Quantitative isoperimetric inequalities and homeomorphisms with finite distortion

Kai Rajala*

Abstract

We prove quantitative isoperimetric inequalities for images of the unit ball under homeomorphisms of exponentially integrable distortion. We show that the metric distortions of such domains can be controlled by their Fraenkel asymmetries. An application of the quantitative isoperimetric inequality proved by Hall and Fusco, Maggi, and Pratelli then shows that for these domains a version of Bonnesen's inequality holds in all dimensions.

Mathematics Subject Classification (2000): 30C65, 46E35.

1 Introduction

The classical Bonnesen inequality states that for a planar Jordan domain Ω the inequality

$$\mathcal{H}^1(\partial\Omega)^2 - 4\pi|\Omega| \geq \pi^2(R-\rho)^2 \tag{1.1}$$

holds, where R and ρ are the circumradius and the inradius of Ω, respectively, see Osserman [12]. There are several related inequalities which show that if a planar Jordan domain is almost a disk in the sense of the isoperimetric inequality, then it is also geometrically close to a disk, with quantitative bounds. Such inequalities are called Bonnesen-style inequalities in [12].

By considering cusp domains, one sees that inequalities like (1.1) do not hold in dimensions higher than two. However, Hall [5] (see also [6]) showed that another natural quantitative isoperimetric inequality holds in all dimensions. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The Fraenkel asymmetry $\lambda(\Omega)$ is

$$\lambda(\Omega) = \min_{x\in\mathbb{R}^n} \frac{|\Omega \setminus B(x,r)|}{r^n},$$

where r is defined by $|B(x,r)| = |\Omega|$. The isoperimetric deficit of Ω is

$$\delta(\Omega) = \frac{\mathcal{H}^{n-1}(\partial\Omega)}{n\alpha_n^{1/n}|\Omega|^{(n-1)/n}} - 1.$$

*Research supported by the Academy of Finland.

Hall proved that the isoperimetric deficit controls the Fraenkel asymmetry, and conjectured that the sharp inequality is

$$\lambda(\Omega) \leq C(n)\sqrt{\delta(\Omega)}. \tag{1.2}$$

A beautiful solution to this problem was given in [3], where it was shown that (1.2) indeed holds. Recently, a lot of progress has been made in understanding related inequalities, cf. the references in [3]. For convex domains, Fuglede [2] proved essentially sharp higher-dimensional versions of (1.1).

In [13] we applied Hall's result in our study of the branching of quasiregular mappings in space. We especially showed that an inequality like (1.1) holds for images of the unit ball under global K-quasiconformal maps. This was done by proving that the Fraenkel asymmetry of such a domain Ω controls its metric distortion

$$\beta(\Omega) = \min\left\{\frac{R-r}{r} : \exists\, x \in \mathbb{R}^n \text{ so that } B(x,r) \subset \Omega \subset B(x,R)\right\} :$$

$$\beta(\Omega)^n \leq C(n,K)\lambda(\Omega), \tag{1.3}$$

and then applying Hall's theorem. In this paper we consider the more general case where the class of quasiconformal maps is replaced by the class of homeomorphisms with exponentially integrable distortion. From the point of view of conformal analysis, this is essentially the largest class for which inequalities like (1.3), and, consequently, inequalities like (1.1) hold true. Our main objectives are to prove fairly sharp extensions of (1.3), and to demonstrate, again relying on (1.2), that besides convex domains there are also other natural classes of domains in the n-space which satisfy Bonnesen-style inequalities.

Denote by $|Df|$ and J_f the operator norm and the Jacobian determinant of the distributional differential of a $W^{1,1}$-homeomorphism f, respectively, and assume that $J_f \geq 0$ almost everywhere. Then $K(x) = K_f(x) = |Df(x)|^n/J_f(x)$ if $J_f(x) > 0$, $K(x) = 1$ if $|Df(x)| = J_f(x) = 0$, and $K(x) = \infty$ otherwise. Our main theorem reads as follows.

Theorem 1.1. *Let $f : B(2) \to fB(2) \subset \mathbb{R}^n$ be a $W^{1,1}$-homeomorphism so that $J_f \geq 0$ almost everywhere, and*

$$\int_{B(2)} \exp(\mu K(x))\, \mathrm{d}x \leq K \tag{1.4}$$

for some K and $\mu > 0$. Then

$$\beta(fB^n)^{n+n^2/\mu} \leq C(n,\mu,K)\lambda(fB^n). \tag{1.5}$$

In Section 4 we prove a similar result for the preimage of the unit ball under a polynomial integrability condition on K. The following example demonstrates that, except for the constant n^2 in (1.5), there is not too much room for improvement in Theorem 1.1.

Theorem 1.2. *There exists $c(n) > 0$ so that if $n \geq 2$ and $\mu > 0$, there exist $K = K(n,\mu) > 0$ and a sequence (f_j), so that each f_j satisfies the assumptions of Theorem 1.1, $\lambda(f_j B^n) \to 0$ as $j \to \infty$, and*

$$\beta(f_j B^n)^{n+c(n)/\mu} \geq \lambda(f_j B^n)$$

for every j.

By combining Theorem 1.1 with the sharp inequality (1.2) proved in [3], we have the following Bonnesen-style inequality. Recall that a homeomorphism f is by definition K-quasiconformal, $1 \leq K < \infty$, if the distortion $K(x)$ defined above satisfies $K(x) \leq K$ almost everywhere.

