GENERALIZED DIMENSION DISTORTION UNDER PLANAR SOBOLEV HOMEOMORPHISMS

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Abstract. We prove essentially sharp dimension distortion estimates for planar Sobolev-Orlicz homeomorphisms.

1. Introduction

Let Ω, $\Omega' \subset \mathbb{R}^2$ be open and connected. We consider homeomorphisms $f\colon \Omega \to \Omega'$ that belong to the Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$, which means that both component functions of f have locally integrable distributional partial derivatives. It is by now well-known that the Luzin condition (N), which requires that f map Lebesgue null sets to Lebesgue null sets, holds if we additionally assume that $|Df| \in L^2_{\mathrm{loc}}(\Omega)$ [18, 15, 14], but may fail if $|Df| \in L^p_{\mathrm{loc}}(\Omega)$ for some $p < 2$ [16, 17]. On the other hand, if $|Df| \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > 2$, then the image of any set of Hausdorff dimension strictly less then two is also of Hausdorff dimension strictly less then two [4, 10]. Recently it was proven [11] that already the local integrability of $|Df|^2 \log^{-1}(e+|Df|)$ suffices for the Luzin condition (N). The motivation for this result and our results below partially arises from the theory of mappings with finite distortion, where the natural regularity assumption is that $|Df|^2 \log^{\lambda-1}(e+|Df|) \in L^1_{\mathrm{loc}}$ for some $\lambda > 0$ [2, 1, 7, 8, 3].

Analogously to the L^p–scale setting, one expects that some kind of a dimension distortion estimate to hold when λ as above is strictly positive. However, it is rather easy to map, for example, a subset of the real line onto a set of Hausdorff dimension two [6, 19] and thus we have to work with a refined scale. Towards this end, we consider the gauge functions $h_\lambda(t) = t^2 \log^\lambda \frac{1}{t}$, $\lambda > 0$. In Section 2 below, we describe a homeomorphism f that maps a Cantor set E of Minkowski (and so also Hausdorff) dimension strictly less than two to a set of positive $\mathcal{H}^{h_\lambda}$-measure, with $|Df|^2 \log^{t-1}(e+|Df|) \in L^1_{\mathrm{loc}}$ for all $t < \lambda$.

Our main result shows that this homeomorphism is critical for our generalized dimension distortion.

Theorem 1. *Let Ω and Ω' be open sets in $\mathbb{R}^2$ and $f\colon \Omega \to \Omega'$ a homeomorphism of class $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ with*

$$|Df|^2 \log^{\lambda-1}(e+|Df|) \in L^1_{\mathrm{loc}}(\Omega)$$

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for some $\lambda > 0$. *Then*

$$\mathcal{H}^{h_\lambda}(f(E)) = 0$$

for every set $E \subset \Omega$ *of lower Minkowski dimension* $\dim_{\mathcal{M}}(E)$ *strictly less than two.*

We conjecture that one may replace the Minkowski dimension in Theorem 1 with the Hausdorff dimension. For a related, weaker result in this direction, see [12].

This note is organized as follows. In Section 2 we recall the necessary definitions and describe the construction for the homeomorphism referred to above. Section 3 contains the proof of Theorem 1.

2. Preliminaries

Let $U \subset \mathbb{R}^2$ be open and connected. We say that a mapping $f \in L^1(U;\mathbb{R}^2)$ has *bounded variation*, $f \in BV(U)$, if the component functions f_1 and f_2 of f are of bounded variation. That is,

$$\sup\Big\{\int_U f_i \operatorname{div}\varphi \, dx \,|\, \varphi \in C_0^1(U;\mathbb{R}^2),\ |\varphi| \le 1\Big\} < \infty, \quad i = 1, 2.$$

We write $f \in BV_{\mathrm{loc}}(U)$ if $f \in BV(G)$ for each open and connected G, compactly contained in U. For each function $g \in BV(U;\mathbb{R})$ of bounded variation we can define a Radon measure $||Dg||$ in the following way: for an open set $V \subset U$ we put

$$||Dg||(V) = \sup\Big\{\int_V g \operatorname{div}\varphi \, dx \,|\, \varphi \in C_0^1(V;\mathbb{R}^2),\ |\varphi| \le 1\Big\},$$

and for $A \subset U$ not necessarily open

$$||Dg||(A) = \inf\big\{||Dg||(V) \,|\, A \subset V \subset U,\ V \text{ is open}\big\}.$$

