SHARP ASYMPTOTIC BEHAVIOR FOR THE SOLUTIONS OF DEGENERATE AND SINGULAR PARABOLIC EQUATIONS

PETRI JUUTINEN AND PETER LINDQVIST

Abstract. We study the asymptotic behavior, as $t \to \infty$, of the solutions to the evolutionary p-Laplace equation

$$v_t = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$$

with time-independent lateral boundary values. We obtain the sharp decay rate of $\max_{x\in\Omega}|v(x,t) - u(x)|$, where u is the stationary solution, both in the degenerate case $p > 2$ and in the singular case $1 < p < 2$. A key tool in the proofs is the Moser iteration, which is applied to the difference $v(x,t) - u(x)$. In the singular case we construct an example proving that the celebrated phenomenon of finite extinction time, valid for $v(x,t)$ when $u \equiv 0$, does not have a counterpart for $v(x,t) - u(x)$.

1. Introduction

The asymptotic behavior of the solutions of the evolutionary p-Laplace equation

$$v_t = \operatorname{div}(|\nabla v|^{p-2}\nabla v) \tag{1.1}$$

as $t \to \infty$ is a much studied question. It is clear that $u(x) = \lim_{t\to\infty} v(x,t)$ should be a solution to the stationary problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1.2}$$

under natural assumptions. The case when the domain is the whole $\mathbb{R}^n \times (0,\infty)$ is well understood, see e.g. [10], [14], [5], [11]. Also the case $\Omega \times (0,\infty)$ when $v = 0$ on the lateral boundary $\partial\Omega \times (0,\infty)$ is much studied. In both cases explicit comparison functions are available.

The objective of our note is the sharp decay rate of

$$\max_{x\in\Omega}|v(x,t) - u(x)|$$

as $t \to \infty$, without the assumption $u = 0$ on $\partial\Omega$. We assume that Ω has finite volume. In the linear case $p = 2$ superposition reduces this to the situation with zero boundary values. However, the nonlinearity of the equation (1.1) causes technical difficulties for other values of p. In the degenerate case $p > 2$ the sharp decay rate is $O(t^{-1/(p-2)})$ as $t \to \infty$. The right decay rate is easy to obtain for the L^m-norm, $2 \le m < \infty$, but the passage to the above L^∞-norm is demanding in the range $p < n$. Our proof is based on the Moser iteration, which, of course, is well known for solutions. However, we will apply the method to the difference $v(x,t) - u(x)$, which itself is not a solution, thus bounding the L^∞ norm in terms of the L^p norm. A special feature in the structure of the equation makes this possible. In passing, we mention that this approach does not appear to work for the rather similar looking equations

$$v_t = \Delta(|v|^{m-1}v)$$

Date: June 19, 2008.

2000 *Mathematics Subject Classification.* 35K55, 35K65, 35B40.

Key words and phrases. evolutionary p-Laplace equation, asymptotic behavior, Moser iteration.

and

$$\frac{\partial |v|^{p-2}v}{\partial t} = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$$

having non-constant lateral boundary values.

Then we turn to the singular case[1] $\max\{1, \frac{2n}{n+2}\} < p < 2$. We first prove the decay estimate

$$\max_{x\in\Omega} |v(x,t) - u(x)| \le Ce^{-\lambda t}.$$

Here we have detected a phenomenon that is surprising at first sight. It is known that with zero lateral boundary values (which implies $u \equiv 0$) there is a finite extinction time: $v(x,t) = 0$ when $t \ge T^*$, see [6, Chapter VII]. However, in general, it is not true that $v(x,t) \equiv u(x)$ after some time T^*. To show this, we have constructed an example with no better than exponential decay. Thus the situation with zero lateral boundary values is, indeed, quite special and different from the general case. This interesting phenomenon indicates that our study completes the existing theory.

Contents

2. Definitions and main results

To be on the safe side, we first define the concept of solutions. They are the usual weak solutions belonging to a Sobolev space. Let Ω be a bounded domain in the Euclidean space $\mathbb{R}^n$ and consider the space-time cylinder $\Omega \times (0,\infty)$. In the case $p > 2$ we say that $v \in L^p_{loc}(0,\infty; W^{1,p}(\Omega))$ is a solution to (1.1) if

$$\int_0^\infty \int_\Omega (-v\varphi_t + |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi)\, dx\, dt = 0 \tag{2.1}$$

for each $\varphi \in C_0^\infty(\Omega \times (0,\infty))$. The singular case $1 < p < 2$ requires an extra a priori summability, for example $v \in L^\infty_{loc}(0,\infty; L^2(\Omega))$ will do. In particular, we always have

[1]The case $p < \max\{1, \frac{2n}{n+2}\}$ seems to be somewhat unsettled, and we do not consider it in our study.

that the integral

$$\int_{t_1}^{t_2}\int_\Omega \left(v^2+|\nabla v|^p\right)\,dx\,dt$$

is finite for any $0<t_1<t_2<\infty$. By the regularity theory we may assume that the solutions are continuous, see [6].

The lateral boundary values are taken in Sobolev's sense. Thus we require that $v(x,t)-u(x)\in W_0^{1,p}(\Omega)$ for a.e. time t. In fact, we may as well have two time dependent solutions $v=v(x,t)$ and $u=u(x,t)$ of (1.1). The decay of $|v(x,t)-u(x,t)|$ is studied under the assumption $v-u\in L^p_{loc}(0,\infty;W_0^{1,p}(\Omega))$. We do not prescribe initial values.

Our main results read as follow:

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $2<p<\infty$. Let u be a solution to the stationary problem* (1.2) *in Ω and let v be a solution to* (1.1) *in $\Omega\times(0,\infty)$ such that $v-u\in L^p_{loc}(0,\infty;W_0^{1,p}(\Omega))$. Then there exists a constant $C=C(p,n,\Omega)$ such that*

$$\sup_{x\in\Omega}|v(x,t)-u(x)|\le Ct^{\frac{1}{2-p}}\qquad \textit{for } t>0.$$

Notice that the costant C above is numerical. However, in the singular case below also a quantity L, depending on the solution itself, is present.

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\max\{1,\frac{2n}{n+2}\}<p<2$. Let u be a solution to the stationary problem* (1.2) *in Ω and let v be a solution to* (1.1) *in $\Omega\times(0,\infty)$ such that $v-u\in L^p_{loc}(0,\infty;W_0^{1,p}(\Omega))$. Then there exists $\lambda=\lambda(p,n,\Omega)$ such that for all $t>2\tau>0$ we have*

$$\sup_{x\in\Omega}|v(x,t)-u(x)|\le C(tL)^{-\frac{p\kappa}{4(p-\kappa)}}\|u-v\|_{L^2(\Omega\times\{\tau\})}e^{-\lambda L(\frac{t}{2}-\tau)},$$

where $\kappa=\frac{2n}{n+2}$ and

$$L=\|\nabla v\|^{p-2}_{L^p(\Omega\times\{\tau\})}.$$

As mentioned in the introduction, both Theorems 2.1 and 2.2 are sharp. The relevant examples are given in Section 6.

Finally, we recall the following elementary vector inequalities that are used frequently: for all $a,b\in\mathbb{R}^n$ we have

$$2^{2-p}|a-b|^p\le\left(|a|^{p-2}a-|b|^{p-2}b\right)\cdot(a-b) \tag{2.2}$$

if $2<p<\infty$, and

$$(p-1)\frac{|a-b|^2}{(|a|+|b|)^{2-p}}\le\left(|a|^{p-2}a-|b|^{p-2}b\right)\cdot(a-b), \tag{2.3}$$

if $1<p<2$. (As a matter of fact, our approach is rather versatile and it works just as well for equations of the form

$$v_t=\operatorname{div}\mathcal{A}(x,\nabla v),$$

where $\mathcal{A}(x,\nabla v)$ satisfies standard growth conditions and a counterpart of (2.2) or (2.3)).