Corollary 1.3. *Let f be as in Theorem 1.1. Then*

$$\beta(fB^n)^{2n+2n^2/\mu} \leq C(n,\mu,K)\delta(fB^n). \tag{1.6}$$

If f is K-quasiconformal, then

$$\beta(fB^n)^{2n} \leq C(n,K)\delta(fB^n).$$

The exponent in (1.6) should not be sharp even for quasiconformal maps, and it would be interesting to find the sharp exponent, as is done in [2]. Also, there should be a geometric characterization for the domains for which inequalities like (1.6) hold true. In [13] we essentially used the quasisymmetry property of quasiconformal maps to deduce (1.3). Such a method does not work in the case of maps unbounded distortion because the quasisymmetries they possess are too weak for the purpose of Theorem 1.1. Inequality (1.2) holds for general Borel sets, and it is stated, as isoperimetric inequalites usually are, in terms of the perimeter measure. One can apply [1], Proposition 3.62, to show that the estimates above can be stated in terms of the Hausdorff $(n-1)$-measure.

2 Proof of Theorem 1.1

We denote an n-ball with center x and radius r by $B(x,r)$, and $B(r) = B(0,r)$, $B^n = B(0,1)$. The corresponding notations for $(n-1)$-spheres are $S(x,r)$ and $S(r) = S(0,r)$. The Lebesgue measure of $E \subset \mathbb{R}^n$ is $|E|$, and $\alpha_n = |B^n|$. The matrix $D^\sharp f^{-1}(x)$ is the adjoint matrix of the differential $Df^{-1}(x)$. Under the assumptions of Theorem 1.1 we have

$$|D^\sharp f^{-1}|^n \leq K J_f^{n-1} \tag{2.1}$$

almost everywhere, see [8].

In this section we assume that f satisfies the assumptions of Theorem 1.1. Let $G \subset fB(2)$ be open, $y \in \mathbb{R}^n$, $E \subset (0,\infty)$ a Borel set, and

$$\Lambda = \{U_t : t \in E\} = \{G \cap S(y,t) : t \in E\}.$$

Moreover, we denote by Y the family of all Borel functions $\rho : \mathbb{R}^n \to [0,\infty]$ for which

$$\int_{U_t} \rho(y)\, \mathrm{d}\mathcal{H}^{n-1}(y) \geq 1 \quad \text{for every } t \in E,$$

and by X the corresponding family with the requirement

$$\int_{f^{-1}(U_t)} \rho(x)\, \mathrm{d}\mathcal{H}^{n-1}(x) \geq 1 \quad \text{for every } t \in E.$$

Lemma 2.1. *We have*

$$\begin{aligned} M\Lambda &:= \inf_{\rho \in Y} \int_{\mathbb{R}^n} \rho(y)^{n/(n-1)}\, \mathrm{d}y \\ &\leq \inf_{\rho \in X} \int_{\mathbb{R}^n} \rho(x)^{n/(n-1)} K(x)^{1/(n-1)}\, \mathrm{d}x =: M_K f^{-1}\Lambda. \end{aligned}$$

Proof. Let $\rho \in X$. Under our assumptions we have $f^{-1} \in W^{1,n}_{\mathrm{loc}}(fB(2), \mathbb{R}^n)$, see [7]. In particular, the restriction of f^{-1} to U_t locally belongs to $W^{1,n}$ for almost every t. In such U_t, the change of variables inequality

$$\int_{U_t} |D^\sharp f^{-1}(y)| \rho(f^{-1}(y))\, \mathrm{d}\mathcal{H}^{n-1}(y) \geq \int_{f^{-1}(U_t)} \rho(x)\, \mathrm{d}\mathcal{H}^{n-1}(x) \geq 1.$$

holds. Thus the function

$$y \mapsto |D^\sharp f^{-1}(y)| \rho(f^{-1}(y)),$$

belongs to Y (after redefining the function in a set of measure zero). The n-dimensional change of variables formula holds under our assumptions, see [8], and by (2.1) we have

$$\begin{aligned} &\int_{fB(2)} \Big(|D^\sharp f^{-1}(y)| \rho(f^{-1}(y)) \Big)^{n/(n-1)}\, \mathrm{d}y \\ = &\int_{fB(2)} \frac{|D^\sharp f^{-1}(y)|^{n/(n-1)}}{J_{f^{-1}}(y)} \rho(f^{-1}(y))^{n/(n-1)} J_{f^{-1}}(y)\, \mathrm{d}y \\ \leq &\int_{fB(2)} K(f^{-1}(y))^{1/(n-1)} \rho(f^{-1}(y))^{n/(n-1)} J_{f^{-1}}(y)\, \mathrm{d}y \\ = &\int_{B(2)} K(x)^{1/(n-1)} \rho(x)^{n/(n-1)}\, \mathrm{d}x. \end{aligned}$$

The lemma follows by taking the infimum with respect to $\rho \in X$. $\square$

We will also apply a distortion estimate. Similar estimates have been proved in [9].

Lemma 2.2. *If $B(x,t) \subset B(3/2)$, then*

$$\frac{L}{l} = \frac{\max_{y\in S(3/2)} |f(x)-f(y)|}{\min_{y\in S(x,t)} |f(x)-f(y)|} \le \exp\Big(C(n,\mu,K)t^{-1/(n-3/2)}\Big).$$

Proof. We first show that

$$\frac{L}{L_t} \le \exp\Big(C(n,\mu,K)t^{-1/(n-3/2)}\Big), \tag{2.2}$$

where $L_t = \max_{y\in S(x,t)} |f(x)-f(y)|$. We choose a point $x_0 \in S(3/2)$ such that $|f(x_0)-f(x)| = L$, and

$$I = f^{-1}(\{f(x_0) + T(f(x_0)-f(x)) : T \ge 0\}).$$

Then there exist $p \in B(2)$ and $t/4 \le r \le 3/2$ such that, for every $r < s < r + t/8$, the sphere $S(p,s)$ contains a spherical cap $Q(s) \subset B(2)$ such that

$$Q(s) \cap I \ne \emptyset \quad \text{and} \quad Q(s) \cap B(x,t) \ne \emptyset. \tag{2.3}$$

