REMARKS ON THE IWANIEC-ŠVERÁK CONJECTURE

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Abstract. We give sufficient conditions that guarantee discreteness and openness of a mapping of finite distortion with integrable n-energy.

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1. Introduction

Let $f : \Omega \to \mathbb{R}^n$ be a $W^{1,1}_{\mathrm{loc}}$-map with locally integrable Jacobian determinant J_f. Then f is K_O-quasiregular, $1 \leq K_O < \infty$, if

$$|Df(x)|^n \leq K_O J_f(x) \quad \text{for almost every } x \in \Omega. \tag{1.1}$$

Moreover, f is a mapping of finite distortion if (1.1) holds for a measurable, almost everywhere finite function K_O. A fundamental theorem, due to Reshetnyak, says that a non-constant quasiregular map has strong topological properties. Namely, the preimage set of every point is discrete, and f is an open map, see [13].

Iwaniec and Šverák [8] proved in the plane that Reshetnyak's theorem remains valid for mappings of finite distortion f as long as $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and (1.1) holds for some

$$K_O \in L^{n-1}_{\mathrm{loc}}(\Omega), \tag{1.2}$$

and conjectured that this is the case in every dimension. A sharper form of this conjecture is stated by replacing assumption (1.2) with $K_I \in L^1_{\mathrm{loc}}(\Omega)$, where K_I is a measurable function satisfying

$$|D^\sharp f(x)|^n \leq K_I(x) J_f(x)^{n-1} \quad \text{for almost every } x \in \Omega. \tag{1.3}$$

Here $D^\sharp f$ is the adjoint matrix of Df. The inequality $K_I \leq K_O^{n-1}$ holds for the smallest possible distortion functions. It is also an open problem whether $K_I \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > 1$ suffices. Assumption (1.3) is very natural, because it is the inner distortion coefficient K_I that controls the relevant properties of the local inverse branches of discrete and open maps, the existence of which is the main content of Reshetnyak's theorem, cf. [7].

Both forms of the Iwaniec-Šverák conjecture remain open, but the case where f is assumed to be essentially finite-to-one is now well understood, see [6], [12], [5], and the proof of Theorem 2.2 below. In the general case Manfredi and Villamor [15] proved that discreteness and openness follow

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when $K_O \in L^q_{\mathrm{loc}}(\Omega)$ for some $q > n-1$, also see [2]. The main purpose of this note is to give the following improvement of their result.

Theorem 1.1. *Let $f : \Omega \to \mathbb{R}^n$, $n \geq 2$, be a non-constant mapping of finite distortion satisfying $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $K_O \in L^{n-1}_{\mathrm{loc}}(\Omega)$ and $K_I \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > 1$. Then f is discrete and open.*

The proof of Theorem 1.1 is given in Section 2. Although the Iwaniec-Šverák conjecture is known to be sharp in terms of the assumptions on K_O or K_I by an example given in [1], it seems that there are no such higher-dimensional examples of maps with infinite multiplicity. In Section 3 we construct planar maps of infinite multiplicity with $K_O \in L^p$ for every $p < 1$. These maps f are local homeomorphisms outside a line segment $E = f^{-1}(0)$. Our next result, the proof of which is given in Section 4, shows that maps with such properties cannot serve as counterexamples to the Iwaniec-Šverák conjecture in any dimension.

Theorem 1.2. *Let $f : \Omega \to \mathbb{R}^n$, $n \geq 2$, be a non-constant mapping of finite distortion satisfying $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $K_I \in L^1_{\mathrm{loc}}(\Omega)$. Assume moreover that f is a local homeomorphism outside a connected set $E = f^{-1}(0)$. Then f is discrete and open (and a local homeomorphism when $n \geq 3$).*

2. Proof of Theorem 1.1

We denote an n-ball with center x and radius r by $B(x,r)$, and $B(r) = B(0,r)$, $B^n = B(0,1)$. The corresponding notations for $(n-1)$-spheres are $S(x,r)$ and $S(r) = S(0,r)$. The Lebesgue measure of $E \subset \mathbb{R}^n$ is $|E|$. The k-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$. We will use the operator norm $|\cdot|$ for matrices. When $G \subset \Omega$, notation $N(y, f, G)$ refers to the number of preimage points of y under f in G.