3. Norm decay estimates

We begin by establishing decay estimates for the norms $\|u-v\|_{L^m(\Omega\times\{t\})}$, $1<m<\infty$. The idea is to show that $t\mapsto \|u-v\|_{L^m(\Omega\times\{t\})}$ obeys a certain differential inequality. Similar reasoning has been used for example in [3] and [6], but under the additional assumption $u\equiv 0$.

3.1. The degenerate case.

Proposition 3.1. *Let u,v be weak solutions of the p-parabolic equation, $p\geq 2$, in $\Omega\times(0,\infty)$ such that $u-v\in L^p_{loc}(0,\infty;W_0^{1,p}(\Omega))$. Then there exists a constant $C=C(p,n,\Omega)$ such that for any $1<m<\infty$*

$$\|u-v\|_{L^m(\Omega\times\{t_2\})}\leq \min\left\{\|u-v\|_{L^m(\Omega\times\{t_1\})}, Cm^{\frac{p-1}{p-2}}(t_2-t_1)^{\frac{1}{2-p}}\right\}$$

for all $0<t_1<t_2<\infty$. In particular, $t\mapsto\|u-v\|_{L^m(\Omega\times\{t\})}$ is non-increasing for any $1<m<\infty$ and also for $m=\infty$.

Proof. For $m>1$, let

$$I_m(t)=\int_\Omega |u(x,t)-v(x,t))|^m\,dx.$$

Then, using the equation and (2.2), we formally obtain

$$\begin{aligned}\frac{d}{dt}I_m(t) &= m\int_\Omega |u-v|^{m-2}(u-v)(u_t-v_t)\,dx\\ &= m(1-m)\int_\Omega |u-v|^{m-2}\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\big)\cdot\nabla(u-v)\,dx \qquad (3.1)\\ &\leq Cm(1-m)\int_\Omega |u-v|^{m-2}|\nabla(u-v)|^p\,dx.\end{aligned}$$

Let us denote $w=|u-v|$. Since

$$|\nabla w^{\frac{m-2}{p}+1}|^p=\left(\frac{m+p-2}{p}\right)^p w^{m-2}|\nabla w|^p,$$

we infer by using the elliptic Sobolev inequality and Hölder's inequality that

$$\begin{aligned}\frac{d}{dt}I_m(t) &\leq Cm(1-m)\left(\frac{p}{m+p-2}\right)^p\int_\Omega w^{m+p-2}\,dx\\ &\leq Cm(1-m)\left(\frac{p}{m+p-2}\right)^p\left(\int_\Omega w^m\,dx\right)^{\frac{m+p-2}{m}}\\ &= Cm(1-m)\left(\frac{p}{m+p-2}\right)^p I_m(t)^{\frac{m+p-2}{m}}\end{aligned}$$

for a positive constant $C=C(p,n,\Omega)$. In other words, we have shown that

$$\frac{d}{dt}\Big[I_m(t)^{\frac{2-p}{m}}\Big]\geq C\frac{m-1}{(m+p-2)^p} \qquad (3.2)$$

for all $t\in(0,\infty)$. This formal computation can be made rigorous by resorting to Steklov averages, see e.g. [6].

Upon integration, (3.2) yields

$$I_m(t_2)^{\frac{2-p}{m}}\geq I_m(t_1)^{\frac{2-p}{m}}+C\frac{m-1}{(m+p-2)^p}(t_2-t_1),$$

that is,

$$I_m(t_2)^{\frac{1}{m}} \le \Big[I_m(t_1)^{\frac{2-p}{m}} + C\frac{m-1}{(m+p-2)^p}(t_2-t_1)\Big]^{\frac{1}{2-p}}.$$

Since the exponent $\frac{1}{2-p}$ is negative, this inequality implies the desired estimate. $\square$

A solution to (1.1) does not, in general, belong to $L^p(0,\infty;W^{1,p}(\Omega))$. For example, if $u \in W^{1,p}(\Omega)$ is a stationary solution, it is obvious that the integral

$$\int_0^\infty \int_\Omega |u(x)|^p \, dx \, dt$$

is finite only if $u \equiv 0$. However, the following is true:

Corollary 3.2. *Let u and v be as in Proposition 3.1. Then*

$$\int_\tau^\infty \int_\Omega \Big(|u-v|^p + |\nabla u - \nabla v|^p\Big) dx \, dt$$

is finite for any $\tau > 0$.

Proof. In view of Proposition 3.1, we have

$$\int_\tau^\infty \int_\Omega |u-v|^p \, dx \, dt = \int_\tau^\infty I_p(t) \, dt \le C \int_\tau^\infty t^{\frac{p}{2-p}} \, dt < \infty.$$

On the other hand, we infer from (3.1) that

$$\frac{d}{dt} I_2(t) \le -C \int_\Omega |\nabla u - \nabla v|^p \, dx,$$

which, upon integration, yields

$$\int_\tau^T \int_\Omega |\nabla u - \nabla v|^p \, dx \, dt \le C\Big(I_2(\tau) - I_2(T)\Big) \le C I_2(\tau) \le C I_p(\tau)^{\frac{2}{p}}$$

for any $0 < \tau < T < \infty$. $\square$

If $u \equiv 0$, then it follows from (3.1) and the monotonicity of $t \mapsto \int_\Omega |\nabla v(x,t)|^p \, dx$ that

$$\|\nabla v\|_{L^p(\Omega \times \{t\})} \le C t^{\frac{1}{2-p}},$$

that is, we have an estimate for the decay of energy. It is not known to us whether a similar estimate holds for $\|\nabla v - \nabla u\|_{L^p(\Omega \times \{t\})}$ in general.

Remark 3.3. It is illuminating to examine the proof of Proposition 3.1 in the case $p = 2$. Then one obtains

$$\frac{d}{dt} I_m(t) = \frac{4(1-m)}{m} \int_\Omega |\nabla |u-v|^{\frac{m}{2}}|^2 \, dx,$$

which, in view of the fact that

$$\frac{\int |\nabla \psi|^2 \, dx}{\int |\psi|^2 \, dx} \ge \lambda_1, \tag{3.3}$$

where $\psi \in W_0^{1,2}(\Omega)$ and λ_1 is the first eigenvalue of the Laplacian, gives

$$\frac{d}{dt} I_m(t) \le \frac{4(1-m)}{m} \lambda_1 I_m(t).$$

Upon integration, this leads to the exponential rate of convergence in L^m-norm. However, the estimate does not remain stable as $m \to \infty$, although it is known by other means that for the heat equation one has exponential rate of convergence also in L^∞-norm. It is important to notice that in the calculation above the only inequality occurs when we apply (3.3) to the function $|u-v|^{\frac{m}{2}}$.

In order to make things even more concrete, let us choose $u \equiv 0$ and $v(x,t) = e^{-\lambda_1 t}\varphi_1(x)$, where φ_1 is a (positive) first eigenfunction of the Laplacian, that is, $\Delta\varphi_1 + \lambda_1\varphi_1 = 0$ in Ω. Then v is a solution to the heat equation, and by the reasoning above we have

$$\frac{d}{dt}I_m(t) = \frac{4(1-m)}{m}e^{-m\lambda_1 t}\int_\Omega |\nabla(\varphi_1^{\frac{m}{2}})|^2\,dx.$$

However, a direct calculation yields

$$\begin{aligned}\frac{d}{dt}I_m(t) = &\frac{d}{dt}\left(\int_\Omega |e^{-\lambda_1 t}\varphi_1|^m\,dx\right)\\ = &-\lambda_1 m e^{-m\lambda_1 t}\int_\Omega |\varphi_1^{\frac{m}{2}}|^2\,dx,\end{aligned}$$

so that

$$\frac{\int_\Omega |\nabla(\varphi_1^{\frac{m}{2}})|^2\,dx}{\int_\Omega |\varphi_1^{\frac{m}{2}}|^2\,dx} = \frac{\lambda_1 m^2}{4(m-1)}.$$

In particular, the Rayleigh quotient of $\varphi_1^{\frac{m}{2}}$ grows linearly as $m \to \infty$. Incidentally, this is the kind of improvement (in the case $p=2$) over (3.3) that would allow us to pass to the limit as $m \to \infty$ in the proof of Proposition 3.1.