Let

$$g = \max\{\min\{\log|f-f(x)|, \log L\}, \log L_t\},$$

and $E_t = \{L_t \le |f-f(x)| \le L\}$. Then, by (2.3) and the Sobolev embedding theorem on spheres,

$$\begin{aligned} \frac{t}{8}\Big(\log\frac{L}{L_t}\Big)^{n-1/2} &\le C(n)\int_r^{r+t/8}\int_{Q(s)\cap E_t} |\nabla g(z)|^{n-1/2}\,\mathrm{d}\mathcal{H}^{n-1}(z)\,\mathrm{d}s \\ &\le C(n)\int_{E_t} \frac{|Df(z)|^{n-1/2}}{|f(z)-f(x)|^{n-1/2}}\,\mathrm{d}z. \end{aligned}$$

By Hölder's inequality and the distortion inequality $|Df|^n \le KJ_f$, the last integral is bounded from above by

$$\Big(\int_{B(2)} K(z)^{2n-1}\,\mathrm{d}z\Big)^{1/(2n)}\Big(\int_{E_t}\frac{J_f(z)}{|f(z)-f(x)|^n}\,\mathrm{d}z\Big)^{(n-1/2)/n}.$$

By Jensen's inequality and (1.4), the first integral is bounded by $C(n,\mu,K)$. Also, by change of variables, the second term is bounded by

$$\Big(\log\frac{L}{L_t}\Big)^{(n-1/2)/n}.$$

Combining the estimates gives (2.2). We also have

$$\frac{L_t}{l} \le \exp\Big(C(n,\mu,K)t^{-1/(n-3/2)}\Big). \tag{2.4}$$

Inequality (2.4) is proved in a similar way as (2.2), and we thus omit the proof. See [9], Theorem 3.6 for a more general result. The lemma follows by combining (2.2) and (2.4). □

The following continuity estimate is proved in [11]

Theorem 2.3. *If x and $y \in B(5/4)$, then*

$$|f(x) - f(y)| \leq \frac{C(n, \mu, K)}{\log^{\mu/n} \frac{1}{|x-y|}} |fB(3/2)|^{1/n}.$$

We now begin to prove Theorem 1.1. We may assume that $|fB^n| = \alpha_n$. We choose x_0 so that $\lambda(fB^n) = |fB^n \setminus B(x_0, 1)|$. Without loss of generality, $x_0 = 0$. Lemma 2.2 applied to the unit ball now shows that $\beta(fB^n) \leq C(n, \mu, K)$. Thus we may assume that

$$\lambda = \lambda(fB^n) < \epsilon = \epsilon(n, \mu, K), \tag{2.5}$$

where $\epsilon > 0$ is to be determined later. Let

$$R = \min\{s : fB^n \subset B(s)\},$$

and

$$r = \max\{s : B(s) \subset fB^n\}.$$

Then $\beta(fB^n) \leq R/r - 1$. We first give an estimate for $R - 1$.

We choose $a \in S^{n-1}$ so that $|f(a)| = R$. Without loss of generality, $a = e_1$ and $f(e_1) = Re_1$.

Lemma 2.4. *There exists $\kappa' = \kappa'(n, \mu, K) > 0$ such that*

$$f^{-1}B(Re_1, \kappa') \subset B(e_1, 1/4).$$

Proof. Since $|fB^n| = \alpha_n$, we have $|f(e_1) - f(x)| \geq 1$ for some $x \in B(3/2)$. Thus by Lemma 2.2,

$$|f(x) - f(e_1)| = |f(x) - Re_1| \geq \kappa'(n, \mu, K)$$

for every $x \notin B(e_1, 1/4)$. $\square$

Now let $\kappa = \min\{R - 1, \kappa'\}$, and $U_t = fB^n \cap S(Re_1, t)$. We may assume that κ is so small that $s < 1/10$ in Lemma 2.5 below.

Lemma 2.5. *There exists $C = C(n, \mu, K) > 0$ so that if $s = \exp(-C\kappa^{-n/\mu})$, then $f^{-1}U_t$ separates $B^n \setminus B(e_1, 1/4)$ and $B^n \cap B(e_1, s)$ in B^n for every $\kappa/2 < t < \kappa$.*

Proof. Let $\kappa/2 < t < \kappa$. From Lemma 2.2 it follows that

$$|fB(3/2)| \leq C(n, \mu, K)|fB^n| = C(n, \mu, K)\alpha_n.$$

Combining this with Theorem 2.3 shows that $|x - e_1| \geq s$ whenever $x \in f^{-1}U_t$. Lemma 2.4 then shows that

$$s \leq |x - e_1| \leq 1/4 \tag{2.6}$$

for every $x \in f^{-1}U_t$. Since U_t separates $B(Re_1, \kappa/2)$ and any point $y \in fB^n \setminus B(Re_1, t)$ in fB^n, the lemma follows by (2.6). $\square$

Lemma 2.6. *Let κ and U_t be as in Lemma 2.5, and*

$$\Lambda = \{U_t : \kappa/2 < t < \kappa\}.$$

Then

$$M\Lambda \geq \frac{\kappa^{n/(n-1)}}{2^{n/(n-1)}\lambda^{1/(n-1)}}.$$

Proof. Let $\rho \in Y$, see Lemma 2.1. Now

$$\begin{aligned} \kappa/2 &\leq \int_{\kappa/2}^{\kappa} \int_{U_t} \rho(z)\,\mathrm{d}\mathcal{H}^{n-1}(z)\,\mathrm{d}t \leq \int_{fB^n \setminus B^n} \rho(y)\,\mathrm{d}y \\ &\leq |fB^n \setminus B^n|^{1/n} \Big(\int_{fB^n \setminus B^n} \rho(y)^{n/(n-1)}\,\mathrm{d}y \Big)^{(n-1)/n} \end{aligned}$$

by Hölder's inequality. The lemma follows, since $\lambda = |fB^n \setminus B^n|$. □

