Denseness of certain smooth Lévy functionals in $\mathbb{D}_{1,2}$

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Abstract

The Malliavin derivative for a Lévy process (X_t) can be defined on the space $\mathbb{D}_{1,2}$ using a chaos expansion or in the case of a pure jump process also via an increment quotient operator [12]. In this paper we define the Malliavin derivative operator $\boldsymbol{D}$ on the class $\mathcal{S}$ of smooth random variables $f(X_{t_1}, \ldots, X_{t_n})$, where f is a smooth function with compact support. We show that the closure of $L_2(\mathbb{P}) \supseteq \mathcal{S} \xrightarrow{\boldsymbol{D}} L_2(\mathbb{m}\otimes\mathbb{P})$ yields to the space $\mathbb{D}_{1,2}$. As an application we conclude that Lipschitz functions operate on $\mathbb{D}_{1,2}$.

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1 Introduction

In the recent years Malliavin calculus for Lévy processes has been developed using various types of chaos expansions. For example, Lee and Shih [7] applied a white noise approach, León et al. [5] worked with certain strongly orthogonal martingales, Løkka [6] and Di Nunno et al. [2] considered multiple integrals with respect to the compensated Poisson random measure and Solé et al. [11] used the chaos expansion proved by Itô [4].

This chaos representation from Itô applies to any square integrable functional of a general Lévy process. It uses multiple integrals like in the well known Brownian motion case but with respect to an independent random measure associated with the Lévy process. Solé et al. propose in [12] a canonical space for a general Lévy process. They define for random variables on the canonical space the increment quotient operator

$$\Psi_{t,x}F(\omega) = \frac{F(\omega_{t,x}) - F(\omega)}{x}, \qquad x \neq 0,$$

in a pathwise sense, where, roughly speaking, $\omega_{t,x}$ can be interpreted as the outcome of adding at time t a jump of the size x to the path ω. They show that on the canonical Lévy space the Malliavin derivative $D_{t,x}F$ defined via the chaos expansion due to Itô and $\Psi_{t,x}F$ coincide a.e. on $\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega$ (where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$) whenever $F \in L_2$ and $\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}_0} |\Psi_{t,x}F|^2 d\mathbb{m}(t,x) < \infty$ (see Section 2 for the definition of $\mathbb{m}$).

On the other hand, on the Wiener space, the Malliavin derivative is introduced as an operator D mapping smooth random variables of the form $F = f(W(h_1), \ldots, W(h_n))$ into $L_2(\Omega; H)$, i.e.

$$DF = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(W(h_1), \ldots, W(h_n))h_i,$$

(see, for example, [8]). Here f is a smooth function mapping from $\mathbb{R}^n$ into $\mathbb{R}$ such that all its derivatives have at most polynomial growth, and $\{W(h), h \in H\}$ is an isonormal Gaussian family associated with a Hilbert space H. The closure of the domain of the operator D is the space $\mathbb{D}_{1,2}$.

In the present paper we proceed in a similar way for a Lévy process $(X_t)_{t \geq 0}$. We will define a Malliavin derivative operator on a class of smooth random variables and determine its closure. The class of smooth random variables we consider consists of elements of the form $F = f(X_{t_1}, \ldots, X_{t_n})$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support.

Analogously to results of Solé et al. [12] about the canonical Lévy space the Malliavin derivative $DF \in L_2(\mathbb{m} \otimes \mathbb{P})$, defined via chaos expansion, can be expressed explicitly as a two-parameter operator $D_{t,x}$. For certain smooth random variables of the form $F = f(X_{t_1}, \dots, X_{t_n})$ it holds

$$\begin{aligned} D_{t,x} f(X_{t_1}, \dots, X_{t_n}) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) \mathbb{1}_{[0,t_i] \times \{0\}}(t, x) \\ &\quad + \Psi_{t,x} f(X_{t_1}, \dots, X_{t_n}) \mathbb{1}_{\{x \neq 0\}}(x), \end{aligned}$$

for $\mathbb{m} \otimes \mathbb{P}$-a.e. (t, x, ω). Here $\Psi_{t,x}$, for $x \neq 0$, is given by

$$\begin{aligned} &\Psi_{t,x} f(X_{t_1}, \dots, X_{t_n}) \\ &:= \frac{f(X_{t_1} + x\mathbb{1}_{[0,t_1]}(t), \dots, X_{t_n} + x\mathbb{1}_{[0,t_n]}(t)) - f(X_{t_1}, \dots, X_{t_n})}{x}. \end{aligned}$$

Our main result is that the smooth random variables $f(X_{t_1}, \dots, X_{t_n})$ are dense in the space $\mathbb{D}_{1,2}$ defined via the chaos expansion. This implies that defining $\boldsymbol{D}$ as an operator on the smooth random variables as in Definition 3.2 below and taking the closure leads to the same result as defining D using Itô's chaos expansion (see Definition 2.1).

The paper is organized as follows. In Section 2 we shortly recall Itô's chaos expansion, the definition of the Malliavin derivative and some related facts. The third and fourth section focus on the introduction of the Malliavin derivative operator on smooth random variables and the determination of its closure. Applying the denseness result from the previous section we show in Section 5 that Lipschitz functions map from $\mathbb{D}_{1,2}$ into $\mathbb{D}_{1,2}$.

2 The Malliavin derivative via Itô's chaos expansion

We assume a càdlàg Lévy process $X = (X_t)_{t \geq 0}$, on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Lévy triplet (γ, σ^2, ν) where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and ν is the Lévy measure. Then X has the Lévy-Itô decomposition

$$X_t = \gamma t + \sigma W_t + \int_{[0,t] \times \{|x| \geq 1\}} x dN(t, x) + \int_{[0,t] \times \{0 < |x| < 1\}} x d\tilde{N}(t, x),$$

where W denotes a standard Brownian motion, N is the Poisson random measure associated with the process X and $\tilde{N}$ the compensated Poisson random measure, $d\tilde{N}(t, x) = dN(t, x) - dt d\nu(x)$. Consider the measures μ on $\mathcal{B}(\mathbb{R})$,

$$d\mu(x) := \sigma^2 d\delta_0(x) + x^2 d\nu(x),$$

and $\mathbb{m}$ on $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$, where $\mathbb{R}_+ := [0, \infty)$,

$$d\mathbb{m}(t,x) := dt d\mu(x).$$

For $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ such that $\mathbb{m}(B) < \infty$ let

$$M(B) = \sigma \int_{\{t \in \mathbb{R}_+ : (t,0) \in B\}} \mathrm{d}W_t + \lim_{n \to \infty} \int_{\{(t,x) \in B : 1/n < |x| < n\}} x d\tilde{N}(t,x),$$

where the convergence is taken in $L_2(\Omega, \mathcal{F}, \mathbb{P})$. Now $\mathbb{E}M(B_1)M(B_2) = \mathbb{m}(B_1 \cap B_2)$ for all B_1, B_2 with $\mathbb{m}(B_1) < \infty$ and $\mathbb{m}(B_2) < \infty$. For $n = 1, 2, \ldots$ write

$$L_2^n := L_2\left((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^{\otimes n}, \mathbb{m}^{\otimes n}\right).$$

For $f \in L_2^n$ Itô [4] defines a multiple integral $I_n(f)$ with respect to the random measure M. It holds $I_n(f) = I_n(\tilde{f})$, a.s., where $\tilde{f}$ is the symmetrization of f,

$$\tilde{f}(z_1, \ldots, z_n) = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} f(z_{\pi(1)}, \ldots, z_{\pi(n)}) \quad \text{for all } z_i = (t_i, x_i) \in \mathbb{R}_+ \times \mathbb{R},$$

and $\mathcal{S}_n$ denotes the set of all permutations on $\{1, \ldots, n\}$.