3.2. **The singular case.** The quantity L in the next proposition depends on the "energy" of the solutions. Further estimates of L are given in Remark 3.5.

Proposition 3.4. *Let u, v be weak solutions of the evolutionary p-Laplace equation, $\frac{2n}{n+2} \le p < 2$, $n \ge 2$, in $\Omega \times (0,\infty)$ such that $u - v \in L^p_{loc}(0,\infty; W_0^{1,p}(\Omega))$. Then there exists a constant $C = C(p,n,\Omega) > 0$ such that for any $m > 1$*

$$\|u-v\|_{L^m(\Omega\times\{t_2\})} \le \|u-v\|_{L^m(\Omega\times\{t_1\})}e^{-\frac{CL}{m}(t_2-t_1)} \qquad \textit{for all } 0 < t_1 < t_2 < \infty,$$

where

$$L = \left[\operatorname*{ess\,sup}_{t_1\le t\le t_2}\left(\|\nabla u\|_{L^p(\Omega\times\{t\})} + \|\nabla v\|_{L^p(\Omega\times\{t\})}\right)\right]^{p-2}.$$

Proof. We proceed as in the proof of Proposition 3.1 and denote

$$I_m(t) = \int_\Omega |u(x,t) - v(x,t)|^m\,dx.$$

Instead of (3.1), we now obtain

$$\begin{aligned}\frac{d}{dt}I_m(t) = &m(1-m)\int_\Omega |u-v|^{m-2}\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot\nabla(u-v)\,dx\\ \le &Cm(1-m)\int_\Omega |u-v|^{m-2}\frac{|\nabla(u-v)|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\,dx\\ = &C\frac{1-m}{m}\int_\Omega \frac{|\nabla|u-v|^{\frac{m}{2}}|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\,dx,\end{aligned}\tag{3.4}$$

where integration by parts and (2.3) were used. Since

$$\int_\Omega |\nabla|u-v|^{\frac{m}{2}}|^p\,dx \le \left(\int_\Omega \frac{|\nabla|u-v|^{\frac{m}{2}}|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\,dx\right)^{\frac{p}{2}}\left(\int_\Omega (|\nabla u| + |\nabla v|)^p\,dx\right)^{\frac{2-p}{2}}$$

by Hölder's inequality, we infer from the elliptic Sobolev inequality that

$$
\begin{aligned}
\int_\Omega \frac{|\nabla |u-v|^{\frac{m}{2}}|^2}{(|\nabla u|+|\nabla v|)^{2-p}}\,dx &\geq \left(\int_\Omega |\nabla |u-v|^{\frac{m}{2}}|^p\,dx\right)^{\frac{2}{p}} \left(\int_\Omega (|\nabla u|+|\nabla v|)^p\,dx\right)^{\frac{p-2}{p}} \\
&\geq C\left(\int_\Omega |u-v|^{\frac{mp^*}{2}}\,dx\right)^{\frac{2}{p^*}} \left(\int_\Omega (|\nabla u|+|\nabla v|)^p\,dx\right)^{\frac{p-2}{p}} \qquad (3.5)\\
&\geq C\left(\int_\Omega |u-v|^m\,dx\right)\Big[\|\nabla u\|_{L^p}+\|\nabla v\|_{L^p}\Big]^{p-2}
\end{aligned}
$$

where $p^* = \frac{pn}{n-p} \geq 2$ because $p \geq \frac{2n}{n+2}$.

Combining (3.4) and (3.5) leads to the estimate

$$\frac{d}{dt} I_m(t) \leq C\frac{1-m}{m} L(t) I_m(t)$$

where

$$L(t) = \Big[\|\nabla u\|_{L^p(\Omega\times\{t\})} + \|\nabla v\|_{L^p(\Omega\times\{t\})}\Big]^{p-2}.$$

Hence we have

$$\frac{d}{dt}\Big[\log I_m(t)\Big] \leq C\frac{1-m}{m}L(t)$$

that upon integration gives

$$
\begin{aligned}
\log I_m(t_2) - \log I_m(t_1) &\leq -C\frac{m-1}{m}\int_{t_1}^{t_2}\Big[\|\nabla u\|_{L^p}+\|\nabla v\|_{L^p}\Big]^{p-2}\,dt \\
&\leq -C\frac{m-1}{m}\Big[\operatorname*{ess\,sup}_{t_1\leq t\leq t_2}\big(\|\nabla u\|_{L^p}+\|\nabla v\|_{L^p}\big)\Big]^{p-2}(t_2-t_1),
\end{aligned}
$$

or, equivalently,

$$I_m(t_2) \leq I_m(t_1)e^{-C\frac{m-1}{m}L(t_2-t_1)}.$$

Raising both sides to the power $\frac{1}{m}$ yields the desired estimate. □

Remark 3.5. (i) If $u = u(x)$ is a stationary solution, then, formally, $v_t \equiv 0$ on $\partial\Omega \times (0,\infty)$ because the lateral boundary values of v are time-independent. This implies that $t \mapsto \int_\Omega |\nabla v(x,t)|^p\,dx$ is non-increasing, cf. [8]. Hence we have

$$
\begin{aligned}
\operatorname*{ess\,sup}_{t_1\leq t\leq t_2}\big(\|\nabla u\|_{L^p(\Omega\times\{t\})}+\|\nabla v\|_{L^p(\Omega\times\{t\})}\big) &= \|\nabla u\|_{L^p(\Omega\times\{t_1\})}+\|\nabla v\|_{L^p(\Omega\times\{t_1\})} \\
&\leq 2\,\|\nabla v\|_{L^p(\Omega\times\{t_1\})},
\end{aligned}
$$

where the last inequality holds because u minimizes the energy $I(\psi) = \int_\Omega |\nabla\psi|^p\,dx$ over all functions ψ for which $u-\psi \in W_0^{1,p}(\Omega)$.

(ii) Proposition 3.4 holds (essentially with the same proof) in the case $n=1$ for all $1<p<2$.

(iii) If both v and u are Lipschitz continuous, then for all $1<p<2$ we obtain the estimate

$$\|u-v\|_{L^m(\Omega\times\{t_2\})} \leq \|u-v\|_{L^m(\Omega\times\{t_1\})}e^{-\frac{CL'}{m}(t_2-t_1)}$$

where $L' = (\|\nabla v\|_\infty + \|\nabla u\|_\infty)^{p-2}$.

4. The Moser iteration

Since the constants in the estimates of Section 3 blow up as $m\to\infty$, the passage to the case $m=\infty$ requires additional work.