In this section we assume that $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ is a non-constant mapping of finite distortion. Then (cf. [3]) f has a continuous representative, is differentiable almost everywhere, sense-preserving, and satisfies the change of variables formula

$$\int_G g(f(x)) J_f(x)\, dx = \int_{fG} g(y) N(y, f, G)\, \mathrm{d}y.$$

By the classical proof of Reshetnyak's theorem, we know that if f has the properties listed above, then it is discrete and open if the preimage set $f^{-1}(y)$ is totally disconnected for every $y \in \mathbb{R}^n$, cf. [3]. Therefore, we will prove Theorems 1.1 and 1.2 by showing that an arbitrary point y has this property. From [12] it follows that, under the assumption $K_I \in L^1$, no nontrivial component of $f^{-1}(y)$ can be compactly contained in the domain Ω.

2.1. **Integrability of the adjoint differential.** We may assume that $\Omega = B^n$ and that our local integrability assumptions in fact global ones. We first show that, under the assumptions of Theorem 1.1,

$$\int_{S(t)} \frac{|D^\sharp f(x)|}{|f(x)|^{n-1}}\, \mathrm{d}\mathcal{H}^{n-1}(x) < \infty \tag{2.1}$$

for almost every $0 < t < 1$. This is a rather straightforward consequence of some previously known results.

Proposition 2.1. *Let f satisfy the assumptions of Theorem 1.1. Then* (2.1) *holds for almost every* $0 < t < 1$.

Proof. By [11], the assumption $K_I \in L^p(B^n)$ for some $p > 1$ implies

$$\int_{B(R)} \frac{|D^\sharp f(x)|^p}{(|f(x)| \log \frac{1}{|f(x)|})^{(n-1)p}}\, \mathrm{d}x < \infty \quad \text{for every } 0 < R < 1 \tag{2.2}$$

(while the authors in [11] do not explicitly state this estimate, it follows from their arguments by replacing $|Df|^{n-1}$ and K_O^{n-1} with $|D^\sharp f|$ and K_I, respectively, and applying the corresponding distortion inequalities). Also, by [15] (see also [11]), the assumption $K_O \in L^{n-1}(B^n)$ yields (we assume that $|f| < 1/2$)

$$u = \log\log \frac{1}{|f|} \in W^{1,n-1}_{\mathrm{loc}}(B^n).$$

Hence the integral (2.2) over $S(t)$ is finite for almost every $0 < t < 1$, and the trace of u in $S(t)$ has $(n-1)$-integrable partial differentials. We fix such t. By Trudinger's inequality, cf. [16, Theorem 2.9.1],

$$\int_{S(t)} \Big(\log \frac{1}{|f(x)|}\Big)^q \, \mathrm{d}\mathcal{H}^{n-1}(x) < \infty$$

for every $1 < q < \infty$. Now Hölder's inequality yields

$$\begin{aligned}
\int_{S(t)} \frac{|D^\sharp f(x)|}{|f(x)|^{n-1}}\, \mathrm{d}\mathcal{H}^{n-1}(x) &= \int_{S(t)} \frac{|D^\sharp f(x)|(\log \frac{1}{|f(x)|})^{n-1}}{(|f(x)| \log \frac{1}{|f(x)|})^{n-1}}\, \mathrm{d}\mathcal{H}^{n-1}(x) \\
&\le \Big(\int_{S(t)} \frac{|D^\sharp f(x)|^p}{(|f(x)| \log \frac{1}{|f(x)|})^{(n-1)p}}\, \mathrm{d}\mathcal{H}^{n-1}(x)\Big)^{1/p} \\
&\times \Big(\int_{S(t)} \Big(\log \frac{1}{|f(x)|}\Big)^q \, \mathrm{d}\mathcal{H}^{n-1}(x)\Big)^{(n-1)/q} < \infty
\end{aligned}$$

for some $q > 1$. The proposition follows. □

From now on we only need to make the minimal assumption $K_I \in L^1(B^n)$. In view of Proposition 2.1, the following, independently interesting result implies Theorem 1.1.

