k t^{k-1} = \sup_{0<t<1} \left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_2}^2, \tag{30}$$

since $\left(\frac{\partial F}{\partial x}(t, W_t)\right)_{t\in[0,1[}$ is a martingale (see (2)).

For $2 < p < \infty$, (23) is clear from (30) and (6).

Inequality (25) is similar to the first part of (22). For (24), assume that

$$\|f\|_{L_p} + \sup_{0<t<1} (1-t)^{\frac{1}{2}} \left\| \frac{\partial^2 F}{\partial x^2}(t, W_t) \right\|_{L_p} \leq C < \infty.$$

As in the proof of the second part of (22) above,

$$\begin{aligned}
\left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_p} &\leq c_p \left(\int_0^t \left\| \frac{\partial^2 F}{\partial x^2}(s, W_s) \right\|_{L_p}^2 ds \right)^{\frac{1}{2}} + \left\| \frac{\partial F}{\partial x}(0,0) \right\|_{L_p} \\
&\leq c_p C \left(\int_0^t (1-s)^{-1} ds \right)^{\frac{1}{2}} + 2\hat{c}_p C \\
&= C \left(c_p (-\log(1-t))^{\frac{1}{2}} + 2\hat{c}_p \right) \\
&\leq C(c_p + 2\hat{c}_p) \left(\left(\log \frac{1}{(1-t)} \right)^{\frac{1}{2}} + 1 \right)
\end{aligned}$$

for any $0 < t < 1$. Thus

$$\sup_{0<t<1} \frac{1}{\left(\log \frac{1}{(1-t)}\right)^{\frac{1}{2}} + 1} \left\| \frac{\partial F}{\partial x}(t, W_t) \right\|_{L_p} \leq C(c_p + 2\hat{c}_p)$$

and the proof is complete. □

4.3 Approximation rate and second derivative

Lemma 4.9. *Let* $2 \leq p < \infty$, $f \in L_p(\gamma)$ *and* $0 < \theta < 1$. *Then there exists a constant* $c_{(4.9)} > 0$ *depending only on* p *and* θ *such that*

$$\sup_n n^{\frac{\theta}{2}} \|C_1(f, \tau_n)\|_{L_p} \sim_{c_{(4.9)}} \sup_{0<t<1} (1-t)^{\frac{2-\theta}{2}} \left\| \frac{\partial^2 F}{\partial x^2}(t, W_t) \right\|_{L_p}.$$

Proof. Assume first that $\sup_n n^{\frac{\theta}{2}} \|C_1(f, \tau_n)\|_{L_p} \leq C < \infty$. Recall that by (2), $\left(\frac{\partial^2 F}{\partial x^2}(t, W_t)\right)_{t\in[0,1[}$ is a p-integrable martingale. For $n \geq 2$ and the equidistant time net $\tau_n = \left(\frac{i}{n}\right)_{i=0}^n$, Lemma 4.6 implies for the penultimate time point $t_{n-1}^n = \frac{n-1}{n}$ that

$$\left\| \frac{\partial^2 F}{\partial x^2}(t_{n-1}^n, W_{t_{n-1}^n}) \right\|_{L_p}$$

$$\leq c_{(4.6)} (1-t^n_{n-1})^{-1} \left\| \int_{t^n_{n-1}}^1 \int_{t^n_{n-1}}^u \frac{\partial^2 F}{\partial x^2}(s, W_s) dW_s dW_u \right\|_{L_p},$$

where we first consider the outer integral over the interval $]t^n_{n-1}, b]$ for $t^n_{n-1} < b < 1$ and then let $b \to 1$. By Itô's formula,

$$\frac{\partial F}{\partial x}(u, W_u) = \frac{\partial F}{\partial x}(t, W_t) + \int_t^u \frac{\partial^2 F}{\partial x^2}(s, W_s) dW_s \text{ a.s.} \tag{31}$$

for $0 \leq t < u < 1$, so that

$$\begin{aligned} & \left\| \int_{t^n_{n-1}}^1 \int_{t^n_{n-1}}^u \frac{\partial^2 F}{\partial x^2}(s, W_s) dW_s dW_u \right\|_{L_p} \\ = & \left\| \int_{t^n_{n-1}}^1 \left[\frac{\partial F}{\partial x}(u, W_u) - \frac{\partial F}{\partial x}(t^n_{n-1}, W_{t^n_{n-1}}) \right] dW_u \right\|_{L_p} \\ = & \left\| C_1(f, \tau_n) - C_{t^n_{n-1}}(f, \tau_n) \right\|_{L_p}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| \frac{\partial^2 F}{\partial x^2}(t^n_{n-1}, W_{t^n_{n-1}}) \right\|_{L_p} & \leq 2c_{(4.6)} (1-t^n_{n-1})^{-1} \| C_1(f, \tau_n) \|_{L_p} \\ & \leq 2c_{(4.6)} (1-t^n_{n-1})^{-1} C \left(\frac{1}{n} \right)^{\frac{\theta}{2}} \\ & \leq 2c_{(4.6)} C (1-t^n_{n-1})^{\frac{\theta}{2}-1}. \end{aligned}$$

To consider time points $\frac{1}{2} < t < 1$ not belonging to any equidistant time net, let $t^n_{n-1} < t < t^{n+1}_n$ and note that $(1-t^{n+1}_n)^{\frac{\theta}{2}-1} = \left(\frac{1}{n}\right)^{\frac{\theta}{2}-1} \left(\frac{n+1}{n}\right)^{1-\frac{\theta}{2}} \leq 2\left(\frac{1}{n}\right)^{\frac{\theta}{2}-1}$. Since $\frac{\partial^2 F}{\partial x^2}$ is a martingale (see (2)), this yields

$$\begin{aligned} \left\| \frac{\partial^2 F}{\partial x^2}(t, W_t) \right\|_{L_p} & \leq \left\| \frac{\partial^2 F}{\partial x^2}(t^{n+1}_n, W_{t^{n+1}_n}) \right\|_{L_p} \\ & \leq 2c_{(4.6)} C (1-t^{n+1}_n)^{\frac{\theta}{2}-1} \\ & \leq 4c_{(4.6)} C (1-t^n_{n-1})^{\frac{\theta}{2}-1} \\ & \leq 4c_{(4.6)} C (1-t)^{\frac{\theta}{2}-1}. \end{aligned}$$

Finally, for $0 \leq t < \frac{1}{2}$,

$$\left\| \frac{\partial^2 F}{\partial x^2}(t, W_t) \right\|_{L_p} \leq \left\| \frac{\partial^2 F}{\partial x^2}(\frac{1}{2}, W_{\frac{1}{2}}) \right\|_{L_p}$$

$$
\begin{aligned}
&\leq 2c_{(4.6)}C(1-\frac{1}{2})^{\frac{\theta}{2}-1} \\
&\leq 4c_{(4.6)}C \\
&\leq 4c_{(4.6)}C(1-t)^{\frac{\theta}{2}-1},
\end{aligned}
$$

as desired.

