PEANO CUBES WITH DERIVATIVES IN A LORENTZ SPACE

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ABSTRACT. We show that for any length-compact metric space Y and any $1 < q \leq n$, there is a continuous surjection in a suitably defined Sobolev-Lorentz space $W^{1,n,q}([0,1]^n, Y)$. On the other hand, we show that mappings in the space $W^{1,n,1}([0,1]^n, Y)$ satisfy condition (N). This implies that the target Y can be at most n-dimensional.

1. INTRODUCTION

The classical Hahn-Mazurkiewicz Theorem characterizes continuous images of the unit interval as precisely the topological spaces that are compact, connected, locally connected, and metrizable. This theorem has been updated by Hajłasz and Tyson to the differential setting, using the language of Sobolev mappings with metric space targets [3].

Theorem 1.1 (Hajłasz-Tyson)**.** *For all $n \geq 2$, each length-compact metric space Y is the image of a continuous surjection in the Sobolev space $W^{1,n}([0,1]^n; Y)$.*

A metric space is *length-compact* if it is compact when equipped with the associated path metric. This condition, though not necessary, can be considered as a differentiable version of the connectedness conditions imposed in the classical Hahn-Mazurkiewicz Theorem. The collection of length-compact metric spaces includes infinite-dimensional spaces such as the Hilbert Cube.

This paper gives a sharp version of Theorem 1.1 by refining the integrability condition on the gradient of the mappings in question to the Lorentz scale.

Theorem 1.2. *For all $n \geq 2$ and $1 < q \leq n$, each length-compact metric space Y is the image of a continuous surjection in the Sobolev-Lorentz space $W^{1,n,q}([0,1]^n; Y)$. However, for all $n \geq 1$, if Y is the image of a continuous surjection in the Sobolev-Lorentz space $W^{1,n,1}([0,1]^n; Y)$, then the Hausdorff dimension of Y is at most n.*

As the Lorentz space $L^{1,1}$ coincides with the Lebesgue space L^1, Theorem 1.2 places the failure of Theorem 1.1 when $n = 1$ in a larger context. Our results may also be applied to Carnot groups, as in [3].

To prove the first part of Theorem 1.2, we generalize the approach of [3]. The proof of the second part is based on the following theorem. The precise meaning of the local integrability and the weak partial derivatives of a metric space-valued mapping is explained in Section 4. Here $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure.

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Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a domain, and let Y be a separable metric space. If $f\colon \Omega \to Y$ is a locally integrable and continuous mapping with weak partial derivatives in $L^{n,1}$, then f satisfies condition (N), meaning that for each set $E \subseteq \Omega$ with $\mathcal{H}^n(E) = 0$, the image satisfies $\mathcal{H}^n(f(E)) = 0$.*

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2. Lorentz spaces

We now define and discuss the Lorentz spaces, which refine the Lebesgue spaces. Lorentz spaces are often used in interpolation theorems; see [1, Chapter 4.4] for more information.

We consider functions from a totally σ-finite measure space (X, μ) to a Banach space $(V, ||\ ||_V)$. An almost-everywhere defined function $f\colon X \to V$ is said to be *Bochner measurable* if it is measurable in the usual sense, and essentially separably valued, i.e., there is a set $E \subseteq X$ of measure 0 such that $f(X \backslash E)$ is a separable subset of V. For more information on Bochner measurability and integrability, see [5]. We assume familiarity with the Bochner-Lebesgue spaces $L^Q(X;V)$ and $L^Q_{\mathrm{loc}}(X;V)$.

For a Bochner measurable function $f\colon X \to V$, we define the distribution function $\omega_f\colon [0,\infty) \to [0,\infty]$ of f by

$$\omega_f(\alpha) = \mu(\{x \in X : ||f(x)||_V > \alpha\}).$$

The non-increasing rearrangement $f^*\colon [0,\infty) \to [0,\infty]$ of f is given by

$$f^*(t) = \inf\{\alpha \geq 0 : \omega_f(\alpha) \leq t\}.$$

Let $1 \leq Q < \infty$ and $1 \leq q \leq Q$. The (Q,q)-Lorentz norm of f is given by

$$||f||_{L^{Q,q}} = \left(\int_0^\infty t^{-1} \left(t^{1/Q} f^*(t)\right)^q \, dt\right)^{1/q}. \tag{2.1}$$

The (Q,q)-Lorentz space $L^{Q,q}(X;V)$ is the set of equivalence classes of Bochner measurable functions $f\colon X \to V$ with $||f||_{L^{Q,q}} < \infty$, where two functions are equivalent if they agree almost everywhere. By [1, Theorem 4.4.3], the normed vector space $L^{Q,q}(X;\mathbb{R})$ is a Banach space. The fact that $L^{Q,q}(X;V)$ is a Banach space follows similarly, essentially because we consider only the value of the norm of f in the definition of the distribution function.

The following statment gives the basic relationships between the Lorentz spaces with different indices. The proof in the case that $V = \mathbb{R}$ can be found at [1, Propositions 2.1.8 and 4.4.2]. The general case follows similarly.