Let $(\mathcal{F}_t^X)_{t \geq 0}$ be the augmented natural filtration of X. Then $(\mathcal{F}_t^X)_{t \geq 0}$ is right continuous ([9, Theorem I 4.31]). Set $\mathcal{F}^X := \bigvee_{t \geq 0} \mathcal{F}_t^X$. By Theorem 2 of Itô [4] it holds the chaos decomposition

$$L_2 := L_2(\Omega, \mathcal{F}^X, \mathbb{P}) = \bigoplus_{n=0}^{\infty} I_n(L_2^n),$$

where $I_0(L_2^0) := \mathbb{R}$ and $I_n(L_2^n) := \{I_n(f_n) : f_n \in L_2^n\}$ for $n = 1, 2, \ldots$. For $F \in L_2$ the representation

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

with $I_0(f_0) = \mathbb{E}F$, a.s., is unique if the functions f_n are symmetric. Furthermore,

$$\|F\|_{L_2}^2 = \sum_{n=0}^{\infty} n! \|\tilde{f}_n\|_{L_2^n}^2.$$

Definition 2.1 Let $\mathbb{D}_{1,2}$ be the space of all $F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2$ such that

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|^2_{L_2^n} < \infty.$$

Set $L_2(\mathbb{m} \otimes \mathbb{P}) := L_2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}^X, \mathbb{m} \otimes \mathbb{P})$. The Malliavin derivative operator $D : \mathbb{D}_{1,2} \to L_2(\mathbb{m} \otimes \mathbb{P})$ is defined by

$$D_{t,x} F := \sum_{n=1}^{\infty} n I_{n-1}(\tilde{f}_n((t,x), \cdot)), \qquad (t,x,\omega) \in \mathbb{R}_+ \times \mathbb{R} \times \Omega. \tag{1}$$

We consider (as Solé et al. [12]) the operators $D_{\cdot,0}$ and $D_{\cdot,x}$, $x \neq 0$ and their domains $\mathbb{D}^0_{1,2}$ and $\mathbb{D}^J_{1,2}$. For $\sigma > 0$ let $\mathbb{D}^0_{1,2}$ consist of random variables $F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2$ such that

$$\|F\|^2_{\mathbb{D}^0_{1,2}} := \|F\|^2_{L_2} + \sum_{n=1}^{\infty} n \cdot n! \|\tilde{f}_n \mathbb{1}_{(\mathbb{R}_+ \times \{0\}) \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}}\|^2_{L_2^n} < \infty.$$

For $\nu \neq 0$, let $\mathbb{D}^J_{1,2}$ be the set of $F \in L_2$ such that

$$\|F\|^2_{\mathbb{D}^J_{1,2}} := \|F\|^2_{L_2} + \sum_{n=1}^{\infty} n \cdot n! \|\tilde{f}_n \mathbb{1}_{(\mathbb{R}_+ \times \mathbb{R}_0) \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}}\|^2_{L_2^n} < \infty,$$

where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. If both $\sigma > 0$ and $\nu \neq 0$, then it holds

$$\mathbb{D}_{1,2} = \mathbb{D}^0_{1,2} \cap \mathbb{D}^J_{1,2}. \tag{2}$$

In case $\nu = 0$, $D_{\cdot,0}$ coincides with the classical Malliavin derivative D^W (see, for example, [8]) except for a multiplicative constant, $D^W_t F = \sigma D_{t,0} F$.

In the next lemma we formulate a denseness result which will be used to determine the closure of the Malliavin operator from Definition 3.1 below.

Lemma 2.2 *Let $\mathcal{L} \subseteq L_2$ be the linear span of random variables of the form*

$$M(T_1 \times A_1) \cdots M(T_n \times A_n), \qquad n = 1, 2, \ldots$$

where the A_i's are finite intervals of the form $(a_i, b_i]$ and the T_i's are finite disjoint intervals of the form $T_i = (s_i, t_i]$. Then $\mathcal{L}$ is dense in L_2, $\mathbb{D}_{1,2}$, $\mathbb{D}^0_{1,2}$ and $\mathbb{D}^J_{1,2}$.

Proof. 1° First we consider the class of all linear combinations of

$$M(B_1)\cdots M(B_n) = I_n(1\!\!1_{B_1\times\cdots\times B_n}),$$

$n = 1, 2, \ldots$, where the sets $B_i \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ are disjoint and fulfill the condition $\mathbb{m}(B_i) < \infty$. It follows from the completeness of the multiple integrals in L_2 (see [4, Theorem 2]) that this class is dense in L_2. Especially, the class of all linear combinations of $1\!\!1_{B_1\times\cdots\times B_n}$ with disjoint sets $B_1, \ldots, B_n$ of finite measure $\mathbb{m}$ is dense in $L_2^n = L_2\left((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^{\otimes n}, \mathbb{m}^{\otimes n}\right)$. Let $\mathcal{H}_n$ be the linear span of $1\!\!1_{(T_1\times A_1)\times\cdots\times(T_n\times A_n)}$ where $A_i = (a_i, b_i]$ and $T_i = (s_i, t_i]$. One can easily see that $\mathcal{H}_n$ is dense in L_2^n as well: Because $\mathbb{m}$ is a Radon measure, there are compact sets $C_i \subseteq B_i$ such that $\mathbb{m}(B_i \setminus C_i)$ is sufficiently small to get

$$\|1\!\!1_{B_1\times\cdots\times B_n} - 1\!\!1_{C_1\times\cdots\times C_n}\|_{L_2^n} < \varepsilon$$

for some given $\varepsilon > 0$. Since the compact sets (C_i) are disjoint one can find disjoint bounded open sets $U_i \supseteq C_i$ such that $\|1\!\!1_{C_1\times\cdots\times C_n} - 1\!\!1_{U_1\times\cdots\times U_n}\|_{L_2^n} < \varepsilon$. For any bounded open set $U_i \subseteq (0,\infty)\times\mathbb{R}$ one can find a sequence of 'half-open rectangles' $Q_{i,k} = (s_k^i, t_k^i] \times (a_k^i, b_k^i] = T_k^i \times A_k^i$, such that $U_i = \bigcup_{k=1}^{\infty} Q_{i,k}$ (taking half-open rectangles $Q_x \subseteq U_i$ with rational 'end points' containing the point $x \in U_i$ gives $U_i = \bigcup_{Q_x \subseteq U_i}^{\infty} Q_x$).