Now, we assume that

$$
\sup_{0<t<1}(1-t)^{\frac{2-\theta}{2}}\left\|\frac{\partial^2 F}{\partial x^2}(t,W_t)\right\|_{L_p}\leq C<\infty.
$$

For any time net $\tau=(t_i)_{i=0}^n$, $n=1,2,\ldots$, we see by (31) that

$$
\begin{aligned}
\|C_1(f,\tau)\|_{L_p} &= \left\|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\left[\frac{\partial F}{\partial x}(u,W_u)-\frac{\partial F}{\partial x}(t_{i-1},W_{t_{i-1}})\right]dW_u\right\|_{L_p} \\
&= \left\|\int_0^1\int_0^u\sum_{i=1}^n\chi_{[t_{i-1},t_i[}(u)\,\chi_{[t_{i-1},u[}(s)\,\frac{\partial^2 F}{\partial x^2}(s,W_s)dW_s dW_u\right\|_{L_p}.
\end{aligned}
$$

Applying the Burkholder-Davis-Gundy inequalities twice, this can be estimated from above by

$$
\begin{aligned}
& c_p\left\|\left(\int_0^1\left(\int_0^u\sum_{i=1}^n\chi_{[t_{i-1},t_i[}(u)\,\chi_{[t_{i-1},u[}(s)\,\frac{\partial^2 F}{\partial x^2}(s,W_s)dW_s\right)^2du\right)^{\frac{1}{2}}\right\|_{L_p} \\
&\leq c_p\left(\int_0^1\left\|\int_0^u\sum_{i=1}^n\chi_{[t_{i-1},t_i[}(u)\,\chi_{[t_{i-1},u[}(s)\,\frac{\partial^2 F}{\partial x^2}(s,W_s)dW_s\right\|_{L_p}^2du\right)^{\frac{1}{2}} \\
&\leq c_p^2\left(\int_0^1\left\|\left(\int_0^u\left[\sum_{i=1}^n\chi_{[t_{i-1},t_i[}(u)\,\chi_{[t_{i-1},u[}(s)\,\frac{\partial^2 F}{\partial x^2}(s,W_s)\right]^2ds\right)^{\frac{1}{2}}\right\|_{L_p}^2du\right)^{\frac{1}{2}} \\
&\leq c_p^2\left(\int_0^1\int_0^u\left\|\sum_{i=1}^n\chi_{[t_{i-1},t_i[}(u)\,\chi_{[t_{i-1},u[}(s)\,\frac{\partial^2 F}{\partial x^2}(s,W_s)\right\|_{L_p}^2dsdu\right)^{\frac{1}{2}} \\
&= c_p^2\left(\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^u\left\|\frac{\partial^2 F}{\partial x^2}(s,W_s)\right\|_{L_p}^2dsdu\right)^{\frac{1}{2}},
\end{aligned}
$$

where c_p is the constant from the Burkholder-Davis-Gundy inequalities. Thus

$$\begin{aligned}
\|C_1(f,\tau)\|_{L_p} &\leq c_p^2 \left(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{u} C^2 (1-s)^{\theta-2} \, ds du \right)^{\frac{1}{2}} \\
&\leq c_p^2 C \left(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{1}{1-\theta} \left[(1-u)^{\theta-1} - (1-t_{i-1})^{\theta-1} \right] du \right)^{\frac{1}{2}} \\
&= c_{p,\theta} \left(\int_0^1 (1-u)^{\theta-1} du - \sum_{i=1}^{n} (t_i - t_{i-1})(1-t_{i-1})^{\theta-1} \right)^{\frac{1}{2}},
\end{aligned}$$

where $c_{p,\theta} = c_p^2 C (1-\theta)^{-\frac{1}{2}}$. Since $(1-t_{i-1})^{\theta-1} \geq (1-s)^{\theta-1}$ when $s < t_{i-1} < 1$, we compute for the equidistant time net that

$$\begin{aligned}
\|C_1(f,\tau_n)\|_{L_p} &\leq c_{p,\theta} \left(\int_0^1 (1-u)^{\theta-1} du - \sum_{i=1}^{n} \frac{1}{n} (1-t_{i-1})^{\theta-1} \right)^{\frac{1}{2}} \\
&\leq c_{p,\theta} \left(\frac{1}{\theta} - \sum_{i=2}^{n} \int_{t_{i-2}}^{t_{i-1}} (1-s)^{\theta-1} ds \right)^{\frac{1}{2}} \\
&\leq c_{p,\theta} \, \theta^{-\frac{1}{2}} \left(1 + (1-t_{n-1}^n)^{\theta} - 1 \right)^{\frac{1}{2}} \\
&= c_{p,\theta} \, \theta^{-\frac{1}{2}} \left(\frac{1}{n} \right)^{\frac{\theta}{2}}.
\end{aligned}$$

This completes the proof. □

4.4 Proof of Theorem 1.2

Combine Lemma 4.7 with Lemma 4.9.

5 Examples

Let us illustrate the results with some simple examples.

By Remark 2.3, all the necessary derivatives in our examples can be computed using the piecewise defined classical derivative.