Lemma 2.7. *Let Λ be as in Lemma 2.6. Then*

$$M_K f^{-1}\Lambda \leq C(n,\mu,K)\kappa^{-n^2/((n-1)\mu)}.$$

Proof. By the separation property in Lemma 2.5 and a simple calculation,

$$\rho(x) = 10^n |x - e_1|^{1-n} \chi_{B^n \setminus B(e_1,s)}(x)$$

satisfies $\rho \in X$, where s is as in Lemma 2.5. We may assume that $\log \frac{1}{s}$ is an integer. Then

$$\begin{aligned} M_K f^{-1}\Lambda &\leq \int_{\mathbb{R}^n} \rho(x)^{n/(n-1)} K(x)^{1/(n-1)}\,\mathrm{d}x \\ &\leq C(n) \sum_{j=0}^{\log \frac{1}{s}} |B_j|^{-1} \int_{B_j} K(x)^{1/(n-1)}\,\mathrm{d}x, \end{aligned}$$

where $B_j = B(e_1, \exp(-j))$. Since the function $t \mapsto \exp(\mu t^{n-1})$ is convex, we can use Jensen's inequality as follows:

$$\begin{aligned} |B_j|^{-1} \int_{B_j} K(x)^{\frac{1}{n-1}}\,\mathrm{d}x &\leq \mu^{\frac{-1}{n-1}} \log^{\frac{1}{n-1}} \Big(|B_j|^{-1} \int_{B_j} \exp(\mu K(x))\,\mathrm{d}x \Big) \\ &\leq \mu^{\frac{-1}{n-1}} \log^{\frac{1}{n-1}} (\alpha_n^{-1} \exp(nj)K) \leq C(n,\mu,K) j^{\frac{1}{n-1}}. \end{aligned}$$

Thus

$$\begin{aligned} M_K f^{-1}\Lambda &\leq C(n,\mu,K) \sum_{j=0}^{\log \frac{1}{s}} j^{\frac{1}{n-1}} \leq C(n,\mu,K) \log^{\frac{n}{n-1}} \frac{1}{s} \\ &= C(n,\mu,K)\kappa^{\frac{-n^2}{(n-1)\mu}}. \end{aligned}$$

□

Combining Lemmas 2.1, 2.6 and 2.7 yields

$$\kappa^{n+n^2/\mu} \leq C(n,\mu,K)\lambda.$$

Thus, when we assume that ϵ in (2.5) is small enough depending on n, μ and K, we have

$$(R-1)^{n+n^2/\mu} \leq C(n,\mu,K)\lambda. \tag{2.7}$$

We now give an estimate for $1-r$. Notice that we could have $r=0$. However, we will soon see that, when λ is small enough, r is close to one. We denote

$$a = \inf\{|f(x)| : x \in B(2) \setminus B(3/2)\}.$$

Lemma 2.8. *We have*

$$1 - a \leq C(n,\mu,K)\lambda^{1/n}.$$

Proof. We may assume that $a < 1$. Let $\eta = \min\{1-a, 1/2\}$. Since $\max_{x\in\overline{B}(3/2)} |f(x)| > 1$, for every $1-\eta < t < 1$ there exists

$$p(t) \in S(t) \cap f(B(2) \setminus B(3/2)).$$

We may assume that $\lambda < \epsilon(n)$, so that $S(t)\cap\overline{fB^n} \neq \emptyset$ for every $1-\eta < t < 1$. We choose a point

$$q(t) \in S(t) \cap \overline{fB^n}$$

so that

$$s(t) = |p(t) - q(t)| = \min\{|p(t) - y| : y \in S(t) \cap \overline{fB^n}\}.$$

Since f^{-1} belongs to $W^{1,n}$ by [7], also the restriction of f^{-1} to $S(t)\cap fB(2)$ belongs to $W^{1,n}$ for almost every t. We denote

$$Q(t) = S(t) \cap B(p(t), s(t)).$$

Then, by the Sobolev embedding theorem in $Q(t)$,

$$\begin{aligned} \frac{1}{2} &\leq |f^{-1}(p(t)) - f^{-1}(q(t))| \\ &\leq C(n)s(t)^{1/n}\Big(\int_{Q(t)} |\nabla|f^{-1}|(y)|^n \,\mathrm{d}\mathcal{H}^{n-1}(y)\Big)^{1/n} \\ &\leq C(n)s(t)^{1/n}\Big(\int_{Q(t)} K_{f^{-1}}(y)J_{f^{-1}}(y) \,\mathrm{d}\mathcal{H}^{n-1}(y)\Big)^{1/n} \end{aligned} \tag{2.8}$$

for almost every t, $1-\eta < t < 1$. Since $K_{f^{-1}}(y) \leq K(f^{-1}(y))^{n-1}$ almost everywhere, see [8], (2.8) yields

$$C(n)^{-1}s(t)^{-1} \leq \int_{Q(t)} K(f^{-1}(y))^{n-1}J_{f^{-1}}(y) \,\mathrm{d}\mathcal{H}^{n-1}(y).$$

By integrating and changing variables (the change of variables formula holds under our assumptions, see [8]), we have

$$\int_{1-\eta}^{1} \frac{\mathrm{d}t}{s(t)} \le C(n) \int_{B(2)} K(x)^{n-1}\,\mathrm{d}x \le C(n,\mu,K). \tag{2.9}$$