For a set V and a number $\delta > 0$, $V + \delta$ denotes the set

$$\{y |\, \operatorname{dist}(y, V) < \delta\}.$$

We write $\mathcal{H}^h(A)$ for the *generalized Hausdorff measure* of a set A, given by

$$\begin{aligned}(1)\quad \mathcal{H}^h(A) &= \lim_{\delta\to 0} \mathcal{H}^h_\delta(A) \\ &= \lim_{\delta\to 0}\Big[\inf\Big\{\sum_{i=1}^\infty h(\operatorname{diam} U_i)\colon A \subset \bigcup_{i=1}^\infty U_i, \operatorname{diam} U_i \le \delta\Big\}\Big],\end{aligned}$$

where h is a dimension gauge (non-decreasing, $h(0) = 0$). If $h(t) = t^\alpha$ for some $\alpha \ge 0$, we put simply $\mathcal{H}^\alpha$ for $\mathcal{H}^{t^\alpha}$ and call it the *Hausdorff α-dimensional measure* and the *Hausdorff dimension* $\dim_{\mathcal{H}}(A)$ of the set A is the smallest $\alpha_0 \ge 0$ such that $\mathcal{H}^\alpha(A) = 0$ for any $\alpha > \alpha_0$. The *lower Minkowski dimension* $\dim_{\mathcal{M}}(A)$ of a bounded set $A \subset \mathbb{R}^2$ is defined as

$$\dim_{\mathcal{M}}(A) = \inf\{s\colon \liminf_{\varepsilon\to 0+} N(A,\varepsilon)\varepsilon^s = 0\},$$

where $N(A,\varepsilon)$, $\varepsilon>0$, denotes the smallest number of balls of radius ε needed to cover A:

$$N(A,\varepsilon)=\min\bigl\{k\colon A\subset\bigcup_{i=1}^{k}B(x_i,\varepsilon)\text{ for some }x_i\in\mathbb{R}^2\bigr\}.$$

Finally, let $a\lesssim b$ mean that there exists some constant $C>0$ such that $a\le Cb$.

In [6] a homeomorphism $h\colon\mathbb{R}^n\to\mathbb{R}^n$ was constructed, which maps a set $\mathcal{C}$ of Minkowski and Hausdorff dimension $n\log 2/\log(1/\sigma)$ for some $0<\sigma<1/2$ onto a set $\mathcal{C}'$ of positive $\mathcal{H}^h$-measure with $h(t)=t^n(\log(1/t))^{pn}$ for given $p>0$. This mapping is the identity outside the cube $[0,1]^n$ and satisfies $|Dh(x)|\le\frac{\tau_1\cdots\tau_k}{\sigma^k}$ in A_{ki}. Here A_{ki}, $k=1,2,\ldots$, $i=1,\ldots,2^{kn}$, are the open "cubical frames" needed to construct the Cantor set $\mathcal{C}$. They are pairwise disjoint with respect to both i and k, that is, $\operatorname{int}(A_{ki})\cap\operatorname{int}(A_{lj})=\emptyset$, when $(k,i)\ne(l,j)$, cover the set $[0,1]^n$ up to a set of zero n-Lebesgue measure, and each A_{ki} is contained in a cube of edge length $(1/2)\sigma^{k-1}$. The numbers τ_k, $k=1,2,\ldots$, used to construct the image Cantor-type set, are defined as follows

$$\tau_1=\frac12\frac{1}{\log^p 4}\quad\text{and}\quad\tau_k=\frac12\bigl(1-\frac1k\bigr)^p\quad\text{for }k=2,3,\ldots.$$

Note, that

$$\tau_1\cdots\tau_k=\frac{1}{2^k}\frac{1}{\log^p 4}\frac{1}{k^p},$$

so, in the case $n=2$, we have

$$\begin{aligned}(2)\quad\int_{[0,1]^2}|Dh|^2\log^s(e+|Dh|)&=\sum_{k=1}^{\infty}\sum_{i=1}^{4^k}\int_{A_{ki}}|Dh|^2\log^s(e+|Dh|)\\&\le\sum_{k=1}^{\infty}4^k\frac14\sigma^{2k-2}\frac{(\tau_1\cdots\tau_k)^2}{\sigma^{2k}}\log^s(e+\frac{\tau_1\cdots\tau_k}{\sigma^k})\\&=\sum_{k=1}^{\infty}\frac{1}{4\sigma^2k^{2p}\log^{2p}4}\log^s\bigl(e+\frac{1}{(2\sigma)^kk^p\log^p 4}\bigr)\lesssim\sum_{k=1}^{\infty}k^{s-2p}<\infty,\end{aligned}$$

when $s+1<2p$.