5.1 Equidistant time nets

Example 5.1. *Let* $2 \le p < \infty$ *and*

$$f(x) := \chi_{[0,\infty[}(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases}.$$

Then

$$\|C_1(f,\tau_n)\|_{L_p} \le cn^{-\frac{1}{2p}}$$

and, for all $\lambda > 0$*,*

$$\mathbb{P}\left(|C_1(f,\tau_n)| \ge \frac{\lambda}{n^{\frac{1}{2p}}}\right) \le c^p \lambda^{-p}$$

for some $c > 0$ *depending only on* p*.*

Proof. For any $0 < t < 1$ we can take the decomposition $f = f_0^t + f_1^t$ with

$$f_0^t(x) := \begin{cases} 0, & x < 0 \\ 1 - \frac{x}{t}, & 0 \le x \le t \\ 0, & x > t \end{cases} \text{ and } f_1^t(x) := \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 \le x \le t \\ 1, & x > t \end{cases}.$$

We see that

$$\begin{aligned} & K(f,t;L_p(\gamma),\mathbb{D}_{1,p}(\gamma)) \\ \le \;& \left|\left|f_0^t\right|\right|_{L_p} + t\left|\left|f_1^t\right|\right|_{\mathbb{D}_{1,p}} \\ \le \;& t^{\frac{1}{p}} + t\left(\int_0^t \left(\frac{x}{t}\right)^p d\gamma(x) + \int_t^\infty 1^p d\gamma(x) + \int_0^t \left(\frac{1}{t}\right)^p d\gamma(x)\right)^{\frac{1}{p}} \\ \le \;& c_p t^{\frac{1}{p}}, \end{aligned}$$

for some $c_p > 0$ depending only on p. The case $t \ge 1$ is majorized by $||f||_{L_p} < \infty$, so that $f \in (L_p(\gamma), \mathbb{D}_{1,p}(\gamma))_{\frac{1}{p},\infty}$.

Theorem 1.2 thus implies that $||C_1(f,\tau_n)||_{L_p} \le cn^{-\frac{1}{2p}}$, and the tail estimate follows using Chebyshev's inequality for $|C_1(f,\tau_n)|^p$. □

Example 5.1 implies that, for any $p \ge 2$, all convex combinations of jump functions are in $(L_p(\gamma), \mathbb{D}_{1,p}(\gamma))_{\frac{1}{p},\infty}$. Improving integrability, i.e. increasing p, makes the fractional smoothness decrease - as well as the convergence rate. For the tail estimate, with a larger p, the decay of the tail is faster, but there are more time points needed for the same treshold.

Example 5.2. *Let* $0 < \alpha < 1$ *and*

$$f_\alpha(x) := \begin{cases} 0, & x < 0 \\ x^\alpha, & x \geq 0 \end{cases} .$$

Then, for $p \geq 2$*, we have the following:*

1. *if* $\alpha > 1 - \frac{1}{p}$ *, then* $f_\alpha \in \mathbb{D}_{1,p}(\gamma)$*, and*
2. *if* $\alpha < 1 - \frac{1}{p}$ *, then* $f_\alpha \in (L_p(\gamma), \mathbb{D}_{1,p}(\gamma))_{\alpha+\frac{1}{p},\infty}$ *.*

For the approximation error, this means that

1. *if* $\alpha > 1 - \frac{1}{p}$ *, then* $\|C_1(f_\alpha, \tau_n)\|_{L_p} \leq \frac{c}{\sqrt{n}}$ *, and*
2. *if* $\alpha < 1 - \frac{1}{p}$ *, then* $\|C_1(f_\alpha, \tau_n)\|_{L_p} \leq c\left(\frac{1}{\sqrt{n}}\right)^{\alpha+\frac{1}{p}}$

with the equidistant time nets $\tau_n = \left(\frac{i}{n}\right)_{i=0}^n$ *and with some* $c > 0$ *depending only on* p *and* α*.*

Notice that in Example 5.2, the first condition, $\alpha > 1 - \frac{1}{p}$, implies that $\frac{1}{2} < \alpha < 1$ because $p \geq 2$. The second statement matches with Example 5.1, where the jump function can be seen as the "$\alpha = 0$" case. The case $\alpha = 1$ is Lipschitz and thus belongs to $\mathbb{D}_{1,p}(\gamma)$ with any $p \geq 2$ by Lemma A.5.

The case $\alpha = 1 - \frac{1}{p}$ is not explicitely covered by Example 5.2; however, by (13), the second statement implies that $f_{1-\frac{1}{p}} \in (L_p(\gamma), \mathbb{D}_{1,p}(\gamma))_{\theta,\infty}$ with any $0 < \theta < 1$.

For $p = 2$, the approximation rates achieved in Examples 5.1 and 5.2 coincide with those computed earlier in [13]; see also [9, Remark 6.6 and examples on p. 254].

As in Example 5.1, the L_p convergence rates in Example 5.2 imply tail estimates with decay λ^{-p}.

Proof of Example 5.2. For $\alpha > 1 - \frac{1}{p}$,

$$\|f_\alpha\|_{\mathbb{D}_{1,p}}^p = \int_0^\infty x^{\alpha p} d\gamma(x) + \int_0^\infty \left(\alpha x^{\alpha-1}\right)^p d\gamma(x) < \infty$$

because $p(1-\alpha) < 1$.

So, let $\alpha < 1 - \frac{1}{p}$. As in the proof of (21) in Lemma 4.7, we need to prove that

$$\sup_{0<t<1} t^{-\left(\alpha+\frac{1}{p}\right)} K(f_\alpha, t; L_p, \mathbb{D}_{1,p}) < \infty.$$

For any $0 < t < 1$ we can take the decomposition

$$g^t(x) := \begin{cases} 0, & x < 0 \\ x^\alpha - t^{\alpha-1}x, & 0 \le x \le t \\ 0, & x > t \end{cases} \quad \text{and} \quad h^t(x) := \begin{cases} 0, & x < 0 \\ t^{\alpha-1}x, & 0 \le x \le t \\ x^\alpha, & x > t \end{cases}.$$