4.1. **The degenerate case.**

Proposition 4.1. *Let u, v be weak solutions of the evolutionary p-Laplace equation, $p \geq 2$, in $\Omega \times (0, \infty)$ such that $u - v \in L^p_{loc}(0, \infty; W^{1,p}_0(\Omega))$. Then there exists a constant $C = C(p, n, \Omega) > 0$ such that for all $T > 0$*

$$\operatorname*{ess\,sup}_{t \geq T} \|u - v\|^{n+p-\frac{2n}{p}}_{L^\infty(\Omega \times \{t\})} \leq \frac{C}{T^{1+\frac{n}{p}}} \int_{T/2}^{T} \int_\Omega |u - v|^p \, dx \, dt. \tag{4.1}$$

The first step of the proof is the following Caccioppoli estimate for the difference $u - v$:

Lemma 4.2. *Let u and v be as above. Then there is $C = C(p) > 0$ such that for any $\alpha > 1$ and $0 < t_1 < t_2 < \infty$, we have*

$$\begin{aligned} &\operatorname*{ess\,sup}_{t_1 \leq t \leq t_2} \left(\int_\Omega \zeta |u - v|^\alpha \, dx \right) + \frac{C(\alpha - 1)\alpha}{(p - 2 + \alpha)^p} \int_{t_1}^{t_2} \int_\Omega \zeta |\nabla |u - v|^{\frac{p-2+\alpha}{p}}|^p \, dx \, dt \\ &\leq 2 \int_{t_1}^{t_2} \int_\Omega |\zeta_t| |u - v|^\alpha \, dx \, dt. \end{aligned} \tag{4.2}$$

Here $\zeta = \zeta(t)$ is any non-negative Lipschitz continuous function so that $\zeta(t_1) = 0$.

Remark 4.3. Notice that the usual integral with $\nabla \zeta$ does not show up. This is due to the fact that $u - v = 0$ on the lateral boundary, which allows us take ζ to be independent of x. This in turn has the effect that under Moser iteration the domain does not shrink in the space variable x, only the time interval is reduced.

The proof of Lemma 4.2. Let

$$\psi(x, t) = |u(x, t) - v(x, t)|^{\alpha - 2}(u(x, t) - v(x, t))\zeta(t),$$

where $\alpha > 1$. By using ψ (after taking a Steklov average) as a test function both for u and for v, and subtracting the resulting equations we obtain

$$\begin{aligned} 0 = &\int_{t_1}^{t_2} \int_\Omega \left(-\psi_t (u - v) + \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla \psi \right) \, dx \, dt \\ &+ \Big/_{t_1}^{t_2} \int_\Omega \psi (u - v) \, dx. \end{aligned} \tag{4.3}$$

Let us first investigate the part involving ψ_t. We have, upon denoting $w = u - v$, that

$$\begin{aligned} \int_{t_1}^{t_2} -\psi_t w \, dt &= -\int_{t_1}^{t_2} w \frac{\partial}{\partial t} (|w|^{\alpha - 2} w \zeta) \, dt \\ &= -\Big/_{t_1}^{t_2} |w|^\alpha \zeta + \int_{t_1}^{t_2} |w|^{\alpha - 2} w \zeta w_t \, dt \\ &= -\Big/_{t_1}^{t_2} |w|^\alpha \zeta + \frac{1}{\alpha} \int_{t_1}^{t_2} \zeta \frac{\partial}{\partial t} |w|^\alpha \, dt \\ &= -\Big/_{t_1}^{t_2} |w|^\alpha \zeta + \frac{1}{\alpha} \Big/_{t_1}^{t_2} \zeta |w|^\alpha - \frac{1}{\alpha} \int_{t_1}^{t_2} \zeta_t |w|^\alpha \, dt. \end{aligned}$$

Hence

$$\int_{t_1}^{t_2}\int_\Omega -\psi_t w\,dx\,dt + \Big/_{t_1}^{t_2}\int_\Omega \psi w\,dx = \frac{1}{\alpha}\left(\Big/_{t_1}^{t_2}\zeta|w|^\alpha - \int_{t_1}^{t_2}\zeta_t|w|^\alpha\,dt\right),$$

and (4.3) and (2.2) then yield

$$\begin{aligned}\frac{1}{\alpha}\Big/_{t_1}^{t_2}\int_\Omega \zeta|w|^\alpha\,dx + C(\alpha-1)\int_{t_1}^{t_2}\int_\Omega |w|^{\alpha-2}\zeta|\nabla w|^p\,dx\,dt \\ \leq \frac{1}{\alpha}\int_{t_1}^{t_2}\int_\Omega |\zeta_t||w|^\alpha\,dx\,dt.\end{aligned} \tag{4.4}$$

Select $\zeta \geq 0$ so that $\zeta(t_1) = 0$. Since for any $t_1 \leq \tau \leq t_2$

$$\int_\Omega \zeta(\tau)|w(x,\tau)|^\alpha dx = \Big/_{t_1}^{\tau}\int_\Omega \zeta(t)|w|^\alpha\,dx \leq \int_{t_1}^{t_2}\int_\Omega |\zeta_t||w|^\alpha\,dx\,dt$$

by (4.4), we infer that

$$\begin{aligned}\frac{1}{\alpha}\operatorname*{ess\,sup}_{t_1\leq t\leq t_2}\left(\int_\Omega \zeta|w|^\alpha\,dx\right) + C(\alpha-1)\int_{t_1}^{t_2}\int_\Omega |w|^{\alpha-2}\zeta|\nabla w|^p\,dx\,dt \\ \leq \frac{2}{\alpha}\int_{t_1}^{t_2}\int_\Omega |\zeta_t||w|^\alpha\,dx\,dt.\end{aligned}$$

After noticing that

$$|w|^{\alpha-2}|\nabla w|^p = \left(\frac{p}{p-2+\alpha}\right)^p |\nabla |w|^{\frac{p-2+\alpha}{p}}|^p,$$

this reads

$$\begin{aligned}\operatorname*{ess\,sup}_{t_1\leq t\leq t_2}\left(\int_\Omega \zeta|w|^\alpha\,dx\right) + \frac{C(\alpha-1)\alpha}{(p-2+\alpha)^p}\int_{t_1}^{t_2}\int_\Omega \zeta|\nabla |w|^{\frac{p-2+\alpha}{p}}|^p\,dx\,dt \\ \leq 2\int_{t_1}^{t_2}\int_\Omega |\zeta_t||w|^\alpha\,dx\,dt.\end{aligned}$$

□

We have arrived at an estimate for w of a kind from which an L^∞-estimate is known to follow via Moser iteration. For the reader's convenience we write down the details, beginning with the parabolic Sobolev inequality. For the proof, see e.g. [6, Chapter I].

Theorem 4.4. *There exists a constant $\gamma = \gamma(n,p,r) > 0$ such that for all $f \in L^\infty(t_1,t_2;L^r(\Omega)) \cap L^p(t_1,t_2;W_0^{1,p}(\Omega))$,*

$$\int_{t_1}^{t_2}\int_\Omega |f|^q\,dx\,dt \leq \gamma^q\left(\int_{t_1}^{t_2}\int_\Omega |\nabla f|^p\,dx\,dt\right)\left(\operatorname*{ess\,sup}_{t_1\leq t\leq t_2}\int_\Omega |f|^r\,dx\right)^{\frac{p}{n}},$$

where

$$q = p\left(1+\frac{r}{n}\right).$$

The proof of Proposition 4.1. We apply the parabolic Sobolev inequality with the following choices: $f = |w|^{\frac{p-2+\alpha}{p}}$, where, as before, $w = u - v$, and $r = \frac{p\alpha}{p-2+\alpha}$, which implies that

$$q = p\left(1+\frac{p\alpha}{n(p-2+\alpha)}\right).$$

Hence we obtain the estimate

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |w|^{p-2+\alpha(1+\frac{p}{n})}\,dx\,dt \le \gamma^q \left(\int_{t_1}^{t_2}\int_\Omega |\nabla|w|^{\frac{p-2+\alpha}{p}}|^p\,dx\,dt\right) \\ \times \left(\operatorname*{ess\,sup}_{t_1\le t\le t_2}\int_\Omega |w|^\alpha\,dx\right)^{\frac{p}{n}}.\end{aligned}\tag{4.5}$$

Notice that $\alpha \in]1,\infty[$ implies $r \in]\frac{p}{p-1}, p[$ and

$$p\left(1+\frac{p}{n(p-1)}\right) \le q \le p\left(1+\frac{p}{n}\right).$$

In particular, this means that above γ^q can be replaced by a constant $S > 0$ that depends only on n and p, and not on α.