Theorem 2.2. *Suppose that $K_I \in L^1(B^n)$, and that* (2.1) *holds true for almost every $0 < t < 1$. Then $f^{-1}(0)$ is totally disconnected.*

2.2. **Multiplicity bound.** The first step in proving Theorem 2.2 is to show that (2.1) implies that the map f is "on the average" finite-to-one around the origin.

Lemma 2.3. *Let f satisfy the assumptions of Theorem 2.2. For almost every $0 < t < 1$ and $0 < r < \infty$,*

$$r^{1-n} \int_{S(r)} N(y, f, B(t))\, \mathrm{d}\mathcal{H}^{n-1}(y) \leq C(n) \int_{S(t)} \frac{|D^\sharp f(x)|}{|f(x)|^{n-1}}\, \mathrm{d}\mathcal{H}^{n-1}(x).$$

Proof. Let $\epsilon > 0$ be small, and

$$\phi(y) = \begin{cases} 0, & |y| < r - \epsilon, \\ \frac{|y|-r+\epsilon}{2\epsilon}, & r - \epsilon < |y| < r + \epsilon, \\ 1, & |y| \geq r + \epsilon. \end{cases}$$

We define a differential $(n-1)$-form

$$\omega(y) = \sum_{j=1}^{n} (-1)^{j-1} \frac{\phi(y) y_j}{|y|^n} dy_1 \wedge .. \wedge \tilde{dy_j} \wedge .. \wedge dy_n.$$

Then the pullback $f^*\omega$ satisfies

$$||f^*\omega(x)|| \leq C(n) \frac{|D^\sharp f(x)|}{|f(x)|^{n-1}} \quad \text{for almost every } x \in B^n,$$

where $|| \cdot ||$ is any fixed norm. We denote

$$A(r, \epsilon) = B(r+\epsilon) \setminus \overline{B(r-\epsilon)}.$$

Then

$$\begin{aligned} d\omega(y) &= \frac{\chi_{A(r,\epsilon)}(y)}{2\epsilon|y|^{n-1}} dy_1 \wedge .. \wedge dy_n, \quad \text{and} \\ df^*\omega(x) &= \frac{\chi_{f^{-1}A(r,\epsilon)}(x) J_f(x)}{2\epsilon|f(x)|^{n-1}} dx_1 \wedge .. \wedge dx_n \end{aligned}$$

almost everywhere. We choose $0 < t < 1$ so that Stokes theorem holds in $B(t)$ (by approximating with smooth forms one sees that it holds for almost every t). By change of variables and Stokes theorem,

$$\begin{aligned} \int_{A(r,\epsilon)} \frac{N(y, f, B(t))}{2\epsilon|y|^{n-1}}\, \mathrm{d}y &= \int_{B(t)} df^*\omega = \int_{S(t)} f^*\omega \\ &\leq C(n) \int_{S(t)} \frac{|D^\sharp f(x)|}{|f(x)|^{n-1}}\, \mathrm{d}\mathcal{H}^{n-1}(x). \end{aligned}$$

When $\epsilon \to 0$, the first integral converges to

$$\frac{1}{r^{n-1}} \int_{S(r)} N(y, f, B(t))\, \mathrm{d}\mathcal{H}^{n-1}(y)$$

for almost every r by the Lebesgue differentiation theorem. The lemma follows. $\square$

2.3. **Bad points.** We need an auxiliary result. Let W' be the set of points y in fB^n for which $f^{-1}(y)$ has a nontrivial component. Moreover, let

$$W = \{(x_1, \dots x_{n-1}) \in \mathbb{R}^{n-1} : \exists\, t \in \mathbb{R} \text{ so that } (x_1, \dots, x_{n-1}, t) \in W'\}.$$