This yields the estimates

$$\left|\left|g^t\right|\right|_{L_p} = \left(\int_0^t (x^\alpha - t^{\alpha-1}x)^p d\gamma(x)\right)^{\frac{1}{p}} \le (t^{\alpha p+1})^{\frac{1}{p}} = t^{\alpha+\frac{1}{p}}$$

and

$$\begin{aligned} \left|\left|h^t\right|\right|_{\mathbb{D}_{1,p}} &= \left(\int_0^t (t^{\alpha-1}x)^p d\gamma(x) + \int_t^\infty x^{\alpha p} d\gamma(x) \right. \\ &\quad \left. + \int_0^t t^{p(\alpha-1)} d\gamma(x) + \int_t^\infty \left(\alpha x^{\alpha-1}\right)^p d\gamma(x)\right)^{\frac{1}{p}} \\ &\le \left(t^{p(\alpha-1)} \Big/_0^t \frac{1}{p+1} x^{p+1} + \hat{c}_{\alpha p} \right. \\ &\quad \left. + t^{p(\alpha-1)+1} + \alpha^p \int_t^\infty x^{-p(1-\alpha)} d\gamma(x)\right)^{\frac{1}{p}}, \end{aligned} \tag{32}$$

where $\hat{c}_{\alpha p} = \int_0^\infty x^{\alpha p} d\gamma(x) < \infty$. By integration by parts, we achieve for any $y > 0$ and any $\beta > 1$ the estimate

$$\int_y^\infty x^{-\beta} d\gamma(x) \le \frac{y^{1-\beta}}{\beta - 1}.$$

Applying this to (32) with $\beta = p(1-\alpha) > 1$, we obtain

$$\begin{aligned} \left|\left|h^t\right|\right|_{\mathbb{D}_{1,p}} &\le \left(\frac{1}{p+1} t^{p(\alpha-1)+p+1} + \hat{c}_{\alpha p} + t^{p(\alpha-1)+1} + \alpha^p \frac{t^{1-p(1-\alpha)}}{p(1-\alpha)-1}\right)^{\frac{1}{p}} \\ &\le c_{\alpha,p} \left(t^{p(\alpha-1)+1}\right)^{\frac{1}{p}} \\ &= c_{\alpha,p} t^{\alpha-1+\frac{1}{p}}, \end{aligned}$$

where $c_{\alpha,p} = \left(2 + \hat{c}_{\alpha p} + \frac{\alpha^p}{p(1-\alpha)-1}\right)^{\frac{1}{p}}$. Therefore,

$$K(f_\alpha, t; L_p, \mathbb{D}_{1,p}) \le \left|\left|g^t\right|\right|_{L_p} + t\left|\left|h^t\right|\right|_{\mathbb{D}_{1,p}} \le t^{\alpha+\frac{1}{p}} + t c_{\alpha,p} t^{\alpha-1+\frac{1}{p}} \le c t^{\alpha+\frac{1}{p}},$$

where $c = c_{\alpha,p} + 1$.

Theorems 1.1 and 1.2 then imply the convergence rates. □

5.2 Remark on non-equidistant time nets

For any $0 < \theta \leq 1$, we define the time nets $\tau_n^\theta = \left(t_i^{n,\theta}\right)_{i=0}^n$ by setting $t_i^{n,\theta} := 1 - (1 - \frac{i}{n})^{\frac{1}{\theta}}$ for $i = 0, 1, \ldots, n$. For $\theta = 1$, this definition yields the equidistant time nets, and for smaller θ, the time nets are denser near 1.

Using a similar interpolation argument as in Section 3, we obtain the following result:

Theorem 5.3. *Let $0 < \eta, \theta < 1$. If $f \in (B_{2,2}^\theta(\gamma), \mathrm{Lip})_{\eta, \frac{2}{1-\eta}}$, where $B_{2,2}^\theta(\gamma) = (L_2(\gamma), \mathbb{D}_{1,2}(\gamma))_{\theta,2}$ and Lip denotes the space of Lipschitz functions equipped with the norm $||g||_{\mathrm{Lip}} := |g(0)| + \sup_{x<y} \frac{|g(y)-g(x)|}{y-x}$, then there exists a constant $c_{(5.3)} > 0$ not depending on n such that*

$$\left|\left|C_1(f, \tau_n^\theta)\right|\right|_{L_p} \leq \frac{c_{(5.3)}}{\sqrt{n}}$$

for $p = \frac{2}{1-\eta}$.

Notice that the constant $c_{(5.3)}$ may depend on p, θ, and f.

Proof. The proof follows the approach of Theorem 1.1 with minor changes. We use interpolation for f, not for f', and for the other endpoint, we use $B_{2,2}^\theta(\gamma)$ instead of $\mathbb{D}_{1,2}(\gamma)$.

For $f \in B_{2,2}^\theta(\gamma)$, Theorem 3.2 and formula (4) of [11] imply that

$$\left|\left|C_1(f, \tau_n^\theta)\right|\right|_{L_2} \leq \frac{c_1}{\sqrt{n}} ||f||_{B_{2,2}^\theta}$$

for some $c_1 > 0$ not depending on n. For Lipschitz functions, $\left|\frac{\partial F}{\partial x}(t, x)\right| \leq ||f||_{\mathrm{Lip}}$ for all $0 < t < 1$ and all $x \in \mathbb{R}$, see e.g. the proof of Lemma A.5. It remains to observe that

$$\left|\left|\tau_n^\theta\right|\right|_\infty \leq \frac{1}{\theta n}$$

for all $n \geq 1$. □

Theorem 5.3 is not necessarily sharp, in the sense that the same optimal L_p convergence rate might be achieved with time nets whose refining index θ is closer to 1, or there could be a larger class of functions that lead to the same convergence rate with the same time nets τ_n^θ.

Example 5.4. *Let $0 < \alpha < 1$ and*

$$f_\alpha(x) := \begin{cases} 0, & x < 0 \\ x^\alpha, & x \geq 0 \end{cases} .$$

Then, for any $0 < p < \frac{2}{1-\alpha}$ *and any* $0 < \theta < \frac{1}{2}$,

$$\left|\left|C_1(f, \tau_n^\theta)\right|\right|_{L_p} \leq \frac{c_{(5.3)}}{\sqrt{n}}.$$

Proof. With a computation similar to Example 5.1, we see that

$$\left|\left|\chi_{[0,t[}\right|\right|_{(L_2, \mathbb{D}_{1,2})_{\frac{1}{2},\infty}} \leq 3 + t^{\frac{1}{2}}.$$

