OPTIMAL APPROXIMATION RATE OF CERTAIN STOCHASTIC INTEGRALS

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Abstract. Given an increasing function $H : [0,1) \to [0,\infty)$ and

$$A_n(H) := \inf_{\tau \in \mathcal{T}_n} \left(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t_i - t) H^2(t) dt \right)^{\frac{1}{2}},$$

where $\mathcal{T}_n := \{\tau = (t_i)_{i=0}^n : \ 0 = t_0 < t_1 < \cdots < t_n = 1\}$, we characterize the property $A_n(H) \leq \frac{c}{\sqrt{n}}$, and give conditions for $A_n(H) \leq \frac{c}{\sqrt{n^\beta}}$ and $A_n(H) \geq \frac{1}{c\sqrt{n^\beta}}$ for $\beta \in (0,1)$, both in terms of integrability properties of H. These results are applied to the approximation of certain stochastic integrals.

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1. Introduction

During the last years quantitative problems in stochastic finance have become more and more important. One typical example consists of replacing a continuously adjusted hedging portfolio in the Black-Scholes option pricing model by a discretely adjusted one, as portfolios are adjusted in practice only finitely many times. If we consider the quadratic error which occurs in this replacement (and which we can interpret as risk in finance), then we obtain the following stochastic approximation problem:

We model the share price by an appropriate positive diffusion $X = (X_t)_{t\in[0,1]}$ such that

$$dX_t = \sigma(X_t)dW_t \quad \text{with } X_0 \equiv x_0 > 0,$$

where $W = (W_t)_{t\in[0,1]}$ is a standard Brownian motion, σ satisfies certain regularity properties, and $(\mathcal{F}_t)_{t\in[0,1]}$ is the augmentation of the filtration generated by W. Assume that $f : (0,\infty) \to (0,\infty)$ is a polynomially bounded and Borel measurable payoff function of a European type option. Setting $Z = f(X_1)$, we consider

$$C(Z, \tau^n, X, v) := Z - \mathbb{E}Z - \sum_{i=1}^{n} v_{i-1}(X_{t_i} - X_{t_{i-1}}), \tag{1}$$

which is the difference between the exact payoff Z and the payoff of the discretely adjusted hedging portfolio $x_0 + \sum_{i=1}^n v_{i-1}(X_{t_i} - X_{t_{i-1}})$ with initial capital $x_0 = \mathbb{E}Z$ and rebalanced at the deterministic time points $\tau^n = (t_i)_{i=0}^n$ with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$. Here

$v_{i-1} : \Omega \to \mathbb{R}$ is the number of shares one holds between t_{i-1} and t_i. We are interested in the minimal quadratic risk under the constraint that one trades only n times, i.e.

$$a_n^X(Z) := \inf_{\tau^n} a_X(Z, \tau^n) \tag{2}$$

where $a_X(Z, \tau^n) := \inf (\mathbb{E}|C(Z, \tau^n, X, v)|^2)^{\frac{1}{2}}$ with the infimum taken over all $v = (v_i)_{i=0}^{n-1}$, where $v_i : \Omega \to \mathbb{R}$ is a $\mathcal{F}_{t_i}$-measurable step function for all $i = 0, ..., n-1$.

Under certain conditions on Z and σ, C. and S. Geiss showed that if $\tau^n = (\frac{i}{n})_{i=0}^n$ is the equidistant time net with cardinality $n+1$, then one has that

$$a_X(Z, \tau^n) \leq \frac{c}{\sqrt{n}}$$

if and only if Z belongs to the Malliavin Sobolev space $\mathbb{D}_{1,2}$ [3, Theorems 2.3, and 2.6]. Furthermore, they proved that there exists a constant $c > 0$ such that $a_n^X(Z) \geq \frac{1}{c\sqrt{n}}$ unless there are constants c_0 and c_1 such that $Z = c_0 + c_1 X_1$ a.s. [3, Theorem 2.5] (if such constants do exists, then $a_n^X(Z) = 0$). It is also known by [3, Theorem 2.9] that there exists a constant $c > 0$ such that $a_n^X(Z) \leq \frac{c}{\sqrt{n}}$, if Z has a certain polynomial smoothness measured by Besov spaces generated by real interpolation. In this case the rate $\frac{1}{\sqrt{n}}$ is obtained by adapted non-equidistant time nets.

M. Hujo showed in [8, Theorem 3], for X being the Brownian motion or the geometric Brownian motion, that there exists random variables $Z = f(X_1) \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\sup_{n \in \mathbb{N}} \sqrt{n} a_n^X(Z) = \infty,$$

which means that the approximation rate is not always $\frac{1}{\sqrt{n}}$ even if the underlying process is the standard Brownian motion. However, in a sense, there are no explicit examples of such functions.

These results lead us to the question of how to characterize those $Z = f(X_1) \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ with

$$a_n^X(Z) \leq \frac{c}{\sqrt{n}} \quad \text{for some } c = c(Z) > 0.$$

Using Theorem 4.4 below, we see that finally the problem can be reduced to a non-stochastic one. Actually we characterize the property

$$\inf_{\tau \in \mathcal{T}_n} \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H^2(t) dt \right)^{\frac{1}{2}} \leq \frac{c}{\sqrt{n}},$$

where the function $H : [0,1) \to [0, \infty)$ is increasing and

$$\mathcal{T}_n := \{\tau = (t_i)_{i=0}^n : \ 0 = t_0 < t_1 < \cdots < t_n = 1\},$$

using the integrability properties of H. This with Theorem 4.4 immediately gives the characterization of $a_n^X(Z) \le \frac{c}{\sqrt{n}}$. Moreover, we give sufficient conditions for

$$\inf_{\tau \in \mathcal{T}_n} \left(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t_i - t) H^2(t) dt \right)^{\frac{1}{2}} \le \frac{c}{\sqrt{n^\beta}}$$

and

$$\inf_{\tau \in \mathcal{T}_n} \left(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t_i - t) H^2(t) dt \right)^{\frac{1}{2}} \ge \frac{1}{c\sqrt{n^\beta}},$$

where $\beta \in [0,1)$, in terms of the growth rate of $H : [0,1) \to [0,\infty)$. As we see, the results do not strictly depend on the setting introduced above and can be applied also to other situations, for example to the quadratic approximation of multi-dimensional stochastic integrals (see [9], [12] and [13]).

In Section 2 we introduce the main results of the paper, their proofs can be found in Section 3. In Section 4 we apply the results of Section 2 to the setting explained above. We also give an example of random variables for which the approximation rate is $a_n^X(Z) \sim_c \frac{1}{c\sqrt{n^\beta}}$, for $\beta \in (0,1)$ in case X is the standard Brownian motion or the geometric Brownian motion. In Section 5 the results of Section 2 are applied to the approximation of d-dimensional stochastic integrals where the underlying diffusion might have a drift.

2. Results

To shorten the notation in the following, we say that $A \sim_c B$ if there exists $c \ge 1$ such that $\frac{1}{c} A \le B \le cA$.