Let $T > 0$ and $T_k = T(1-2^{-k})$, and choose ζ_k to be a piece-wise linear function that satisfies

$$\begin{cases}\zeta_k(t) = 0 & \text{for } t \le T_k,\\ \zeta_k(t) = 1 & \text{for } t \ge T_{k+1},\\ |\zeta_k'(t)| \le T_{k+1} - T_k = \frac{2^{k+1}}{T}. \end{cases}$$

Let also $t_2 > T$. By combining (4.5) and (4.2), we have

$$\begin{aligned}\int_{T_{k+1}}^{t_2}\int_\Omega |w|^{p-2+\alpha(1+\frac{p}{n})}\,dx\,dt &\le SC\alpha^{p-2}\left(\int_{T_k}^{t_2}\int_\Omega |\zeta_k'||w|^\alpha\,dx\,dt\right)^{1+\frac{p}{n}} \\ &\le SC\alpha^{p-2}\left(\frac{2^{k+1}}{T}\right)^{1+\frac{p}{n}}\left(\int_{T_k}^{t_2}\int_\Omega |w|^\alpha\,dx\,dt\right)^{1+\frac{p}{n}}.\end{aligned}$$

Thus, denoting $\beta = 1+\frac{p}{n}$,

$$\left(\int_{T_{k+1}}^{t_2}\int_\Omega |w|^{p-2+\alpha\beta}\,dx\,dt\right)^{\frac{1}{\beta}} \le \left(SC\alpha^{p-2}\right)^{\frac{1}{\beta}}\frac{2^{k+1}}{T}\int_{T_k}^{t_2}\int_\Omega |w|^\alpha\,dx\,dt \tag{4.6}$$

for $k = 1, 2, 3, \ldots$.

We start the iteration from $k = 1$, $\alpha_1 = p$, in which case (4.6) reads

$$\left(\int_{T_2}^{t_2}\int_\Omega |w|^{p-2+p\beta}\,dx\,dt\right)^{\frac{1}{\beta}} \le \left(SCp^{p-2}\right)^{\frac{1}{\beta}}\frac{2^2}{T}\int_{T_1}^{t_2}\int_\Omega |w|^p\,dx\,dt.$$

Then for $k = 2$, $\alpha_2 = p-2+p\beta < n\beta^2$, we have

$$\begin{aligned}\left(\int_{T_3}^{t_2}\int_\Omega |w|^{(p-2)(1+\beta)+p\beta^2}\,dx\,dt\right)^{\frac{1}{\beta^2}} &\le \left(SCp^{p-2}\right)^{\frac{1}{\beta}}\frac{2^2}{T}\left(SC(n\beta^2)^{p-2}\right)^{\frac{1}{\beta^2}}\left(\frac{2^3}{T}\right)^{\frac{1}{\beta}} \\ &\quad \times \int_{T_1}^{t_2}\int_\Omega |w|^p\,dx\,dt \\ &\le \left(SCn^{p-2}\right)^{\frac{1}{\beta}+\frac{1}{\beta^2}}\frac{2^{2+\frac{3}{\beta}}}{T^{1+\frac{1}{\beta}}}\beta^{(p-2)(\frac{1}{\beta}+\frac{2}{\beta^2})} \\ &\quad \times \int_{T_1}^{t_2}\int_\Omega |w|^p\,dx\,dt,\end{aligned}$$

and for $k = 3$, $\alpha_3 = (p-2)(1+\beta) + p\beta^2 < n\beta^3$ the estimate takes the form

$$\left(\int_{T_4}^{t_2}\int_\Omega |w|^{(p-2)(1+\beta+\beta^2)+p\beta^3}\,dx\,dt\right)^{\frac{1}{\beta^3}} \le \left(SCn^{p-2}\right)^{\frac{1}{\beta}+\frac{1}{\beta^2}+\frac{1}{\beta^3}} \frac{2^{2+\frac{3}{\beta}+\frac{4}{\beta^2}}}{T^{1+\frac{1}{\beta}+\frac{1}{\beta^2}}}$$
$$\times\, \beta^{(p-2)(\frac{1}{\beta}+\frac{2}{\beta^2}+\frac{3}{\beta^3})}\int_{T_1}^{t_2}\int_\Omega |w|^p\,dx\,dt.$$

Since $\beta = 1 + \frac{p}{n} > 1$, the series $\sum \frac{1}{\beta^k}$ and $\sum \frac{k}{\beta^k}$ are convergent. Moreover,

$$\alpha_{k+1} = (p-2)(1+\beta+\beta^2+\cdots+\beta^{k-1}) + p\beta^k = \frac{n}{p}(p-2)(\beta^k-1) + p\beta^k,$$

so that

$$\frac{\alpha_{k+1}}{\beta^k} \to n + p - \frac{2n}{p} \qquad \text{as } k \to \infty.$$

Thus we finally arrive at the estimate

$$\operatorname*{ess\,sup}_{T\le t\le t_2} \|w\|^{n+p-\frac{2n}{p}}_{L^\infty(\Omega\times\{t\})} \le C\left(\frac{1}{T}\right)^{\frac{n+p}{p}} \int_{\frac{T}{2}}^{t_2}\int_\Omega |w|^p\,dx\,dt, \tag{4.7}$$

where $C > 0$ depends only on p and n, and $t_2 > T > 0$. Observe that the power of $\frac{1}{T}$ comes from

$$1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \cdots = \frac{1}{1-\frac{1}{\beta}} = \frac{n+p}{p}.$$

The estimate (4.1) is obtained from (4.7) by letting $t_2 \to T$ and using the fact that $t \mapsto \|w\|_{L^\infty(\Omega\times\{t\})}$ is non-increasing (see Proposition 3.1). □

4.2. **The singular case.**

Proposition 4.5. *Let u, v be weak solutions of the evolutionary p-Laplace equation, $\max\{1, \frac{2n}{n+2}\} < p < 2$, in $\Omega \times (0,\infty)$ such that $u - v \in L^p_{loc}(0,\infty; W^{1,p}_0(\Omega))$. Then there exists a constant $C = C(p,n,\Omega) > 0$ such that for all $T > 0$*

$$\operatorname*{ess\,sup}_{t\ge T} \|u-v\|^2_{L^\infty(\Omega\times\{t\})} \le \frac{C}{L^{\frac{p\kappa}{2(p-\kappa)}}} T^{\frac{(2-p)\kappa-2p}{2(p-\kappa)}} \int_{\frac{T}{2}}^{T}\int_\Omega |u-v|^2\,dx\,dt,$$

where $\kappa = \frac{2n}{n+2}$ and

$$L = \left[\operatorname*{ess\,sup}_{t\ge\frac{T}{2}} \left(\|\nabla u\|_{L^p(\Omega\times\{t\})} + \|\nabla v\|_{L^p(\Omega\times\{t\})}\right)\right]^{p-2}.$$

The Caccioppoli estimate reads in the singular case as follows:

Lemma 4.6. *Let u and v be as above, and $w = u - v$. Then for any $\alpha > 1$ and $0 < t_1 < t_2 < \infty$, we have*

$$\begin{aligned}&\operatorname*{ess\,sup}_{t_1\le t\le t_2}\left(\int_\Omega \zeta|w|^\alpha\,dx\right) + C\frac{\alpha-1}{\alpha}L\left(\int_{t_1}^{t_2}\int_\Omega \zeta|\nabla w^{\frac{\alpha}{2}}|^p\,dx\,dt\right)^{\frac{2}{p}}\left(\int_{t_1}^{t_2}\zeta\,dt\right)^{\frac{p-2}{p}}\\ &\quad\le\; 2\int_{t_1}^{t_2}\int_\Omega |\zeta_t||w|^\alpha\,dx\,dt,\end{aligned} \tag{4.8}$$