Proposition 2.1. *For all $1 \leq r \leq q \leq Q$, there is a constant c depending only on Q, q, and r, such that for all Bochner measurable functions $f\colon X \to V$,*

$$||f||_{L^{Q,q}} \leq c||f||_{L^{Q,r}}.$$

In particular, there is a bounded embedding $L^{Q,r}(X;V) \hookrightarrow L^{Q,q}(X;V)$. Moreover, $L^{Q,Q}(X;V) = L^Q(X;V)$ and

$$||f||_{L^{Q,Q}} = ||f||_{L^Q}.$$

Corollary 2.2. *For all $1 \leq q \leq Q$, there is a bounded embedding $L^{Q,q}(X;V) \hookrightarrow L^Q(X;V)$.*

Remark 2.3. If $f, g\colon X \to V$ are Bochner measurable functions, and $||f(x)||_V \leq ||g(x)||_V$ for almost every $x \in X$, then

$$||f||_{L^{Q,q}} \leq ||g||_{L^{Q,q}}.$$

This property is true of every Banach function norm; see [1, Chapter 1 and Proposition 4.4.2].

We now discuss a characterization of real-valued Lorentz spaces given in [6]. We say that a *gauge* is a non-increasing function $\phi\colon (0,\infty) \to [0,\infty)$. Given $1 \leq q \leq Q$ and a gauge ϕ, we define functions $T_\phi^{Q,q}, F_\phi^{Q,q}\colon [0,\infty) \to [0,\infty)$ by

$$T_\phi^{Q,q}(r) = \begin{cases} r^{q-1}\phi^{q/Q}(r) & r > 0, \\ 0 & r = 0, \end{cases} \quad \text{and} \quad F_\phi^{Q,q}(r) = \begin{cases} r^q \phi^{(q-Q)/Q}(r) & r > 0, \\ 0 & r = 0. \end{cases}$$

We denote by $\mathcal{A}^{Q,q}$ the collection of gauges satisfying

$$\int_0^\infty T_\phi^{Q,q}(r)\; dr < \infty.$$

The following theorem states that the Lorentz spaces are determined by a family of Orlicz conditions [6, Corollary 2.4].

Theorem 2.4 (Kauhanen-Koskela-Malý)**.** *A measurable function $f\colon X \to \mathbb{R}$ is in $L^{Q,q}(X)$ if and only if there is $\phi \in \mathcal{A}^{Q,q}$ such that $\phi(|f(x)|) > 0$ for almost every $x \in X$ with $|f(x)| > 0$, and*

$$\int_X F_\phi^{Q,q}(|f(x)|)\; d\mu(x) < \infty.$$

In addition, there is a constant C depending only on ϕ, Q, and q such that

$$||f||_{L^{Q,q}}^Q \leq C \int_X F_\phi^{Q,q}(|f(x)|)\; d\mu(x). \tag{2.2}$$

3. Sobolev-Lorentz spaces

In this section, we follow the approach of [3] in defining Sobolev-Lorentz spaces with Banach space targets. We first give a definition based on integration by parts, and then an equivalent definition based on the approach of Reshetnyak [7]. The reason for this is that the second definition is more suited to metric space targets, but the computations are more difficult.

From now on, we let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, equipped with the Hausdorff measure $\mathcal{H}$. Moreover, we let V be a Banach space and $1 \leq q \leq n$.

Let $f \in L^1_{\text{loc}}(\Omega; V)$. A Bochner measurable function $v\colon \Omega \to V$ is called an i^{th} *weak partial derivative* of f if for every $\phi \in \mathcal{C}_0^\infty(\Omega)$,

$$\int_\Omega \frac{\partial \phi}{\partial x_i} f\; d\mathcal{H}^n = -\int_\Omega \phi v\; d\mathcal{H}^n.$$

If an i^{th} weak partial derivative of f exists, then it is unique, and we denote it by $\partial_i f$.

Definition 3.1. A function $f\colon \Omega \to V$ belongs to the *Sobolev-Lorentz space* $W^{1,n,q}(\Omega; V)$ if:

(i) $f \in L^n(\Omega; V)$,
(ii) f has weak partial derivatives $\partial_1 f, \dots, \partial_n f$ in the space $L^{n,q}(\Omega; V)$.

The usual Sobolev space $W^{1,n}(\Omega; V)$ is obtained by replacing the space $L^{n,q}(\Omega; V)$ in (ii) with the larger space $L^n(\Omega; V)$. We denote by $\underline{W}^{1,n,q}(\Omega; V)$ the space obtained by replacing (i) by the weaker condition that $f \in L^1_{\mathrm{loc}}(\Omega; V)$.