Here the last inequality follows from Jensen's inequality and our assumption on K. By Hölder's inequality, and (2.9),

$$\begin{aligned}
\eta &= \int_{1-\eta}^{1} \frac{s(t)^{(n-1)/n}}{s(t)^{(n-1)/n}}\,\mathrm{d}t \\
&\le \Big(\int_{1-\eta}^{1} s(t)^{n-1}\,\mathrm{d}t\Big)^{1/n}\Big(\int_{1-\eta}^{1}\frac{\mathrm{d}t}{s(t)}\Big)^{(n-1)/n} \\
&\le C(n,\mu,K)\Big(\int_{1-\eta}^{1}\mathcal{H}^{n-1}(S(t)\cap B^n \setminus fB^n)\,\mathrm{d}t\Big)^{1/n} \qquad (2.10)\\
&\le C(n,\mu,K)\lambda^{1/n}.
\end{aligned}$$

Thus, when λ is small enough depending on n, μ and K, $\eta = 1-a$ and the lemma follows from (2.10). □

Now we continue in a similar way as in the proof of (2.7). We claim that

$$(1-r)^{n+n^2/\mu} \le C(n,\mu,K)\lambda. \tag{2.11}$$

We may assume that $f(e_1) = \min\{|f(x)| : x \in S(1)\} = r'$. Notice that if r' is close to one and if λ is small, then $r' = r$. The argument given below will show that r' is indeed close to one. Hence we will from now on abuse notation and denote r' by r. By Lemma 2.8, we may assume that $a \ge (1+r)/2$. Let $b = \min\{1,a\}$. Then, if

$$r_0 = r + (1-r)/4,$$

$b - r_0 \ge (1-r)/4$. We denote

$$W_t = S(t) \cap f(B(2) \setminus B^n).$$

Then

$$f^{-1}W_t \subset B(3/2) \setminus B(e_1,s)$$

for every $r_0 < t < b$ by Theorem 2.3 and Lemma 2.8, where

$$s = \exp(-C(n,\mu,K)(1-r)^{-n/\mu}).$$

Therefore, $f^{-1}W_t$ separates $B(e_1,s)\setminus B^n$ and $S(3/2)$ in $B(3/2)\setminus B^n$ for every such t. We denote $\Lambda = \{W_t : r_0 < t < b\}$. Then Lemma 2.1 gives

$$M\Lambda \le M_K f^{-1}\Lambda. \tag{2.12}$$

We estimate $M\Lambda$ from below as follows: if $\rho \in Y$, then

$$\begin{aligned}\frac{1-r}{4} &\leq \int_{r_0}^{b}\int_{W_t} \rho(z)\,\mathrm{d}\mathcal{H}^{n-1}(z)\,\mathrm{d}t \leq \int_{B^n\setminus fB^n} \rho(y)\,\mathrm{d}y \\ &\leq \lambda^{1/n}\Big(\int_{\mathbb{R}^n} \rho(y)^{n/(n-1)}\Big)^{(n-1)/n},\end{aligned}$$

and so

$$M\Lambda \geq \frac{(1-r)^{n/(n-1)}}{4^{n/(n-1)}\lambda^{1/(n-1)}}. \tag{2.13}$$

In order to give an upper bound for $M_K f^{-1}\Lambda$, we notice that the separation property mentioned above implies that the function $\rho : B(2) \to [0,\infty)$,

$$\rho(x) = 100^n|x-e_1|^{1-n}\chi_{B(3/2)\setminus B(e_1,s)}(x)$$

belongs to the test function space X. By calculating as in Lemma 2.7, we see that

$$M_K f^{-1}\Lambda \leq C(n,\mu,K)(1-r)^{-n^2/((n-1)\mu)}. \tag{2.14}$$

Combining (2.12), (2.13) and (2.14) then yields (2.11). Since $r \geq 1/2$ for small enough λ, Theorem 1.1 follows by combining (2.7) and (2.11).

3 Proof of Theorem 1.2

We fix $\mu > 0$ and a small $a > 0$, and denote $\epsilon = a^{\epsilon_1(n)/\mu} << 1$, where $\epsilon_1(n)$ will be determined later. The main part of the proof will be the construction of a Lipschitz continuous homeomorphism $g : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ with the following properties. Denote $\mathbb{H}_a = \{x_1 \leq -a\}$, and by V the truncated cone

$$\{-a \leq x_1 \leq -|x|\cos\epsilon\}.$$

Then we require that

$$g(\mathbb{H}_0) = \mathbb{H}_a \cup V \quad \text{and} \quad g(0) = 0. \tag{3.1}$$

Also, we require that

$$\int_{B^n} \exp(\mu K_g(x))\,\mathrm{d}x \leq C(\mu,n), \tag{3.2}$$

and that g is $K(n)$-quasiconformal in $\overline{R}^n \setminus \overline{B}^n$.

Suppose for the moment that such a g exists. Denote $\tau_a(x) = x + ae_1$, and let $\mathcal{M} : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a Möbius transformation that maps B^n onto $\mathbb{H}_0$ and e_1 to 0. Then consider

$$f = \mathcal{M}^{-1}\circ\tau_a\circ g\circ\mathcal{M}.$$

Then, since $\mathcal{M}_{|B(e_1,1/2)}$ is bi-Lipschitz continuous, f maps B^n onto the union of B^n and a bi-Lipschitz image of V when a is small enough. Thus

$$\lambda(fB^n) \leq C(n)a^n\epsilon^{n-1} = C(n)a^{n+(n-1)\epsilon_1(n)/\mu} \leq C(n)\beta(fB^n)^{n+(n-1)\epsilon_1(n)/\mu},$$

which is the desired estimate (when $a \to 0$). Also, assuming (3.2), we can use the conformality and the local bi-Lipschitz property of $\mathcal{M}$ to show that

$$\int_{B(2)} \exp(\mu K_f(x))\, \mathrm{d}x \leq C(\mu, n).$$

We conclude that in order to prove Theorem 1.2 it suffices to construct, for any given small $a > 0$, a homeomorphism g as above.