Lemma 2.4. *Let f satisfy the assumptions of Theorem 2.2. Then*

$$\mathcal{H}^{n-1}(W) = 0.$$

Proof. Let $Z \in B^n$ be the set of points that belong to some nontrivial component of $f^{-1}(y)$ for some $y \in \mathbb{R}^n$. Then Z is a closed set, and f is discrete and open outside Z. Let W_j, $j = 1, 2, \dots$, be the set of points y in W for which $\mathcal{H}^1(f^{-1}(y \times \mathbb{R}) \cap Z) \geq 1/j$. Then $W \subset \cup_j W_j$. By the coarea formula (see [10]),

$$\begin{aligned} j^{-1}\mathcal{H}^{n-1}(W_j) &\leq \int_{W_j} \mathcal{H}^1(f^{-1}(y \times \mathbb{R}) \cap Z) = y\})\, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &\leq C(n) \int_Z |D^\sharp f(x)|\, \mathrm{d}x. \end{aligned} \tag{2.3}$$

Since f is differentiable almost everywhere, $J_f(x) = 0$ for almost every $x \in Z$. Moreover, since f is a mapping of finite distortion, it follows that also $|D^\sharp f(x)| = 0$ for almost every such x. We conclude that the last integral in (2.3) equals 0. The proof is complete. $\square$

2.4. **Modulus of path families.** Let $g \geq 0$ be measurable and Γ a family of paths in an open set Ω. Then the weighted conformal modulus $M_g\Gamma$ is

$$M_g\Gamma = \inf_{\rho \in X} \int_\Omega g(x)\rho(x)^n\, \mathrm{d}x,$$

where X is the set of all Borel functions $\rho : \Omega \to [0, \infty]$ for which $\int_\gamma \rho\, \mathrm{d}s \geq 1$ for every locally rectifiable $\gamma \in \Gamma$. The conformal modulus $M\Gamma$ corresponds to the function $g = 1$.

We will use the following familiar estimate. Fix, for every $R \in E \subset (0, \infty)$, disjoint points p_R and q_R in $S(R)$. Moreover, let Γ_R be the family of paths joining p_R and q_R in $S(R)$, and Γ the union of the Γ_R:s. Then

$$M\Gamma \geq C_n \int_E \frac{\mathrm{d}R}{R}. \tag{2.4}$$

This estimate also holds if instead of spheres $S(R)$ we consider only spherical caps $C_R \subset S(R)$ for which both p_R and $q_R \in C_R$. See [14, Chapter 10] for these facts. In fact, (2.4) is equivalent to the Sobolev embedding theorem for $W^{1,n}(C_R)$. By examining the standard proof, we see that (2.4) remains valid if we define Γ in the following way. Let p_R and $q_R \in C_R$, and let T_R be the intersection of C_R with the set of points a that satisfy $|a - p_R| = |a - q_R|$. Now we define Γ'_R as the family of paths γ such that each γ first connects

p_R to some $a \in T_R$ with a spherical geodesic, and then a to q_R with another spherical geodesic. Let $\Gamma_R^0 \subset \Gamma_R'$ be a subfamily of paths so that the union

$$\bigcup_{\gamma \in \Gamma_R^0} |\gamma|$$

covers at most a set of vanishing $(n-1)$-measure in C_R, and define

$$\Gamma_R = \Gamma_R' \setminus \Gamma_R^0.$$

If Γ is again the union of the Γ_R:s, then (2.4) holds.

In [9] it was shown that a weak version of the conformal invariance of modulus remains valid (basically) in our setting, in the following sense.

Lemma 2.5. *Let W' be as in 2.3 and let Γ be a family of paths in B^n so that $f(|\gamma|) \cap W' = \emptyset$ for every $\gamma \in \Gamma$. Then $Mf\Gamma \leq M_{K_I}\Gamma$.*

Proof. By [9], the claim holds if f is discrete and open (in [9] it is assumed that $K_O \in L^p$ for some $p > n-1$, but only to have discreteness and openness. If this is already known, the arguments there go through with the assumption $K_I \in L^1$). The lemma follows since $f^{-1}(\mathbb{R}^n \setminus W')$ is discrete and open. $\square$