Here $\zeta = \zeta(t)$ is any non-negative Lipschitz continuous function so that $\zeta(t_1) = 0$ and

$$L = \left[\operatorname*{ess\,sup}_{t_1\le t\le t_2} \left(\|\nabla u\|_{L^p(\Omega\times\{t\})} + \|\nabla v\|_{L^p(\Omega\times\{t\})}\right)\right]^{p-2}.$$

Proof. We proceed as in the degenerate case $p \geq 2$ and obtain

$$
\begin{aligned}
0 &= \int_{t_1}^{t_2} \int_\Omega \left(-\psi_t (u-v) + \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla \psi \right) dx\, dt \\
&\quad + \Big/_{t_1}^{t_2} \int_\Omega \psi (u-v)\, dx
\end{aligned}
\tag{4.9}
$$

with $\psi(x,t) = |u(x,t) - v(x,t)|^{\alpha-2}(u(x,t) - v(x,t))\zeta(t)$. The part involving ψ_t is treated as before, so let us focus on the part involving $\nabla\psi$. Following the proof of Proposition 3.4, we have

$$
\begin{aligned}
&\int_{t_1}^{t_2} \int_\Omega \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla \psi \, dx\, dt \\
&= (\alpha - 1) \int_{t_1}^{t_2} \zeta(t) \left(\int_\Omega |u-v|^{\alpha-2} \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla (u-v)\, dx \right) dt \\
&\geq C(\alpha-1) \int_{t_1}^{t_2} \zeta(t) \left(\int_\Omega |u-v|^{\alpha-2} \frac{|\nabla(u-v)|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\, dx \right) dt \\
&= C\frac{\alpha-1}{\alpha^2} \int_{t_1}^{t_2} \zeta(t) \left(\int_\Omega \frac{|\nabla |u-v|^{\frac{\alpha}{2}})|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\, dx \right) dt.
\end{aligned}
\tag{4.10}
$$

Since

$$
\int_\Omega \frac{|\nabla |u-v|^{\frac{\alpha}{2}})|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\, dx \geq \Big[\operatorname*{ess\,sup}_{t_1 \leq t \leq t_2} \left(\|\nabla u\|_{L^p} + \|\nabla v\|_{L^p} \right) \Big]^{p-2} \left(\int_\Omega |\nabla |u-v|^{\frac{\alpha}{2}}|^p\, dx \right)^{\frac{2}{p}}
$$

for all $t_1 \leq t \leq t_2$, and

$$
\left(\int_{t_1}^{t_2} \int_\Omega \zeta |\nabla |u-v|^{\frac{\alpha}{2}}|^p\, dx\, dt \right)^{\frac{2}{p}} \leq \left(\int_{t_1}^{t_2} \zeta\, dt \right)^{\frac{2-p}{p}} \left(\int_{t_1}^{t_2} \zeta \left(\int_\Omega |\nabla |u-v|^{\frac{\alpha}{2}}|^p\, dx \right)^{\frac{2}{p}} dt \right)
$$

by Hölder's inequality, (4.10) implies that

$$
\begin{aligned}
&\int_\Omega \int_{t_1}^{t_2} \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla \psi \, dx\, dt \\
&\geq C\frac{\alpha-1}{\alpha^2} L \left(\int_{t_1}^{t_2} \int_\Omega \zeta |\nabla |u-v|^{\frac{\alpha}{2}}|^p\, dx\, dt \right)^{\frac{2}{p}} \left(\int_{t_1}^{t_2} \zeta\, dt \right)^{\frac{p-2}{p}}
\end{aligned}
$$

where $L = \Big[\operatorname*{ess\,sup}_{t_1 \leq t \leq t_2} \left(\|\nabla u\|_{L^p(\Omega\times\{t\})} + \|\nabla v\|_{L^p(\Omega\times\{t\})} \right) \Big]^{p-2}$. By combining this with the estimates on the two remaining terms in (4.9) obtained already in the degenerate case, we conclude that

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t_1 \leq t \leq t_2} \left(\int_\Omega \zeta |w|^\alpha\, dx \right) + C\frac{\alpha-1}{\alpha} L \left(\int_{t_1}^{t_2} \int_\Omega \zeta |\nabla w^{\frac{\alpha}{2}}|^p\, dx\, dt \right)^{\frac{2}{p}} \left(\int_{t_1}^{t_2} \zeta\, dt \right)^{\frac{p-2}{p}} \\
&\leq 2 \int_{t_1}^{t_2} \int_\Omega |\zeta_t| |w|^\alpha\, dx\, dt,
\end{aligned}
$$

as claimed. $\square$

The proof of Proposition 4.5. Let $T > 0$ and $T_k = T(1 - 2^{-k})$, and choose the cut-off functions ζ_k as in the proof of Proposition 4.1, that is,

$$\begin{cases} \zeta_k(t) = 0 & \text{for } t \leq T_k, \\ \zeta_k(t) = 1 & \text{for } t \geq T_{k+1}, \\ |\zeta_k'(t)| \leq T_{k+1} - T_k = \frac{2^{k+1}}{T}. \end{cases}$$

Let also $t_2 \in (T, \frac{3}{2}T)$.

We begin the iteration with $\alpha_1 = 2$. Since $p > \frac{2n}{n+2}$, we have $\frac{p}{2} + \frac{p}{n} > 1$, and the Sobolev inequality, Theorem 4.4, yields

$$\int_{t_1}^{t_2} \int_\Omega |w|^{2(\frac{p}{2}+\frac{p}{n})}\, dx\, dt \leq S \left(\int_{t_1}^{t_2} \int_\Omega |\nabla w|^p \, dx\, dt \right) \left(\operatorname*{ess\,sup}_{t_1 \leq t \leq t_2} \int_\Omega |w|^2\, dx \right)^{\frac{p}{n}},$$

where again $w = u - v$. In view of (4.8), we thus have

$$\int_{T_2}^{t_2} \int_\Omega |w|^{2(\frac{p}{2}+\frac{p}{n})}\, dx\, dt \leq S \frac{C}{L^{\frac{p}{2}}} \left(\int_{T_1}^{t_2} \int_\Omega |\zeta_1'||w|^2 \, dx\, dt \right)^{\frac{p}{2}+\frac{p}{n}} \left(\int_{T_1}^{t_2} \zeta_1\, dt \right)^{\frac{2-p}{2}}. \tag{4.11}$$

Let $\beta = \frac{p}{2} + \frac{p}{n}$ and $\gamma = 1 + \frac{2}{n}$, and observe that $2\beta = p\gamma$. Using this new notation (4.11) can be rewritten as

$$\left(\int_{T_2}^{t_2} \int_\Omega |w|^{2\beta}\, dx\, dt \right)^{\frac{1}{\beta}} \leq \frac{(CS)^{\frac{1}{\beta}}}{L^{\frac{1}{\gamma}}} \frac{4}{T} T^{\frac{2-p}{2\beta}} \int_{T_1}^{t_2} \int_\Omega |w|^2\, dx\, dt.$$

Next take $\alpha_2 = 2\beta$, whence

$$\begin{aligned} \int_{T_3}^{t_2} \int_\Omega |w|^{2\beta^2}\, dx\, dt &\leq S \left(\int_\Omega \int_{T_3}^{t_2} |\nabla |w|^\beta|^p \, dx\, dt \right) \left(\operatorname*{ess\,sup}_{T_3 \leq t \leq t_2} \int_\Omega |w|^{2\beta}\, dx \right)^{\frac{p}{n}} \\ &\leq \frac{CS}{L^{\frac{\beta}{\gamma}}} \left(\int_{T_2}^{t_2} \int_\Omega \zeta_2' |w|^{2\beta}\, dx\, dt \right)^{\beta} \left(\int_{T_2}^{t_2} \zeta_2\, dt \right)^{\frac{2-p}{2}} \end{aligned}$$

by Sobolev inequality and (4.8), and hence

$$\left(\int_{T_3}^{t_2} \int_\Omega |w|^{2\beta^2}\, dx\, dt) \right)^{\frac{1}{\beta^2}} \leq \frac{(CS)^{\frac{1}{\beta}+\frac{1}{\beta^2}}}{L^{\frac{1}{\gamma}(1+\frac{1}{\beta})}} \frac{2^{2+\frac{3}{\beta}}}{T^{1+\frac{1}{\beta}}} T^{\frac{2-p}{2}(\frac{1}{\beta}+\frac{1}{\beta^2})} \int_{T_1}^{t_2} \int_\Omega |w|^2\, dx\, dt.$$