Let $\mu_0 = C_0(n)\mu$, where $C_0(n)$ is determined later. We will consider the case $n \geq 3$; the case $n = 2$ is an easy modification. Let $x = (r, \varphi, \phi)$, where $r = |x|$ and $0 \leq \varphi \leq \pi$ is the angle $\angle(x, 0, e_1)$. Also, $\phi \in S^{n-2}$, $\phi = \hat{x}/|\hat{x}|$ when $\hat{x} \neq 0$, where $\hat{x} = (x_2, \ldots, x_n)$. Then the map g is of the form

$$g(r, \varphi, \phi) = (g_r, \eta(r, \varphi), \phi)$$

when $\hat{x} \neq 0$, $g(0) = 0$, and $g(x) = g_r x_1/|x_1|$ otherwise. Here

$$g_r = \begin{cases} r\exp(1) & \exp(-1) \leq r \leq \infty, \\ \log^{-\mu_0}\frac{1}{r}, & \exp(-(2a)^{-1/\mu_0}) \leq r \leq \exp(-1), \\ 2a\exp((2a)^{-1/\mu_0})r & 0 \leq r \leq \exp(-(2a)^{-1/\mu_0}). \end{cases}$$

In order to define η, we first set

$$\eta_0 = \begin{cases} \pi - \arccos\left(\frac{a}{g_r}\right), & r \geq \exp\left(-\left(\frac{a}{\cos\epsilon}\right)^{-1/\mu_0}\right), \\ \pi - \epsilon & \text{otherwise.} \end{cases}$$

Then

$$\eta = \begin{cases} \frac{2\eta_0\varphi}{\pi}, & 0 \leq \varphi \leq \pi/2, \\ (2 - 2\eta_0/\pi)\varphi - \pi + 2\eta_0, & \pi/2 \leq \varphi \leq \pi. \end{cases}$$

Now g is a Lipschitz homeomorphism and satisfies (3.1). We next estimate K_g. The Jacobian determinant J_g is given by

$$J_g = \partial_r g_r \cdot \frac{g_r\partial_\varphi\eta}{r} \cdot \left(\frac{g_r\sin\eta}{r\sin\varphi}\right)^{n-2},$$

and

$$|Dg| \leq C(n)\max\left\{\partial_r g_r, \frac{g_r\partial_\varphi\eta}{r}, \frac{g_r\sin\eta}{r\sin\varphi}\right\}.$$

Thus g is $K(n)$-quasiconformal when $r \geq \exp(-1)$. Let A_1 be the set where $g_r = \log^{-\mu_0}\frac{1}{r}$. Then

$$\begin{aligned} \partial_r g_r &= \mu_0\log^{-1}\frac{1}{r}\cdot\frac{g_r}{r}, \\ |Dg| &\leq C(n)\frac{g_r}{r} \end{aligned}$$

in A_1. Also, since $\eta_0 \le 2\pi/3$ in A_1,

$$\partial_\varphi \eta \Big(\frac{\sin\eta}{\sin\varphi}\Big)^{n-2} \ge C(n)^{-1}.$$

Therefore,

$$K_g \le \max\Big\{C(n),\, \frac{C(n)}{\mu_0}\log\frac{1}{r}\Big\},$$

and

$$\int_{A_1} \exp(\mu K_g(x))\,\mathrm{d}x \le 100^{n+\mu} + \int_{A_1} |x|^{-\alpha}\,\mathrm{d}x,$$

where $\alpha = C(n)/C_0(n)$. Thus, when C_0 is chosen to be large enough so that $\alpha \le 1$, the integral is bounded by $200^{n+\mu}$.

Let A_2 be the set where $g_r = 2a\exp((2a)^{-1/\mu_0})r$. Then

$$|Dg| \le \frac{C(n)g_r}{r}, \quad \partial_r g_r = \frac{g_r}{r}, \quad \text{and} \quad \partial_\varphi \eta \Big(\frac{\sin\eta}{\sin\varphi}\Big)^{n-2} \ge C(n)^{-1}\epsilon^{n-1}$$

in A_2. Therefore,

$$K_g \le C(n)\epsilon^{1-n},$$

and

$$\begin{aligned}\int_{A_2} \exp(\mu K_g(x))\,\mathrm{d}x &\le \exp\Big(C(n)\mu\epsilon^{1-n}\Big)|A_2| \\ &\le \exp\Big(C(n)\mu\epsilon^{1-n} - C(n,\mu)a^{-1/\mu_0}\Big).\end{aligned}$$

If we now choose $\epsilon_1(n)$ to be small enough depending on $C_0(n)$, then the integral is smaller than 1 for small a. By combining the estimates we see that (3.2) holds when a is small. The proof is complete.

4 Theorem 1.1 for inverse images

In this section we show that, when a suitable polynomial integrability condition on K is assumed, an estimate similar to Theorem 1.1 holds for the inverse of a ball under f.