For any $0 < t < 1$, using the decomposition $f_\alpha = f_\alpha^{0,t} + f_\alpha^{1,t}$ with

$$f_\alpha^{0,t}(x) := \begin{cases} 0, & x < 0 \\ x^\alpha - t^{\alpha-1}x, & 0 \leq x \leq t \\ 0, & x > t \end{cases} \quad \text{and} \quad f_\alpha^{1,t}(x) := \begin{cases} 0, & x < 0 \\ t^{\alpha-1}x, & 0 \leq x \leq t \\ x^\alpha, & x > t \end{cases}$$

we obtain the estimate

$$\begin{aligned} K(f_\alpha, t; (L_2(\gamma), \mathbb{D}_{1,2}(\gamma))_{\frac{1}{2},\infty}, \mathrm{Lip}) &\leq \left|\left|f_\alpha^{0,t}\right|\right|_{(L_2, \mathbb{D}_{1,2})_{\frac{1}{2},\infty}} + t\left|\left|f_\alpha^{1,t}\right|\right|_{\mathrm{Lip}} \\ &\leq \left|\left|t^\alpha \chi_{[0,t[}\right|\right|_{(L_2, \mathbb{D}_{1,2})_{\frac{1}{2},\infty}} + t \cdot t^{\alpha-1} \\ &\leq t^\alpha \left(3 + t^{\frac{1}{2}}\right) + t^\alpha \\ &\leq 5t^\alpha. \end{aligned}$$

This means that

$$f_\alpha \in \left((L_2(\gamma), \mathbb{D}_{1,2}(\gamma))_{\frac{1}{2},\infty}, \mathrm{Lip}\right)_{\alpha,\infty}$$

and, by (13) and (14),

$$f_\alpha \in ((L_2(\gamma), \mathbb{D}_{1,2}(\gamma))_{\theta,2}, \mathrm{Lip})_{\alpha,\infty}$$

for all $0 < \theta < \frac{1}{2}$. Recall the notation $B_{2,2}^\theta(\gamma) = (L_2(\gamma), \mathbb{D}_{1,2}(\gamma))_{\theta,2}$. By (14) and Lemma A.5,

$$||g||_{B_{2,2}^\theta} \leq c_\theta ||g||_{\mathbb{D}_{1,2}} \leq c_\theta c_{(A.5)} ||g||_{\mathrm{Lip}}$$

for any $g \in \mathrm{Lip}$. Thus by (14) again, for any $0 < \beta < \alpha$ and any $0 < \theta < \frac{1}{2}$,

$$f_\alpha \in (B_{2,2}^\theta(\gamma), \mathrm{Lip})_{\beta, \frac{2}{1-\beta}}$$

and by Theorem 5.3, there exists a $c_{(5.3)} > 0$ not depending on n such that

$$\left|\left|C_1(f, \tau_n^\theta)\right|\right|_{L_p} \leq \frac{c_{(5.3)}}{\sqrt{n}}$$

for $p = \frac{2}{1-\beta}$. □

Example 5.4 shows that we can increase the power p over the limit $\frac{1}{1-\alpha}$ (or, equivalently, decrease the regularity condition from $\alpha > 1 - \frac{1}{p}$ to $\alpha > 1 - \frac{2}{p}$) discovered in Example 5.2 and still get the optimal convergence rate if we adjust the time nets by some index $0 < \theta < \frac{1}{2}$. However, the limit is only raised by factor 2; we do not know whether this can be improved further.

6 Concluding remarks

There are several questions to consider for further research. Are these kind of convergence results true for processes other than Brownian motion? In particular, to what extent can the same techniques be used for more general processes? Could we generalize other known results from L_2 to L_p, $p > 2$, such as employing non-equidistant time nets to achieve optimal convergence rates? What about $p < 2$, or stronger smoothness conditions?

It seems that Theorem 1.1 also has an analogue for geometric Brownian motion. The results under interpolation are almost the same for both processes, allowing for a transformation of the function f; only instead of bounded mean oscillation (BMO), geometric Brownian motion leads to a *weighted* BMO space. Interpolation between this particular type of weighted BMO and L_2 is not known to the author, but by stepping back to L_q and interpolating between L_2 and L_q with arbitrary large q, the convergence result should follow for the $L_{p-\epsilon}$ norm.

The techniques developed in Section 4 might carry us further. For $p = 2$, diffusions satisfying $dY_t = \sigma(Y_t)dW_t$, $Y_0 = y_0$, with some conditions on σ, are treated in [7], and there extension to diffusions with drift is mentioned but not considered in detail. Though the techniques of [7] apply only to L_2, this gives hope that similar results for more general processes might be true in L_p, $p > 2$, too.

In [7], the L_2 convergence rate is improved by using special non-equidistant time nets (see also [11]). Improvement is also possible in L_p, $p > 2$, as seen in Theorem 5.3; however, this result is only a first observation to that direction, and by no means sharp like the results in L_2. This improvement in convergence rate is not contradictory to Corollary 1.3, since there the optimal convergence rate is required to hold when using any kind of time nets, in particular equidistant time nets. It would be interesting to check if a connection as in [7] between fractional smoothness and refinement of time nets required for the optimal convergence rate could be proved for $p > 2$.

In this paper, approximations of stochastic integrals are of the first order, like the Euler scheme in simulations of SDE's. It might be of interest to also consider higher order approximations, and see if connections between convergence properties and higher order fractional smoothness could be revealed. One might expect to see improved convergence rates when considering functions of higher smoothness.

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A Appendix

Proofs of Theorems 2.8 and 2.12 are presented here for the convenience of the reader.

Assume the notation and setting of Section 1.1 also through this part.

A.1 Proof of Theorem 2.8

Proof of Theorem 2.8. Our aim is to show that there exists a constant $c > 0$ such that

$$\mathbb{E}\left([C_1(f,\tau) - C_s(f,\tau)]^2 \mid \mathcal{F}_s \right) \le c \, \|\tau\|_\infty \sup_{\substack{0<t<1 \\ x\in R}} \left| \frac{\partial F}{\partial x}(t,x) \right|^2$$

$\mathbb{P}$-a.s. for all $0 \le s \le 1$. The endpoint $s = 1$ is trivial.