Hence for sufficiently large K_i's one has

$$\|1\!\!1_{U_1\times\cdots\times U_n} - 1\!\!1_{\bigcup_{k=1}^{K_1} Q_{1,k}\times\cdots\times\bigcup_{k=1}^{K_n} Q_{n,k}}\|_{L_2^n} < \varepsilon$$

where the $Q_{i,k}$'s can now be chosen such that they are disjoint. This implies that the linear span of $1\!\!1_{Q_1\times\cdots\times Q_n}$ where the Q_i's are of the form $T_i \times A_i$ is dense in L_2^n.

2° For the convenience of the reader we recall the idea of the proof of Lemma 2 [4] to show that the intervals T_i can be chosen disjoint. Consider

$$1\!\!1_{(T_1\times A_1)\times\cdots\times(T_n\times A_n)}, \tag{3}$$

with $\mu(A_1) > 0, \ldots, \mu(A_n) > 0$, where $(T_i \times A_i) \cap (T_j \times A_j) = \emptyset$ for $i \neq j$. Assume, for example (all other cases can be treated similarly), that $T_1 = \cdots = T_m =: T$ while $T_m, \ldots, T_n$ are pairwise disjoint. Given the expression

$$1\!\!1_{(T\times A_1)\times\cdots\times(T\times A_m)\times(T_{m+1}\times A_{m+1})\times\cdots\times(T_n\times A_n)}$$

choose an equidistant partition $(E_j)_{j=1}^k$ of T so that $|E_j| = \frac{|T|}{k}$ and set

$$c := \mu(A_1)\cdots\mu(A_n)|T_{m+1}|\cdots|T_n|.$$

Now

$$
\begin{aligned}
&1\!\!1_{(T\times A_1)\times\cdots\times(T\times A_m)\times(T_{m+1}\times A_{m+1})\times\cdots\times(T_n\times A_n)} \\
= &\sum_{\substack{j_1,\ldots,j_m=1\\ \text{all } j_i \text{ distinct}}}^{k} 1\!\!1_{(E_{j_1}\times A_1)\times\cdots\times(E_{j_m}\times A_m)\times(T_{m+1}\times A_{m+1})\times\cdots\times(T_n\times A_n)} \\
&+ \sum_{\substack{j_1,\ldots,j_m=1\\ j_i \text{ not distinct}}}^{k} 1\!\!1_{(E_{j_1}\times A_1)\times\cdots\times(E_{j_m}\times A_m)\times(T_{m+1}\times A_{m+1})\times\cdots\times(T_n\times A_n)} \\
= &S_1 + S_2,
\end{aligned}
$$

where S_1 is a sum of indicator functions with disjoint time intervals. We complete the proof by observing that

$$
\begin{aligned}
\|S_2\|^2_{L_2^n} &= \sum_{\substack{j_1,\ldots,j_m=1\\ j_i \text{ not distinct}}}^{k} \mathbb{m}(E_{j_1}\times A_1)\cdots\mathbb{m}(E_{j_m}\times A_m) \\
&\qquad\qquad \times\mathbb{m}(T_{m+1}\times A_{m+1})\cdots\mathbb{m}(T_n\times A_n) \\
&= c\sum_{\substack{j_1,\ldots,j_m=1\\ j_i \text{ not distinct}}}^{k} |E_{j_1}|\cdots|E_{j_m}| \\
&= c\left(k^m - m!\binom{k}{m}\right)\left(\frac{|T|}{k}\right)^m \\
&= c|T|^m\left(1-\left(1-\frac{1}{k}\right)\cdots\left(1-\frac{m-1}{k}\right)\right) \to 0
\end{aligned}
$$

for $k\to\infty$.

3° The denseness of $\mathcal{H}_n$ in L_2^n implies that $\mathcal{L}$ is dense in L_2 and $\mathbb{D}_{1,2}$. The remaining cases follow from the fact that

$$\|f_n 1\!\!1_{(\mathbb{R}_+\times\{0\})\times(\mathbb{R}_+\times\mathbb{R})^{n-1}}\|_{L_2^n} \le \|f_n\|_{L_2^n}$$

and

$$\|f_n 1\!\!1_{(\mathbb{R}_+\times\mathbb{R}_0)\times(\mathbb{R}_+\times\mathbb{R})^{n-1})}\|_{L_2^n} \le \|f_n\|_{L_2^n}.$$

□

3 The Malliavin derivative as operator on $\mathcal{S}$

Let $C_c^\infty(\mathbb{R}^n)$ denote the space of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ with compact support.

Definition 3.1 A random variable of the form $F = f(X_{t_1}, \dots, X_{t_n})$, where $f \in C_c^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, and $t_1, \dots, t_n \geq 0$, is said to be a *smooth random variable*. The set of all smooth random variables is denoted by $\mathcal{S}$.

Definition 3.2 For $F = f(X_{t_1}, \dots, X_{t_n}) \in \mathcal{S}$ we define the Malliavin derivative operator $\boldsymbol{D}$ as a map from $\mathcal{S}$ into $L_2(\mathbb{m} \otimes \mathbb{P})$ by

$$
\begin{aligned}
&\boldsymbol{D}_{t,x} f(X_{t_1}, \dots, X_{t_n}) \\
:=\ & \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) \mathbb{1}_{[0,t_i]\times\{0\}}(t,x) \\
&+ \frac{f(X_{t_1} + x\mathbb{1}_{[0,t_1]}(t), \dots, X_{t_n} + x\mathbb{1}_{[0,t_n]}(t)) - f(X_{t_1}, \dots, X_{t_n})}{x} \mathbb{1}_{\mathbb{R}_0}(x)
\end{aligned}
$$

for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

We have the following result.

Lemma 3.3 *It holds $\boldsymbol{D}F = DF$ in $L_2(\mathbb{m} \otimes \mathbb{P})$ for all $F \in \mathcal{S}$.*

Since for $f(X_{t_1}, \dots, X_{t_n}) \in \mathcal{S}$ one has

$$\mathbb{E} \int_{\mathbb{R}_+} |\boldsymbol{D}_{t,0} f(X_{t_1}, \dots, X_{t_n})|^2 dt < \infty$$

and

$$\mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}_0} |\boldsymbol{D}_{t,x} f(X_{t_1}, \dots, X_{t_n})|^2 d\mathbb{m}(t,x) < \infty$$

Lemma 3.3 follows for the canonical Lévy space from [12, Proposition 3.5] and [12, Proposition 5.5].

A proof of Lemma 3.3 for the situation where the Lévy process (X_t) is a square integrable pure jump process which has an absolutely continuous distribution can be found in [6].

