

# Spiraling spectra of geodesic lines in negatively curved manifolds

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## Abstract

Given a negatively curved geodesic metric space  $M$ , we study the asymptotic penetration behaviour of geodesic lines of  $M$  in small neighbourhoods of closed geodesics and of other compact convex subsets of  $M$ . We define a *spiraling spectrum* which gives precise information on the asymptotic spiraling lengths of geodesic lines around these objects. We prove analogs of the theorems of Dirichlet, Hall and Cusick in this context. As a consequence, we obtain Diophantine approximation results of elements of  $\mathbb{R}, \mathbb{C}$  or the Heisenberg group by irrational quadratic ones. <sup>1</sup>

## 1 Introduction

Let  $M$  be a finite volume connected complete Riemannian manifold with dimension  $n$  at least 2 and sectional curvature at most  $-1$ . Let  $e$  be an end of  $M$ , and let  $C$  be a closed geodesic in  $M$ . One of the aims of this paper is to study the asymptotic spiraling behavior of the (locally) geodesic lines in  $M$  starting from  $e$  around the closed geodesic  $C$ .

Just for the sake of normalization, fix a Margulis neighbourhood  $N$  of the cusp  $e$  in  $M$  (see for instance [BK]). Let  $\text{Lk}_N(M)$  be the set of geodesic lines starting from  $e$  that first meet  $\partial N$  at time 0, and do not converge to a cusp of  $M$ . Let  $d_N$  be the Hamenstädt distance on  $\text{Lk}_N(M)$  (see [HP2]), which is a natural distance inducing the compact-open topology on  $\text{Lk}_N(M)$ , and which coincides with the induced Riemannian distance on the first intersection points with  $\partial N$  if  $N$  has constant curvature.

Let  $\text{Lk}_{N,C}(M)$  be the (countable, dense) set of elements  $\rho$  in  $\text{Lk}_N(M)$  that spiral indefinitely around  $C$ , i.e. such that  $\lim_{t \rightarrow +\infty} d(\rho(t), C) = 0$ . For every  $r$  in  $\text{Lk}_{N,C}(M)$ , let  $D(r)$  be the shortest length of a path between  $\partial N$  and  $C$  which is homotopic (while its endpoints stay in  $\partial N$  and  $C$  respectively), for any  $t$  big enough, to the path obtained by following  $r$  from  $r(0)$  to  $r(t)$ , and then a geodesic between  $r(t)$  and its closest point on  $C$ . This number  $D(r)$  naturally measures the wandering of  $r$  in  $M$  before  $r$  seriously starts to spiral indefinitely around  $C$ . See the end of Section 3 for explicit computations when  $M$  is locally symmetric.

We define the *spiraling constant* around  $C$  of  $\xi \in \text{Lk}_N(M)$  by

$$c(\xi) = \liminf_{r \in \text{Lk}_{N,C}(M), D(r) \rightarrow +\infty} e^{D(r)} d_N(\xi, r),$$

which measures how well  $\xi$  is approximated by geodesic lines spiraling indefinitely around  $C$ , and, when small, says that, asymptotically,  $\xi$  has long periods of time during which it

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spirals around  $C$ . We define the *spiraling spectrum* around  $C$  in  $M$  by

$$\mathrm{Sp}_{N,C}(M) = \{c(\xi) : \xi \in \mathrm{Lk}_N(M) - \mathrm{Lk}_{N,C}(M)\} .$$

Here is a sample of our results.

**Theorem 1.1 (Dirichlet-type theorem)** *The spiraling spectrum  $\mathrm{Sp}_{N,C}(M)$  is a bounded subset of  $[0, +\infty[$ .*

**Theorem 1.2 (Cusick-type and Hall-type theorem)** *If  $M$  has constant curvature, then the spiraling spectrum  $\mathrm{Sp}_{N,C}(M)$  is closed. If, in addition, the dimension of  $M$  is at least 3, then the spectrum contains an interval  $[0, c]$  for some  $c > 0$ .*

When  $C$  is replaced by a cusp (and spiraling a long time around  $C$  is replaced by having a long excursion in a fixed cusp neighborhood), the analogous results are motivated by Diophantine approximation results: see for instance [For, Coh, Pat, Sul, Ser, Haa, CF, Vul, Dal], below in this introduction, and Remark 3.3. In that context, the boundedness of the spectrum was proved in [HP2, Theorem 1.1], the closedness of the spectrum was shown in [Mau], and the existence of a Hall ray was proved in [PP2, Theorem 1.6].

Although our arithmetic applications are going to be in the setting defined at the beginning of this introduction, our results are true in much more general situations (see the beginning of Section 3). In particular,  $M$  does not need to have a cusp (and for instance could be compact): we may replace  $e$  by a point  $x_0$  in  $M$ , and then consider the geodesic rays starting from  $x_0$ . Or  $M$  could be allowed to have a compact totally geodesic boundary, and we may replace  $e$  by a connected component  $\partial_0 M$  of  $\partial M$ , considering the geodesic rays starting from a point of  $\partial_0 M$  perpendicularly to  $\partial_0 M$ . Furthermore,  $C$  can be replaced by a connected embedded totally geodesic submanifold of positive nonmaximal dimension, or by the convex core of a precisely invariant quasifuchsian subgroup (see for instance [MT] for definitions). The theorems 1.1 and 1.2 remain valid under certain more general hypotheses on  $M$  (see the theorems 4.4, 4.8 and Corollary 5.5 for statements). Section 5, where we prove the existence of Hall rays in spiraling spectra, relies on [PP2]. In Section 4.4, we also give upper bounds on the spiraling spectra in several classical examples.