Fix a time net $\tau = (t_i)_{i=0}^n$, and a number $s \in [0,1[$. Set i_0 such that $s \in [t_{i_0-1}, t_{i_0}[$, and define a new time net on $[s,1]$ by setting $r_{i_0-1} := s$ and $r_i = t_i$ for $i = i_0, \ldots, n$. Then

$$[C_1(f,\tau) - C_s(f,\tau)]^2 \le 2(I_1^2 + I_2^2),$$

where

$$I_1 := \int_s^1 \frac{\partial F}{\partial x}(u, W_u) dW_u - \sum_{i=i_0}^n \frac{\partial F}{\partial x}(r_{i-1}, W_{r_{i-1}}) \left(W_{r_i} - W_{r_{i-1}} \right)$$

and

$$I_2 := \left(\frac{\partial F}{\partial x}(s, W_s) - \frac{\partial F}{\partial x}(t_{i_0-1}, W_{t_{i_0-1}}) \right) \left(W_{t_{i_0}} - W_s \right).$$

First we observe that for I_2, almost surely,

$$\begin{aligned} & \mathbb{E}\left(I_2^2 \mid \mathcal{F}_s \right) \\ = & \left(\frac{\partial F}{\partial x}(s, W_s) - \frac{\partial F}{\partial x}(t_{i_0-1}, W_{t_{i_0-1}}) \right)^2 \mathbb{E}\left(\left(W_{t_{i_0}} - W_s \right)^2 \mid \mathcal{F}_s \right) \\ = & \left(\frac{\partial F}{\partial x}(s, W_s) - \frac{\partial F}{\partial x}(t_{i_0-1}, W_{t_{i_0-1}}) \right)^2 |t_{i_0} - s| \\ \le & \; 4 \sup_{\substack{0<t<1 \\ x\in R}} \left| \frac{\partial F}{\partial x}(t,x) \right|^2 \|\tau\|_\infty . \end{aligned}$$

For I_1, we condition the problem by W_s, i.e. start anew at $W_s =: y_0$. In case $s = 0$ we require that $y_0 = 0$. Set $\tilde{f}(y) := f(y_0 + y)$, $\tilde{T} := 1 - s$ and $\tilde{F}(t,x) := \mathbb{E}\left(\tilde{f}(x + W_{\tilde{T}-t}) \right) = F(t+s, y_0 + x)$. Here we use the fact that if $f \in L_2(\gamma)$, then $\mathbb{E}\left(f(y + W_s) \right)^2 < \infty$ for all $y \in \mathbb{R}$ and $0 < s < 1$ (see (11) for computation). From Theorem 4.4 of [9] we obtain, with some constant $c_1 > 0$, that for any $y_0 \in \mathbb{R}$,

$$\mathbb{E}\left(I_1^2 \mid W_s = y_0 \right)$$

$$
\begin{aligned}
&= \mathbb{E}\Bigg(\int_s^1 \frac{\partial F}{\partial x}(u, y_0 + W_{u-s})dW_{u-s} \\
&\quad - \sum_{i=i_0}^{n} \frac{\partial F}{\partial x}(r_{i-1}, y_0 + W_{r_{i-1}-s})(W_{r_i-s} - W_{r_{i-1}-s})\Bigg)^2 \\
&= \mathbb{E}\Bigg(\int_0^{\tilde{T}} \frac{\partial \tilde{F}}{\partial x}(v, W_v)dW_v \\
&\quad - \sum_{i=i_0}^{n} \frac{\partial \tilde{F}}{\partial x}(r_{i-1} - s, W_{r_{i-1}-s})(W_{r_i-s} - W_{r_{i-1}-s})\Bigg)^2 \\
&\leq c_1^2 \sum_{i=i_0}^{n} \int_{r_{i-1}-s}^{r_i-s} (r_i - s - v)\mathbb{E}\left(\frac{\partial^2 \tilde{F}}{\partial x^2}(v, W_v)\right)^2 dv \\
&\leq c_1^2 \|\tau\|_\infty \int_0^{\tilde{T}} \mathbb{E}\left(\frac{\partial^2 F}{\partial x^2}(v + s, y_0 + W_v)\right)^2 dv.
\end{aligned}
$$

By monotone convergence, this is the same as

$$
\begin{aligned}
&c_1^2 \|\tau\|_\infty \sup_{0<\tilde{S}<\tilde{T}} \int_0^{\tilde{S}} \mathbb{E}\left(\frac{\partial^2 F}{\partial x^2}(v + s, y_0 + W_v)\right)^2 dv \\
&= c_1^2 \|\tau\|_\infty \sup_{0<\tilde{S}<\tilde{T}} \mathbb{E}\left(\int_0^{\tilde{S}} \left[\frac{\partial^2 F}{\partial x^2}(v + s, y_0 + W_v)\right]^2 dv\right) \\
&= c_1^2 \|\tau\|_\infty \sup_{0<\tilde{S}<\tilde{T}} \mathbb{E}\left(\int_0^{\tilde{S}} \frac{\partial^2 F}{\partial x^2}(v + s, y_0 + W_v)dW_v\right)^2.
\end{aligned}
$$

Since $\tilde{S} + s < 1$, we can continue as in (31) and achieve

$$
\begin{aligned}
&\mathbb{E}\left(I_1^2 \mid W_s = y_0\right) \\
&\leq c_1^2 \|\tau\|_\infty \sup_{0<\tilde{S}<\tilde{T}} \mathbb{E}\left(\frac{\partial F}{\partial x}(\tilde{S} + s, y_0 + W_{\tilde{T}}) - \frac{\partial F}{\partial x}(0 + s, y_0 + 0)\right)^2 \\
&\leq 4c_1^2 \|\tau\|_\infty \sup_{\substack{0<t<1 \\ x\in R}} \left|\frac{\partial F}{\partial x}(t, x)\right|^2
\end{aligned}
$$

so that the proof is complete. □

A.2 Proof of Theorem 2.12

Recall that the assumptions of Section 1.1 imply that every square-integrable martingale $M = (M)_{0\leq t\leq 1}$ with mean zero can be written as a stochastic integral: $M_0 = 0$ and $M_t = \int_0^t L_s dW_s$ for $0 < t \leq 1$ with some progressively measurable process $(L_t)_{0 \leq t \leq 1}$ satisfying $\int_0^1 \mathbb{E}\left(L_t^2\right) dt < \infty$. In this section we may assume that all martingales start at zero a.s.

Definition A.1 (Square bracket/quadratic variation)**.** *For a square-integrable martingale $M = (M_t)_{0 \leq t \leq 1}$ we define the quadratic variation $[M]_t := \int_0^t L_s^2 ds$ for $0 < t \leq 1$ and $[M]_0 := 0$.*

Definition A.2 (Hardy spaces)**.** *Let M be a square-integrable martingale and $2 \leq p < \infty$. Then $M \in H_p$ if*

$$||M||_{H_p} := \left|\left|[M]_1^{\frac{1}{2}}\right|\right|_{L_p} < \infty,$$

i.e. if

$$\mathbb{E}\left(\int_0^1 L_t^2 dt\right)^{\frac{p}{2}} < \infty.$$

Notice that the Burkholder-Davies-Gundy inequalities with Doob's maximal inequality say that for $2 \leq p < \infty$, the Hardy and L_p spaces are equivalent, when considering only mean zero random variables:

$$||M||_{H_p} \sim_{c_p} ||M_1||_{L_p^0}, \tag{33}$$

for all square-integrable martingales $M = (M)_{0 \leq t \leq 1}$, where the constant $c_p > 0$ depends only on p.