Finally, we will lift paths, see [13, II.3]. Let $f : \Omega \to \mathbb{R}^n$ be a continuous, discrete, open, and sense-preserving map. Moreover, let $\gamma : [0, 1) \to \mathbb{R}^n$ be a path, and $x \in \Omega$ such that $f(x) = \gamma(0)$. Then, by [13, Theorem II.3.2], there exists a so-called maximal f-lifting γ' of γ starting at x_0. That is, γ' is a path that has the following properties: $\gamma'(0) = x$, $f \circ \gamma' = \gamma|[0, c)$, and if $c < c' < 1$, then there does not exist a path $\gamma'' : [0, c') \to \Omega$ so that $\gamma' = \gamma''|[0, c)$ and $f \circ \gamma'' = \gamma|[a, c')$. If Γ and Γ' are path families so that every $\gamma \in \Gamma$ has a maximal f-lifting $\gamma' \in \Gamma'$, then clearly $Mf\Gamma' \geq M\Gamma$.

2.5. **Proof of Theorem 2.2.** We will prove that $f^{-1}(y)$ is totally disconnected for every $y \in \mathbb{R}^n$. Without loss of generality, $y = 0$. We choose a nontrivial component E of $f^{-1}(0)$. By restricting f to a small ball centered at some point in E and rotating, we may assume that a connected subset of E joins 0 and $e_n/2$ in $B(1/2)$. We fix an even integer M, to be determined later. We denote

$$\Omega_j = \{x \in B^n : (j-1)/M < x_n < j/M\}, \quad j = 1, \ldots, M/2.$$

Also, for each j we choose a line segment $I_j : [0, 1] \to V_j$ so that $|f(I_j(0))| = s$ and $f(I_j(1)) = 0$. Here $s > 0$ is some fixed number not depending on j, and $V_j = \{x \in B(1/2) : x_n = (j - 1/2)/M\}$.

We now fix j, and let Φ_j be the set of spheres $S(r)$, $0 < r < s$, for which there exists

$$q_r \in S(r) \setminus f(\overline{\Omega_j \cap B(2/3)}) = S(r) \setminus F_j. \tag{2.5}$$

We will prove a modulus estimate for paths on these spheres. This estimate is not affected if we modify Φ_j in a set of vanishing 1-measure.

Because $\mathbb{R}^n \setminus F_j$ is open, there exists for every $r \in \Phi_j$ a line segment

$$\xi_r^\epsilon = \{tq_r : 1-\epsilon < t < 1+\epsilon\} \subset \mathbb{R}^n \setminus F_j. \tag{2.6}$$

Also, there exists $p_r \in f(I_j[0,1]) \cap S(r)$. If $p_r \notin W'$, then the discreteness and openness of f outside $f^{-1}W'$ imply that there exists a line segment

$$\psi_r^\delta = \{tp_r : 1-\delta < t < 1+\delta\} \subset fQ_M \setminus W', \tag{2.7}$$

where Q_M is the $M/10$-neighborhood of $I_j[0,1]$. If $p_r \in W'$, we can guarantee the validity of (2.7) on almost every $0 < r < s$ by choosing p_r to be any point in $fQ_M \cap S(r) \setminus W'$ (that this set is nonempty for almost every $0 < r < s$ can be seen using Lemma 2.4 and the absolute continuity of f on almost every segment in Q_M parallel to I_j). We denote $\epsilon_r = \min\{\epsilon, \delta\}$, where ϵ and δ are as in (2.6) and (2.7), respectively.

Next, by using suitable coverings we may assume that

$$\Phi_j = \cup_{k=1}^\infty \Delta_k, \quad \Delta_k = (r_k(1-\epsilon_k), r_k(1+\epsilon_k)), \quad \epsilon_k = \epsilon_{r_k},$$

where the intervals Δ_k have bounded overlap. Let

$$A_k = \{x \in S(r) : r \in \Delta_k\}.$$

We will consider paths in A_k that join $\psi_k = \psi_{r_k}^{\epsilon_k}$ to $\xi_k = \xi_{r_k}^{\epsilon_k}$. By using local coverings of $A_k \setminus (\psi_k \cup \xi_k)$ and applying Lemma 2.4 in the coverings (together with bi-Lipschitz maps) in connection with the paths we constructed after (2.4), we see that if Γ_k is the family of all paths joining ψ_k to ξ_k in $A_k \setminus W'$, then $M\Gamma_k \geq \int_{\Delta_k} \frac{\mathrm{d}r}{r}$, and so

$$M\Gamma \geq \int_{\Phi_j} \frac{\mathrm{d}r}{Cr}, \quad \text{where} \quad \Gamma = \cup_{k=1}^\infty \Gamma_k. \tag{2.8}$$