If a function $f\colon \Omega \to V$ has weak partial derivatives, we define the *weak gradient* $\nabla f\colon \Omega \to V^n$ by

$$\nabla f(x) = (\partial_1 f(x), \dots, \partial_n f(x)).$$

Remark 3.2. We will always employ the following norm on V^n when considering the gradient of a map:

$$||\nabla f(x)||_{V^n} := \max_{i=1,\dots,n} ||\partial_i f(x)||_V. \tag{3.1}$$

This is potentially confusing when dealing with a real valued function $f\colon \Omega \to \mathbb{R}$, because we employ the standard 2-norm when considering a point $x \in \Omega \subseteq \mathbb{R}^n$, but not when considering the gradient $\nabla f(x) \in \mathbb{R}^n$ at a point $x \in \Omega$. However, if we did not make this convention, we would have a factor of $\sqrt{n}$ appearing in many formulas.

Proposition 3.3. *Let $f \in \underline{W}^{1,n,q}(\Omega, V)$. Then*

$$\max_{i=1,\dots,n} ||\partial_i f||_{L^{n,q}} \leq ||\nabla f||_{L^{n,q}} \leq n^{(1/n+1/q)} \max_{i=1,\dots,n} ||\partial_i f||_{L^{n,q}}.$$

Proof. The definitions imply that for $\alpha \geq 0$,

$$\omega_{\nabla f}(\alpha) = \mathcal{H}^n(\{x \in \Omega : ||\nabla f(x)||_{V^n} > \alpha\}) \leq n \max_i \ \omega_{\partial_i f}(\alpha).$$

As a result, for all $t \geq 0$,

$$(\nabla f)^*(t) \leq \inf\{\alpha \geq 0 : \max_i \ \omega_{\partial_i f}(\alpha) \leq t/n\}. \tag{3.2}$$

Let $\epsilon > 0$, and set

$$\beta = \max_i \ (\partial_i f)^*(t/n) + \epsilon.$$

Then for each index i,

$$\omega_{\partial_i f}(\beta) \leq t/n,$$

and so

$$\inf\{\alpha \geq 0 : \max_i \ \omega_{\partial_i f}(\alpha) \leq t/n\} \leq \beta.$$

Letting ϵ tend to 0, inequality (3.2) shows that for all $t \geq 0$

$$(\nabla f)^*(t) \leq \max_i \ (\partial_i f)^*(t/n).$$

We now see that

$$||\nabla f||_{L^{n,q}} \leq n^{1/n} \left(\int_0^{\mathcal{H}^n(\Omega)/n} \max_i \left\{ t^{-1} \left(t^{1/n} (\partial_i f)^*(t) \right)^q \right\} \ dt \right)^{1/q}. \tag{3.3}$$

It is easy to see that for functions $h_1, \dots, h_n\colon \mathbb{R} \to \mathbb{R}$,

$$\int \max_i h_i(t) \ dt \leq n \max_i \int h_i(t) \ dt.$$

Thus (3.3) implies the second inequality in the statement. The first inequality follows easily from the fact that

$$\omega_{\nabla f}(\alpha) \geq \omega_{\partial_i f}(\alpha) \quad \text{and} \quad (\nabla f)^*(t) \geq (\partial_i f)^*(t)$$

for all $i = 1, \dots, n$. $\square$

By Proposition 3.3, we may make the following definition. For $f \in W^{1,n,q}(\Omega, V)$, set

$$||f||_{W^{1,n,q}} := ||f||_{L^n} + ||\nabla f||_{L^{n,q}}.$$

The space $W^{1,n,q}(\Omega; V)$ is a Banach space. The proof of this fact mimics the proof that $W^{1,n}(\Omega; V)$ is a Banach space, and uses the fact that $L^{n,q}(\Omega; V)$ is a Banach space.

We now introduce the Reshetnyak approach to Sobolev-Lorentz spaces.

Definition 3.4. The *Reshetnyak-Sobolev-Lorentz space* $R^{1,n,q}(\Omega; V)$ is the class of functions $f \in L^n(\Omega, V)$ for which there exists a non-negative function $g \in L^{n,q}(\Omega; \mathbb{R})$ such that for every $v^* \in V^*$ with $||v^*|| \leq 1$, we have

(i) $\langle v^*, f\rangle \in W^{1,n,q}(\Omega; \mathbb{R})$,
(ii) $||\nabla\langle v^*, f\rangle||_{\mathbb{R}^n} \leq g$ a.e.

We denote by $\underline{R}^{1,n,q}(\Omega; V)$ the space obtained if we replace the condition that $f \in L^n(\Omega; V)$ with the condition that $f \in L^1_{\mathrm{loc}}(\Omega; V)$, and replace (i) with the condition that $\langle v^*, f\rangle \in \underline{W}^{1,n,q}(\Omega; \mathbb{R})$.

A function g satisfying condition (ii) above is called a *Reshetnyak upper gradient* of f. We equip $R^{1,n,q}(\Omega; V)$ with the norm

$$||f||_{R^{1,n,q}} = ||f||_{L^n} + \inf ||g||_{L^{n,q}},$$

where the infimum is taken over all Reshetnyak upper gradients g of f.