2.1. **Definition.** Let $H : [0,1) \to [0,\infty)$ be a measurable function. If $\tau = (t_i)_{i=0}^n \in \mathcal{T}_n$, we define

$$\begin{cases} A(H,\tau) := \left(\sum_{i=0}^{n} \int_{t_{i-1}}^{t_i} (t_i - t) H^2(t) dt \right)^{\frac{1}{2}}, \\ A_n(H) := \inf_{\tau \in \mathcal{T}_n} A(H,\tau). \end{cases}$$

2.2. **Definition.** We say that an increasing function $H : [0,1) \to [0,\infty)$ belongs to the set $\mathcal{A}$ if and only if

$$\|H\|_{\mathcal{A}} := \sup_{n \in \mathbb{N}} \sqrt{n} A_n(H) < \infty,$$

and to the set $\mathcal{H}$ if and only if

$$\|H\|_{\mathcal{H}} := \int_0^1 H(t) dt < \infty.$$

2.3. **Theorem.** *Let $H : [0,1) \to [0,\infty)$ be an increasing function. Then*

$$\sup_{n\in\mathbb{N}} \sqrt{n} A_n(H) < \infty$$

if and only if $\int_0^1 H(t)dt < \infty$. In particular, one has that

$$\|H\|_{\mathcal{A}} \sim_{\sqrt{2}} \|H\|_{\mathcal{H}}.$$

2.4. *Remark.* Theorem 2.3 implies that $I := \int_0^1 H(t)dt < \infty$ gives

$$A_n(H) \leq \frac{I}{\sqrt{n}} \quad \textit{for all } n \in \mathbb{N}.$$

This rate can be obtained by regular sequences (see [10] and [11]) generated by H. Regular sequences generated by H are time nets $\tau^n = (t_i^n)_{i=0}^n$ for which

$$\int_0^{t_i^n} H(t)dt = \frac{i}{n}\int_0^1 H(t)dt$$

for all $i \in \{0,...,n\}$.

2.5. **Theorem.** *Let $H : [0,1) \to [0,\infty)$ be an increasing function and $\alpha \in (\frac{1}{2},1)$. Then one has the following:*

(1) *If there exists a constant $c_1 \geq 1$ such that*

$$H(t) \leq c_1 \frac{(1-\log(1-t))^{-\alpha}}{1-t} \quad \textit{for all } t \in [0,1),$$

then

$$A_n(H) \leq \frac{c}{\sqrt{n^{2\alpha-1}}} \quad \textit{for all } n \in \mathbb{N},$$

where $c = c(\alpha) \geq 1$.

(2) *If there exists $s \in [0,1)$ and a constant $c_2 \geq 1$ such that*

$$H(t) \geq \frac{(1-\log(1-t))^{-\alpha}}{c_2(1-t)} \quad \textit{for all } t \in [s,1),$$

then

$$A_n(H) \geq \frac{1}{c\sqrt{n^{2\alpha-1}}} \quad \textit{for all } n \in \mathbb{N},$$

where $c = c(\alpha, c_2) \geq 1$.

2.6. *Remark.* Geiss showed in [5, Lemma 4.14, Proposition 4.16] that if H is increasing and there are $C \in (0,\infty)$, $\alpha \in (1,\infty)$ with

$$H(t) \leq \frac{C}{[\alpha + \log(1+\frac{1}{1-t})]^{\alpha}(1-t)}$$

for all $t \in [0,1)$, then one has that

$$\sup_n \sqrt{n} A_n(H) < \infty \quad \text{for all } n \in \mathbb{N}.$$

2.7. *Remark.* Let $H : [0,1) \to [0,\infty)$ be a measurable function such that $H^*(t) := \sup_{s\in[0,t]} H(s) < \infty$ for all $t \in [0,1)$. Then the monotonicity properties of $A_n(\cdot)$ imply the following:

(1) $\sup_{n\in\mathbb{N}} \sqrt{n} A_n(H) \le \|H^*\|_{\mathcal{H}}$ as a consequence of Lemma 3.1.

(2) If $H^*(t) \le c_1 \frac{(1-\log(1-t))^{-\alpha}}{1-t}$ for all $t \in [0,1)$, then

$$A_n(H) \le \frac{c}{\sqrt{n^{2\alpha-1}}} \quad \textit{for all } n \in \mathbb{N}.$$

3. Proof

In this chapter we prove Theorems 2.3 and 2.5. To prove Theorem 2.3 we need two lemmas concerning the connection between $A_n(H)$ and $\int_0^1 H(t)dt$, where H is a non-negative and increasing function.

3.1. **Lemma.** *Let $H : [0,T) \to [0,\infty)$, $T > 0$, be an increasing function such that*

$$I = \int_0^T H(t)dt < \infty.$$

Then for all $n \in \mathbb{N}$ there exists a sequence $\tau^n = (t_i^n)_{i=0}^n$, $0 = t_0^n < t_1^n < \cdots < t_n^n = T$ such that

$$\int_0^{t_i^n} H(t)dt = \frac{i}{n} I$$

for all $i \le n$ and for this sequence it holds that

$$A(H, \tau^n) \le \frac{I}{\sqrt{n}}.$$

Proof. The existence of the sequence $(t_i^n)_{i=0}^n$ for which

$$\int_0^{t_i^n} H(t)dt = \frac{i}{n} I$$

follows from the continuity of the integral. Now we have that

$$\begin{aligned}\sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i^n - t) H^2(t)dt &= \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} [(t_i^n - t)H(t)]H(t)dt \\ &\le \frac{I}{n} \sum_{i=1}^n \sup_{t\in[t_{i-1}^n, t_i^n)} (t_i^n - t)H(t).\end{aligned}$$

Since H is increasing, it is clear that

$$(t_i^n - t)H(t) \le \int_t^{t_i^n} H(s)ds \le \int_{t_{i-1}^n}^{t_i^n} H(s)ds$$

for all $t \in [t_{i-1}^n, t_i^n)$. Hence

$$A^2(H, \tau^n) \le \frac{I}{n} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} H(t)dt = \frac{I^2}{n}.$$

□

3.2. **Lemma.** *Let $H : [0,1) \to [0,\infty)$ be an increasing function. If for all $n \in \mathbb{N}$ there exists a time net $\tau^n = (t_i^n)_{i=0}^n \in \mathcal{T}_n$ such that*

$$A(H,\tau^n) \leq \frac{c}{\sqrt{n}}$$

for some fixed $c > 0$, then H is integrable and

$$\int_0^1 H(t)dt \leq \sqrt{2}c.$$

Proof. If $H \equiv 0$, then the claim is trivial. Assume then that $H(t) > 0$ for some $t \in [0,1)$. Let $a := \inf\{t \in [0,1); H(t) > 0\}$ and $\tilde{\tau}^n = \{a\} \cup \{t_i^n \in \tau^n, t_i^n > a\}$. Since H is positive on $(a,1)$, our assumption implies that $||\tilde{\tau}^n||_\infty \to 0$ as $n \to \infty$. Using the Cauchy-Schwartz inequality and assumption $A^2(H,\tau^n) \leq \frac{c^2}{n}$ we see that

$$\begin{aligned} \left[\sum_{i=1}^{n-1} H(t_{i-1}^n)(t_i^n - t_{i-1}^n)\right]^2 &\leq n \sum_{i=1}^{n-1} H^2(t_{i-1}^n)(t_i^n - t_{i-1}^n)^2 \\ &\leq 2n \sum_{i=1}^{n} \int_{t_{i-1}^n}^{t_i^n} (t_i^n - t)H^2(t)dt \leq 2c^2. \end{aligned} \tag{3}$$