Theorem 4.1. *Let $f : f^{-1}B(2) \to B(2)$ be a $W^{1,1}$-homeomorphism so that $J_f \ge 0$ almost everywhere, and*

$$\int_{f^{-1}B(2)} K(x)^p\,\mathrm{d}x \le K|f^{-1}B^n| \tag{4.1}$$

for some $K > 0$ and $p > n-1$. Then

$$\beta(f^{-1}B^n)^{n+n^2/(p-n+1)} \le C(n,p,K)\lambda(f^{-1}B^n)$$

whenever

$$\lambda(f^{-1}B^n) < \epsilon(n). \tag{4.2}$$

We do not know if assumption (4.2) is really needed. By (4.1) and Hölder's inequality,

$$\int_{f^{-1}B(2)} |Df(x)|^q \,\mathrm{d}x \le C(n,p,K,|f^{-1}B^n|), \tag{4.3}$$

where $q = np/(p+1) > n-1$ when $p > n-1$.

The proof of Theorem 4.1 is similar to the proof of Theorem 1.1. Therefore, we will leave out some details to avoid unnecessary repetition. Let $G \subset f^{-1}B(2)$ be open, $x \in \mathbb{R}^n$, and $E \subset (0,\infty)$ a Borel set. As in the beginning of the proof of Theorem 1.1, we consider the family

$$\Lambda = \{U_t : t \in E\} = \{S(x,t) \cap G : t \in E\}, \tag{4.4}$$

and define the quantities

$$M_{1/K}\Lambda = \inf_{\rho \in X} \int_{\mathbb{R}^n} \rho(x)^{n/(n-1)} K(x)^{-1} \,\mathrm{d}x,$$

where X is the family of all Borel functions for which

$$\int_{U_t} \rho(x) \,\mathrm{d}\mathcal{H}^{n-1}(x) \ge 1 \quad \text{for every } t \in E,$$

and $Mf\Lambda$ as before.

Lemma 4.2. *We have*

$$M_{1/K}\Lambda \le Mf\Lambda.$$

Proof. Since $f \in W^{1,q}(f^{-1}B(2), \mathbb{R}^n)$ for some $q > n-1$ by (4.3), the $(n-1)$-dimensional change of variables formula holds on almost every U_t. The proof can now be carried out as the proof of Lemma 2.1. Notice that the n-dimensional change of variables formula is not needed here. The inequality

$$\int_U g(f(x)) J_f(x) \,\mathrm{d}x \le \int_{fU} g(y) \,\mathrm{d}y,$$

valid for all $W^{1,1}$-homeomorphisms, is sufficient. □

We will use the following continuity estimate for the inverse. In the case $n = 2$ this was proved in [10], and the case $n \ge 3$ can be proved similarly.

Theorem 4.3. *Let f be as in Theorem 4.1, and x and y in $B(3/2)$. Then*

$$|f^{-1}(x) - f^{-1}(y)| \le \frac{C(n,p,K,|f^{-1}B^n|)}{\log^\alpha \frac{1}{|x-y|}},$$

where $\alpha = p(n-1)/n$.

We may assume that $|f^{-1}B^n| = \alpha_n$ and that

$$\lambda = \lambda(f^{-1}B^n) = |f^{-1}(B^n) \setminus B^n|.$$

Let

$$R = \min\{s : f^{-1}B^n \subset B(s)\},$$

and

$$r = \max\{s : B(s) \subset f^{-1}B^n\}.$$

We first claim that

$$(R-1)^{n+n^2/(p-n+1)} \leq C(n,p,K)\lambda. \tag{4.5}$$

We may assume that $Re_1 = f^{-1}(e_1)$. We choose U_t and Λ in (4.4) so that $x = 0$, $G = f^{-1}B^n$, and $E = (1, 1 + (R-1)/2)$.

Lemma 4.4. *There exist $\kappa = \kappa(n,p,K) > 0$ and a continuum γ in $B^n \cap f^{-1}B^n$ so that* $\operatorname{diam} f\gamma \geq \kappa$.

Proof. If λ is small enough, then there exists $p \in B(1/2) \cap f^{-1}B^n$. Consequently, p and $S(1) \cap f^{-1}B^n$ can be connected in $\overline{B^n} \cap f^{-1}B^n$ by a continuum γ. Since $\operatorname{diam}\gamma \geq 1/2$, $\operatorname{diam} f\gamma \geq \kappa(n,p,K)$ by Theorem 4.3. $\square$

Lemma 4.5. *There exists $C = C(n,p,K) > 0$ so that if*

$$s = \exp(-C(R-1)^{-n/((n-1)p)}),$$

then fU_t separates $f\gamma$ and $F = B^n \cap B(e_1, s)$ in B^n for every $1 < t < 1 + (R-1)/2$.

Proof. Apply Theorem 4.3 as in the proof of Lemma 2.5. $\square$

Lemma 4.6. *We have*

$$C(n,p,K)M_{1/K}\Lambda \geq (R-1)^{n/(n-1)}\lambda^{(n-1-p)/(p(n-1))}.$$

Proof. Let $\rho \in X$. Then, by polar coordinates and Hölder's inequality,

$$\begin{aligned}
\frac{R-1}{2} &\leq \int_{f^{-1}(B^n)\setminus B^n} \rho(x)K(x)^{-(n-1)/n}K(x)^{(n-1)/n}\,\mathrm{d}x \\
&\leq \Big(\int_{f^{-1}B(2)} \rho(x)^{n/(n-1)}K(x)^{-1}\,\mathrm{d}x\Big)^{(n-1)/n} \\
&\times \Big(\int_{f^{-1}B(2)} K(x)^p\,\mathrm{d}x\Big)^{(n-1)/(np)} |f^{-1}(B^n)\setminus B^n|^{\tau},
\end{aligned}$$

where $\tau = (p-n+1)/(np)$. The lemma follows. $\square$

Lemma 4.7. *We have*

$$Mf\Lambda \le C(n,p,K)\min\{1,(R-1)^{-n/((n-1)p)}\}.$$

Proof. We use the duality of the modulus $M\Lambda$ of separating surfaces and conformal capacity, and apply a classical estimate for conformal capacity. Namely, by Lemma 4.5 and [14], and the so-called Loewner property of the unit ball (cf. [4]), we have