To conclude this introduction, we give Diophantine approximation results which follow from the above theorems in Riemannian geometry. Recall that for  $x \in \mathbb{R} - \mathbb{Q}$ , the *approximation constant* of  $x$  by rational numbers is

$$c(x) = \liminf_{p,q \in \mathbb{Z}, q \rightarrow +\infty} q^2 \left| x - \frac{p}{q} \right|,$$

and that the *Lagrange spectrum* is  $\mathrm{Sp}_{\mathbb{Q}} = \{c(\xi) : \xi \in \mathbb{R} - \mathbb{Q}\}$ . Numerous properties of the Lagrange spectrum are known (see for instance [CF]). In particular,  $\mathrm{Sp}_{\mathbb{Q}}$  is bounded (Dirichlet 1842), has maximum  $\frac{1}{\sqrt{5}}$  (Korkine-Zolotareff 1873, Hurwitz 1891), is closed (Cusick 1975), contains a *Hall ray*, i.e. a maximal non trivial interval  $[0, \mu]$  (Hall 1947), with  $\mu = 491993569/(2221564096 + 283748\sqrt{468})$  (Freiman 1975). Also, recall Khintchine's result [Khi] saying that almost every real number is badly approximable by rational numbers. The following result, which is a quite particular case of the results of Section 6, gives analogous Diophantine approximation results of real numbers by (families of) irrational quadratic elements.

For every real irrational quadratic number  $\alpha$  over  $\mathbb{Q}$ , let  $\alpha^\sigma$  be its Galois conjugate. Let  $\alpha_0$  be a fixed real irrational quadratic number over  $\mathbb{Q}$ . Let  $\mathcal{E}_{\alpha_0} = \text{PSL}_2(\mathbb{Z}) \cdot \{\alpha_0, \alpha_0^\sigma\}$  be its (countable, dense in  $\mathbb{R}$ ) orbit for the action by homographies and anti-homographies of  $\text{PSL}_2(\mathbb{Z})$  on  $\mathbb{R} \cup \{\infty\}$ . For instance, if  $\phi$  is the Golden Ratio  $\frac{1+\sqrt{5}}{2}$ , then  $\mathcal{E}_\phi$  is the set of real numbers whose continued fraction expansion ends with an infinite string of 1's.

For every  $x \in \mathbb{R} - (\mathbb{Q} \cup \mathcal{E}_{\alpha_0})$ , define the *approximation constant* of  $x$  by elements of  $\mathcal{E}_{\alpha_0}$ , as

$$c_{\alpha_0}(x) = \liminf_{\alpha \in \mathcal{E}_{\alpha_0} : |\alpha - \alpha^\sigma| \rightarrow 0} \frac{|x - \alpha|}{|\alpha - \alpha^\sigma|},$$

and the corresponding *approximation spectrum*, by

$$\text{Sp}_{\alpha_0} = \{c_{\alpha_0}(x) : x \in \mathbb{R} - (\mathbb{Q} \cup \mathcal{E}_{\alpha_0})\}.$$

**Theorem 1.3** *Let  $\alpha_0$  be a real irrational quadratic number over  $\mathbb{Q}$ . Then  $\text{Sp}_{\alpha_0}$  is a closed bounded subset of  $[0, +\infty[$ .*

*Furthermore, let  $\psi : ]0, +\infty[ \rightarrow ]0, +\infty[$  be a map such that  $t \mapsto \log(\psi(e^{-t}))$  is Lipschitz. If  $\int_0^1 \psi(t) dt$  diverges (resp. converges), then for Lebesgue almost all  $x \in \mathbb{R}$ ,*

$$\liminf_{\alpha \in \mathcal{E}_{\alpha_0} : |\alpha - \alpha^\sigma| \rightarrow 0} \frac{|x - \alpha|}{\psi(|\alpha - \alpha^\sigma|)} = 0 \quad (\text{resp.} = +\infty).$$

In this particular case, the last statement can be derived from [BV] or [DMPV]. Except for the following result, we do not know the maximum  $K_{\alpha_0}$  of  $\text{Sp}_{\alpha_0}$  (an analog of Hurwitz's constant), though  $K_{\alpha_0} \leq (1 + \sqrt{2})\sqrt{3} \approx 4.19$  for any  $\alpha_0$ , see Section 4.4.

**Proposition 1.4** *For the Golden Ratio  $\phi = \frac{1+\sqrt{5}}{2}$ , we have  $K_\phi = 1 - 1/\sqrt{5} \approx 0.55$ , and  $K_\phi$  is not isolated in  $\text{Sp}_{\alpha_0}$ .*

There are many papers on the Diophantine approximation of real numbers by algebraic numbers. After pioneering work by Roth and Wirsing, the seminal papers by Davenport and Schmidt [DS1, DS2] proved, in particular, Dirichlet type theorems, on which Bugeaud-Teulié [BT] improve to show the following: Let  $\mathbb{Q}_{\text{quad}}$  be the set of real irrational quadratic numbers over  $\mathbb{Q}$ , and denote by  $H(\alpha)$  the naive height of an algebraic number  $\alpha$  (the maximal absolute value of the coefficients of its minimal polynomial over  $\mathbb{Z}$ ). For every real irrational nonquadratic number  $x$ , they proved that

$$\liminf_{\alpha \in \mathbb{Q}_{\text{quad}} : H(\alpha) \rightarrow +\infty} \left( \frac{H(\alpha)}{\log(H(\alpha))} \right)^{\frac{3+\sqrt{5}}{2}} |x - \alpha| < +\infty,$$

and that for every  $\epsilon > 0$ , for Lebesgue almost every  $x$  in  $\mathbb{R}$ ,

$$\liminf_{\alpha \in \mathbb{Q}_{\text{quad}} : H(\alpha) \rightarrow +\infty} H(\alpha)^{2-\epsilon} |x - \alpha| = 0.$$

We refer to [Bug] and its impressive bibliography for further references. But note that none of the works that we know of is approximating by elements in the orbit under integral homographies of a given algebraic number; almost all of them are approximating using (a simple function of) the naive height as a complexity, but none using our complexity

$h(\alpha) = 1/|\alpha - \alpha^\sigma|$ . This complexity (see [BPP] for an algebraic interpretation) behaves very differently from the naive height  $H(\alpha)$ , even in such an orbit, see Section 6.1.