Using an analogous definition of the Reshetnyak-Sobolev space $R^{1,n}(\Omega; V)$, it was shown in [3, Theorem 2.14] that it coincides with the space $W^{1,n}(\Omega; V)$ in the case that V is the dual of a separable Banach space. The proof relies only on pointwise estimates, and it passes without difficulty to the Lorentz setting.

Proposition 3.5. *Suppose that V is the dual of a separable Banach space. Then $f \in \underline{W}^{1,n,q}(\Omega; V)$ if and only $f \in \underline{R}^{1,n,q}(\Omega; V)$. Moreover, if f is in either of these spaces, then*

$$||\nabla f||_{L^{n,q}} = \inf ||g||_{L^{n,q}},$$

where the infimum is taken over all Reshetnyak upper gradients g of f.

Remark 3.6. Proposition 3.5 implies that the identity map between $R^{1,n,q}(\Omega; V)$ and $W^{1,n,q}(\Omega; V)$ is isometric. The factor of $\sqrt{n}$ appearing in [3, Theorem 2.14] does not occur here because of our choice of norm on V^n; see Remark 3.2.

The following lemma is crucial to the construction of space filling mappings with controlled derivatives. The proof is essentially the same as the proof of [3, Lemma 2.15], but we include it as it is brief and illustrates the minor differences between our definitions and those in [3].

Lemma 3.7. *Suppose that Ω is bounded and that V is the dual of a separable Banach space. If $f \in \underline{W}^{1,n,q}(\Omega; \mathbb{R})$ satisfies $f(\Omega) \subseteq [a, b] \subseteq \mathbb{R}$, and $\gamma\colon [a, b] \to V$ is a 1-Lipschitz map, then $\gamma \circ f \in \underline{W}^{1,n,q}(\Omega; V)$ and*

$$||\nabla(\gamma \circ f)||_{L^{n,q}} \leq ||\nabla f||_{L^{n,q}}.$$

Proof. It is clear that $\gamma \circ f \in L^1_{\mathrm{loc}}(\Omega; V)$. Let $v^* \in V^*$ with $||v^*|| \leq 1$, and define $\phi\colon [a,b] \to \mathbb{R}$ by

$$\phi(t) = \langle v^*, \gamma(t) \rangle.$$

Then ϕ is a 1-Lipschitz function, and for each $x \in \Omega$,

$$\phi \circ f(x) = \langle v^*, \gamma \circ f(x) \rangle.$$

A standard result [9, Theorem 2.1.11] now implies that as Ω is bounded, $\langle v^*, \gamma \circ f \rangle$ is in $L^1_{\mathrm{loc}}(\Omega; \mathbb{R})$ and has a weak gradient satisfying

$$||\nabla \langle v^*, \gamma \circ f(x) \rangle||_{\mathbb{R}^n} \leq ||\nabla f(x)||_{\mathbb{R}^n}$$

for almost every point $x \in \Omega$. Thus Proposition 3.3 shows that $\gamma \circ f \in R^{1,n,q}(\Omega, V)$ and that $||\nabla f||_{\mathbb{R}^n}$ is a Reshetnyak upper gradient of $\gamma \circ f$. Proposition 3.5 now yields the desired result. $\square$

4. Mappings to a metric space

We define Sobolev-Lorentz mappings with metric targets in terms of isometric embeddings into a Banach space. In this section, we let (Y, d) be a separable metric space. Recall that every separable metric space may be isometrically embedded in the Banach space l^∞ [4, Exercise 12.6], which is the dual of the separable space l^1. We begin by defining integrability conditions for a metric space-valued mapping.

Definition 4.1. Let $1 \leq p \leq \infty$. A measurable mapping $f\colon \Omega \to Y$ is in the space $\mathcal{L}^p(\Omega; Y)$ if there is an isometric embedding $\iota\colon Y \hookrightarrow l^\infty$ such that $\iota \circ f \in L^p(\Omega; l^\infty)$. Similarly, $f \in \mathcal{L}^p_{\mathrm{loc}}(\Omega; Y)$ if there is an isometric embedding $\iota\colon Y \hookrightarrow l^\infty$ such that $\iota \circ f \in L^p_{\mathrm{loc}}(\Omega; l^\infty)$.

Remark 4.2. The condition that $\iota \circ f \in L^p_{\mathrm{loc}}(\Omega; l^\infty)$ is independent of the choice of the isometric embedding ι. Similarly, if Ω has finite measure, then the condition that $\iota \circ f \in L^p(\Omega; l^\infty)$ is independent of the choice of ι. This need not be true if Ω has infinite measure.

Definition 4.3. We say that a map $f\colon \Omega \to Y$ belongs to the *Sobolev-Lorentz space* $W^{1,n,q}(\Omega; Y)$ if there is an isometric embedding $\iota\colon Y \hookrightarrow l^\infty$ so that $\iota \circ f \in W^{1,n,q}(\Omega; l^\infty)$. Similarly, f belongs to the space $\underline{W}^{1,n,q}(\Omega; Y)$ if there is an isometric embedding $\iota\colon Y \hookrightarrow l^\infty$ so that $\iota \circ f \in \underline{W}^{1,n,q}(\Omega; l^\infty)$.