Let $b \in (a,1)$ and $0 < \epsilon < \sqrt{c}$. Choose n such that $b < t_{n-1}^n$ and

$$\int_0^b H(t)dt < \sum_{i=1}^{n-1} H(t_{i-1}^n)(t_i^n - t_{i-1}^n) + \epsilon.$$

(We can choose n satisfying this, since the positivity of the function H on the interval $(a,1)$ implies that $t_{n-1}^n \to 1$ and $||\tilde{\tau}^n||_\infty \to 0$ as $n \to \infty$.) Now (3) gives that

$$\int_0^b H(t)dt \leq \sqrt{2}c + \epsilon$$

and since this is true for any $b \in (a,1)$ and any $\epsilon > 0$, we finally have that

$$\int_0^1 H(t)dt \leq \sqrt{2}c.$$

□

3.3. **Lemma.** *Let $H : [0,1) \to [0,\infty)$ be an increasing function. Then*

$$A_n(H) \leq \inf_{T \in (0,1)} \left[\frac{\left(\int_0^T H(t)dt\right)^2}{n-1} + \int_T^1 (1-t)H^2(t)dt\right]^{\frac{1}{2}}$$

for all $n \geq 2$.

Proof. Let $T \in [0,1)$ and let $\tau^n = (t_i)_{i=0}^n \in \mathcal{T}_n$ be a time net such that $0 = t_0 < t_1 < \cdots < t_{n-1} = T < t_n = 1$ and

$$\int_0^{t_i} H(t)dt = \frac{i}{n-1} \int_0^T H(t)dt \quad \text{for all } i = 1, ..., n-1.$$

Using Lemma 3.1 we get that

$$\begin{aligned} A^2(H, \tau^n) &= \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (t_i - t)H^2(t)dt + \int_{t_{n-1}}^1 (1-t)H^2(t)dt \\ &\leq \frac{\left(\int_0^T H(t)dt\right)^2}{n-1} + \int_T^1 (1-t)H^2(t)dt. \end{aligned}$$

By definition, we have that $A_n(H) \leq A(H, \tau_n)$ and we are done. $\square$

3.4. *Remark.* The best rate that Lemma 3.3 can give, is obtained by choosing T such that

$$\int_0^T H(t)dt = \sqrt{n-1} \left(\int_T^1 (1-t)H^2(t)dt \right)^{1/2}.$$

However, it is not known if Lemma 3.3 gives the optimal upper bound, i.e. we do not know whether the inequality

$$A_n^2(H) \geq \frac{1}{c} \inf_{T \in (0,1)} \left[\frac{\left(\int_0^T H(t)dt\right)^2}{n-1} + \int_T^1 (1-t)H^2(t)dt \right] \tag{4}$$

holds. What we have is

$$A_n^2(H) = \inf_{T \in (0,1)} \left[A_{n-1}^2(H|[0,T]) + \int_T^1 (1-t)H^2(t)dt \right],$$

where

$$A_{n-1}^2(H|[0,T]) := \inf_{0=t_0<\cdots<t_{n-1}=T} \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} (t_i - t)H^2(t)dt.$$

In order to obtain inequality (4) we would need to know that there exists a constant $c > 0$ such that

$$A_{n-1}^2(H|[0,T]) \geq \frac{1}{c} \frac{\left(\int_0^T H(t)dt\right)^2}{n-1},$$

for all $n \geq 2$, but we do not know whether this is true.

Proof of Theorem 2.3. Assume first that $H \in \mathcal{H}$. Then $I := \int_0^1 H(t)dt < \infty$ and Lemma 3.1 implies that

$$\sqrt{n} A_n(H) \leq I \quad \text{for all } n \in \mathbb{N}$$

and $\|H\|_{\mathcal{A}} \leq \|H\|_{\mathcal{H}}$.

Assume now that $H \in \mathcal{A}$, which means that

$$\sup_{n\in\mathbb{N}} \sqrt{n} A_n(H) < \infty.$$

Lemma 3.2 implies that

$$\int_0^1 H(t)dt \le \sqrt{2} \sup_{n\in\mathbb{N}} \sqrt{n} A_n(H)$$

and $\|H\|_{\mathcal{H}} \le \sqrt{2}\|H\|_{\mathcal{A}}$.

The computations above imply that

$$\int_0^1 H(t)dt < \infty \iff \sup_{n\in\mathbb{N}} \sqrt{n} A_n(H)$$

and that $\|H\|_{\mathcal{H}} \sim_{\sqrt{2}} \|H\|_{\mathcal{A}}$. □

For the proof of Theorem 2.5, we need the following lemmas.

3.5. **Lemma.** *Let $\beta \in (0,1)$. Then there exists a constant $c > 0$ such that*

$$\frac{(1-\log(1-t))^{-(1+\beta)}}{(1-t)^2} \sim_c \int_1^\infty z^{-\beta-2}(1-t)^{\frac{1}{z}-2}dz \quad \textit{for all } t \in [0,1).$$

Proof. Let $\psi_\beta(t) = \frac{(1-\log(1-t))^{-(1+\beta)}}{(1-t)^2}$ and
$\varphi_\beta(t) = \int_1^\infty z^{-\beta-2}(1-t)^{\frac{1}{z}-2}dz = \frac{(-\log(1-t))^{-(\beta+1)}}{(1-t)^2}\int_0^{-\log(1-t)} x^\beta e^{-x}dx.$

The statement follows from

$$\lim_{t\to 1} \frac{\varphi_\beta(t)}{\psi_\beta(t)} = \int_0^\infty x^\beta e^{-x}dx \in (0,\infty).$$

□

3.6. **Lemma.** [8, Lemma 7] *Let $\theta \in [1,2)$ and $H_\theta : [0,1) \to [0,\infty)$, be given by*

$$H_\theta(t) = \sqrt{(2-\theta)(1-t)^{-\theta}} \quad \textit{for all } t \in [0,1).$$

Then

$$\inf_{(t_i)_{i=0}^n \in \mathcal{T}_n} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t)H_\theta^2(t)dt \ge (\theta-1)^{n-1}$$

for all $n \in \{1,2,...\}$.

3.7. **Lemma.** *Let $H : [0,1) \to [0,\infty)$ be an increasing function and $\beta \in (0,1)$. If*

$$H^2(t) \ge \int_1^\infty z^{-\beta-2}(1-t)^{\frac{1}{z}-2}dz \quad \textit{for all } t \in [0,1),$$

then

$$A_n(H) \ge \frac{1}{c_\beta\sqrt{n^\beta}} \quad \textit{for all } n \in \mathbb{N}$$

where $c_\beta = \sqrt{2^{\beta+2}e}$.