In Section 6, expanding Theorem 1.3, we will give arithmetic applications analogous to the results of Dirichlet, Cusick, and Khintchine for the Diophantine approximation of points of  $\mathbb{R}$  (resp.  $\mathbb{C}$ , the Heisenberg group  $\text{Heis}_{2n-1}$ ) by classes of irrational quadratic elements over  $\mathbb{Q}$  (resp. over imaginary quadratic extensions of  $\mathbb{Q}$ ), and Hall's result in  $\mathbb{C}$  and  $\text{Heis}_{2n-1}$ .

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## 2 Preliminaries

Throughout the paper,  $(X, d)$  will be a proper  $\text{CAT}(-1)$  geodesic metric space, and  $\partial_\infty X$  its boundary at infinity. We use [BH] as a general reference for this section. Unless otherwise stated, balls and horoballs are closed.

Let  $\Gamma$  be a discrete group of isometries of  $X$ . The limit set of  $\Gamma$  is denoted by  $\Lambda\Gamma$ . The conical limit set of  $\Gamma$  is denoted by  $\Lambda_c\Gamma$ . When  $\Lambda\Gamma$  contains at least two points, the convex hull of  $\Lambda\Gamma$  is denoted by  $\mathcal{C}\Gamma$ . The group  $\Gamma$  is *convex-cocompact* if  $\Lambda\Gamma$  contains at least two points, and if the action of  $\Gamma$  on  $\mathcal{C}\Gamma$  has compact quotient.

We will say that a subgroup  $H$  of a group  $G$  is *almost malnormal* if, for every  $g$  in  $G - H$ , the subgroup  $gHg^{-1} \cap H$  is finite. We refer for instance to [HP5, Prop. 2.6] for a proof of the following well known result.

**Proposition 2.1** *Let  $\Gamma_0$  be a convex-cocompact subgroup of  $\Gamma$ . The following assertions are equivalent.*

- (1)  $\Gamma_0$  is almost malnormal in  $\Gamma$ ;
- (2) the limit set of  $\Gamma_0$  is precisely invariant under  $\Gamma_0$ , i.e. for every  $\gamma \in \Gamma - \Gamma_0$ , the set  $\Lambda\Gamma_0 \cap \gamma\Lambda\Gamma_0$  is empty;
- (3)  $\mathcal{C}\Gamma_0 \cap \gamma\mathcal{C}\Gamma_0$  is compact for every  $\gamma \in \Gamma - \Gamma_0$ ;
- (4) for every  $\epsilon > 0$ , there exists  $\kappa = \kappa(\epsilon) > 0$  such that  $\text{diam}(\mathcal{N}_\epsilon\mathcal{C}\Gamma_0 \cap \gamma\mathcal{N}_\epsilon\mathcal{C}\Gamma_0) \leq \kappa$  for every  $\gamma \in \Gamma - \Gamma_0$ . □

For every  $\xi$  in  $\partial_\infty X$ , the *Busemann function at  $\xi$*  is the map  $\beta_\xi$  from  $X \times X$  to  $\mathbb{R}$  defined by

$$\beta_\xi(x, y) = \lim_{t \rightarrow +\infty} d(x, \xi_t) - d(y, \xi_t),$$

for any geodesic ray  $t \mapsto \xi_t$  ending at  $\xi$ .

Let  $C$  be a nonempty closed convex subset of  $X$ . We denote by  $\partial_\infty C$  its set of points at infinity, and by  $\partial C$  its boundary in  $X$ . The *closest point map* of  $C$  is the map  $\pi_C$  :

$(X \cup \partial_\infty X) \rightarrow (C \cup \partial_\infty C)$  which associates to a point  $x \in X$  its closest point in  $C$  in the usual sense, which fixes all points of  $\partial_\infty C$ , and which associates to a point  $\xi \in \partial_\infty X - \partial_\infty C$  the point of  $C$  which minimizes the map  $x \mapsto \beta_\xi(x, x_0)$  for any  $x_0$  in  $X$ . This map is continuous.

As in [HP5], we define the *distance-like map*  $d_C : (\partial_\infty X - \partial_\infty C)^2 \rightarrow [0, +\infty[$  associated to  $C$  as follows: For  $\xi, \xi' \in \partial_\infty X - \partial_\infty C$ , let  $x = \pi_C(\xi), x' = \pi_C(\xi')$  be their closest points in  $C$ . Let  $\xi_t, \xi'_t : [0, +\infty[ \rightarrow X$  be the geodesic rays starting at  $x, x'$  and converging to  $\xi, \xi'$  as  $t \rightarrow \infty$ . Let

$$d_C(\xi, \xi') = \lim_{t \rightarrow +\infty} e^{\frac{1}{2}d(\xi_t, \xi'_t) - t}. \quad (-1-)$$

The distance-like map is invariant by the diagonal action of the isometries of  $X$  preserving  $C$ , and generalizes the visual and Hamenstädt distances: If  $C$  consists of a single point  $x$ , then  $d_C$  is the *visual distance* on  $\partial_\infty X$  based at  $x$  (see for instance [Bou]), and we denote it by  $d_x$ . For balls, the distance-like map is a positive constant multiple of the visual distance based at their center. If  $C$  is a horoball with point at infinity  $\xi_0$ , then  $d_C$  is the *Hamenstädt distance* on  $\partial_\infty X - \{\xi_0\}$ , and we also denote it by  $d_{\xi_0, \partial C}$  to put the emphasis on  $\xi_0$ .