The following definition makes precise the statement of Theorem 1.3 in the introduction.

Definition 4.4. A mapping $f\colon \Omega \to Y$ is said to *be locally integrable and have weak partial derivatives in* $L^{n,q}$ if it is in the space $\underline{W}^{1,n,q}(\Omega; Y)$.

Note that $f \in W^{1,n,q}(\Omega; Y)$ implies that $f \in \mathcal{L}^n(\Omega; Y)$, and that $f \in \underline{W}^{1,n,q}(\Omega; Y)$ implies that $f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega; Y)$. It is possible that given two isometric embeddings $\iota, \iota'\colon Y \hookrightarrow l^\infty$, one of $\iota \circ f$ and $\iota' \circ f$ is in $W^{1,n,q}(\Omega; l^\infty)$ while the other is not. In particular, if Y happens to be a Banach space, the space given by Definition 4.3 may be larger than the space given by Definition 3.1. The following proposition indicates that this occurs only because of the integrability of the mappings themselves, not the integrability of the gradients.

Proposition 4.5. *Let $\iota, \iota' \colon Y \hookrightarrow l^\infty$ be isometric embeddings and let $f\colon \Omega \to Y$ be a measurable function. Then $\iota\circ f \in \underline{W}^{1,n,q}(\Omega; l^\infty)$ if and only if $\iota'\circ f \in \underline{W}^{1,n,q}(\Omega; l^\infty)$. Moreover,*

$$||\nabla(\iota' \circ f)||_{L^{n,q}} = ||\nabla(\iota \circ f)||_{L^{n,q}}.$$

Remark 4.6. A simple modification of the proof shows that there is a vector $v_0 \in l^\infty$ depending only ι and ι' such that if we set $\iota'_0 = \iota' - v_0$, the following statement holds: $\iota \circ f \in W^{1,n,q}(\Omega; l^\infty)$ if and only if $\iota'_0 \circ f \in W^{1,n,q}(\Omega; l^\infty)$. If the measure of Ω is finite, we may take $v_0 = 0$.

We will need the following lemma, which is an analogue of [3, Proposition 2.16]. The proof given there is valid in this setting, with the obvious modifications.

Lemma 4.7. *Suppose that $f \in \underline{R}^{1,n,q}(\Omega, V)$ and that g is a Reshetnyak upper gradient of f. If $\phi\colon V \to \mathbb{R}$ is a 1-Lipschitz function, then $\phi\circ f \in \underline{W}^{1,n,q}(\Omega; \mathbb{R})$, and for almost every $x \in \Omega$,*

$$||\nabla(\phi \circ f)(x)||_{\mathbb{R}^n} \le g(x). \tag{4.1}$$

Conversely, suppose that $f \in L^1_{\text{loc}}(\Omega; V)$ and that $g \in L^{n,q}(\Omega; \mathbb{R})$ is a non-negative function with the property that for every 1-Lipschitz function $\phi\colon V \to \mathbb{R}$, the function $\phi \circ f$ is in the space $\underline{W}^{1,n,q}(\Omega; \mathbb{R})$, and (4.1) holds for almost every $x \in \Omega$. Then $f \in \underline{R}^{1,n,q}(\Omega; V)$ and g is a Reshetnyak upper gradient of f.

Remark 4.8. A statement similar to Lemma 4.7 is true for the space $R^{1,n,q}(\Omega; V)$, with the obvious additional assumptions on the integrability of f.

Proof of Proposition 4.5. We first mention that as f is measurable and Y is separable, both $\iota\circ f$ and $\iota'\circ f$ are Bochner measurable. The map $\iota'\circ\iota^{-1}\colon \iota(Y) \to \iota'(Y)$ is an isometry, and so by the McShane extension theorem [4, Exercise 6.7], there is a 1-Lipschitz mapping $I\colon l^\infty \to l^\infty$ such that $I|_{\iota(Y)} = \iota'\circ\iota^{-1}$.

Assume that $\iota\circ f \in \underline{W}^{1,n,q}(\Omega; l^\infty)$, and hence that $\iota\circ f \in L^1_{\text{loc}}(\Omega; l^\infty)$. For each $x \in \Omega$,

$$||\iota' \circ f(x)||_{l^\infty} = ||I\circ\iota\circ f(x) - I(0)||_{l^\infty} + ||I(0)||_{l^\infty} \le ||\iota\circ f(x)||_{l^\infty} + ||I(0)||_{l^\infty},$$

showing that $\iota'\circ f \in L^1_{\text{loc}}(\Omega; l^\infty)$ as well; see Remarks 4.2 and 4.6.