Proof. Let $g : [1,\infty) \times [0,1) \to (0,\infty)$ be given by

$$g(z,t) = z^{-\beta-2}(1-t)^{\frac{1}{z}-2}.$$

Then

$$\frac{g(k,t)}{g(k+1,t)} = \left(1+\frac{1}{k}\right)^{\beta+2}(1-t)^{\frac{1}{k(k+1)}} \leq 2^{\beta+2}$$

for all $k \geq 1$ and $t \in [0,1)$. We have that

$$\frac{d}{dz}g(z,t) = (-\log(1-t) - (2+\beta)z)\frac{(1-t)^{\frac{1}{z}-2}}{z^{\beta+4}}$$

and it is easy to see that for any fixed $t \in [0,1)$ there exists $k_t \in \mathbb{N}$ such that $g(z,t)$ is increasing for all $z \leq k_t - 1$ and decreasing for all $z \geq k_t$. Hence

$$\int_1^\infty g(z,t)dz \geq \sum_{k=1}^{k_t-2} g(k,t) + \sum_{k=k_t+1}^{\infty} g(k,t),$$

where we treat an empty sum as zero. Since $g(k,t) \leq 2^{\beta+2}g(k+1,t)$ for all $k \geq 1$, we have that

$$g(k_t-1,t) + g(k_t,t) + g(k_t+1,t) \leq c_\beta g(k_t+1,t),$$

with $c_\beta := (4^{\beta+2} + 2^{\beta+2} + 1)$, (where we omit $g(k_t-1,t)$ if $k_t = 1$) and therefore

$$\begin{aligned}\sum_{k=(k_t-1)\vee 1}^{\infty} g(k,t) &= g(k_t-1,t) + g(k_t,t) + g(k_t+1,t) + \sum_{k=k_t+1}^{\infty} g(k+1,t) \\ &\leq c_\beta \sum_{k=k_t}^{\infty} g(k+1,t).\end{aligned}$$

This implies that

$$\begin{aligned}\int_1^\infty g(z,t)dz &\geq \sum_{k=1}^{k_t-2} g(k,t) + \sum_{k=k_t}^{\infty} g(k+1,t) \\ &\geq \sum_{k=1}^{k_t-2} g(k,t) + \frac{1}{c_\beta}\sum_{k=(k_t-1)\vee 1}^{\infty} g(k,t) \\ &\geq \frac{1}{c_\beta}\sum_{k=1}^{\infty} g(k,t)\end{aligned}$$

for all $t \in [0,1)$.

Let $a_k = 2 - \frac{1}{k}$ and $p_k = k^{-(1+\beta)}$. By assumption,

$$\begin{aligned} H^2(t) &\geq \int_1^\infty g(z,t)dz \\ &\geq \frac{1}{c_\beta}\sum_{k=1}^\infty g(k,t) \\ &= \frac{1}{c_\beta}\sum_{k=1}^\infty \frac{1}{k^{\beta+1}}\frac{1}{k}(1-t)^{\frac{1}{k}-2} \\ &= \frac{1}{c_\beta}\sum_{k=1}^\infty p_k(2-a_k)(1-t)^{-a_k}. \end{aligned}$$

Now

$$\begin{aligned} A_n^2(H) &= \inf_{\tau\in\mathcal{T}_n}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}(t_i-t)H^2(t)dt \\ &\geq \inf_{\tau\in\mathcal{T}_n}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}(t_i-t)\frac{1}{c_\beta}\sum_{k=1}^\infty p_k(2-a_k)(1-t)^{-a_k}dt \\ &= \frac{1}{c_\beta}\inf_{\tau\in\mathcal{T}_n}\sum_{k=1}^\infty p_k\sum_{i=1}^n\int_{t_{i-1}}^{t_i}(t_i-t)(2-a_k)(1-t)^{-a_k}dt \\ &\geq \frac{1}{c_\beta}\sum_{k=1}^\infty p_k\inf_{\tau\in\mathcal{T}_n}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}(t_i-t)(2-a_k)(1-t)^{-a_k}dt. \end{aligned}$$

To prove our claim it is enough to consider $n \geq 2$. We set

$$H_{a_k}(t) = \sqrt{(2-a_k)(1-t)^{-a_k}},$$

and now Lemma 3.6 implies that

$$\begin{aligned} A_n^2(H) &\geq \frac{1}{c_\beta}\sum_{k=1}^\infty p_k(a_k-1)^{n-1} \\ &= \frac{1}{c_\beta}\sum_{k=1}^\infty k^{-(1+\beta)}\left(1-\frac{1}{k}\right)^{n-1} \\ &\geq \frac{1}{c_\beta e}\sum_{k=n}^\infty k^{-(1+\beta)} \\ &\geq \frac{1}{c_\beta e\beta}n^{-\beta} \\ &= \frac{1}{\tilde{c}_\beta n^\beta}, \end{aligned}$$

where $\tilde{c}_\beta = e\beta c_\beta$. □

3.8. **Lemma.** *Let $\beta \in (0,1)$ and $H : [0,1) \to [0,\infty)$ be an increasing function such that there exists a constant $c_1 \geq 1$ for which*

$$A_n(H+1) \geq \frac{1}{c_1\sqrt{n^\beta}} \quad \textit{for all } n \in \mathbb{N}.$$

Then there exists a constant $c_2 \geq 1$ such that

$$A_n(H) \geq \frac{1}{c_2\sqrt{n^\beta}} \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Assume first that $n \geq \tilde{n} := 2^{\frac{\beta}{1-\beta}} c_1^{\frac{2}{1-\beta}}$, then we have that $\frac{1}{c_1^2(2n)^\beta} \geq \frac{1}{n}$ and since

$$A_{2n-1}^2(H+1) \leq A_n^2(H) + A_n^2(1) \leq A_n^2(H) + \frac{1}{2n} \quad \text{for all } n \in \mathbb{N},$$

we get that

$$A_n^2(H) \geq \frac{1}{c_1^2(2n-1)^\beta} - \frac{1}{2n} \geq \frac{1}{2c_1^2(2n)^\beta} = \frac{1}{\tilde{c}_2^2 n^\beta}$$

for all $n \geq \tilde{n}$, where $\tilde{c}_2 = 2^{\frac{1+\beta}{2}} c_1$.

If $n < \tilde{n}$, the computations above imply that

$$A_n^2(H) \geq A_{\lceil \tilde{n} \rceil}^2(H) \geq \frac{1}{\tilde{c}_2^2 \lceil \tilde{n} \rceil^\beta} \geq \frac{1}{c_2^2 n^\beta},$$

where $c_2 = \tilde{c}_2 \lceil \tilde{n} \rceil^{\frac{\beta}{2}}$ and $\lceil \tilde{n} \rceil := \inf\{k \in \mathbb{Z} : \tilde{n} \leq k\}$. $\square$

Proof of Theorem 2.5.

(1) Let $T = 1 - e^{c_\alpha(n)}$, where $c_\alpha(n) = 1 - ((1-\alpha)n^{1-\alpha} + 1)^{\frac{1}{1-\alpha}}$. Then

$$\begin{aligned}\int_0^T H(t)dt &\leq c_1 \int_0^T \frac{(1-\log(1-t))^{-\alpha}}{1-t} dt \\ &= \frac{c_1}{1-\alpha}[(1-\log(1-T))^{1-\alpha} - 1] \\ &= c_1 n^{1-\alpha}\end{aligned}$$

and

$$\begin{aligned}\int_T^1 (1-t)H^2(t)dt &\leq c_1^2 \int_T^1 \frac{(1-\log(1-t))^{-2\alpha}}{1-t} dt \\ &= \frac{c_1^2}{2\alpha-1}(1-\log(1-T))^{1-2\alpha} \\ &= \frac{c_1^2}{2\alpha-1}((1-\alpha)n^{1-\alpha}+1)^{\frac{1-2\alpha}{1-\alpha}} \\ &\leq \frac{c_1^2(1-\alpha)^{\frac{1-2\alpha}{1-\alpha}}}{2\alpha-1} n^{1-2\alpha}\end{aligned}$$