3. Proofs

Clearly, we may assume in the rest of this note that Ω is an open and connected subset of $\mathbb{R}^2$. We begin with the following lemma.

Lemma 1. *Let $f\colon\Omega\to f(\Omega)\subset\mathbb{R}^2$ be a homeomorphism, $f\in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$. Then there exists a set $F\subset f(\Omega)$ such that $\mathcal{H}^{3/2}(F)=0$ and for all $y\in f(\Omega)\setminus F$ there exist constants $C_y>0$ and $r_y>0$ such that*

$$(3)\qquad \operatorname{diam}(f^{-1}(B(y,r)))\le C_y r^{1/2},$$

for all $0<r<r_y$.

Proof. First, note that by Theorem 1.2 in [5], f^{-1} is in $BV_{\mathrm{loc}}(f(\Omega))$. Next, fix $y \in f(\Omega)$ and $r > 0$ such that $B(y, 3r) \subset f(\Omega)$. Let $Q(y,t)$ be the square, centered in y and having the edge length $2t$. As f^{-1} is a homeomorphism, for $t \in (r, 2r)$ we have

$$\begin{aligned}(4)\quad \operatorname{diam} f^{-1}(B(y,r)) < \operatorname{diam} f^{-1}(Q(y,t)) &\le \operatorname{diam} f^{-1}(\partial Q(y,t)) \\ &\le \operatorname{diam} f_1^{-1}(\partial Q(y,t)) + \operatorname{diam} f_2^{-1}(\partial Q(y,t)),\end{aligned}$$

where f_i^{-1}, $i = 1,2$, denotes the i-th component function of f^{-1}. Integrating this inequality over the interval $[r, 2r]$ with respect to t, we obtain

$$(5)\qquad r \operatorname{diam} f^{-1}(B(y,r)) < \sum_{i=1}^{2} \int_{[r,2r]} \operatorname{diam} f_i^{-1}(\partial Q(y,t))dt.$$

Let us consider the smooth approximation $f_i^\varepsilon = \eta_\varepsilon * f_i^{-1}$ of f_i^{-1}, $i = 1,2$, on the cube $Q(y, 2r)$. Here η_ε is a standard bump function. As f^{-1} is continuous, the convergence $f_i^\varepsilon \to f_i^{-1}$ is pointwise and uniform on each compact set $K \subset Q(y, 2r)$. So, for $t \in (r, 2r)$ and $i = 1,2$ we have $\operatorname{diam} f_i^{-1}(\partial Q(y,t)) = \lim_{\varepsilon\to 0} \operatorname{diam} f_i^\varepsilon(\partial Q(y,t))$. Put $a_i = y_i - 2r$, $b_i = y_i + 2r$, $i = 1,2$, where $y = (y_1, y_2)$. Fatou's Lemma implies

$$\begin{aligned}(6)\quad \int_{[r,2r]} \operatorname{diam} f_i^{-1}(\partial Q(y,t))dt &= \int_{[r,2r]} \lim_{\varepsilon\to 0} \operatorname{diam} f_i^\varepsilon(\partial Q(y,t))dt \\ &\le \liminf_{\varepsilon\to 0} \int_{[r,2r]} \operatorname{diam} f_i^\varepsilon(\partial Q(y,t))dt \\ \le \liminf_{\varepsilon\to 0} \int_{[r,2r]} \Bigg\{ \int_{[a_2,b_2]} \left| \frac{\partial f_i^\varepsilon}{\partial \xi}(y_1 - t, \xi) \right| d\xi &+ \int_{[a_2,b_2]} \left| \frac{\partial f_i^\varepsilon}{\partial \xi}(y_1 + t, \xi) \right| d\xi \\ + \int_{[a_1,b_1]} \left| \frac{\partial f_i^\varepsilon}{\partial \xi}(\xi, y_2 - t) \right| d\xi &+ \int_{[a_1,b_1]} \left| \frac{\partial f_i^\varepsilon}{\partial \xi}(\xi, y_2 + t) \right| d\xi \Bigg\} dt \\ = \liminf_{\varepsilon\to 0} \Bigg\{ \int_{[a_1,y_1-r]\times[a_2,b_2]} \left| \frac{\partial f_i^\varepsilon}{\partial x_2}(x) \right| dx &+ \int_{[y_1+r,b_1]\times[a_2,b_2]} \left| \frac{\partial f_i^\varepsilon}{\partial x_2}(x) \right| dx \\ + \int_{[a_1,b_1]\times[a_2,y_2-r]} \left| \frac{\partial f_i^\varepsilon}{\partial x_1}(x) \right| dx &+ \int_{[a_1,b_1]\times[y_2+r,b_2]} \left| \frac{\partial f_i^\varepsilon}{\partial x_1}(x) \right| dx \Bigg\} \\ &\le \liminf_{\varepsilon\to 0} \sum_{j=1}^{2} \int_{Q(y,2r)} \left| \frac{\partial f_i^\varepsilon}{\partial x_j}(x) \right| dx,\end{aligned}$$