An outline of the proof in the general case is given in the appendix. Like in [6, Proposition 8] one can derive from the proof an explicit form for the functions (f_n) of the chaos expansion $f(X_{t_1}, \ldots, X_{t_k}) = \sum_{n=0}^{\infty} I_n(f_n)$,

$$\begin{aligned} & f_n((s_1, x_1), \ldots, (s_n, x_n)) \\ = \; & \mathbb{E} \sum_{I \subset \{1,\ldots,n\} \cup \emptyset} \frac{(-1)^{n-|I|}}{n!} \frac{f(X_{t_1} + \sum_{i \in I} x_i \mathbb{1}_{[0,t_1]}(s_i), \ldots, X_{t_k} + \sum_{i \in I} x_i \mathbb{1}_{[0,t_k]}(s_i))}{x_1 \cdots x_n}, \end{aligned}$$

with the convention that to get $f_n((s_1, x_1), \ldots, (s_i, 0), \ldots, (s_n, x_n))$ one has to take the limit $\lim_{|x_i| \downarrow 0} f_n((s_1, x_1), \ldots, (s_n, x_n))$.

Especially, we conclude from the fact that any $F \in L_2 \supseteq \mathcal{S}$ has a unique chaos expansion that also $\boldsymbol{D}F$ does not depend on the representation $F = f(X_{t_1}, \ldots, X_{t_n})$.

Using the equality of $\boldsymbol{D}$ and D on $\mathcal{S}$ and the fact that $\mathcal{S}$ is closed with respect to multiplication we are now able to reformulate Proposition 5.1 of [12] for our situation:

Corollary 3.4 *For F and G in $\mathcal{S}$ it holds*

$$D_{t,x}(FG) = GD_{t,x}F + FD_{t,x}G + xD_{t,x}FD_{t,x}G$$

for $\mathbb{m} \otimes \mathbb{P}-a.e.$ $(t, x, \omega) \in \mathbb{R}_+ \times \mathbb{R} \times \Omega$.

4 The closure of the Malliavin derivative operator

The operator $\boldsymbol{D} : \mathcal{S} \to L_2(\mathbb{m} \otimes \mathbb{P})$ is closable, if for any sequence $(F_n) \subseteq \mathcal{S}$ which converges to 0 in L_2 such that $\boldsymbol{D}(F_n)$ converges in $L_2(\mathbb{m}\otimes\mathbb{P})$, it follows that $(\boldsymbol{D}F_n)$ converges to 0 in $L_2(\mathbb{m}\otimes\mathbb{P})$. As we know from the previous section that D and $\boldsymbol{D}$ coincide on $\mathcal{S} \subseteq \mathbb{D}_{1,2}$, it is clear that $\boldsymbol{D}$ is closable and the closure of the domain of definition of $\boldsymbol{D}$ with respect to the norm

$$\|F\|_{\boldsymbol{D}} := \left[\mathbb{E}|F|^2 + \mathbb{E}\|\boldsymbol{D}F\|^2_{L_2(\mathbb{m})}\right]^{\frac{1}{2}},$$

is contained in $\mathbb{D}_{1,2}$. What remains to show is that the closure is equal to $\mathbb{D}_{1,2}$.

Theorem 4.1 *The closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\boldsymbol{D}} = \|\cdot\|_{\mathbb{D}_{1,2}}$ is the space $\mathbb{D}_{1,2}$.*

Theorem 4.1 implies that the Malliavin derivative D defined via Itô's chaos expansion and the closure of the operator $L_2 \supseteq \mathcal{S} \xrightarrow{\boldsymbol{D}} L_2(\mathbb{m} \otimes \mathbb{P})$ coincide. Before we start with the proof we formulate a Lemma for later use.

Lemma 4.2 *For $\varphi \in C_c^\infty(\mathbb{R})$ and partitions $\pi_n := \{s = t_0^n < t_1^n < \cdots < t_n^n = u\}$ of the interval $[s,u]$ it holds for $\psi(x) := x\varphi(x)$ that*

$$\mathbb{D}_{1,2} - \lim_{|\pi_n| \to 0} \left(\sum_{j=1}^n \psi(X_{t_j^n} - X_{t_{j-1}^n}) - \mathbb{E} \sum_{j=1}^n \psi(X_{t_j^n} - X_{t_{j-1}^n}) \right) = \int_{(s,u]\times\mathbb{R}} \varphi(x)\, dM(t,x),$$

where $|\pi_n| := \max_{1 \le i \le n} |t_i^n - t_{i-1}^n|$.

Proof. To keep the notation simple, we will drop the n of the partition points t_j^n. Notice that $\int_{(s,u]\times\mathbb{R}} \varphi(x)\, dM(t,x) = I_1(1\!\!1_{(s,u]} \otimes \varphi)$. We set

$$G^n := \sum_{j=1}^n \psi(X_{t_j} - X_{t_{j-1}}) - \mathbb{E} \sum_{j=1}^n \psi(X_{t_j} - X_{t_{j-1}})$$

and

$$G := \int_{(s,u]\times\mathbb{R}} \varphi(x)\, dM(t,x).$$

Choose functions $\beta_m \in C_c^\infty(\mathbb{R})$ such that $0 \le \beta_m \le 1$ and $\beta_m(x) = 1$ for $|x| \le m$, the support of β_m is contained in $\{x; |x| \le m+2\}$ and $\|\beta_m'\|_\infty \le 1$. As $\psi(X_{t_j} - X_{t_{j-1}})\beta_m(X_{t_{j-1}}) \in \mathcal{S}$, we get from Lemma 3.3 that

$$\boldsymbol{D}_{t,x}\psi(X_{t_j} - X_{t_{j-1}})\beta_m(X_{t_{j-1}}) = D_{t,x}\psi(X_{t_j} - X_{t_{j-1}})\beta_m(X_{t_{j-1}}). \tag{4}$$

One easily checks that $L_2 - \lim_{m\to\infty} \psi(X_{t_j} - X_{t_{j-1}})\beta_m(X_{t_{j-1}}) = \psi(X_{t_j} - X_{t_{j-1}})$ and the limit of (4) in $L_2(\mathbb{m} \otimes \mathbb{P})$ is $\boldsymbol{D}_{t,x}\psi(X_{t_j} - X_{t_{j-1}}) = D_{t,x}\psi(X_{t_j} - X_{t_{j-1}})$. So we can write $D_{t,x}G^n$ explicitly as

$$\begin{aligned} D_{t,x} & \left(\sum_{j=1}^n \psi(X_{t_j} - X_{t_{j-1}}) \right) \\ &= \sum_{j=1}^n \psi'(X_{t_j} - X_{t_{j-1}}) 1\!\!1_{(t_{j-1},t_j]\times\{0\}}(t,x) \\ &\quad + \sum_{j=1}^n \frac{\psi(X_{t_j} - X_{t_{j-1}} + x) - \psi(X_{t_j} - X_{t_{j-1}})}{x} 1\!\!1_{(t_{j-1},t_j]\times\mathbb{R}_0}(t,x). \end{aligned}$$