and hence Lemma 3.3 says that, for $n \geq 2$,

$$
\begin{aligned}
A_n(H) &\leq \left[\frac{1}{n-1}\left(\int_0^T H(t)dt\right)^2 + \left(\int_T^1 (1-t)H^2(t)dt\right)\right]^{1/2} \\
&\leq \left[\frac{c_1^2}{n-1}n^{2-2\alpha} + c_1^2\tilde{c}_\alpha n^{1-2\alpha}\right]^{1/2} \\
&\leq c_1\frac{(2+\tilde{c}_\alpha)^{\frac{1}{2}}}{\sqrt{n^{2\alpha-1}}},
\end{aligned}
$$

where $\tilde{c}_\alpha = \frac{(1-\alpha)^{\frac{1-2\alpha}{1-\alpha}}}{2\alpha-1}$.
(2) Assume that there exists a constant $c_2 \geq 1$ such that

$$
H(t) \geq \frac{(1-\log(1-t))^{-\alpha}}{c_2(1-t)} \quad \text{for all } t \in [s,1).
$$

Then there exists a constant $c_3 \geq 1$ such that

$$
H(t) + 1 \geq \frac{(1-\log(1-t))^{-\alpha}}{c_3(1-t)} \quad \text{for all } t \in [0,1).
$$

If we write $\beta = 2\alpha - 1 \in (0,1)$, Lemma 3.5 implies that there exists a constant $c_4 \geq 1$ such that

$$
(H(t)+1)^2 \geq \frac{1}{c_4}\int_1^\infty z^{-\beta-2}(1-t)^{\frac{1}{z}-2}dz \quad \text{for all } t \in [0,1),
$$

and Lemma 3.7 implies that there exists $c_5 \geq 1$ such that

$$
A_n(H+1) \geq \frac{1}{c_5\sqrt{n^\beta}} \quad \textit{for all } n \in \mathbb{N}.
$$

Now Lemma 3.8 implies the existence of a constant $c \geq 1$ such that

$$
A_n(H) \geq \frac{1}{c\sqrt{n^\beta}} \quad \textit{for all } n \in \mathbb{N}.
$$

□

4. Application: Optimal approximation rate of certain stochastic integrals

Throughout the section, we assume a standard Brownian motion $W = (W_t)_{t\in[0,1]}$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,1]})$, where $(\mathcal{F}_t)_{t\in[0,1]}$ is the augmentation of the natural filtration of W and $\mathcal{F} = \mathcal{F}_1$. We let the process $S = (S_t)_{t\in[0,1]}$ be the geometric Brownian motion, i.e. $S_t = e^{W_t - \frac{t}{2}}$ for all $t \in [0,1]$. Moreover, we let $X = (X_t)_{t\in[0,1]}$ be a diffusion such that

$$
dX_t = \sigma(X_t)dW_t \quad \text{with } X_0 \equiv x_0 \in \mathbb{R}, \tag{5}
$$

where the process X is obtained through $Y = (Y_t)_{t\in[0,1]}$ given as unique continuous solution of

$$dY_t = \hat{\sigma}(Y_t)dW_t + \hat{b}(Y_t)dt \quad \text{with } Y_0 \equiv y_0 \in \mathbb{R},$$

with $0 < \epsilon_0 \le \hat{\sigma} \in \mathcal{C}_b^\infty(\mathbb{R})$ and $\hat{b} \in \mathcal{C}_b^\infty(\mathbb{R})$, in the following two ways:

(a) $y_0 = x_0 \in \mathbb{R}$, $\hat{\sigma} := \sigma$, $\hat{b} := 0$, $X_t := Y_t$,

(b) $y_0 = \log x_0$ with $x_0 > 0$,

$$\hat{\sigma}(y) := \frac{\sigma(e^y)}{e^y}, \quad \hat{b}(y) := -\frac{1}{2}\hat{\sigma}(y)^2, \quad \textit{and } X_t = e^{Y_t}.$$

Moreover, we let γ be the Gaussian measure on $\mathbb{R}$, i.e.

$$d\gamma(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

4.1. **Definition.** Let $\mathcal{C}_e$ be the linear space of Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that there exists $m > 0$ for which

$$\sup_{x\in\mathbb{R}} e^{-m|x|} \mathbb{E} f^2(x + tg) < \infty$$

for all $t > 0$, where g is a centered standard normal random variable. Moreover, we define

$$\mathcal{C} := \{Z := f(Y_1) : \Omega \to \mathbb{R} \mid f \in \mathcal{C}_e \text{ and } Y \text{ as above}\}.$$

The main tool for investigating the approximation problem in papers of C. Geiss, S. Geiss, and Hujo was the H-functional defined in the following way.

4.2. **Definition.** Let X be a stochastic process as in (5) and assume that $Z \in \mathcal{C}$ (or $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ if $X \in \{W, S\}$). Then we set

$$H_X Z(t) := \left\| \left(\sigma^2 \frac{\partial^2 F}{\partial x^2} \right)(t, X_t) \right\|_{L_2} \qquad \text{for all } t \in [0,1), \tag{6}$$

where $F : [0,1) \times I \to \mathbb{R}$ is given by $F(t,x) = \mathbb{E}(Z|X_t = x)$, with $I = \mathbb{R}$ in case of (a) and $I = (0,\infty)$ in case of (b).

4.3. **Lemma.** [3, Lemma 5.3] *The function $H_X Z : [0,1) \to [0,\infty)$ is continuous and increasing.*

In order to deduce from Theorem 2.3 a characterization of the approximation rate

$$a_n^X(Z) \le \frac{c}{\sqrt{n}},$$

we need the following theorem.

4.4. **Theorem.** [3, Lemma 3.2] [6, Theorem 4.4] *Let X be a stochastic process as in* (5), *$Z \in \mathcal{C}$ (or $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ if $X \in \{W, S\}$) and $\tau = (t_i)_{i=0}^n \in \mathcal{T}_n$. Then*

$$a_X(Z,\tau) \sim_c \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t_i - t) H_X^2 Z(t) dt \right)^{\frac{1}{2}}$$

where $c \geq 1$ is an absolute constant depending on σ only. Consequently,

$$a_n^X(Z) \sim_c A_n(H_X Z).$$

4.5. **Corollary.** *Let X be as in* (5) *and $Z \in \mathcal{C}$ (or $Z \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ if $X \in \{W, S\}$). Then*

$$\sup_{n \in \mathbb{N}} \sqrt{n} a_n^X(Z) \sim_c \int_0^1 \left\| \left(\sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2} dt,$$

where $F : [0,1) \times I \to \mathbb{R}$ is given by $F(t,x) = \mathbb{E}(Z | X_t = x)$, with $I = \mathbb{R}$ in case of (a) and $I = (0, \infty)$ in case of (b).

Proof. Theorem 2.3 together with Lemma 4.3 and Theorem 4.4 gives the result immediately. □

4.6. *Remark.* Remark 2.4 implies that if $\left\| \left(\sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2}$ is integrable, then the regular sequences generated by $\left\| \left(\sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2}$ give the rate $\frac{1}{\sqrt{n}}$. Using these sequences, denoted by τ_r^n, we have that if $A := \int_0^1 \left\| \left(\sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2} dt < \infty$, then

$$a_n^X(Z) \leq a_X(Z, \tau_r^n) \leq \frac{c_{(4.4)} A}{\sqrt{n}} \quad \textit{for all } n \in \mathbb{N},$$

where $c_{(4.4)} > 0$ is taken from Theorem 4.4 above.