for $i = 1,2$. Let us show that

$$(7)\qquad \int_{Q(y,2r)} \left| \frac{\partial f_i^\varepsilon}{\partial x_j}(x) \right| dx \le ||Df_i^{-1}||(Q(y,2r))$$

for $i, j = 1, 2$. Given $\varphi \in C_0^1(Q(y, 2r))$, $|\varphi| \le 1$, we may write

(8)
$$\begin{aligned}\int_{Q(y,2r)} \frac{\partial f_i^\varepsilon}{\partial x_j}\varphi dx &= -\int_{Q(y,2r)} f_i^\varepsilon \frac{\partial \varphi}{\partial x_j} dx = -\int_{Q(y,2r)} (\eta_\varepsilon * f_i^{-1}) \frac{\partial \varphi}{\partial x_j} dx \\ &= -\int_{Q(y,2r)} f_i^{-1} \frac{\partial \eta_\varepsilon * \varphi}{\partial x_j} dx \le ||Df_i^{-1}||(Q(y, 2r)).\end{aligned}$$

This implies (7), and combining it with (5) and (6), we finally obtain

$$\operatorname{diam} f^{-1}(B(y, r)) < \frac{2}{r}(||Df_1^{-1}||(Q(y, 2r)) + ||Df_2^{-1}||(Q(y, 2r)))$$

for all $y \in f(\Omega)$ and $r > 0$ such that $B(y, 3r) \subset f(\Omega)$. That is, the inequality (3) holds for all $y \in f(\Omega)$ such that

$$\frac{||Df_k^{-1}||(Q(y, 2r))}{r^{3/2}} < M_y \tag{9}$$

is valid for $k = 1, 2$, all small enough $r > 0$ and some constant M_y, depending on y. Let F_1 be the set of those y for which (9) does not hold for $k = 1$. Let $K \subset f(\Omega)$ be a compact set and fix some $\delta > 0$ such that $\operatorname{dist}(K, \partial f(\Omega)) > \delta$. For every $i \in \mathbb{N}$ and every $y \in F_1 \cap K$ there exists $r_{i,y} < \delta\sqrt{2}/20$ such that $||Df_1^{-1}||(Q(y, 2r_{i,y})) \ge i(r_{i,y})^{3/2}$. Consider the collection of all balls $\mathcal{B}_i = \{B(y, 2\sqrt{2}r_{i,y})\colon y \in F_1 \cap K\}$ for every $i \in \mathbb{N}$. Using Vitali's covering theorem, we obtain for every $i \in \mathbb{N}$ a countable subcollection of disjoint balls $B_{i,j}$, $j = 1, 2, \ldots$, centered in $F_1 \cap K$, having radii $2\sqrt{2}r_j^i < \delta/5$ and with $5B_{i,j}$ covering $F_1 \cap K$. As $Q(y, 2r_j^i) \subset B_{i,j}$, we have

(10)
$$\begin{aligned}\mathcal{H}_\delta^{3/2}(F_1 \cap K) &\le \sum_{j=1}^\infty (10\sqrt{2}r_j^i)^{3/2} \le \frac{(10\sqrt{2})^{3/2}}{i} \sum_{j=1}^\infty ||Df_1^{-1}||(Q(y, 2r_j^i)) \\ &\le \frac{(10\sqrt{2})^{3/2}}{i} \sum_{j=1}^\infty ||Df_1^{-1}||(B_{i,j}) \le \frac{(10\sqrt{2})^{3/2}||Df_1^{-1}||(K + \delta/5)}{i}\end{aligned}$$

for all $i \in \mathbb{N}$. Letting $i \to \infty$ and $\delta \to 0$, we obtain $\mathcal{H}^{3/2}(F_1 \cap K) = 0$. □

The previous lemma implies the following result.