Moreover, we have $D_{t,x} I_1(\mathbb{1}_{(s,u]} \otimes \varphi) = \mathbb{1}_{(s,u]}(t)\varphi(x)$ $\mathbb{m}-$a.e. Using the general fact that for any $F \in \mathbb{D}_{1,2}$ with expectation zero it holds $\|F\|^2_{\mathbb{D}_{1,2}} \leq 2\|DF\|^2_{L_2(m\otimes\mathbb{P})}$ we obtain

$$
\begin{aligned}
& \|G - G^n\|^2_{\mathbb{D}_{1,2}} \\
\leq\ & 2\|DG - DG^n\|^2_{L_2(\mathbb{m}\otimes\mathbb{P})} \\
=\ & 2\sigma^2 \mathbb{E} \int_{\mathbb{R}_+} \sum_{j=1}^n \mathbb{1}_{(t_{j-1},t_j]}(t) \big[\varphi(0) - \psi'(X_{t_j} - X_{t_{j-1}})\big]^2 dt \\
& + 2\mathbb{E} \int_{\mathbb{R}_+\times\mathbb{R}_0} \sum_{j=1}^n \mathbb{1}_{(t_{j-1},t_j]}(t) \big[\varphi(x) - D_{t,x}\psi(X_{t_j} - X_{t_{j-1}})\big]^2 d\mathbb{m}(t,x) \\
=\ & 2\sigma^2 \mathbb{E} \int_{\mathbb{R}_+} \sum_{j=1}^n \mathbb{1}_{(t_{j-1},t_j]}(t) \big[\varphi(0) - \psi'(X_{t_j} - X_{t_{j-1}})\big]^2 dt \\
& + 2\mathbb{E} \int_{\mathbb{R}_+\times\mathbb{R}_0} \sum_{j=1}^n \mathbb{1}_{(t_{j-1},t_j]}(t) [\psi(x) - \psi(X_{t_j} - X_{t_{j-1}} + x) \\
& \qquad\qquad + \psi(X_{t_j} - X_{t_{j-1}})]^2 dt d\nu(x) \\
& \to 0
\end{aligned}
$$

as $n \to \infty$ because of dominated convergence and the a.s. càdlàg property of the paths of (X_t). Indeed, one can use the estimates

$$|\varphi(0) - \psi'(y)| \leq \|\varphi\|_\infty + \|\psi'\|_\infty,$$

$$|\psi(x) - \psi(y + x) + \psi(y)| \leq (\|\varphi\|_\infty + \|\psi'\|_\infty + 3\|\psi\|_\infty)(|x| \wedge 1)$$

and the fact that $\int_{\mathbb{R}}(x^2 \wedge 1)d\nu(x) < \infty$. Moreover, for $|\pi_n| \to 0$ the càdlàg property of the paths implies the pointwise convergence in $t \in (s,u]$ of the expressions

$$\sum_{j=1}^n \mathbb{1}_{(t^n_{j-1},t^n_j]}(t) \big[\varphi(0) - \psi'(X_{t^n_j} - X_{t^n_{j-1}})\big]^2 \to \big[\varphi(0) - \psi'(X_t - X_{t-})\big]^2$$

(note that $\varphi(0) - \psi'(0) = 0$), and

$$
\begin{aligned}
& \sum_{j=1}^n \mathbb{1}_{(t^n_{j-1},t^n_j]}(t) [\psi(x) - \psi(X_{t^n_j} - X_{t^n_{j-1}} + x) + \psi(X_t - X_{t-})]^2 \\
& \qquad \to [\psi(x) - \psi(X_t - X_{t-} + x) + \psi(X_t - X_{t-})]^2.
\end{aligned}
$$

Because the set $\{t > 0; \quad X_t - X_{t-} \neq 0\}$ is at most countable for càdlàg paths the assertion follows. $\square$

Proof of Theorem 4.1. According to Lemma 2.2 it is sufficient to show that an expression like $M(T_1 \times A_1) \cdots M(T_n \times A_n)$, where the $A_i's$ are bounded Borel sets and the $T_i's$ finite disjoint intervals, can be approximated in $\mathbb{D}_{1,2}$ by a sequence $(F_k)_k \subseteq \mathcal{S}$.

1° In this step we want to show that it is enough to approximate

$$I_1(\mathbb{1}_{T_1} \otimes \varphi_1) \cdots I_1(\mathbb{1}_{T_n} \otimes \varphi_n), \tag{5}$$

by $(F_k)_k \subseteq \mathcal{S}$ where $\varphi_i \in C_c^\infty(\mathbb{R})$. Since the intervals T_i are disjoint the definition of the multiple integral implies that

$$M(T_1 \times A_1) \cdots M(T_n \times A_n) = I_n(\mathbb{1}_{T_1 \times A_1} \otimes \cdots \otimes \mathbb{1}_{T_n \times A_n}) \quad \text{a.s.}$$

By the same reason,

$$I_1(\mathbb{1}_{T_1} \otimes \varphi_1) \cdots I_1(\mathbb{1}_{T_n} \otimes \varphi_n) = I_n((\mathbb{1}_{T_1} \otimes \varphi_1) \otimes \cdots \otimes (\mathbb{1}_{T_n} \otimes \varphi_n)) \quad \text{a.s.}$$

We have

$$\begin{aligned} & \| I_n(\mathbb{1}_{(T_1 \times A_1) \times \cdots \times (T_n \times A_n)}) - I_n((\mathbb{1}_{T_1} \otimes \varphi_1) \otimes \cdots \otimes (\mathbb{1}_{T_n} \otimes \varphi_n)) \|_{\mathbb{D}_{1,2}}^2 \\ \leq\ & (n+1)! \| \mathbb{1}_{(T_1 \times A_1) \times \cdots \times (T_n \times A_n)} - (\mathbb{1}_{T_1} \otimes \varphi_1) \otimes \cdots \otimes (\mathbb{1}_{T_n} \otimes \varphi_n) \|_{L_2^n}^2 \\ \leq\ & (n+1)! |T_1| \cdots |T_n| \| \mathbb{1}_{A_1 \times \cdots \times A_n} - \varphi_1 \otimes \cdots \otimes \varphi_n \|_{L_2^n(\mu^{\otimes n})}^2 . \end{aligned}$$

The last expression can be made arbitrarily small by choosing φ_i such that $\| \mathbb{1}_{A_i} - \varphi_i \|_{L_2^1(\mu)}$ is small. For example, for each i there are compact sets $C_1^i \subseteq C_2^i \subseteq \cdots \subseteq A_i$ and open sets $U_1^i \supseteq U_2^i \supseteq \cdots \supseteq A_i$ such that

$$\mu(U_k^i \setminus C_k^i) \to 0$$

as $k \to \infty$. By the C^∞ Urysohn Lemma ([3], p. 237) there is for each k a function $\varphi_k^i \in C_c^\infty(\mathbb{R})$ such that $0 \leq \varphi_k^i \leq 1$, $\varphi_k^i = 1$ on C_k^i and $\operatorname{supp}(\varphi_k^i) \subset U_k^i$. Then

$$\| \mathbb{1}_{A_i} - \varphi_i^k \|_{L_2^1(\mu)}^2 \leq \mu(U_k^i \setminus C_k^i) \to 0$$

as $k \to \infty$.