On the other hand, we have constructed the ψ_k:s and ξ_k:s so that for every $\gamma \in \Gamma$ there exists a maximal f-lifting γ' that starts at a point in Q_M and leaves Ω_j (see (2.5)). This implies that every such γ' has length at least $1/(8M)$. Let Γ' be the family of all these lifts. Then

$$M_{K_I}\Gamma' \leq (8M)^n \int_{\Omega_j} K_I(x)\, \mathrm{d}x < \infty. \tag{2.9}$$

By combining Lemma 2.5, (2.8), and (2.9), we see that the logarithmic measure of Φ_j is finite. Consequently, the logarithmic measure of the union $\Phi = \cup_j \Phi_j$ is finite. This means that most of the spheres $S(r)$ for small r are covered at least $M/2$ times by $fB(2/3)$. It follows that for a set of r:s with infinite logarithmic measure,

$$\int_{S(r)} N(y, f, B(2/3))\, \mathrm{d}\mathcal{H}^{n-1}(y) \geq M/2.$$

This contradicts Lemma 2.3 and (2.1) when M is chosen to be large enough. The proof of Theorem 2.2 (and consequently the proof of Theorem 1.1) is complete.

3. Examples

The sharpness of the condition $K_O \in L^{n-1}$ (or $K_I \in L^1$) in the Iwaniec-Šverák conjecture is shown by a simple example that squeezes a line segment to a point with a map that is homeomorphic outside the line segment, see [1]. Other examples with more interesting preimage sets have been constructed in [4]. These examples are not homeomorphisms outside the non-trivial preimage sets, but they are finite-to-one. Here we construct some planar examples in which the image of a square spirals around the origin.

We define a map $f : (1,2) \times (-1,1) \to \mathbb{R}^2$ so that $f((1,2) \times \{0\}) = \{0\}$. It suffices to give the construction in $(1,2) \times [0,1)$, because one can then extend f to the rest of the domain by the reflecting:

$$f(x_1, x_2) = (f_1(x_1, -x_2), -f_2(x_1, -x_2)).$$

Let

$$f(x_1, x_2) = (x_1 r(x_2) \cos(\log x_2), x_1 r(x_2) \sin(\log x_2)),$$

where $r(x_2) \to 0$ as $x_2 \to 0$. Then

$$Df(x) = \begin{bmatrix} r(x_2)\cos(\log x_2) & x_1\left(\partial_2 r(x_2)\cos(\log x_2) - \frac{r(x_2)\sin(\log x_2)}{x_2}\right) \\ r(x_2)\sin(\log x_2) & x_1\left(\partial_2 r(x_2)\sin(\log x_2) + \frac{r(x_2)\cos(\log x_2)}{x_2}\right) \end{bmatrix}.$$

Moreover, $J_f(x) = \frac{x_1 r(x_2)^2}{x_2}$,

$$|Df(x)|^2 \le C \max\left\{\frac{r(x_2)^2}{x_2^2}, (\partial_2 r(x_2))^2\right\},$$

and

$$K_O(x) \le C \max\left\{\frac{1}{x_2}, \frac{x_2(\partial_2 r(x_2))^2}{r(x_2)^2}\right\}.$$

Now, if we choose $r(x_2) = x_2^\alpha$ for some $\alpha \ge 1$, then f is Lipschitz, and satisfies $K_O \in L^p$ for every $p < 1$. However, the restriction of f to $(1,2) \times (0,1)$ is then finite-to-one. If, on the other hand,

$$r(x_2) = \left(\log \frac{2}{x_2}\right)^{-\alpha} \quad \text{for some } \alpha > 1,$$

then again $K_O \in L^p$ for every $p < 1$, but now f has infinite multiplicity. This map f does not satisfy the assumption $f \in W^{1,2}$ in Theorem 1.1, but it belongs to $W^{1,1}$, and it has all the essential analytic properties listed in the beginning of Section 2.