Proposition 3.5 implies that $\iota \circ f \in \underline{R}^{1,n,q}(\Omega; l^\infty)$. Let $g \in L^{n,q}(\Omega, \mathbb{R})$ be a Reshetnyak upper gradient of $\iota\circ f$. By Lemma 4.7, if $\phi\colon l^\infty \to \mathbb{R}$ is 1-Lipschitz map, then $\phi\circ\iota\circ f \in \underline{W}^{1,n,q}(\Omega, \mathbb{R})$, and for almost every $x \in \Omega$,

$$||\nabla(\phi\circ\iota\circ f)(x)||_{\mathbb{R}^n} \le g(x).$$

Let $\psi\colon l^\infty \to \mathbb{R}$ be a 1-Lipschitz function. Then $\psi\circ I$ is also a 1-Lipschitz function. As

$$\psi\circ\iota'\circ f = \psi\circ I\circ\iota\circ f,$$

the previous paragraph shows that $\psi\circ\iota'\circ f \in \underline{W}^{1,n,q}(\Omega, \mathbb{R})$, and for almost every $x \in \Omega$,

$$||\nabla(\psi\circ\iota'\circ f)(x)||_{\mathbb{R}^n} \le g(x).$$

As ψ is arbitrary, Lemma 4.7 yields that $\iota'\circ f \in \underline{R}^{1,n,q}(\Omega; l^\infty)$, and that g is a Reshetnyak upper gradient of $\iota'\circ f$. By Proposition 3.5, $\iota'\circ f \in \underline{W}^{1,n,q}(\Omega; l^\infty)$, and

$$||\nabla(\iota'\circ f)||_{L^{n,q}} \le ||g||_{L^{n,q}}.$$

Taking the infimum over all Reshetnyak upper gradients g of $\iota \circ f$, and again applying Proposition 3.5, we see that

$$||\nabla(\iota' \circ f)||_{L^{n,q}} \leq ||\nabla(\iota \circ f)||_{L^{n,q}}.$$

Interchanging the roles of ι and ι' completes the proof. $\square$

5. The Sobolev-Lorentz capacity of a point

A key ingredient in the proof of Theorem 1.1 is the fact that a point in $\mathbb{R}^n$ has Sobolev n-capacity zero. To prove Theorem 1.2, we give a similar statement for a Sobolev-Lorentz (n,q)-capacity, $1 < q \leq n$.

Theorem 5.1. *Suppose that $1 < q \leq n$. For all $\epsilon, \tau > 0$, there is a Lipschitz function $\eta_{\epsilon,\tau} \colon \mathbb{R}^n \to [0,\infty)$ such that*

(i) $\operatorname{supp} \eta_{\tau,\epsilon} \subseteq B^n(0,\epsilon)$,
(ii) $\eta_{\tau,\epsilon} \equiv \tau$ *on a neighborhood of the origin, and*
(iii) $||\nabla \eta_{\tau,\epsilon}||_{L^{n,q}} < \epsilon$.

Proof. In the case $q = n$, Proposition 2.1 shows that the statement reduces to the Sobolev n-capacity case, which is discussed in [3]. Thus we assume that $1 < q < n$. We find a number α such that

$$\frac{n-q}{q} < \alpha < n - 1.$$

Define a gauge

$$\psi(r) = \begin{cases} r^{-n} \log^{\alpha n/(q-n)}(e + r) & r \geq 1, \\ \log^{\alpha n/(q-n)}(e+1) & r \leq 1. \end{cases}$$

An easy calculation shows that because $\alpha > (n-q)/q$, we have $\psi \in \mathcal{A}^{n,q}$.

Define $\eta \colon B^n(0, e^{-1}) \to [0,\infty]$ by

$$\eta(x) = \log\log(|x|^{-1}).$$

Then η is smooth except at the origin, and for all $x \neq 0$ and $i = 1, \dots, n$,

$$|\partial_i \eta(x)| = \frac{|x_i|}{|x|^2 \log(|x|^{-1})} \leq \frac{1}{|x| \log(|x|^{-1})}. \tag{5.1}$$

For $0 < s < t < \infty$, define $\eta_s^t \colon \mathbb{R}^n \to \mathbb{R}$ by

$$\eta_s^t(x) = \begin{cases} t - s & |x| \leq e^{-e^t}, \\ \eta(x) - s & e^{-e^t} \leq |x| \leq e^{-e^s}, \\ 0 & |x| \geq e^{-e^s}. \end{cases}$$

Then η_s^t is Lipschitz, and smooth except at the origin and on two spheres.

Using (5.1) and the fact that ψ is non-increasing, we calculate that for each $i = 1, \ldots, n$,

$$\begin{aligned}\int_{\mathbb{R}^n} F_\psi^{n,q}(|\partial_i \eta_s^t(x)|)\, d\mathcal{H}^n(x) =& \int_{e^{-e^t} \le |x| \le e^{-e^s}} |\partial_i \eta(x)|^q \left(\psi(|\partial_i \eta(x)|)\right)^{(q-n)/n}\, d\mathcal{H}^n(x) \\ \le& c(n) \int_{e^{-e^t}}^{e^{-e^s}} \frac{1}{r \log^n(r^{-1})} \log^\alpha \left(e + \frac{1}{r\log(r^{-1})}\right) dr \\ =& c(n) \int_{e^s}^{e^t} u^{-n} \log^\alpha \left(e + \frac{e^u}{u}\right) du \\ \le& c(\alpha, n) \int_{e^s}^{e^t} u^{\alpha - n}\, du.\end{aligned}$$

As $\alpha < n-1$, we may choose s so large that the above integral is as small as desired.