2° Now we use Lemma 4.2 to approximate the expression (5) by a sequence $(F_k)_k \subseteq \mathcal{S}$. For $i = 1, \ldots, n$ set $\psi_i(x) := x\varphi_i(x)$ and

$$G_i^k := \sum_{j=1}^{k} 1\!\!1_{\{t_j, t_{j-1} \in \bar{T}_i\}} \psi_i(X_{t_j} - X_{t_{j-1}}) - \mathbb{E} \sum_{j=1}^{k} 1\!\!1_{\{t_j, t_{j-1} \in \bar{T}_i\}} \psi_i(X_{t_j} - X_{t_{j-1}}).$$

The partition $\pi_k = \{0 \le t_0^k \le \ldots \le t_k^k\}$ can be chosen such that all end points of the closed intervals $\bar{T}_i$ belong to π_k. Put

$$f_k(X_{t_0}, \ldots, X_{t_k}) := \Pi_{i=1}^n G_i^k$$

and notice that $f_k \in C^\infty(\mathbb{R}^{k+1})$. Setting $x_{-1} := 0$ and $\alpha_m(x_0, \ldots, x_k) := \Pi_{i=0}^k \beta_m(x_i - x_{i-1})$ where the (β_m) are taken from the proof of Lemma 4.2 we have $f_k(x)\alpha_m(x) \in C_c^\infty(\mathbb{R}^{k+1})$. By dominated convergence one can show that $\mathbb{D}_{1,2} - \lim_{m\to\infty} f_k(X_{t_0}, \ldots, X_{t_k})\alpha_m(X_{t_0}, \ldots, X_{t_k}) = f_k(X_{t_0}, \ldots, X_{t_k})$. Because the intervals (T_i) are disjoint it follows that the product rule holds in our case:

$$D\Pi_{i=1}^n G_i^k = \sum_{i=1}^{n} G_1^k \cdots G_{i-1}^k (DG_i^k) G_{i+1}^k \cdots G_n^k \qquad \mathrm{m} \otimes \mathbb{P} - \text{a.e.} \tag{6}$$

Indeed, because of $D_{t,x} G_i^k = (D_{t,x} G_i^k) 1\!\!1_{T_i}(t)$ it follows

$$x(D_{t,x} G_i^k) 1\!\!1_{T_i}(t) (D_{t,x} G_j^k) 1\!\!1_{T_j}(t) = 0 \qquad \mathrm{m} \otimes \mathbb{P} - \text{a.e.}$$

for any $i \neq j$. Equation (6) follows then by induction. Let

$$G_i := I_1(1\!\!1_{T_i} \otimes \varphi_i).$$

We observe that $G_1^k, \ldots, G_n^k$ as well as $G_1^k, \ldots, G_{i-1}^k, DG_i^k, G_{i+1}^k, \ldots, G_n^k$ are mutually independent by construction. Hence in order to show L_2- convergence of these products it is enough to prove L_2-convergence for each factor. From Lemma 4.2 we obtain that $G_i^k \to G_i$ in $\mathbb{D}_{1,2}$ for all $i = 1, \ldots, n$, so that

$$\begin{aligned} L_2(\mathrm{m} \otimes \mathbb{P}) - \lim_{|\pi_k| \to 0} G_1^k \cdots G_{i-1}^k (DG_i^k) G_{i+1}^k \cdots G_n^k \\ = G_1 \cdots G_{i-1} (DG_i) G_{i+1} \cdots G_n. \end{aligned}$$

Consequently, we have found a sequence $(F_k) \subseteq \mathcal{S}$ given by

$$F_k = f_k(X_{t_0}, \ldots, X_{t_k}) \alpha_{m_k}(X_{t_0}, \ldots, X_{t_k})$$

where the m_k's are chosen in a suitable way that converges to expression (5) in $\mathbb{D}_{1,2}$. $\square$

Corollary 4.3 *The set $\mathcal{S}$ of smooth random variables is dense in L_2, $\mathbb{D}^0_{1,2}$ and $\mathbb{D}^J_{1,2}$.*

Proof. The denseness in L_2 is clear. To show that $\mathcal{S}$ is dense in $\mathbb{D}^0_{1,2}$ assume $F \in \mathbb{D}^0_{1,2}$ has the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$. For a given $\varepsilon > 0$ fix N_ε such that $\| \sum_{n=N_\varepsilon}^{\infty} I_n(f_n)\|_{\mathbb{D}^0_{1,2}} < \varepsilon$. From $F \in L_2$ we conclude $F^{N_\varepsilon} := \sum_{n=0}^{N_\varepsilon} I_n(f_n) \in \mathbb{D}_{1,2}$. By Theorem 4.1 we can find a sequence $(F_k) \subseteq \mathcal{S}$ converging to F^{N_ε} in $\mathbb{D}_{1,2}$ and therefore also in $\mathbb{D}^0_{1,2}$. In the same way one can see that $\mathcal{S}$ is dense in $\mathbb{D}^J_{1,2}$. $\square$

5 Lipschitz functions operate on $\mathbb{D}_{1,2}$

Lemma 5.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constant L_g.*

(a) If $\sigma > 0$, then $g(F) \in \mathbb{D}^0_{1,2}$ for all $F \in \mathbb{D}^0_{1,2}$ and

$$D_{t,0}g(F) = GD_{t,0}F \quad dt \otimes \mathbb{P} - a.e., \tag{7}$$

where G is a random variable which is a.s. bounded by L_g.

(b) If $\nu \neq 0$, then $g(F) \in \mathbb{D}^J_{1,2}$ for all $F \in \mathbb{D}^J_{1,2}$, where

$$D_{t,x}g(F) = \frac{g(F + xD_{t,x}F) - g(F)}{x} \tag{8}$$

for $\mathbb{m} \otimes \mathbb{P}$-a.e. $(t, x, \omega) \in \mathbb{R}_+ \times \mathbb{R}_0 \times \Omega$.

Proof. We will adapt the proof of Proposition 1.2.4 [8] to our situation. Corollary 4.3 implies that there exists a sequence $(F_n) \subseteq \mathcal{S}$ of the form $F_n = f_n(X_{t_1}, \ldots, X_{t_n})$ which converges to F in $\mathbb{D}^0_{1,2}$. Like in [8] we choose a non-negative $\psi \in C_c^\infty(\mathbb{R})$ such that $\mathrm{supp}(\psi) \subseteq [-1, 1]$ and $\int_{\mathbb{R}} \psi(x)dx = 1$ and set $\psi_m(x) := m\psi(mx)$.

Then $g_m := g * \psi_m$ is smooth and converges to g uniformly. Moreover, $\|g_m'\|_\infty \leq L_g$. Hence $g_m \circ F_n = g_m(f_n(X_{t_1}, \ldots, X_{t_n})) \in \mathcal{S}$ and $(g_n(F_n))$ converges to $g(F)$ in L_2.