It should by now be clear how the iteration proceeds. Since

$$1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \cdots = \frac{\beta}{\beta - 1} = \frac{p}{p - \kappa}$$

and

$$\frac{1}{\beta} + \frac{1}{\beta^2} + \cdots = \frac{\kappa}{p - \kappa},$$

where $\kappa = \frac{2n}{n+2}$, the final estimate takes the form

$$\operatorname*{ess\,sup}_{T \leq t \leq t_2} \|w\|^2_{L^\infty(\Omega \times \{t\})} \leq \frac{C}{L^{\frac{p\kappa}{2(p-\kappa)}}} T^{\frac{(2-p)\kappa - 2p}{2(p-\kappa)}} \int_{\frac{T}{2}}^{T} \int_\Omega w^2\, dx\, dt,$$

where we have also used the relation $\gamma = \frac{2}{\kappa}$. □

5. The end game

We now complete the proofs of Theorems 2.1 and 2.2.

5.1. **The case $2 < p \le n$.** Let u and v be as in Theorem 2.1. By Proposition 3.1,

$$\int_{\Omega\times\{t\}} |u-v|^p\,dx \le Ct^{-\frac{p}{p-2}}.$$

Thus we infer from Proposition 4.1 that

$$\begin{aligned}\|u-v\|_{L^\infty(\Omega\times\{t\})}^{n+p-\frac{2n}{p}} &\le C\left(\frac{1}{t}\right)^{1+\frac{n}{p}}\int_{\frac{t}{2}}^{t}\int_\Omega |u-v|^p\,dx\,ds\\ &\le C\left(\frac{1}{t}\right)^{1+\frac{n}{p}}\int_{\frac{t}{2}}^{t} s^{-\frac{p}{p-2}}\,ds\\ &\le Ct^{-\left(\frac{n}{p}+\frac{p}{p-2}\right)}.\end{aligned}$$

Since

$$\frac{n}{p}+\frac{p}{p-2}=\frac{n+p-\frac{2n}{p}}{p-2},$$

we obtain

$$\|u-v\|_{L^\infty(\Omega\times\{t\})} \le Ct^{-\frac{1}{p-2}},$$

as desired. □

5.2. **The case $p > \max\{n,2\}$.** Here we do not need the Moser iteration. Let u and v be as in Theorem 2.1, and let

$$I_m(t)=\int_\Omega |u(x)-v(x,t))|^m\,dx$$

as before. By the estimate (3.1) in the proof of Proposition 3.1,

$$\frac{d}{dt}I_2(t) \le -C\int_\Omega |\nabla(u-v)|^p\,dx.$$

Since $p > n$,

$$\|u-v\|_{L^\infty(\Omega\times\{t\})}^p \le C\int_\Omega |\nabla(u-v)|^p\,dx,$$

and thus we have

$$\frac{d}{dt}I_2(t) \le -C\|u-v\|_{L^\infty(\Omega\times\{t\})}^p.$$

Upon recalling that $t\mapsto \|u-v\|_{L^\infty(\Omega\times\{t\})}$ is non-increasing, an integration yields

$$\begin{aligned}C(2t-t)\|u-v\|_{L^\infty(\Omega\times\{2t\})}^p &\le C\int_t^{2t}\|u-v\|_{L^\infty(\Omega\times\{s\})}^p\,ds\\ &\le I_2(t)-I_2(2t)\\ &\le I_2(t)\end{aligned}$$

and hence

$$\|u-v\|_{L^\infty(\Omega\times\{2t\})}^p \le \frac{C}{t}I_2(t).$$

Combining this estimate with Proposition 3.1 finally gives

$$\|u-v\|_{L^\infty(\Omega\times\{2t\})} \le C\left(\frac{1}{t}t^{\frac{2}{2-p}}\right)^{\frac{1}{p}} = Ct^{\frac{1}{2-p}}$$

as claimed.

5.3. **The case** $\max\{1, \frac{2n}{n+2}\} < p < 2$**.** Let u and v be as in Theorem 2.2, and let $\tau > 0$. By Proposition 3.4 (see also Remark 3.5 **(ii)**),

$$\int_\Omega |u(x) - v(x,t)|^2\,dx \le \left(\int_\Omega |u(x) - v(x,\tau)|^2\,dx\right) e^{-CL(t-\tau)}$$

for all $t > 2\tau$. Since $t \mapsto \|u - v\|_{L^\infty(\Omega\times\{t\})}$ is non-increasing, we thus infer from Proposition 4.5 that

$$\begin{aligned}\|u - v\|^2_{L^\infty(\Omega\times\{t\})} &\le \frac{C}{L^{\frac{p\kappa}{2(p-\kappa)}}} t^{\frac{(2-p)\kappa-2p}{2(p-\kappa)}} \|u - v\|^2_{L^2(\Omega\times\{\tau\})} \int_{\frac{t}{2}}^t e^{-CL(s-\tau)}\,ds \\ &\le C(tL)^{-\frac{p\kappa}{2(p-\kappa)}} \|u - v\|^2_{L^2(\Omega\times\{\tau\})} e^{-CL(\frac{t}{2}-\tau)},\end{aligned}$$

where $\kappa = \frac{2n}{n+2}$ and, in view of Remark 3.5,

$$L = \left[\operatorname*{ess\,sup}_{s\ge\tau} \left(\|\nabla u\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega\times\{s\})}\right)\right]^{p-2} \ge 2^{p-2}\|\nabla v\|^{p-2}_{L^p(\Omega\times\{\tau\})}.$$

6. The optimality of the decay rates

This section is devoted to some examles. We use the abbreviation

$$\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v).$$

6.1. **The degenerate case.** Let $2 < p < \infty$ and consider a function v of the form

$$v(x,t) = w(x)T(t),$$

where $w = 0$ on $\partial\Omega$. Then, formally,

$$v_t(x,t) = w(x)T'(t), \qquad \Delta_p v(x,t) = |T(t)|^{p-2}T(t)\,\Delta_p w(x),$$

and thus $v_t = \Delta_p v$ if

$$-\Delta_p w = \lambda w \qquad \text{and} \qquad \frac{T'}{|T|^{p-2}T} = -\lambda$$

for some $\lambda \in \mathbb{R}$. The second differential equation can be written as

$$\frac{d}{dt}\left[\frac{1}{2-p}|T(t)|^{2-p}\right] = -\lambda,$$

which leads to

$$T(t) = \pm\Big[(p-2)\lambda t + C\Big]^{\frac{1}{2-p}}, \qquad t > \frac{C}{\lambda(2-p)}.$$

In order to find a non-trivial solution to the first equation $-\Delta_p w = \lambda w$, we minimize the functional

$$I(\phi) = \frac{1}{p}\int_\Omega |\nabla\phi|^p\,dx$$

over

$$K_p := \{\phi \in W_0^{1,p}(\Omega)\colon \|\phi\|_{L^2(\Omega)} = 1\}.$$

By standard theory, a minimizer w satisfies $-\Delta_p w = \lambda w$ for some $\lambda > 0$. Moreover, $w \ne 0$ since $\|w\|_{L^2(\Omega)} = 1$.