Set $\eta_{\epsilon,\tau} = \eta_s^{s+\tau}$. Then for sufficiently large values of s, conditions (i) and (ii) of the statement are satisfied. Theorem 2.4, the above discussion, and Proposition 3.3 show for sufficiently large s, condition (iii) is satisfied as well. □

6. Space filling maps

We have now presented all the tools needed to prove the first part Theorem 1.2. Given these tools, the proof is nearly identical to the proof of Theorem 1.1 presented in [3], and so we provide only a sketch.

Proof of the first part of Theorem 1.2. Let Y be a length-compact metric space. Then Y is compact and hence separable, and we may assume without loss of generality that $\operatorname{diam} Y = 1$. The assumption that Y is length-compact provides a sequence of finite subsets $\{y_0\} = Y_0 \subseteq Y_1 \subseteq \ldots$ with dense union, such that for each $l \in \mathbb{N}$, each point in Y_{l+1} may be joined to some point in Y_l by a path of length no greater than 2^{-l}.

Let $\iota\colon Y \hookrightarrow l^\infty$ be an isometric embedding. We construct a sequence of continuous mappings $\{f_l\colon [0,1]^n \to Y\}_{l\in\mathbb{N}}$ so that the sequence $\{\iota \circ f_l\}_{l\in\mathbb{N}}$ converges uniformly and in $W^{1,n,q}([0,1]^n; l^\infty)$. The limit mapping $\widetilde{f} \in W^{1,n,q}([0,1]^n; l^\infty)$ will be continuous and have image precisely $\iota(Y)$, and so $\iota^{-1} \circ \widetilde{f}$ yields the desired continuous surjection. The mapping f_0 is defined to be the constant map with value $y_0 \in Y$. The map f_1 agrees with f_0 except on a ball $B \subseteq (0,1)^n$ of arbitrarily small size. Choose a bijection $Y_1 \to \{x_i\}_{i\in I_1} \subseteq B$. Using Theorem 5.1, define f_1 to map a small ball around each x_i first to an interval, and then to a curve of length at most 1 connecting the corresponding point y_i to y_0. Lemma 3.7 guarantees that the weak gradient of $\iota \circ f_1$ has controlled Lorentz-norm. We continue this process iteratively near each x_i, and the geometrically decreasing length of the paths connecting Y_{l+1} to Y_l provides the desired convergence. □

Remark 6.1. The construction shows that the continuous surjection $f\colon [0,1]^n \to Y$ can be chosen to be constant off of an arbitrarily small set, and so in particular we may assume that it is constant near the boundary of $[0,1]^n$. This resolves the ambiguity inherent in our notation $W^{1,n,q}([0,1]^n; Y)$; the set $[0,1]^n$ is not a domain, and so *a priori* this space was not actually defined. The fastidious reader can understand $W^{1,n,q}([0,1]^n; Y)$ to be the collection of maps in $W^{1,n,q}((0,1)^n; Y)$ that are constant on a neighborhood of the boundary.

7. The dimension of the target and condition (N)

In this section we additionally allow the case $n = 1$. The requirement that a continuous mapping from Ω to $\mathbb{R}$ has a gradient in the Lorentz space $L^{n,1}(\Omega)$ is known to be a sharp condition guaranteeing a variety of desirable mapping properties, including differentiability almost everywhere [8] and condition (N) [6]. Theorem 1.3 can be seen as an extension of this principle to mappings with metric space targets. Our main tool is the following result [6, Corollary 2.4 and Theorem 3.2].

Theorem 7.1 (Kauhanen-Koskela-Malý)**.** *Suppose that* $u\colon \Omega \to \mathbb{R}$ *is a locally integrable and continuous function with weak partial derivatives in the space* $L^{n,1}(\Omega;\mathbb{R})$. *Then* u *satisfies the Rado-Reichelderfer condition, i.e., there is a weight* $\theta \in L^1(\Omega;\mathbb{R})$ *such that for any ball* B *compactly contained in* Ω,

$$\left(\sup_{x,y\in B} |u(x) - u(y)|\right)^n \leq \int_B \theta \; d\mathcal{H}^n. \tag{7.1}$$

Remark 7.2. It follows from the proof of Theorem 7.1 that if there is a function $g \in L^{n,1}(\Omega;\mathbb{R})$ such that for almost every point $x \in \Omega$,

$$||\nabla u(x)||_{\mathbb{R}^n} \leq g(x),$$

then the weight function θ in (7.1) can be chosen to depend only on g. In short, the weight does not depend on u itself, but rather on the magnitude of the weak gradient of u.