Lemma 2. *Let $E \subset \Omega$ and $f\colon \Omega \to f(\Omega) \subset \mathbb{R}^2$ be a homeomorphism of the class $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$. Then there exists a decomposition $f(E) = \bigcup_{i=0}^\infty F_i$, where $\mathcal{H}^{3/2}(F_0) = 0$ and for each F_i, $i = 1, 2, \ldots$, there exist constants $C_i < \infty$ and $r_i > 0$ such that*

$$f^{-1}(F_i + r) \subset E + C_i r^{1/2}$$

for every $r \in (0, r_i)$.

Proof. We choose $F_0 = F$, where F is the set from the previous lemma for F_0. Moreover, by this lemma we may represent the set $f(E)$ as

$$(11) \quad f(E) = F_0 \cup \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \{y \in f(E) \mid \operatorname{diam}(f^{-1}(B(y,r))) \leq kr^{\frac{1}{2}} \text{ for all } r \in (0, \frac{1}{j})\}.$$

So, putting $C_i = C_{i(j,k)} = k$ and $r_i = r_{i(j,k)} = \frac{1}{j}$, we complete the proof. □

Proof of Theorem 1. As $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ is a homeomorphism, its Jacobian J_f is either non-negative almost everywhere in Ω or non-positive almost everywhere in Ω [13]. We may assume that $J_f \geq 0$ almost everywhere in Ω. Recalling that $|Df|^2 \log^{\lambda-1}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega)$, by Corollary 9.1 in [9], we have $J_f \log^{\lambda}(e + J_f) \in L^1_{\mathrm{loc}}$. Next, as $\dim_{\mathcal{M}}(E) < 2$, there exist constants $C, \varepsilon > 0$ and a sequence of numbers r_j, $j = 1, 2, \ldots$, tending to zero as $j \to \infty$, such that $\mathcal{L}^2(E + r_j) \leq C r_j^{\varepsilon}$ for all $j = 1, 2, \ldots$. By Lemma 2, we have $f(E) = \bigcup_{i=0}^{\infty} F_i$, where $\mathcal{H}^{3/2}(F_0) = 0$ and $f^{-1}(F_i + R_{i,j}) \subset E + r_j$ for all large enough j ($j \geq j_i$ for some $j_i \in \mathbb{N}$). Here $R_{i,j} = (r_j/C_i)^2$ and C_i are the constants from Lemma 2. It suffices to show that $\mathcal{H}^h(F_i) = 0$ for all $i \in \mathbb{N}$. We use the fact that $\mathcal{L}^2(f(A)) \leq \int_A J_f$ for each open $A \subset \Omega$ [11, Lemma 3.2]. Thus, for a fixed $i \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{L}^2(F_i + R_{i,j}) &\leq \int_{f^{-1}(F_i + R_{i,j})} J_f(x)dx \leq \int_{E + r_j} J_f(x)dx \\
&\leq \int_{\{x \in E + r_j :\, J_f(x) < r_j^{-\varepsilon/2}\}} J_f + \int_{\{x \in E + r_j :\, J_f(x) \geq r_j^{-\varepsilon/2}\}} J_f \\
&\leq r_j^{-\varepsilon/2} \mathcal{L}^2(E + r_j) + \log^{-\lambda}(e + r_j^{-\varepsilon/2}) \int_{E + r_j} J_f \log^{\lambda}(e + J_f) \\
&\leq C r_j^{\varepsilon/2} + M(r_j) \log^{-\lambda} \frac{1}{r_j} \qquad (12)
\end{aligned}$$

for big enough j, where $M(r) \to 0$ as $r \to 0$. In other words, $\mathcal{L}^2(F_i + R_{i,j}) = o(\log^{-\lambda} \frac{1}{r_j})$ as $j \to \infty$. Using the Besicovitch covering theorem, for each large enough $j \in \mathbb{N}$, we can cover the set F_i with N countable families of pairwise disjoint balls centered in F_i and of radius $R_{i,j}$ (N is independent of both i and j). It is obvious that each of these families is finite. Let $l_{i,j}$ denote the total number of covering balls. We have $\mathcal{L}^2(F_i + R_{i,j}) \geq C l_{i,j} R_{i,j}^2$, where C is a constant independent of i and j. So, for each fixed $i \in \mathbb{N}$ and all big enough $j \geq j_i$ we have

$$\mathcal{H}^h_{R_{i,j}}(F_i) \leq l_{i,j} R_{i,j}^2 \log^{\lambda}(1/R_{i,j}) \leq \frac{2^{\lambda}}{C} \mathcal{L}^2(F_i + R_{i,j}) \log^{\lambda}(C_i/r_j),$$

and thus (12) shows that $\mathcal{H}^h(F_i) = 0$. It follows that $\mathcal{H}^h(F) = 0$. $\square$

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