Moreover,

$$\mathbb{E} \int_{\mathbb{R}_+} |D_{t,0}g_n(F_n)|^2 dt$$

$$
\begin{aligned}
&= \mathbb{E} g_n'(F_n)^2 \int_{\mathbb{R}_+} \Big| \sum_{i=1}^n \frac{\partial}{\partial x_i} f_n(X_{t_1}, \ldots, X_{t_n}) 1\!\!1_{[0,t_i]}(t) \Big|^2 dt \\
&\leq L_g^2 \, \|F_n\|^2_{\mathbb{D}^0_{1,2}}.
\end{aligned}
$$

Since we have convergence of $(g_n(F_n))$ to $g(F)$ in L_2 and it holds

$$
\sup_n \|g_n(F_n)\|^2_{\mathbb{D}^0_{1,2}} < \infty,
$$

Lemma 1.2.3 [8] states that this implies that $g(F) \in \mathbb{D}^0_{1,2}$ and that $(D_{\cdot,0}\, g_n(F_n))$ converges to $D_{\cdot,0}\, g(F)$ in the weak topology of $L_2(\Omega; L_2(\mathbb{R}_+ \times \{0\}))$. Now $\mathbb{E}|g_n'(F_n)|^2 \leq L_g^2$ implies the existence of a subsequence $(g_{n_k}'(F_{n_k}))_k$ which converges to some $G \in L_2$ in the weak topology of L_2. One can show that $|G| \leq L_g$ a.s. Hence for any element $\alpha \in L_\infty(\Omega; L_2(\mathbb{R}_+ \times \{0\}))$ we have

$$
\begin{aligned}
&\lim_{k\to\infty} \mathbb{E} \int_{\mathbb{R}_+} g_{n_k}'(F_{n_k})(D_{t,0}\, F_{n_k})\alpha(t)dt \\
&\quad = \lim_{k\to\infty} \mathbb{E} \left(g_{n_k}'(F_{n_k}) \int_{\mathbb{R}_+} (D_{t,0}\, F_{n_k})\alpha(t)dt \right) \\
&\quad = \lim_{k\to\infty} \mathbb{E} \left(g_{n_k}'(F_{n_k}) \int_{\mathbb{R}_+} (D_{t,0}\, F)\alpha(t)dt \right) = \mathbb{E}\left(G \int_{\mathbb{R}_+} (D_{t,0}\, F)\alpha(t)dt \right),
\end{aligned}
$$

since $|\mathbb{E} g_{n_k}'(F_{n_k}) \int_{\mathbb{R}_+} (D_{t,0}\, F_{n_k} - D_{t,0}\, F)\alpha(t)dt| \leq L_g \|\alpha\|_{L_2(\Omega;L_2(\mathbb{R}_+))} \|F_{n_k} - F\|_{\mathbb{D}^0_{1,2}}$ converges to zero for $k \to \infty$ and $\int_{\mathbb{R}_+} (D_{t,0}\, F)\alpha(t)dt \in L_2$ because $\|\alpha\|_{L_2(\mathbb{R}_+ \times \{0\})}$ is finite. Consequently,

$$
\mathbb{E} \int_{\mathbb{R}_+} \big(D_{t,0}\, g(F)\big)\alpha(t)dt = \mathbb{E} \int_{\mathbb{R}_+} G(D_{t,0}\, F)\alpha(t)dt
$$

which implies $D_{t,0}\, g(F) = G D_{t,0}\, F \quad dt \otimes \mathbb{P} - \text{a.e.}$

(b) Let $(F_n)_n \subseteq \mathcal{S}$ be a sequence such that $\mathbb{D}^J_{1,2} - \lim F_n = F$. Since the expression $Z(t,x) := \frac{g(F + xD_{t,x}F) - g(F)}{x} 1\!\!1_{\mathbb{R}_+ \times \mathbb{R}_0}(t,x)$ is in $L_2(\mathbb{m} \otimes \mathbb{P})$ it is enough to show that $(Dg_n(F_n) 1\!\!1_{\mathbb{R}_+ \times \mathbb{R}_0})$ converges in $L_2(\mathbb{m} \otimes \mathbb{P})$ to Z where (g_n) is the sequence constructed in (a). Choose $T > 0$ and $L > 0$ large enough and $\delta > 0$ sufficiently small such that

$$
\limsup_n \mathbb{E} \int_{([0,T] \times \{\delta \leq |x| \leq L\})^c} |Z(t,x)|^2 + |D_{t,x} g_n(F_n)|^2 d\mathbb{m}(t,x) < \varepsilon.
$$

Then, for $n \geq n_0$,

$$
\|Z - Dg_n(F_n) 1\!\!1_{\mathbb{R}_+ \times \mathbb{R}_0}\|^2_{L_2(\mathbb{m} \otimes \mathbb{P})}
$$

$$
\begin{aligned}
&\leq \ \varepsilon \\
&\quad +2\mathbb{E}\int_{[0,T]\times\{\delta\leq|x|\leq L\}} |Z(t,x) - \frac{g(F_n + xD_{t,x}F_n) - g(F_n)}{x}|^2 d\mathbb{m}(t,x) \\
&\quad +8\delta^{-2}T\mu(\{\delta \leq |x| \leq L\})\|g - g_n\|_\infty^2 .
\end{aligned}
$$

Hence we obtain (8) from the Lipschitz continuity of g and the uniform convergence of g_n to g. □

Proposition 5.2 *Let $g : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then $F \in \mathbb{D}_{1,2}$ implies $g(F) \in \mathbb{D}_{1,2}$, where $Dg(F)$ is given by* (7) *and* (8).

Proof. The assertion is an immediate consequence from Lemma 5.1 and (2). □

6 Appendix

Proof of Lemma 3.3. We denote by $J_n(f_n)$ the multiple integral

$$
\int_{\mathbb{R}_+\times\mathbb{R}} \int_{[0,t_n)\times\mathbb{R}} \cdots \int_{[0,t_2)\times\mathbb{R}} f_n((t_1,x_1),\ldots,(t_n,x_n))\, dM(t_1,x_1)\cdots dM(t_n,x_n),
$$

where for the definition of a stochastic integral with respect to M we refer to [1]. It holds

$$
I_n(\tilde{f}_n) = n! J_n(\tilde{f}_n). \tag{9}
$$

Let us first prove on $\mathcal{S}$ a Clark-Ocone-Haussman type formula that uses $\boldsymbol{D}$. For $u \in \mathbb{R}^k$ and $s \in \mathbb{R}_+$ denote $\xi(u,s) := 2\pi i \sum_{j=1}^k u_j \mathbb{1}_{[0,t_j]}(s)$ and

$$
\begin{aligned}
\eta(u,t) \ &:= \ \frac{\sigma^2}{2}\int_0^t \xi^2(u,s)ds + \gamma\int_0^t \xi(u,s)ds \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left(e^{x\xi(u,s)} - 1 - x\xi(u,s)\mathbb{1}_{\{|x|<1\}}\right) d\nu(x)ds.
\end{aligned}
$$