One can also optimize over random time nets instead of deterministic ones considered here. The result [4, Theorem 1.1.] from C. and S. Geiss implies that $\frac{1}{\sqrt{n}}$ is the best possible approximation rate also for the random time nets in case the underlying diffusion X is the Brownian motion W or the geometric Brownian motion S and Z is not equal to $c_0 + c_1 X_1$ a.s. for some $c_0, c_1 \in \mathbb{R}$. This means that if $X \in \{W, S\}$, the random time nets do not improve the approximation if the deterministic time nets already give the rate $\frac{1}{\sqrt{n}}$. According to this, Corollary 4.5 implies that if

$$\int_0^1 \left\| \left(\sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2} dt < \infty,$$

then the optimal approximation rate is $\frac{1}{\sqrt{n}}$ also for the random time nets and this rate is obtained by using the regular sequences generated by $\left\| \left(\sigma^2 \frac{\partial^2 F}{\partial x^2} \right) (t, X_t) \right\|_{L_2}$.

Now we give for $\beta \in (0,1)$ an example such that

$$a_n^X(Z) \sim_c \frac{1}{\sqrt{n^\beta}} \quad \text{for all } n \in \mathbb{N},$$

in case X is a standard Brownian motion or the geometric Brownian motion. According to Theorem 2.5, Lemma 4.3 and Theorem 4.4 it is sufficient to find a random variable $Z = f_\alpha(W_1)$ such that

$$H_X Z(t) \sim_c \frac{(1-\log(1-t))^{-\alpha}}{1-t},$$

where $\alpha = \frac{\beta+1}{2}$.

4.7. **Example.** Let $\alpha \in (1/2, 1)$ and $f_\alpha = \sum_{k=0}^\infty a_k h_k \in L_2(\gamma)$, where $a = (a_k)_{k=0}^\infty$ is given by

$$a_k = \begin{cases} 0 & \text{if } k \in \{0,1,3\} \\ \frac{1}{\sqrt{2}} & \text{if } k = 2 \\ \sqrt{\frac{k-2}{k(k-1)}} \log^{-\alpha}(k-2) & \text{if } k \geq 4 \end{cases}$$

and $(h_k)_{k=0}^\infty \subset L_2(\gamma)$ is the complete orthonormal system of Hermite polynomials,

$$h_k(x) = \frac{(-1)^k}{\sqrt{k!}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}.$$

Then $Z_\alpha := f_\alpha(W_1) \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ and it can be shown that

$$H_W Z_\alpha(t) = \left(1 + \sum_{k=2}^\infty k \log^{-2\alpha}(k) t^k\right)^{1/2} \sim_{c_1} \frac{(1-\log(1-t))^{-\alpha}}{1-t}$$

for all $t \in [0,1)$ (according to Lemmas 4.9 and 4.8 below). Using Lemma 4.9 it is easy to show that there exists a constant $c_2 > 0$ such that

$$H_W Z_\alpha(t) \sim_{c_2} H_S Z_\alpha(t) \quad \textit{for all } t \in (0,1).$$

Theorem 2.5 implies that there exists a constant $c_3 \geq 1$ such that

$$\frac{1}{c_3 \sqrt{n^{2\alpha-1}}} \leq a_n^X(Z_\alpha) \leq \frac{c_3}{\sqrt{n^{2\alpha-1}}}$$

for all $n \in \mathbb{N}$, where $X \in \{W, S\}$. In other words, letting $\beta \in (0,1)$ and defining $\alpha := \frac{\beta+1}{2}$ we have

$$a_n^X(Z_\alpha) \sim_{c_3} \frac{1}{\sqrt{n^\beta}} \quad \text{for all } n \in \mathbb{N}.$$

The following lemma should be folklore. For completeness and convinience of the reader we include a proof.

4.8. **Lemma.** *Let $\beta > 1$. Then for all $t \in [0, 1)$, one has that*

$$\frac{(1 - \log(1-t))^{-\beta}}{(1-t)^2} \sim_c 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k, \tag{7}$$

where the constant $c \geq 1$ depends at most on β.

Proof. Let $n \geq e^{\beta}$ be an integer, $\epsilon \in [\frac{1}{n+1}, \frac{1}{n})$, and $t = e^{-\epsilon}$. Since $k \log^{-\beta}(k)$ is increasing if $k \geq e^{\beta}$ and we assumed that $n \geq e^{\beta}$, we have that

$$\begin{aligned} 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k &\geq \sum_{k=n}^{2n} k \log^{-\beta}(k) (e^{-1/n})^k \\ &\geq \sum_{k=n}^{2n} n \log^{-\beta}(n) e^{-2} \\ &\geq e^{-2} n^2 \log^{-\beta}(n). \end{aligned}$$

Moreover,

$$\begin{aligned} 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k &\leq 1 + \sum_{k=2}^{n} k \log^{-\beta}(k) + \sum_{m=1}^{\infty} \sum_{k=mn+1}^{(m+1)n} k \log^{-\beta}(k) e^{-\frac{mn}{n+1}} \\ &\leq c_{\beta} \sum_{k=2}^{n} n \log^{-\beta}(n) + \sum_{m=1}^{\infty} (m+1) n^2 \log^{-\beta}(n) e^{-\frac{mn}{n+1}} \\ &\leq c_{\beta} n^2 \log^{-\beta}(n) + n^2 \log^{-\beta}(n) \sum_{m=1}^{\infty} (m+1) e^{-m/2} \\ &\leq (c_{\beta} + c) n^2 \log^{-\beta}(n), \end{aligned}$$

where c_{β} depends at most on β and $c = \sum_{m=1}^{\infty} (m+1) e^{-m/2}$. This implies that

$$1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k \sim_{c_1} n^2 \log^{-\beta}(n) \quad \text{for all } n \geq e^{\beta},$$

where $c_1 \geq 1$ is a constant depending at most on β. Adapting the constant $c_1 > 0$, we get this for $n \geq 2$.
Now we show that if $n \geq 4$, then

$$\frac{(1 - \log(1-t))^{-\beta}}{(1-t)^2} \sim_{c_2} n^2 \log^{-\beta}(n),$$

where $c_2 \geq 2$ is a constant depending at most on β. Firstly, we have that $\log(\frac{1}{t}) \sim_{c_3} \frac{1}{n}$, where $c_3 = \frac{5}{4}$. Moreover

$$\log(u^{-1}) \sim_{c_4} 1 - u,$$

for all $u \in [e^{-1/2}, 1]$, where $c_4 = [2(1-e^{-\frac{1}{2}})]^{-1}$. Hence

$$1 - t \sim_{c_5} \frac{1}{n},$$

where $c_5 = \frac{5}{8}[1-e^{-\frac{1}{2}}]^{-1}$. Furthermore,

$$\frac{\log n}{2} \leq \log(n/c_5) \leq \log((1-t)^{-1}) \leq \log(c_5 n) \leq 2\log n$$

since $c_5 < 2$ and $n \geq 4$. Now

$$1 + \log\left(\frac{1}{1-t}\right) \sim_3 \log(n)$$

and hence

$$\frac{(1-\log(1-t))^{-\beta}}{(1-t)^2} \sim_{c_2} n^2 \log^{-\beta} n,$$

where $c_2 = 4^\beta c_5^2$.