Recall equation (16) from Section 2 on interpolation between L_2 and L_∞. Here we will use it for centered random variables:

$$\left(L_2^0, L_\infty^0\right)_{\theta,p} = L_p^0 \tag{34}$$

for $0 < \theta < 1$ and $p = \frac{2}{1-\theta}$, and $||X||_{\left(L_2^0, L_\infty^0\right)_{\theta,p}} \sim_{C_p} ||X||_{L_p^0}$ for any random variable $X \in L_p^0$ and some $C_p > 0$ depending only on p.

The following result is included in [20, Proof of Theorem 2]:

Lemma A.3. *Let M be a square-integrable martingale. If*

$$\mathbb{E}\left(([M]_1 - [M]_t)^{\frac{1}{2}} \mid \mathcal{F}_t\right) \leq \gamma \text{ a.s.}$$

for all $0 < t < 1$ and for some random variable γ, then

$$\left|\left|[M]_1^{\frac{1}{2}}\right|\right|_{L_p} \leq \frac{2^p p}{p-1} ||\gamma||_{L_p}$$

for all $2 < p < \infty$.

Proof. We can estimate

$$\begin{aligned}
\left\|[M]_1^{\frac{1}{2}}\right\|_{L_p}^p &= 2^p \int_0^\infty p\alpha^{p-1}\mathbb{P}\left(\frac{[M]_1^{\frac{1}{2}}}{2} > \alpha\right) d\alpha \\
&= 2^p \int_0^\infty p\alpha^{p-2}\alpha\mathbb{E}\left(\chi\left(\frac{[M]_1^{\frac{1}{2}}}{2} > \alpha\right)\right) d\alpha.
\end{aligned}$$

By [20, Lemma 1],

$$\alpha\mathbb{E}\left(\chi\left(\frac{[M]_1^{\frac{1}{2}}}{2} > \alpha\right)\right) \leq \mathbb{E}\left(\chi\left([M]_1^{\frac{1}{2}} > \alpha\right)\gamma\right)$$

for all $\alpha \geq 0$. Fubini's theorem and Hölder's inequality thus imply that

$$\begin{aligned}
\left\|[M]_1^{\frac{1}{2}}\right\|_{L_p}^p &= 2^p \int_0^\infty p\alpha^{p-2}\mathbb{E}\left(\chi\left([M]_1^{\frac{1}{2}} > \alpha\right)\gamma\right) d\alpha \\
&= 2^p\mathbb{E}\left(p\gamma\int_0^\infty \alpha^{p-2}\chi\left([M]_1^{\frac{1}{2}} > \alpha\right) d\alpha\right) \\
&= 2^p\frac{p}{p-1}\mathbb{E}\left(\gamma[M]_1^{\frac{p-1}{2}}\right) \\
&\leq 2^p\frac{p}{p-1}\|\gamma\|_{L_p}\left\|[M]_1^{\frac{1}{2}}\right\|_{L_p}^{p-1},
\end{aligned}$$

which completes the proof. $\square$

Theorem A.4. *Let* $0 < \theta < 1$. *Then, for* $p = \frac{2}{1-\theta}$,

$$\left(L_2^0, \mathrm{BMO}\right)_{\theta,p} \subset L_p^0$$

and there exists a constant $c_{(A.4)} > 0$ *depending only on* p *such that*

$$\|X\|_{L_p^0} \leq c_{(A.4)}\|X\|_{\left(L_2^0,\mathrm{BMO}\right)_{\theta,p}}$$

for any random variable $X \in \left(L_2^0, \mathrm{BMO}\right)_{\theta,p}$.

Proof. Let $X \in L_2^0$. For any $t > 0$ and any $\epsilon > 0$ we find random variables $X_0^{t,\epsilon} \in L_2$ and $X_1^{t,\epsilon} \in \mathrm{BMO}$ such that $X = X_0^{t,\epsilon} + X_1^{t,\epsilon}$ and

$$\left\|X_0^{t,\epsilon}\right\|_{L_2} + t\left\|X_1^{t,\epsilon}\right\|_{\mathrm{BMO}} \leq K(X, t; L_2^0, \mathrm{BMO}) + \epsilon.$$

To avoid the discussion of the measurability of the sharp function $\sup_{0\le s\le 1}\mathbb{E}\left(([X]_1-[X]_s)^{\frac12}\mid\mathcal{F}_s\right)$, we define

$$\gamma_X := \inf\left\{\sup_{0\le s\le 1}\mathbb{E}\left([X_0^{t,\epsilon}]_1^{\frac12}\mid\mathcal{F}_s\right)+\left|\left|X_1^{t,\epsilon}\right|\right|_{BMO}\right\},$$

where the infimum extends over all $t>0$, $t\in\mathbb{Q}$ and all $\epsilon>0$, $\epsilon\in\mathbb{Q}$. Notice that by Jensen's inequality and Itô isometry,

$$\begin{aligned}
&\mathbb{E}\left(([X]_1-[X]_s)^{\frac12}\mid\mathcal{F}_s\right)\\
&\le \mathbb{E}\left(([X_0^{t,\epsilon}]_1-[X_0^{t,\epsilon}]_s)^{\frac12}\mid\mathcal{F}_s\right)+\mathbb{E}\left(([X_1^{t,\epsilon}]_1-[X_1^{t,\epsilon}]_s)^{\frac12}\mid\mathcal{F}_s\right)\\
&\le \mathbb{E}\left([X_0^{t,\epsilon}]_1^{\frac12}\mid\mathcal{F}_s\right)+\left(\mathbb{E}\left([X_1^{t,\epsilon}]_1-[X_1^{t,\epsilon}]_s\mid\mathcal{F}_s\right)\right)^{\frac12}\\
&\le \gamma_X
\end{aligned}$$

almost surely for any $s>0$. By Burkholder-Davis-Gundy inequalities and Lemma A.3,

$$||X||_{L_p}\le c_p\left|\left|[X]_1^{\frac12}\right|\right|_{L_p}\le\frac{2^p p c_p}{p-1}||\gamma_X||_{L_p}. \tag{35}$$