$$Mf\Lambda \le C(n)\log\frac{1}{s} \tag{4.6}$$

when $s < \kappa$ (s is as in Lemma 4.5 and κ as in Lemma 4.4), and

$$Mf\Lambda \le C(n,p,K)$$

otherwise. The lemma follows. □

The estimate (4.5) now follows by combining Lemmas 4.2, 4.6 and 4.7. Now we give an estimate for $1-r$. We claim that

$$(1-r)^{n+n^2/(p-n+1)} \le C(n,p,K)\lambda. \tag{4.7}$$

We denote

$$a = \inf\{|f^{-1}(x)| : x \in B(2)\setminus B(3/2)\}.$$

Lemma 4.8. *We have*

$$1-a \le C(n,p,K)\lambda^{(p-n+1)/(np)}.$$

Proof. We may assume that $a<1$. Let $\eta=\min\{1-a,1/2\}$ and $s(t)=\mathcal{H}^{n-1}(S(t)\setminus f^{-1}B^n)$. As in the proof of Lemma 2.8, we apply the Sobolev embedding theorem on spheres to conclude that

$$1 \le C(n)s(t)^{q-n+1}\int_{S(t)\cap f^{-1}B(2)} |\nabla|f|(x)|^q \,\mathrm{d}\mathcal{H}^{n-1}(x)$$

for almost every $1-\eta \le t \le 1$, where $q = np/(p+1)$. Therefore, by integration with respect to t, and (4.3),

$$\int_{1-\eta}^{1} s(t)^{n-1-q}\,\mathrm{d}t \le C(n,p,K).$$

By Hölder's inequality,

$$\begin{aligned}\eta &\le \Big(\int_{1-\eta}^{1} s(t)^{n-1-q}\,\mathrm{d}t\Big)^{1/\tau}\Big(\int_{1-\eta}^{1} s(t)^{n-1}\,\mathrm{d}t\Big)^{(\tau-1)/\tau}\\ &\le C(n,p,K)\lambda^{(\tau-1)/\tau},\end{aligned}$$

where $\tau = q/(n-1)$. The lemma follows. □

We now prove (4.7). Notice that

$$\frac{np}{p-n+1} \leq n + \frac{n^2}{p-n+1}.$$

Thus we may assume that $a \geq (1+r)/2$ by Lemma 4.8. Also, as in the proof of (2.11), we may assume that

$$r = \min\{|f^{-1}(y)| : y \in S(1)\}.$$

Let $b = \min\{1, a\}$ and $r_0 = r + (1-r)/4$. Moreover, let

$$\Lambda = \{W_t : r_0 < t < b\},$$

where

$$W_t = S(t) \cap f^{-1}(B(2) \setminus B^n).$$

Then, as in the proofs of (2.11) and (4.5), we have

$$C(n,p,K) M_{1/K} \Lambda \geq (1-r)^{n/(n-1)} \lambda^{(n-1-p)/(p(n-1))},$$

and

$$M f\Lambda \leq C(n,p,K)(1-r)^{-n/(p(n-1))}.$$

Combining these estimates with Lemma 4.2 gives (4.7). The proof is complete.

References

[1] Ambrosio, L., Fusco, N., Pallara, D.: Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs, Oxford University Press, 2000.

[2] Fuglede, B.: *Stability in the isoperimetric problem for convex or nearly spherical domains in* $\mathbb{R}^n$, Trans. Amer. Math. Soc., 314 (1989), no. 2, 619–638.

[3] Fusco, N., Maggi, F., Pratelli, A.: *The sharp quantitative isoperimetric inequality*, Ann. of Math. (2), 168 (2008), no. 3, 941–980.

[4] Gehring, F. W.: *Symmetrization of rings in space*, Trans. Amer. Math. Soc., 101 (1961), 499–519.

[5] Hall, R. R.: *A quantitative isoperimetric inequality in n-dimensional space*, J. Reine Angew. Math., 428 (1992), 161–176.

[6] Hall, R. R., Hayman, W. K., Weitsman, A.: *On asymmetry and capacity*, J. Analyse Math., 56 (1991), 87–123.

[7] Hencl, S., Koskela, P., Malý, J.: *Regularity of the inverse of a Sobolev homeomorphism in space*, Proc. Roy. Soc. Edinburgh Sect. A, 136 (2006), no. 6, 1267–1285.

[8] Iwaniec, T., Martin, G.: Geometric Function Theory and Non-linear Analysis, Oxford University Press, 2001.

[9] Kallunki, S.: *Mappings of finite distortion: the metric definition*, Ann. Acad. Sci. Fenn. Math. Diss. No. 131 (2002).

[10] Koskela, P., Takkinen, J.: *Mappings of finite distortion: formation of cusps III*, preprint.

[11] Onninen, J., Zhong, X.: *A note on mappings of finite distortion: the sharp modulus of continuity*, Michigan Math. J., 53 (2005), no. 2, 329–335.

[12] Osserman, R.: *Bonnesen-style isoperimetric inequalities*, Amer. Math. Monthly, 86 (1979), no. 1, 1–29.

[13] Rajala, K.: *The local homeomorphism property of spatial quasiregular mappings with distortion close to one*, Geom. Funct. Anal., 15 (2005), no. 5, 1100–1127.

[14] Ziemer, W. P.: *Extremal length and conformal capacity*, Trans. Amer. Math. Soc., 126 (1967), 460–473.

University of Jyväskylä
Department of Mathematics and Statistics
P.O. Box 35 (MaD)
FI-40014 University of Jyväskylä
Finland
e-mail: kirajala@maths.jyu.fi