Although  $d_C$  is not always an actual distance on  $\partial_\infty X - \partial_\infty C$ , it follows from [HP5, Lemma 2.3 (4)] that for every  $\epsilon > 0$ , there exists  $\eta > 0$  such that for every  $\xi, \xi', \xi''$  in  $\partial_\infty X - \partial_\infty C$ , if  $d_C(\xi, \xi') < \eta$  and  $d_C(\xi', \xi'') < \eta$ , then  $d_C(\xi, \xi'') < \epsilon$ . In particular, the family of subsets

$$\left\{ W_n = \left\{ (\xi, \xi') \in (\partial_\infty X - \partial_\infty C)^2 : d_C(\xi, \xi') < \frac{1}{n+1} \right\} \right\}_{n \in \mathbb{N}}$$

is a countable separating system of entourages of a metrisable uniform structure on  $\partial_\infty X - \partial_\infty C$  (see [Bou, TG II.1]), whose induced topology is the usual one, by [HP5, Lemma 2.3 (1)], and which is invariant by the diagonal action of the isometries of  $X$  preserving  $C$ .

The *crossratio* of four pairwise distinct points  $a, b, c, d \in \partial_\infty X$  is

$$[a, b, c, d] = \frac{1}{2} \lim_{t \rightarrow +\infty} d(a_t, c_t) - d(c_t, b_t) + d(b_t, d_t) - d(d_t, a_t), \quad (-2-)$$

where  $a_t, b_t, c_t, d_t$  are any geodesic rays converging to  $a, b, c, d$ , respectively. For the existence of the limit and the properties of the crossratios, see [Ota] where the order convention is different, and [Bou] whose crossratio is the exponential of ours; we will be using the same expression as in [HP1, PP2]. If  $x_0 \in X$ , then

$$[a, b, c, d] = \log \frac{d_{x_0}(c, a)}{d_{x_0}(c, b)} \frac{d_{x_0}(d, b)}{d_{x_0}(d, a)}.$$

If  $H$  is a horosphere with center  $\xi \in \partial_\infty X$ , then for  $a, b, c, d \in \partial_\infty X - \{\xi\}$ ,

$$[a, b, c, d] = \log \frac{d_H(c, a)}{d_H(c, b)} \frac{d_H(d, b)}{d_H(d, a)}.$$

If  $\xi$  and  $a$  coincide, the above expression simplifies as follows, see [PP2, Section 3.1]:

$$[\xi, b, c, d] = \log \frac{d_H(d, b)}{d_H(c, b)}. \quad (-3-)$$

Let  $\xi \in X \cup \partial_\infty X$ . We say that a geodesic line  $\rho : ] - \infty, +\infty[ \rightarrow X$  (resp. geodesic ray  $\rho : [\iota_0, +\infty[ \rightarrow X$ ) starts from  $\xi$  if  $\xi = \rho(-\infty)$  (resp.  $\xi = \rho(\iota_0)$ ). We denote by  $T_\xi^1 X$  the space of geodesic rays or lines starting from  $\xi$ , endowed with the compact-open topology. If  $\epsilon > 0$  and  $A$  is a subset of  $X$ , we denote by  $\mathcal{N}_\epsilon A$  the (closed)  $\epsilon$ -neighborhood of  $A$  in  $X$ .

In [PP2], the penetration of geodesic rays and lines in neighbourhoods of convex subsets of  $X$  was studied by means of penetration maps. We now recall the definitions of three classes of such maps  $\ell_C, \mathbf{crp}_L, \mathbf{ftp}_L : T_\xi^1 X \rightarrow [0, +\infty]$ , where  $C$  is the (closed)  $\epsilon$ -neighborhood of a closed convex subset in  $X$  for some  $\epsilon > 0$ , and  $L$  is a geodesic line in  $X$ , with endpoints  $L_1, L_2$ .

- (1) The *penetration length map*  $\ell_C$  associates to every  $\rho$  in  $T_\xi^1 X$  the length of the intersection of  $C$  and of the image of  $\rho$ . (This intersection is connected by convexity; there was the assumption in [PP2] that  $\xi \notin C \cup \partial C$ , which is not necessary here.)
- (2) The *fellow-traveller penetration map*  $\mathbf{ftp}_L$  is defined by

$$\mathbf{ftp}_L : \rho \mapsto d(\pi_L(\xi), \pi_L(\rho(+\infty))) ,$$

with the convention that this distance is  $+\infty$  if  $\pi_L(\xi)$  or  $\pi_L(\rho(+\infty))$  is in  $\partial_\infty X$  (there was the assumption in [PP2] that  $\xi \notin \mathcal{N}_\epsilon L \cup \partial L$  where  $\epsilon > 0$  was arbitrary but fixed, which is not necessary here).

- (3) When  $\xi \in \partial_\infty X$ , the *crossratio penetration map*  $\mathbf{crp}_L$  is defined by

$$\mathbf{crp}_L : \rho \mapsto \max \{0, [\xi, L_1, \rho(+\infty), L_2], [\xi, L_2, \rho(+\infty), L_1]\} ,$$

if  $\xi, \rho(+\infty) \notin \{L_1, L_2\}$ , and  $\mathbf{crp}_L(\rho) = +\infty$  otherwise (there was the assumption in [PP2] that  $\xi \notin \partial L$ , which is not necessary here).