Furthermore, for all $t>0$, $t\in\mathbb{Q}$ and all $\epsilon>0$, $\epsilon\in\mathbb{Q}$,

$$\begin{aligned}
K(\gamma_X,t;L_2,L_\infty) &\le K\left(\sup_{0\le s\le 1}\mathbb{E}\left([X_0^{t,\epsilon}]_1^{\frac12}\mid\mathcal{F}_s\right)+\left|\left|X_1^{t,\epsilon}\right|\right|_{BMO},t;L_2,L_\infty\right)\\
&\le \left|\left|\sup_{0\le s\le 1}\mathbb{E}\left([X_0^{t,\epsilon}]_1^{\frac12}\mid\mathcal{F}_s\right)\right|\right|_{L_2}+t\left|\left|\left|\left|X_1^{t,\epsilon}\right|\right|_{BMO}\right|\right|_{L_\infty}\\
&\le 2\left|\left|[X_0^{t,\epsilon}]_1^{\frac12}\right|\right|_{L_2}+t\left|\left|X_1^{t,\epsilon}\right|\right|_{BMO}\\
&\le 2\left(\left|\left|X_0^{t,\epsilon}\right|\right|_{L_2}+t\left|\left|X_1^{t,\epsilon}\right|\right|_{BMO}\right)\\
&\le 2K(X,t;L_2^0,BMO)+2\epsilon,
\end{aligned}$$

where we have employed Doob's maximal inequality and Itô isometry. Letting $\epsilon\to\infty$, $\epsilon\in\mathbb{Q}$, we obtain

$$K(\gamma_X,t;L_2,L_\infty)\le 2K(X,t;L_2^0,BMO)$$

for all $t>0$, $t\in\mathbb{Q}$ and, since $t\mapsto K(x,t;X_0,X_1)$ is continuous for any compatible couple (X_0,X_1), for all $t>0$. This means that

$$||\gamma_X||_{L_p}\le C_p||\gamma_X||_{(L_2,L_\infty)_{\theta,p}}\le 2C_p||X||_{\left(L_2^0,BMO\right)_{\theta,p}}$$

and, by (35),

$$\|X\|_{L_p} \le c_{(A.4)} \|X\|_{\left(L_2^0, \mathrm{BMO}\right)_{\theta,p}}$$

with $c_{(A.4)} = c_p C_p \frac{2^{p+1}p}{p-1}$. $\square$

Proof of Theorem 2.12. Since $L_\infty^0 \subset \mathrm{BMO}$ with $\|M\|_{\mathrm{BMO}} \le 2\|M_1\|_{L_\infty^0}$ for any $M_1 \in L_\infty^0$, we see by (34) and (13) that

$$L_p^0 = \left(L_2^0, L_\infty^0\right)_{\theta,p} \subset \left(L_2^0, \mathrm{BMO}\right)_{\theta,p}$$

for all $0 < \theta < 1$ and $p = \frac{2}{1-\theta}$. Theorem 2.12 thus follows from Theorem A.4. $\square$

Lemma A.5. *If* $f \in \mathrm{Lip}$ *with*

$$\|f\|_{\mathrm{Lip}} := |f(0)| + \sup_{x<y} \frac{|f(y) - f(x)|}{y - x},$$

then $f \in \mathbb{D}_{1,p}(\gamma)$ *for all* $2 \le p < \infty$, *and*

$$\|f\|_{\mathbb{D}_{1,p}} \le c_{(A.5)} \|f\|_{\mathrm{Lip}}$$

for some constant $c_{(A.5)}$ *depending only on* p.

Proof. For some constant $d_p > 0$, we have $(a + b)^p \le d_p(a^p + b^p)$ for all $a, b \ge 0$. Since $|f(x)| \le |f(0)| + |x| \cdot \frac{|f(x)-f(0)|}{|x|}$ for all $x \ne 0$, we obtain

$$\begin{aligned}
\|f\|_{L_p}^p &= \int_{\mathbb{R}} [f(x)]^p \, d\gamma(x) \\
&\le d_p \left(\int_{\mathbb{R}\setminus\{0\}} |f(0)|^p d\gamma(x) + \int_{\mathbb{R}\setminus\{0\}} \left[|x| \cdot \frac{|f(x) - f(0)|}{|x|} \right]^p d\gamma(x) \right) \\
&\le d_p \left(|f(0)|^p + \|f\|_{\mathrm{Lip}}^p \int_{\mathbb{R}\setminus\{0\}} |x|^p d\gamma(x) \right) \\
&\le d_p(\hat{c}_p + 1) \|f\|_{\mathrm{Lip}}^p ,
\end{aligned}$$

where $\hat{c}_p = \int_{\mathbb{R}} |x|^p d\gamma(x)$.

Equation (7) states that

$$\frac{\partial F}{\partial x}(t, x) = \mathbb{E}\left(f(x + W_{1-t}) \frac{W_{1-t}}{1-t} \right)$$

for all $x \in \mathbb{R}$ and all $0 < t < 1$. Therefore,

$$\left| \frac{\partial F}{\partial x}(t, W_t) \right| = \left| \mathbb{E}\left(f(x + W_{1-t}) \frac{W_{1-t}}{1-t} \right) - \mathbb{E}\left(f(x) \frac{W_{1-t}}{1-t} \right) \right|$$

$$
\begin{aligned}
&\leq \mathbb{E}\left|[f(x+W_{1-t})-f(x)]\,\frac{W_{1-t}}{1-t}\right| \\
&\leq \mathbb{E}\left(\frac{|f(x+W_{1-t})-f(x)|}{|W_{1-t}|}\cdot\frac{W_{1-t}^2}{1-t}\right) \\
&\leq \|f\|_{\mathrm{Lip}}\,\mathbb{E}\left|\frac{W_{1-t}^2}{1-t}\right| = \|f\|_{\mathrm{Lip}}
\end{aligned}
$$

for all $0 < t < 1$. This means that

$$
\sup_{0<t<1}\left\|\frac{\partial F}{\partial x}(t, W_t)\right\|_{L_p} \leq \|f\|_{\mathrm{Lip}}
$$

so that Lemma 4.8 completes the proof. □