Proof of Theorem 1.3. As per Definition 4.4, we assume that f is continuous and in $\underline{W}^{1,n,1}(\Omega;Y)$. Let $\iota\colon Y \hookrightarrow l^\infty$ be an isometric embedding so that $\iota \circ f \in \underline{W}^{1,n,1}(\Omega;l^\infty)$; by Proposition 4.5, any such isometric embedding has this property. By Proposition 3.5, $\iota \circ f \in \underline{R}^{1,n,1}(\Omega;l^\infty)$. Accordingly, let $g \in L^{n,1}(\Omega;\mathbb{R})$ be a Reshetnyak upper gradient of $\iota \circ f$.

For each $k \in \mathbb{N}$, define $T_k\colon l^\infty \to \mathbb{R}$ to be the projection onto the k^{th} coordinate, which is a 1-Lipschitz function. By Lemma 4.7, for each $k \in \mathbb{N}$ the mapping $T_k \circ \iota \circ f\colon \Omega \to \mathbb{R}$ is continuous, locally integrable, and has weak partial derivatives in the space $L^{n,1}(\Omega;\mathbb{R})$. Moreover, for almost every $x \in \Omega$,

$$||\nabla(T_k \circ \iota \circ f)(x)||_{\mathbb{R}^n} \leq g(x).$$

It now follows from Theorem 7.1 and Remark 7.2 that there is a weight $\theta \in L^1(\Omega;\mathbb{R})$ such that for each ball B compactly contained in Ω,

$$\sup_{k\in\mathbb{N}} \left(\sup_{x,y\in B} |T_k \circ \iota \circ f(x) - T_k \circ \iota \circ f(y)|\right)^n \leq \int_B \theta \; d\mathcal{H}^n.$$

For any such ball B and any pair of points $x, y \in B$,

$$d_Y(f(x), f(y)) = ||\iota \circ f(x) - \iota \circ f(y)||_{l^\infty} = \sup_{k\in\mathbb{N}} |T_k \circ \iota \circ f(x) - T_k \circ \iota \circ f(y)|.$$

This implies that f also satisfies the Rado-Reichelderfer condition with weight θ. It is now an easy excercise to show that f satisfies condition (N). □

Remark 7.3. The restriction to separable targets is not necessary here, as any metric space Y can be isometrically embedded into the space $l^\infty(Y)$. If one defines Sobolev-Lorentz spaces based on embedding in this space, the same proof yields an analogous result.

We now give the proof of the second part of Theorem 1.2. The following lemma, which states that Sobolev functions on the unit cube are Lipschitz off small sets, is well known in the real-valued setting and is valid in a variety of settings. See, for example, [5, Sections 4 and 6].

Lemma 7.4. *Suppose that $f\colon [0,1]^n \to l^\infty$ is a continuous mapping with weak partial derivatives in $L^1([0,1]^n; l^\infty)$. Then there is a sequence $E_1 \supseteq E_2 \supseteq \dots$ of subsets of $[0,1]^n$ such that $\mathcal{H}^n(E_k) \le 1/k$ and $f|_{X\setminus E_k}$ is Lipschitz for each $k \in \mathbb{N}$.*

Proof. From the proof of [3, Proposition 2.3], we see that the mapping f has a Reshetnyak upper gradient $g \in L^1([0,1]^n)$. Moreover, the proof of [3, Lemma 2.13] implies that f is absolutely continuous on compact intervals in $[0,1]^n$, and for almost every pair of points $x, y \in [0,1]^n$,

$$||f(x) - f(y)||_{l^\infty} \le \int_0^1 g(x + t(y-x))|y-x|\ dt.$$

A standard argument now shows that the pair (f,g) satisfies a 1-Poincaré inequality. As in [5, Proposition 4.6] and [2, Theorem 3.2], a consequence of this is the following pointwise estimate: for almost every pair of points $x, y \in [0,1]^n$,

$$||f(x) - f(y)||_{l^\infty} \le C|x-y|\left(Mg(x) + Mg(y)\right),$$

where $C > 0$ depends only on n and M denotes the Hardy-Littlewood maximal operator. The result now follows from the fact that M maps L^1 to weak-L^1. □

Proof of the second part of Theorem 1.2. Let f be a continuous surjection in the space $W^{1,n,1}([0,1]^n; Y)$. By definition, we may find an isometric embedding $\iota\colon Y \hookrightarrow l^\infty$ such that $\iota \circ f \in W^{1,n,1}([0,1]^n; l^\infty)$; by Remark 4.6, any isometric embedding has this property. Corollary 2.2 and Hölder's inequality imply that $\iota \circ f$ has integrable weak partial derivatives. Thus by Lemma 7.4, may find a set E with $\mathcal{H}^n(E) = 0$ such that the Hausdorff dimension of $f([0,1]^n \setminus E)$ is at most n. By Theorem 1.3, the mapping f satisfies condition (N), and so $\mathcal{H}^n(f(E)) = 0$. This implies that Y has Hausdorff dimension at most n. □

Remark 7.5. The same proof shows that if Y is any metric space, and f is a continuous surjection in the space $\underline{W}^{1,n,1}(\Omega; Y)$, defined as in Remark 7.3, then Y has Hausdorff dimension at most n.

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