In conclusion, the function

$$v(x,t) = w(x)\Big[(p-2)\lambda t\Big]^{\frac{1}{2-p}}$$

is a solution to (1.1) in $\Omega \times (0,\infty)$ and

$$\sup_{x\in\Omega} |v(x,t)| = \sup_{x\in\Omega} |v(x,t) - 0| = O(t^{\frac{1}{2-p}}).$$

Thus the decay rate obtained in Theorem 2.1 cannot be improved.

Remark 6.1. The separation of variables can be carried out also for $\frac{2n}{n+2} < p < 2$. This results in the function

$$v(x,t) = w(x)\Big((2-p)\Big[C-\lambda t\Big]\Big)_+^{\frac{1}{2-p}},$$

where w is again a minimizer of $I(\cdot)$ over K_p. Notice that v becomes extinct in finite time.

6.2. **The singular case.** Let $\frac{2n}{n+2} < p < 2$, $n \geq 2$, and consider the function

$$u(x) = |x|^{\frac{p-n}{p-1}}, \qquad x \in \Omega := B(0,R) \setminus \overline{B}(0,r),$$

where, for given $0 < a < b$, R and r are chosen to satisfy $R^{\frac{p-n}{p-1}} = a$ and $r^{\frac{p-n}{p-1}} = b$. Then u is a stationary solution to (1.1) in Ω, and it takes constant values a and b on the two spheres that comprise $\partial\Omega$.

Define

$$h(s) = \frac{b-a}{2\pi}\sin\left(\frac{\pi}{b-a}(s-a)\right), \quad s \in [a,b].$$

Then

(1) $h(a) = h(b) = 0$, $h(s) \geq 0$,
(2) $|h'(s)| \leq \frac{1}{2}$,
(3) $h''(s) = -\frac{\pi^2}{(b-a)^2}h(s)$

for all $s \in [a,b]$. Now set

$$w(x,t) = u(x) + e^{-\lambda t}h(u(x)), \qquad (x,t) \in \Omega \times (0,\infty),$$

where $\lambda > 0$ is determined later. We have

$$w_t = -\lambda e^{-\lambda t}h(u),$$
$$\nabla w = \Big[1 + e^{-\lambda t}h'(u)\Big]\nabla u,$$

and

$$D^2 w = \Big[1 + e^{-\lambda t}h'(u)\Big]D^2 u + e^{-\lambda t}h''(u)\nabla u \otimes \nabla u.$$

Thus

$$\begin{aligned}\Delta_p w =& |1+e^{-\lambda t}h'(u)|^{p-2}\Big[(1+e^{-\lambda t}h'(u))\Delta_p u + (p-1)|\nabla u|^p e^{-\lambda t}h''(u)\Big]\\ =& -\frac{\pi^2}{(b-a)^2}(p-1)|\nabla u|^p e^{-\lambda t}h(u)\,|1+e^{-\lambda t}h'(u)|^{p-2},\end{aligned}$$

and consequently

$$\begin{aligned}w_t - \Delta_p w =& e^{-\lambda t}h(u)\Big[-\lambda + \frac{\pi^2}{(b-a)^2}(p-1)|\nabla u|^p\,|1+e^{-\lambda t}h'(u)|^{p-2}\Big]\\ \leq& e^{-\lambda t}h(u)\Big[-\lambda + \frac{\pi^2}{(b-a)^2}(p-1)\left(\frac{n-p}{p-1}\right)^p b^{p\frac{1-n}{p-n}}\frac{1}{2^{p-2}}\Big]\\ \leq& 0\end{aligned}$$

if $\lambda > 0$ is chosen large enough. Here the first inequality follows from

$$\sup_{x\in\Omega}|\nabla u(x)| = \left(\frac{n-p}{p-1}\right)r^{\frac{1-n}{p-1}} = \left(\frac{n-p}{p-1}\right)b^{\frac{1-n}{p-n}}$$

and

$$|1 + e^{-\lambda t} h'(u)| \geq 1 - e^{-\lambda t}|h'(u)| \geq \frac{1}{2}.$$

Observe that since $h(a) = h(b) = 0$, $v(x,t) = u(x)$ on $\partial\Omega \times (0,\infty)$. Now let v be the solution to the evolutionary p-Laplace equation (1.1) in $\Omega \times (0,\infty)$ such that $v(x,t) = u(x)$ on $\partial\Omega \times (0,\infty)$ and $v(x,0) = w(x,0)$ for $x \in \Omega$. Then, by the comparison principle,

$$v(x,t) \geq w(x,t) = u(x) + \frac{b-a}{2\pi} e^{-\lambda t}$$

for all x for which $|x|^{\frac{p-n}{p-1}} = \frac{a+b}{2}$. That is,

$$\sup_{x\in\Omega} |v(x,t) - u(x)| \geq \frac{b-a}{2\pi} e^{-\lambda t}. \tag{6.1}$$

This shows that the decay rate obtained in Theorem 2.2 cannot be improved.

In the case $n = 1$ we use the same argument with $u(x) = |x|$. The proof is even simpler since $\sup_{x\in\Omega} |u'(x)| = 1$.

References

[1] N. D. Alikakos, and R. Rostamian, *Gradient estimates for degenerate diffusion equations. I.*, Math. Ann. 259 (1982), no. 1, 53–70.

[2] D. G. Aronson, and L. A. Peletier, *Large time behaviour of solutions of the porous medium equation in bounded domains*, J. Differential Equations 39 (1981), no. 3, 378–412.

[3] P. Bénilan, and M. G. Crandall, *The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$*, Indiana Univ. Math. J. 30 (1981), no. 2, 161–177.

[4] C. Chen, and R. Wang, *L^∞ estimates of solution for the evolution m-Laplacian equation with initial value in $L^q(\Omega)$*, Nonlinear Anal. 48 (2002), no. 4, Ser. A: Theory Methods, 607–616.

[5] M. Del Pino, and J. Dolbeault, *Asymptotic behavior of nonlinear diffusions*, Math. Res. Lett. 10 (2003), no. 4, 551–557.

[6] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag, New York, 1993.

[7] H. Engler, B. Kawohl, and S. Luckhaus, *Gradient estimates for solutions of parabolic equations and systems*, J. Math. Anal. Appl. 147 (1990), no. 2, 309–329.

[8] P. Goldstein, *On the long-time behavior of solutions of the p-Laplacian parabolic system*, preprint, (2007).

[9] C. J. Holland, and J. G. Berryman, *Exponential convergence for nonlinear diffusion problems with positive lateral boundary conditions*, J. Math. Phys. 26 (1985), no. 4, 660–663.

[10] S. Kamin, and J. L. Vázquez, *Fundamental solutions and asymptotic behaviour for the p-Laplacian equation*, Rev. Mat. Iberoamericana 4 (1988), no. 2, 339–354.

[11] K.-A. Lee, A. Petrosyan, and J. L. Vázquez, *Large-time geometric properties of solutions of the evolution p-Laplacian equation*, J. Differential Equations 229 (2006), no. 2, 389–411.

[12] M. Nakao, and C. Chen, *Global existence and gradient estimates for the quasilinear parabolic equations of m-Laplacian type with a nonlinear convection term*, J. Differential Equations 162 (2000), no. 1, 224–250.

[13] F. Salvarani, and G. Toscani, *Large-time asymptotics for nonlinear diffusions: the initial-boundary value problem*, J. Math. Phys. 46 (2005), no. 2,

[14] J. L. Vázquez, *Singular solutions and asymptotic behaviour of nonlinear parabolic equations*, International Conference on Differential Equations, Vol. 1, 2 (Barcelona, 1991), 234–249, World Sci. Publ., River Edge, NJ, 1993.

Department of Mathematics and Statistics, P.O.Box 35, FIN-40014 University of Jyväskylä, Finland

E-mail address: peanju@maths.jyu.fi

Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

E-mail address: lqvist@math.ntnu.no