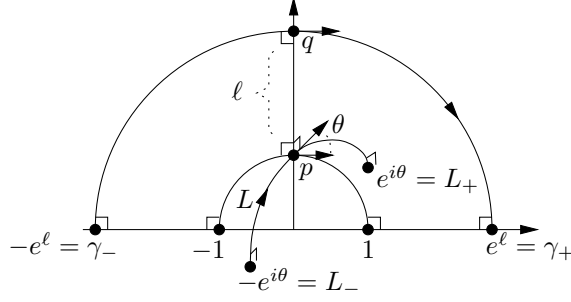
It is shown in Section 3.1 of [PP2] (and it is easy to see that the result is still true if  $\xi \in \mathcal{N}_\epsilon L \cup \partial L$ ) that the above maps are continuous and that, for every  $\epsilon > 0$ , we have, with the convention that  $x - y = 0$  if  $x = y = +\infty$ , the following inequalities

$$\|\mathbf{ftp}_L - \ell_{\mathcal{N}_\epsilon L}\|_\infty = \sup_{\rho \in T_\xi^1 X} |\mathbf{ftp}_L(\rho) - \ell_{\mathcal{N}_\epsilon L}(\rho)| \leq 2c'_1(\epsilon) + 2\epsilon , \quad (-4-)$$

where  $c'_1(\epsilon) = 2 \operatorname{argsinh}(\coth \epsilon)$ , and when  $\xi \in \partial_\infty X$ ,

$$\|\mathbf{crp}_L - \mathbf{ftp}_L\|_\infty = \sup_{\rho \in T_\xi^1 X} |\mathbf{crp}_L(\rho) - \mathbf{ftp}_L(\rho)| \leq 4 \log(1 + \sqrt{2}) . \quad (-5-)$$

In constant curvature, the crossratio penetration map has the following geometric interpretation (see for instance [Bea, §7.23–7.24], and [Fen, §V.3], for related formulas). Recall that the *complex distance*  $\ell + i\theta$  between two oriented geodesic lines  $\gamma$  and  $L$  (in this order) in  $\mathbb{H}_\mathbb{R}^n$  is defined as follows. It is  $0 + i0$  if they are simultaneously asymptotic at  $+\infty$  or at  $-\infty$ , and  $0 + i\pi$  if the terminal point at infinity of one is the original point at infinity of the other. Otherwise, if  $[p, q]$  is the common perpendicular arc (with  $p = q$  the common intersection point of  $\gamma$  and  $L$  if they intersect), where  $p \in L$ , then  $\ell = d(p, q)$  and  $\theta$  is the angle at  $p$  between the parallel transport of  $\gamma$  along  $[p, q]$  and  $L$ .



**Lemma 2.2** *Let  $\gamma$  and  $L$  be oriented geodesic lines in  $\mathbb{H}_{\mathbb{R}}^n$  with pairwise distinct endpoints  $\gamma_-, \gamma_+$  and  $L_-, L_+$ , respectively, and with complex distance  $\ell + i\theta$ . Then*

$$[\gamma_-, L_-, \gamma_+, L_+] = -\log \frac{\cosh \ell + \cos \theta}{2}.$$

*In particular,*

$$\text{crp}_L(\gamma) = \max \left\{ 0, -\log \frac{\cosh \ell \pm \cos \theta}{2} \right\}.$$

**Proof.** Using isometries, we may assume that  $\gamma$  and  $L$  are both contained in the upper halfspace model of  $\mathbb{H}_{\mathbb{R}}^3$ , that the common perpendicular segment  $[p, q]$  is on the vertical axis, with  $p$  at Euclidean height one and  $q$  above  $p$ , and that  $\gamma_+$  is a positive real number. By an easy computation, we then have

$$[\gamma_-, L_-, \gamma_+, L_+] = \log \frac{|\gamma_+ - \gamma_-| |L_+ - L_-|}{|\gamma_+ - L_-| |L_+ - \gamma_-|} = \log \frac{4e^\ell}{|e^\ell + e^{i\theta}|^2} = -\log \frac{\cosh \ell + \cos \theta}{2}.$$

The result follows.  $\square$

Note that in  $\mathbb{H}_{\mathbb{R}}^2$ , we have  $\ell > 0$  if and only if  $\theta = 0$  or  $\theta = \pi$ .

### 3 The approximation and spiraling spectra

In this section, we set up the general framework for our approximation results. We begin by the definition of the quadruples of data that we study.

**The definition of  $\mathcal{D}$ .** *Let  $\Gamma$  be a discrete group of isometries of a proper  $\text{CAT}(-1)$  geodesic space  $X$ . Let  $\Gamma_0$  be an almost malnormal convex-cocompact subgroup of infinite index in  $\Gamma$  and let  $C_0 = \mathcal{C}\Gamma_0$ . Let  $C_\infty$  be a nonempty closed convex subset of  $X$ , with stabilizer  $\Gamma_\infty$  in  $\Gamma$ . Assume that  $C_\infty$  does not contain  $\mathcal{C}\Gamma$ , that  $\Gamma_\infty \backslash \partial C_\infty$  is compact, and that the intersection of  $C_\infty$  and  $\gamma C_0$  is nonempty for only finitely many classes  $[\gamma]$  in  $\Gamma_\infty \backslash \Gamma/\Gamma_0$ . We will denote the quadruple of data  $(X, \Gamma, \Gamma_0, C_\infty)$  by  $\mathcal{D}$ .*

As  $\Lambda\Gamma_0$  has at least two points and since  $\Gamma_0$  has infinite index in  $\Gamma$ , note that  $\Gamma$  is nonelementary. By Proposition 2.1 (3) and since  $C_0$  is noncompact, the subgroup  $\Gamma_0$  is the stabilizer of  $C_0$  in  $\Gamma$ . By the discreteness of  $\Gamma$  and the cocompactness of  $\Gamma_0$ , a compact subset of  $X$  intersects only finitely many  $\gamma C_0$  for  $\gamma \in \Gamma/\Gamma_0$ .