4. Proof of Theorem 1.2

By [8], we may assume that $\Omega = B^n$ and $n \ge 3$. Moreover, by [12] we may assume that the connected set $f^{-1}(0)$ intersects $B(1/2)$ but is not compactly contained in B^n. Let

$$R = \max_{x \in \overline{B(1/2)}} |f(x)| > 0.$$

We claim that for every $0 < r < R$ there exist $x_r \in B(1/2)$ and $q_r \in S(r)$ so that $p_r = f(x_r) \in S(r)$ and so that every path γ joining p_r and q_r in some spherical cap $C_r \subset S(r)$ has a maximal f-lifting γ' starting at x_r and leaving $B(2/3)$ (see 2.4). Suppose for the moment that this is true. Let Γ be the family of all these γ:s, and Γ' the family of the corresponding lifts. Then

$$M\Gamma \leq M_{K_I}\Gamma'$$

by Lemma 2.5 and the discussion at the end of 2.4. Since the lengths of all γ':s are at least 1/6,

$$M_{K_I}\Gamma' \leq 6^n \int_{B(2/3)} K_I(x)\,\mathrm{d}x < \infty.$$

But by (2.4) (and the discussion afterwoods), $M\Gamma = \infty$. This is a contradiction. Thus it suffices to verify the claim above. The argument is basically the same as the main step in the proof of the Zorich Global Homeomorphism Theorem, see [13, III.3]. Therefore we omit most details.

We fix $0 < r < R$, and a point $p_r = f(x_r)$ so that $x_r \in B(1/2)$. Let $C_\varphi \subset S(r)$ be the relatively open spherical cap centered at p_r with opening angle $0 < \varphi \leq \pi$ (that is, $C_\varphi = B(p_r, t) \cap S(r)$ for some $0 < t < 2r$). Denote by φ_0 the supremum of all the φ:s with the property that the restriction of f to the x_r-component U_φ of $f^{-1}(C_\varphi)$ is a homeomorphism onto C_φ. Since f is a local homeomorphism outside $f^{-1}(0)$, $\varphi_0 > 0$. Also, if $\varphi_0 < \pi$, then $\overline{U_{\varphi_0}}$ must intersect $S(1)$. In this case the required point

$$q_r \in B^n \setminus B(2/3)$$

can be found in a suitable C_φ, $\varphi < \varphi_0$; q_r has the desired properties because the restriction of f to U_φ is a homeomorphism. Thus we may assume that $\varphi_0 = \pi$. Then C_{φ_0} is a punctured sphere. The main point of the argument now is that when $n \geq 3$, and if U_{φ_0} is compactly contained in B^n, then the restriction of f extends to a homeomorphism of $\overline{U_{\varphi_0}}$ onto $S(r)$, cf. [13, III.3]. We denote the bounded component of $\mathbb{R}^n \setminus \overline{U_{\varphi_0}}$ by V. We choose $a \in V \setminus f^{-1}(0)$. Then, if $|f(a)| \geq r$, there exists a path γ starting at $f(a)$ and converging to infinity, so that $f(a)$ is the only point in $|\gamma| \cap \overline{B}(r)$. Thus the maximal f-lifting γ' of γ, starting at a, stays in V. This is a contradiction, and so we must have $a \in B(r)$. Then if we lift a line segment joining $f(a)$ and 0, starting at a, we see that $V \cap f^{-1}(0) \neq \emptyset$. This is a contradiction because $f^{-1}(0)$ is a connected set not compactly contained in B^n, and

$$\partial V \cap f^{-1}(0) = \overline{U_{\varphi_0}} \cap f^{-1}(0) = \emptyset.$$

We conclude that f is discrete and open. That f is a local homeomorphism follows from the fact that the set $\mathcal{B}_f$ where f is not a local homeomorphism is either empty or satisfies $\mathcal{H}^{n-2}(f\mathcal{B}_f) > 0$, see [13, III 5.3].

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