If $t > e^{-\frac{1}{4}}$, the computations above imply that

$$\frac{(1-\log(1-t))^{-\beta}}{(1-t)^2} \sim_{c_2} n^2 \log^{-\beta} n \sim_{c_1} 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k,$$

where n is such that $e^{-\frac{1}{n}} < t \leq e^{-\frac{1}{n+1}}$. If $0 \leq t < e^{-\frac{1}{4}}$, then one has that

$$1 \leq 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k \leq c_\beta,$$

where the constant $c_\beta > 0$ depends only on β, and

$$\frac{1}{d_\beta} \leq \frac{(1-\log(1-t))^{-\beta}}{(1-t)^2} \leq d_\beta,$$

where the constant $d_\beta > 0$ depends only on β. Hence

$$\frac{(1-\log(1-t))^{-\beta}}{(1-t)^2} \sim_c 1 + \sum_{k=2}^{\infty} k \log^{-\beta}(k) t^k$$

for all $t \in [0,1)$, where the constant $c \geq 1$ depends on β. □

4.9. **Lemma.** [7, Lemma 3.9] *For $f = \sum_{k=0}^{\infty} a_k h_k \in L_2(\gamma)$, $t \in [0,1)$ and $Z = f(W_1)$ one has that*

$$H_W Z^2(t) = \sum_{k=0}^{\infty} a_{k+2}^2 (k+2)(k+1) t^k,$$

$$H_S Z^2(t) = \sum_{k=0}^{\infty} \left(a_{k+2} - \frac{a_{k+1}}{\sqrt{k+2}}\right)^2 (k+2)(k+1) t^k,$$

where W is a standard Brownian motion and S is the geometric Brownian motion. Moreover

$$\frac{1}{12}H_W Z^2(t) - \frac{2}{3}(a_1^2 + a_2^2) \leq H_S Z^2(t) \leq 4H_W Z^2(t) + 2a_1^2.$$

5. Application: Approximation of certain d-dimensional stochastic integrals with drift

We can apply Theorems 2.3 and 2.5 also to the discrete time approximation of d-dimensional stochastic integrals considered by Zhang [13], Temam [12] and Hujo [9]. Our setting is the same as in [9], which generalizes the 1-dimensional setting of Section 4 to d dimensions.

We assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,1]})$, where $(\mathcal{F}_t)_{t\in[0,1]}$ is the augmentation of the natural filtration generated by the d-dimensional standard Brownian motion $W = (W_t)_{t\in[0,1]}$ with $\mathcal{F} = \mathcal{F}_1$.

We consider a diffusion $X = (X^1, ..., X^d)$, where

$$X_t^i = x_0^i + \int_0^t b_i(X_u)du + \sum_{j=1}^d \int_0^t \sigma_{ij}(X_u)dW_u^j, \quad t \in [0,1], \ a.s. \tag{8}$$

for all $i = 1, ..., d$ and $x_0 = (x_0^1, ..., x_0^d)$. We assume that X is obtained through Y given as unique path-wise continuous solution of

$$Y_t^i = y_0^i + \int_0^t \hat{b}_i(Y_u)du + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Y_u)dW_u^j, \quad t \in [0,1], \ a.s. \tag{9}$$

for all $i = 1, ..., d$, where $\hat{b}_i, \hat{\sigma}_{ij} \in \mathcal{C}_b^\infty(\mathbb{R}^d)$ and $(\hat{\sigma}\hat{\sigma}^T)_{ij}(x) = \sum_{k=1}^d \hat{\sigma}_{ik}(x)\hat{\sigma}_{jk}(x)$ is uniformly elliptic, i.e.

$$\sum_{i,j=1}^d (\hat{\sigma}\hat{\sigma}^T)_{ij}(x)\xi_i\xi_j \geq \lambda\|\xi\|^2, \quad \text{for all } x, \xi \in \mathbb{R}^d \text{ and some } \lambda > 0,$$

where $\|\cdot\|$ is the Euclidean norm. We assume that X is obtained through Y by one of the following two ways:

(a) $x_0 = y_0 \in \mathbb{R}^d$, $\hat{b}_i(x) := b_i(x)$, $\hat{\sigma}_{ij}(x) := \sigma_{ij}(x)$, and $X_t = Y_t$,
(b) $x_0 = e^{y_0} \in (0,\infty)^d$, $\hat{b}_i(y) := \frac{b_i(e^y)}{e^{y_i}} - \frac{1}{2}\sum_{j=1}^d \hat{\sigma}_{ij}^2(y)$, $\hat{\sigma}_{ij}(y) := \frac{\sigma_{ij}(e^y)}{e^{y_i}}$, and $X_t = e^{Y_t}$.

Here and in the following $e^y = (e^{y_1}, ..., e^{y_d})$ for $y = (y_1, ..., y_d)$. As in one dimensional case, (a) is related to the standard Brownian motion and (b) is related to the geometric Brownian motion.

Moreover, we assume that $f : E \to \mathbb{R}$ is a Borel-function such that for some $q \in [2,\infty)$ and $C > 0$ it holds that

$$|f(x)| \leq C(1 + \|x\|^q), \ x \in E, \tag{10}$$

where the set E is defined by

$$E := \begin{cases} \mathbb{R}^d & \text{in case } (a) \\ (0,\infty)^d & \text{in case } (b). \end{cases}$$

Finally, we define the function $g : \mathbb{R}^d \to \mathbb{R}$ by

$$g(y) := \begin{cases} f(y) & \text{in case } (a) \\ f(e^y) & \text{in case } (b). \end{cases}$$

5.1. **Theorem.** [1, Theorem 8 on p. 263], [2, Theorem 5.4 on p. 149] *For $\hat{b}$, $\hat{\sigma}$ with $\hat{\sigma}\hat{\sigma}^T$ uniformly elliptic, there exists a transition density $\Gamma : (0,1]\times\mathbb{R}^d\times\mathbb{R}^d \to [0,\infty) \in \mathcal{C}^\infty$ such that $\mathbb{P}(Y_t \in B) = \int_B \Gamma(t,y,\xi)d\xi$, for $t \in (0,1]$ and $B \in \mathcal{B}(\mathbb{R}^d)$, where $Y = (Y_t)_{t\in[0,1]}$ is the strong solution of SDE (9) starting from y. Moreover, the following is satisfied:*

(i) *For $(s,y,\xi) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ we have that*

$$\frac{\partial}{\partial s}\Gamma(s,y,\xi) = \frac{1}{2}\sum_{k,l=1}^d \sum_{j=1}^d \hat{\sigma}_{kj}\hat{\sigma}_{lj}\frac{\partial^2}{\partial y_k \partial y_l}\Gamma(s,y,\xi) + \sum_{i=1}^d \hat{b}_i(y)\frac{\partial}{\partial y_i}\Gamma(s,y,\xi).$$

(ii) *For $a \in \{0,1,2,...\}$ and multi-indices b and c there exists positive constants C and D, depending only on a, b, c and d such that*

$$\left| \frac{\partial^{a+|b|+|c|}}{\partial^a t \partial^b y \partial^c \xi}\Gamma(t,y,\xi) \right| \le \frac{C}{t^{(d+2a+|b|+|c|)/2}} e^{-D\frac{\|y-\xi\|^2}{t}}.$$