Recall that the *distance* between two subsets  $A, B$  of  $X$  is  $d(A, B) = \inf_{a \in A, b \in B} d(a, b)$ . For every  $r = [\gamma]$  in  $\Gamma_\infty \backslash \Gamma/\Gamma_0$ , define

$$D(r) = d(C_\infty, \gamma C_0),$$

which does not depend on the choice of the representative  $\gamma$  of  $r$ . Using a convexity argument, the cocompactness of the action of  $\Gamma_\infty$  on  $\partial C_\infty$  and the fact that only finitely many translates of  $C_0$  meet a given compact subset, it follows that  $C_\infty \cap \gamma C_0$  is empty if and only if  $D(r) > 0$ , and that then  $\partial_\infty C_\infty \cap \gamma \partial_\infty C_0$  is also empty. For the same reasons, the following result also holds, see [HP5, Lemma 4.1] for a proof in the case  $\Gamma_0 = \Gamma_\infty$ .

**Lemma 3.1** *For every  $T \geq 0$ , there are only finitely many elements  $r$  in  $\Gamma_\infty \backslash \Gamma / \Gamma_0$  such that  $D(r) \leq T$ .  $\square$*

**Lemma 3.2** *The set of double cosets  $\Gamma_\infty \backslash \Gamma / \Gamma_0$  is infinite.*

**Proof.** We first claim that there exists a hyperbolic element  $\gamma$  in  $\Gamma$  whose attractive fixed point  $\gamma_+$  does not belong to  $\partial_\infty C_\infty$ . The limit set  $\Lambda\Gamma$  is the closure of the set of attractive fixed points of elements of  $\Gamma$ , since  $\Gamma$  is nonelementary. Thus, if no such  $\gamma_+$  exists,  $\Lambda\Gamma$  is contained in  $\partial_\infty C_\infty$ , which contradicts the hypotheses on  $\mathscr{D}$  by the convexity of  $C_\infty$ .

By Lemma 2.1 (2), and since  $\Gamma \neq \Gamma_0$ , there exists  $\gamma' \in \Gamma$  such that the repulsive fixed point  $\gamma_-$  of  $\gamma$  does not belong to  $\gamma' \Lambda \Gamma_0$ . Hence the sequence of closed subsets  $(\gamma^n \gamma' C_0)_{n \in \mathbb{N}}$  of the compact space  $X \cup \partial_\infty X$  converges to the singleton  $\{\gamma_+\}$  as  $n$  goes to  $+\infty$ . This implies that  $D([\gamma^n \gamma']) = d(C_\infty, \gamma^n \gamma' C_0)$  converges to  $+\infty$  as  $n$  goes to  $+\infty$ . In particular, the set  $\{D(r) : r \in \Gamma_\infty \backslash \Gamma / \Gamma_0\}$  is infinite, and the result follows.  $\square$

The *link* of  $\mathscr{D}$  (which depends only on  $X, \Gamma$  and  $C_\infty$ ) is

$$\text{Lk}_\infty = \Gamma_\infty \backslash (\Lambda_c \Gamma - \partial_\infty C_\infty).$$

The quotient space  $\Gamma_\infty \backslash (\Lambda\Gamma - \partial_\infty C_\infty)$ , which contains  $\text{Lk}_\infty$ , is compact, since the closest point map from  $\partial_\infty X - \partial_\infty C_\infty$  to  $\partial C_\infty$  is continuous and  $\Gamma_\infty$ -equivariant. Furthermore,  $\text{Lk}_\infty$  is dense in  $\Gamma_\infty \backslash (\Lambda\Gamma - \partial_\infty C_\infty)$ . For every  $r = [\gamma]$  in  $\Gamma_\infty \backslash \Gamma / \Gamma_0$  such that  $D(r) > 0$ , let

$$\Lambda_r = \pi_\infty(\gamma \partial_\infty C_0)$$

be the image by the canonical projection  $\pi_\infty : \Lambda_c \Gamma - \partial_\infty C_\infty \rightarrow \text{Lk}_\infty$  of  $\gamma \partial_\infty C_0 = \gamma \Lambda \Gamma_0$ . Note that  $\gamma \partial_\infty C_0$  is indeed contained in  $\Lambda_c \Gamma$  since  $\Gamma_0$  is convex-cocompact, and that  $\partial_\infty C_\infty$  is disjoint from  $\gamma \partial_\infty C_0$  if  $D(r) > 0$ . Furthermore, the sets  $\Lambda_r$  are compact subsets of  $\text{Lk}_\infty$ , that are pairwise disjoint by Lemma 2.1 (2), and the union

$$\text{Lk}_{\infty,0} = \bigsqcup_{r \in \Gamma_\infty \backslash \Gamma / \Gamma_0, D(r) > 0} \Lambda_r \tag{-6-}$$

is dense in  $\text{Lk}_\infty$ . In this this paper, we study how dense  $\text{Lk}_{\infty,0}$  is in  $\text{Lk}_\infty$ .

Let  $\tilde{d}_\infty : (\partial_\infty X - \partial_\infty C_\infty)^2 \rightarrow [0, +\infty[$  be the distance-like map of  $C_\infty$ , and let  $d_\infty$  be its quotient map on  $\text{Lk}_\infty$ , which defines, as in Section 2, a metrisable uniform structure on  $\Gamma_\infty \backslash (\partial_\infty X - \partial_\infty C_\infty)$ , inducing the quotient topology. We endow the double coset space  $\Gamma_\infty \backslash \Gamma / \Gamma_0$  with the Fréchet filter of the complements of the finite subsets, and denote by  $\liminf_r f(r)$  the lower limit of a real valued map  $f$  along this filter. The *approximation constant* of  $\xi \in \text{Lk}_\infty - \text{Lk}_{\infty,0}$  is

$$c(\xi) = \liminf_r e^{D(r)} d_\infty(\xi, \Lambda_r), \tag{-7-}$$



and the subset of  $[0, +\infty]$  defined by

$$\mathrm{Sp}(\mathcal{D}) = \{c(\xi) : \xi \in \mathrm{Lk}_\infty - \mathrm{Lk}_{\infty,0}\}$$

is called the *approximation spectrum* of points of  $\mathrm{Lk}_\infty$  by points of  $\mathrm{Lk}_{\infty,0}$ . We define the *Hurwitz constant* of  $\mathcal{D}$  as