By the Fourier inversion formula (see, for example, [1] or [3]) it holds for $f \in C_c^\infty(\mathbb{R}^k)$ that

$$
\begin{aligned}
f(X_{t_1},\ldots,X_{t_k}) \ &= \ \int_{\mathbb{R}^k} \hat{f}(u) e^{2\pi i \sum_{j=1}^k u_j X_{t_j}} du \\
&= \ \int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u,T)} Y_T(u) du,
\end{aligned}
$$

where $T = \max\{t_1, \dots, t_k\}$ and

$$Y_t(u) = e^{2\pi i \sum_{j=1}^k u_j X_{t_j \wedge t} - \eta(u,t)}, \qquad \text{for } t \in [0,T].$$

From Itô's formula we obtain

$$\begin{aligned} Y_t(u) &= 1 + \int_0^t Y_{s-}(u)\xi(u,s)\, \sigma dW_s \\ &\quad + \int_{(0,t]\times\mathbb{R}_0} Y_{s-}(u)\left(e^{x\xi(u,s)} - 1\right)\, d\tilde{N}(s,x), \end{aligned}$$

so that

$$\begin{aligned} & f(X_{t_1}, \dots, X_{t_k}) \\ &= \int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u,T)}\, du && (10) \\ &\quad + \int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u,T)} \left(\int_0^T Y_{s-}(u)\xi(u,s)\, \sigma dW_s \right) du && (11) \\ &\quad + \int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u,T)} \left(\int_{(0,T]\times\mathbb{R}_0} Y_{s-}(u)\left(e^{x\xi(u,s)} - 1\right) d\tilde{N}(s,x) \right) du. && (12) \end{aligned}$$

Since $\mathbb{E}\exp\{2\pi i \sum_{j=1}^k \left(u_j X_{t_j}\right)\} = e^{\eta(u,T)}$, it follows by Fubini's theorem that

$$(10) = \mathbb{E}\int_{\mathbb{R}^k} \hat{f}(u) e^{2\pi i \sum_{j=1}^k u_j X_{t_j}}\, du = \mathbb{E} f(X_{t_1}, \dots, X_{t_k}).$$

Using the fact that $(Y_t)_{t\in[0,T]}$ is a square integrable martingale, we obtain by the conditional theorem of Fubini (see, e.g., [1]) and Fubini's theorem for stochastic integrals (see, e.g., [10]) that

$$(11) = \int_0^T \mathbb{E}\left[\int_{\mathbb{R}^k} Y_T(u)\hat{f}(u) e^{\eta(u,T)} \xi(u,s)\, du \middle| \mathcal{F}_{s-} \right]\, \sigma dW_s.$$

Applying Theorem 8.22(e) of [3] and the Fourier inversion formula we rewrite the inner integral as follows.

$$\begin{aligned} & \int_{\mathbb{R}^k} Y_T(u)\hat{f}(u) e^{\eta(u,T)} \xi(u,s)\, du \\ &= \sum_{j=1}^k \mathbb{I}_{[0,t_j]}(s) \int_{\mathbb{R}^k} 2\pi i u_j \hat{f}(u) \exp\{2\pi i \sum_{j=1}^k u_j X_{t_j}\}\, du \\ &= \sum_{j=1}^k \mathbb{I}_{[0,t_j]}(s) \frac{\partial f}{\partial x_j}(X_{t_1}, \dots, X_{t_k}). \end{aligned}$$

Similarly we can see that

$$(12) = \int_{(0,T]\times\mathbb{R}_0} \mathbb{E}\left[\int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u,T)} Y_T(u)\left(e^{x\xi(u,s)}-1\right)\, du \middle| \mathcal{F}_{s^-}\right] d\tilde{N}(s,x),$$

where

$$\begin{aligned}
&\int_{\mathbb{R}^k} \hat{f}(u) e^{\eta(u,T)} Y_T(u)\left(e^{x\xi(u,s)}-1\right)\, du \\
&= \int_{\mathbb{R}^k} \hat{f}(u)\left(e^{2\pi i \sum_{j=1}^k u_j (X_{t_j}+x\mathbb{1}_{[0,t_j]}(s))} - e^{2\pi i \sum_{j=1}^k u_j X_{t_j}}\right)\, du \\
&= f\left(X_{t_1}+x\mathbb{1}_{[0,t_1]}(s),\ldots,X_{t_k}+x\mathbb{1}_{[0,t_k]}(s)\right) - f\left(X_{t_1},\ldots,X_{t_k}\right).
\end{aligned}$$

Consequently, for $F = f(X_{t_1},\ldots,X_{t_k}) \in \mathcal{S}$ it holds

$$F = \mathbb{E}F + \int_{\mathbb{R}_+\times\mathbb{R}} \mathbb{E}\left[\boldsymbol{D}_{t,x}F|\mathcal{F}_{t^-}\right]\, dM(t,x). \tag{13}$$

Since $\boldsymbol{D}_{t,x}f(X_{t_1},\ldots,X_{t_k}) \in \mathcal{S}$ for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, we obtain by iterating equation (13) that

$$f(X_{t_1},\ldots,X_{t_k}) = \mathbb{E}f(X_{t_1},\ldots,X_{t_k}) + \sum_{n=1}^\infty J_n\left(\mathbb{E}\boldsymbol{D}^n f(X_{t_1},\ldots,X_{t_k})\right),$$

where

$$\boldsymbol{D}^n f(X_{t_1},\ldots,X_{t_k}) := \boldsymbol{D}\cdots\boldsymbol{D}f(X_{t_1},\ldots,X_{t_k}).$$

Notice that $\mathbb{E}\boldsymbol{D}^n f(X_{t_1},\ldots,X_{t_k})$ is a symmetric function on $(\mathbb{R}_+\times\mathbb{R})^n$. The relation (9) between the multiple and the iterated integral and equation (1) imply

$$D_{t,x}J_n\left(\mathbb{E}\boldsymbol{D}^n f(X_{t_1},\ldots,X_{t_k})\right) = J_{n-1}\left(\mathbb{E}\boldsymbol{D}^{n-1}\boldsymbol{D}_{t,x}f(X_{t_1},\ldots,X_{t_k})\right).$$

From $\boldsymbol{D}_{t,x}f(X_{t_1},\ldots,X_{t_k}) \in L_2(\mathbb{m}\otimes\mathbb{P})$ and

$$\boldsymbol{D}_{t,x}f(X_{t_1},\ldots,X_{t_k}) = \sum_{n=1}^\infty J_{n-1}\left(\mathbb{E}\boldsymbol{D}^{n-1}\boldsymbol{D}_{t,x}f(X_{t_1},\ldots,X_{t_k})\right)$$

it follows $f(X_{t_1},\ldots,X_{t_k}) \in \mathbb{D}_{1,2}$ and

$$\boldsymbol{D}_{t,x}f(X_{t_1},\ldots,X_{t_k}) = D_{t,x}f(X_{t_1},\ldots,X_{t_k}), \quad \mathbb{m}\otimes\mathbb{P}-a.e.$$

□

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