If we apply Theorem 5.1 to the stochastic differential equation

$$\begin{cases} Z_t^i = Z_0^i + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Z_u)dW_u^j & \text{in case } (a) \\ Z_t^i = Z_0^i - \int_0^t \left(\frac{1}{2}\sum_{j=1}^d \hat{\sigma}_{ij}^2(Z_u)\right)du + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Z_u)dW_u^j & \text{in case } (b), \end{cases}$$

we obtain a transition density Γ_0 such that we can define the function $G \in \mathcal{C}^\infty([0,1] \times \mathbb{R}^d)$ by

$$G(t,y) := \int_{\mathbb{R}^d} \Gamma_0(1-t,y,\xi)g(\xi)d\xi, \ 0 \le t < 1$$

so that

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{1}{2}\sum_{k,l=1}^d \left(\hat{\sigma}\hat{\sigma}^T(y)\right)_{kl}\frac{\partial^2}{\partial y_k \partial y_l}\right)G(t,y) = 0 & \text{(a)} \\ \left(\frac{\partial}{\partial t} - \sum_{i=1}^d \left(\frac{1}{2}\sum_{j=1}^d \hat{\sigma}_{ij}^2(y)\right)\frac{\partial}{\partial y_i} + \frac{1}{2}\sum_{k,l=1}^d \left(\hat{\sigma}\hat{\sigma}^T(y)\right)_{kl}\frac{\partial^2}{\partial y_k \partial y_l}\right)G(t,y) = 0 & \text{(b).} \end{cases}$$

We define the function $F : E \to \mathbb{R}$ by setting

$$F(t,x) := \begin{cases} G(t,x), & \text{in case } (a) \\ G(t,\log(x)), & \text{in case } (b), \end{cases}$$

where $\log x = (\log(x_1), ..., \log(x_d))$, and the operator $\mathcal{L}$ by

$$\mathcal{L} := \frac{\partial}{\partial t} + \frac{1}{2}\sum_{k,l=1}^d L_{kl}(x)\frac{\partial^2}{\partial x_k \partial x_l},$$

where $L_{kl} = \sum_{j=1}^{d} \sigma_{kj}(x)\sigma_{lj}(x)$. Now we have that

$$\mathcal{L}F(t,x) = 0 \ \text{ on } [0,1) \times E,$$

and Itô's formula implies that

$$F(t,X_t) = F(0,X_0) + \sum_{k=1}^{d} \int_0^t \frac{\partial}{\partial x_k} F(u,X_u) dX_u^k, \ \text{a.s. } t \in [0,1).$$

From Theorem 5.1 we get that

$$F(t,X_t) \to f(X_1) \ \text{ in } L_2 \text{ as } t \nearrow 1$$

and

$$f(X_1) = F(0,X_0) + \sum_{k=1}^{d} \int_0^1 \frac{\partial}{\partial x_k} F(u,X_u) dX_u^k \ \text{a.s.}$$

5.2. **Definition.** For f, F and X as above we define

$$a_X^{\text{sim}}(f(X_1),\tau,s) := \left\| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1}^n \wedge s}^{t_i^n \wedge s} \left(\frac{\partial}{\partial x_k} F(u,X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^n, X_{t_{i-1}^n}) \right) dX_u^k \right\|_{L_2},$$

for all $\tau = (t_i)_{i=1}^n \in \mathcal{T}_n$ and $s \in [0,1)$.

5.3. **Definition.** We define $H_X f, H_X^* f : [0,1) \to [0,\infty)$ by setting

$$H_X f(t) := \left(\sup_{\alpha,\beta} \mathbb{E}\left[V_{\alpha\alpha}(X_t) V_{\beta\beta}(X_t) \left| \frac{\partial^2}{\partial x_\alpha \partial x_\beta} F(t,X_t) \right|^2 \right] \right)^{\frac{1}{2}} \quad \text{and}$$
$$H_X^* f(t) := \sup_{s \in [0,t]} H_X f(s).$$

5.4. *Remark.* According to [9], one can check that $H_X^* f(t) < \infty$ for $t \in [0,1)$.

Finally, we define functions $Q_i : \mathbb{R}^d \to \mathbb{R}$ for $i = 1, ..., d$ by

$$Q_i(x) := \begin{cases} 1, & \text{in case } (a) \\ x_i & \text{in case } (b). \end{cases}$$

In this setting we have the following theorem, which refines [9, Theorem 1].

5.5. **Theorem.** *Assume that for all $x \in E$*

$$\left| \frac{\partial^s}{\partial_{x_\beta}^q \partial_{x_\alpha}^r} \sigma_{ij}(x) \right| \le C_1 \frac{Q_i(x)}{Q_\beta^q(x) Q_\alpha^r(x)}, \quad \textit{where } q + r = s,\ q,r,s \in \{0,1,2\},$$

$|b_i(x)| \le C_1 Q_i(x)$ and $V_{ii} \ge \frac{1}{C_1} Q_i^2(x)$ for $i \in \{1, ..., d\}$ and some fixed $C_1 > 0$.

(1) *If one has that*

$$\int_0^1 H_X^* f(t)dt < \infty,$$

then

$$\inf_{\tau \in \mathcal{T}_n} \sup_{s\in[0,1]} a_X^{\text{sim}}(f(X_1),\tau,s) \le \frac{D_1}{\sqrt{n}} \quad \textit{for all } n \in \mathbb{N},$$

where $D_1 = D_1(C_1, d) > 0$.

(2) *If there exists* $C_2 > 0$ *and* $\alpha \in (\frac{1}{2}, 1)$ *such that*

$$H_X^* f(t) \le C_2 \frac{(1-\log(1-t))^{-\alpha}}{1-t} \quad \textit{for all } t \in [0,1),$$

then

$$\inf_{\tau \in \mathcal{T}_n} \sup_{s\in[0,1]} a_X^{\text{sim}}(f(X_1),\tau,s) \le \frac{D_2}{\sqrt{n^{2\alpha-1}}} \quad \textit{for all } n \in \mathbb{N},$$

where $D_2 = D_2(C_1, C_2, d) > 0$.

Proof of Theorem 5.5 . Hujo showed in the proof of [9, Theorem 1, p. 18] that under the assumptions of Theorem 5.5 we have that

$$\begin{aligned}
&\mathbb{E}\left|\sum_{i=1}^n \sum_{k=1}^d \int_{t_{i-1}^n \wedge s}^{t_i^n \wedge s} \left(\frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^n, X_{t_{i-1}^n})\right) dX_u^k\right|^2 \\
&\le c \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \int_{t_{i-1}^n}^{t} \sup_{\alpha,\beta} \mathbb{E}\left[V_{\alpha\alpha}(X_u)V_{\beta\beta}(X_u)\left|\frac{\partial^2}{\partial x_\alpha \partial x_\beta} F(u, X_u)\right|^2\right] dudt \\
&\le c \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (t_i - t)[H_X^* f(t)]^2 dt
\end{aligned}$$

for any $s \in [0,1)$ and any time net $\tau = (t_i^n)_{i=0}^n$. Hence we can conclude by Theorems 2.3 and 2.5. $\square$

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