$$K_{\mathcal{D}} = \sup \mathrm{Sp}(\mathcal{D}) \in [0, +\infty].$$

**Remark 3.3** Let us give some background and motivations for the terminology introduced in this paper: In the definition of the quadruple of data  $\mathcal{D}$ , let us specialise to the situation when  $X$  is a Riemannian manifold with pinched negative curvature and  $\Gamma$  is geometrically finite. If we change the assumptions on  $\Gamma_0$  and  $\Gamma_\infty$  such that  $\Gamma_0 = \Gamma_\infty$  is the stabilizer of a parabolic fixed point  $\xi_\infty$  of  $\Gamma$  and  $C_\infty$  is the maximal precisely invariant horoball centered at  $\xi_\infty$ , then we recover the framework of Diophantine approximation in negatively curved manifolds that was developed in [HP2, HP3, HP4, PP2, PP1]. In this situation,  $\Gamma_0$  is not convex-cocompact and  $C_0 = C_\infty$ , and the new quadruple does not have the properties we require of the quadruples of data in this paper. However, if we take  $\mathrm{Lk}_\infty = \Gamma_\infty \backslash \Lambda_c \Gamma$ ,  $\Lambda_{[\gamma]} = \Gamma_\infty \gamma(+\infty)$ ,  $\mathrm{Lk}_{\infty,0} = \pi_\infty(\Gamma \cdot \infty)$ , all the constructions in Section 3 are still valid.

In particular, let  $X$  be the upper halfplane model of the real hyperbolic plane  $\mathbb{H}_{\mathbb{R}}^2$ , let  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ , let  $C_\infty$  be the horoball in  $X$  of points having Euclidean height at least 1, let  $\Gamma_0 = \Gamma_\infty$  be the cyclic group generated by  $z \mapsto z + 1$ , and let  $\mathcal{D} = (X, \Gamma, \Gamma_0, \Gamma_\infty)$ . Then (see [HP2, section 2.3], [PP1])  $\mathrm{Lk}_\infty = (\mathbb{R} - \mathbb{Q})/\mathbb{Z}$ ; for every  $r = [\gamma] \in \Gamma_\infty \backslash (\Gamma - \Gamma_\infty)/\Gamma_\infty$ , we have  $D(r) = 2 \log q$  if  $\gamma\infty = p/q$  with  $p \in \mathbb{Z}$  and  $q \in \mathbb{N} - \{0\}$  relatively prime; for every  $\xi \in \mathbb{R} - \mathbb{Q}$ , the approximation constant  $c(\xi \bmod \mathbb{Z})$  is the classical approximation constant of the irrational number  $\xi$  by rational numbers; the approximation spectrum  $\mathrm{Sp}(\mathcal{D})$  is the classical Lagrange spectrum, and the Hurwitz constant  $K_{\mathcal{D}}$  is the classical Hurwitz constant  $\frac{1}{\sqrt{5}}$  (see the introduction for the definition of these objects).

The end of this section is devoted to the study of geometric examples.

Let  $M$  be a nonelementary complete connected Riemannian manifold with sectional curvature at most  $-1$ , and dimension at least 2. Let  $A_0$  be a closed geodesic in  $M$ , not necessarily simple (for more general  $A_0$ 's, as for instance in the introduction, we refer to the general setup). Let  $A_\infty$  be a closed codimension 0 submanifold of  $M$  with smooth connected compact locally convex boundary, disjoint from  $A_0$ .

Recall that a locally geodesic ray  $\rho$  in  $M$  is *recurrent* if, as a map from  $[0, +\infty[$  to  $M$ , it is not proper, i.e. if there exist a compact subset  $K$  of  $M$  and a sequence  $(t_n)_{n \in \mathbb{N}}$  in  $[0, +\infty[$  converging to  $+\infty$  such that  $\rho(t_n) \in K$  for every  $n$ . We say that a locally geodesic ray  $\rho$  in  $M$  *spirals around*  $A_0$  if  $d(\rho(t), A_0)$  converges to 0 as  $t$  goes to  $+\infty$ .

Let  $\mathrm{Lk}_{A_\infty}(M)$  be the set of recurrent locally geodesic rays starting perpendicularly from  $\partial A_\infty$  and exiting  $A_\infty$ , and let  $\mathrm{Lk}_{A_\infty, A_0}(M)$  be the subset of elements of  $\mathrm{Lk}_{A_\infty}(M)$  that spiral around  $A_0$ . Recall that a geodesic line in a complete simply connected manifold that crosses a horosphere perpendicularly starts from (up to time reversal) the point at infinity of this horosphere. Hence when  $A_\infty$  is a small Margulis neighborhood  $N_\infty$  of a cusp  $e_\infty$  with compact boundary, it is equivalent to require that a geodesic ray exits perpendicularly from  $A_\infty$  or that the negative subray of the geodesic line containing it is a minimizing geodesic ray starting from the boundary of  $N_\infty$  and converging to  $e_\infty$ .

For every  $\rho, \rho'$  in  $\mathrm{Lk}_{A_\infty}(M)$ , and every  $t \in [0, +\infty[$ , let  $\ell_t$  be the shortest length of a path homotopic (relative to the endpoints) to the path obtained by following (the inverse

of)  $\rho$  from  $\rho(t)$  to  $\rho(0)$ , then following a shortest path contained in  $A_\infty$  between  $\rho(0)$  and  $\rho'(0)$ , then following  $\rho'$  from  $\rho'(0)$  to  $\rho'(t)$ ; define

$$\delta_{A_\infty}(\rho, \rho') = \lim_{t \rightarrow +\infty} e^{\frac{1}{2}\ell_t - t}. \quad (-8-)$$

(We will show below that the limit does exist). For every  $\bar{r}$  in  $\text{Lk}_{A_\infty, A_0}(M)$ , let  $D(\bar{r})$  be the shortest length of a geodesic segment  $[a, b]$  with  $a$  in  $\partial A_\infty$ ,  $b$  in  $A_0$  such that there exists a (locally) geodesic ray  $\rho$  starting from  $b$ , contained in  $A_0$ , such that the path obtained by following  $[a, b]$  from  $a$  to  $b$  and then  $\rho$  is properly homotopic to  $\bar{r}$  while its origin remains in  $\partial A_\infty$ . The *spiraling constant* around  $A_0$  of an element  $\bar{\xi}$  of  $\text{Lk}_{A_\infty}(M)$  is

$$c(\bar{\xi}) = \liminf_{\bar{r} \in \text{Lk}_{A_\infty, A_0}(M), D(\bar{r}) \rightarrow +\infty} e^{D(\bar{r})} \delta_{A_\infty}(\bar{\xi}, \bar{r}),$$

and the subset of  $[0, +\infty]$  defined by

$$\text{Sp}_{A_\infty, A_0}(M) = \{c(\bar{\xi}) : \bar{\xi} \in \text{Lk}_{A_\infty}(M) - \text{Lk}_{A_\infty, A_0}(M)\}$$

is called the *spiraling spectrum* of geodesic rays in  $\text{Lk}_{A_\infty}(M)$  around  $A_0$ . These notions coincide with the similarly named ones in the introduction if  $M$  has finite volume and  $A_\infty$  is the chosen Margulis neighborhood of the cusp  $e$ .

To see the connection with the framework outlined at the beginning of this section, we may define a quadruple of data

$$\mathcal{D}_{M, A_0, A_\infty} = (X, \Gamma, \Gamma_0, C_\infty)$$

as follows. If  $\widetilde{M} \rightarrow M$  is a universal Riemannian covering of  $M$  with covering group  $\Gamma$ , let  $X = \mathcal{C}\Gamma$  be the convex hull of  $\Gamma$ , let  $\Gamma_0$  be the cyclic group which is the stabilizer in  $\Gamma$  of a fixed lift of  $A_0$  to  $\widetilde{M}$ , and let  $C_\infty$  be the intersection with  $X$  of a fixed connected component of the preimage of  $A_\infty$  in  $\widetilde{M}$ .

Note that the image in  $M$  of a geodesic ray  $\rho$  in  $\widetilde{M}$  is recurrent if and only if the endpoint at infinity of  $\rho$  is a conical limit point of  $\Gamma$ . Consider the map  $\widetilde{\Phi}$  from  $\Lambda_c \Gamma - \partial_\infty C_\infty$  to  $\text{Lk}_{A_\infty}(M)$ , which associates to an element  $\xi$  of  $\Lambda_c \Gamma - \partial_\infty C_\infty$  the image in  $M$  of the geodesic ray in  $X$  starting from the closest point on  $C_\infty$  to  $\xi$  and converging to  $\xi$ . By taking the quotient by  $\Gamma_\infty$ , this map induces a homeomorphism  $\Phi : \text{Lk}_\infty \rightarrow \text{Lk}_{A_\infty}(M)$ , which maps  $\text{Lk}_{\infty, 0}$  to  $\text{Lk}_{A_\infty, A_0}(M)$ . By construction, the map  $\Phi$  preserves the maps  $d_\infty$  and  $\delta_{A_\infty}$  (which proves that the limit in (-8-) exists).

For every  $\bar{r}$  in  $\text{Lk}_{A_\infty, A_0}(M)$ , by definition of  $\text{Lk}_{\infty, 0}$  (see Equation (-6-)), there exists a unique  $r$  in  $\Gamma_\infty \backslash \Gamma / \Gamma_0$  such that  $\Phi^{-1}(\bar{r})$  belongs to  $\Lambda_r$ . The map  $\bar{r} \mapsto r$  from  $\text{Lk}_{A_\infty, A_0}(M)$  to  $\Gamma_\infty \backslash \Gamma / \Gamma_0$  satisfies  $D(r) = D(\bar{r})$ ; the complementary subset in  $\Gamma_\infty \backslash \Gamma / \Gamma_0$  of its image is finite, since there are only finitely many  $r \in \Gamma_\infty \backslash \Gamma / \Gamma_0$  such that  $D(r) \leq 0$ ; every point in the image has at most two preimages, since  $\Lambda_r$  has at most two points.

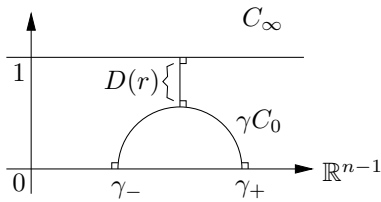
Hence, for every  $\xi$  in  $\text{Lk}_\infty$ , we have by construction  $c(\xi) = c(\Phi(\xi))$ . Therefore, as  $\Phi$  is surjective,

$$\text{Sp}(\mathcal{D}_{M, A_0, A_\infty}) = \text{Sp}_{A_\infty, A_0}(M), \quad (-9-)$$

and we conclude that to obtain results on the spiraling spectrum, it is sufficient to prove results on the approximation spectrum.

### Example 1: Spiraling around a closed geodesic in a real hyperbolic manifold

We will use the upper halfspace model of the real hyperbolic  $n$ -space  $\mathbb{H}_{\mathbb{R}}^n$ , so that  $\partial_{\infty}\mathbb{H}_{\mathbb{R}}^n = \mathbb{R}^{n-1} \cup \{\infty\}$ . Let  $\Gamma$  be a non elementary discrete subgroup of isometries of  $\mathbb{H}_{\mathbb{R}}^n$ . Assume that  $\infty$  is a parabolic fixed point of  $\Gamma$ , with stabilizer  $\Gamma_{\infty}$ , and that the interior of the horoball  $\mathcal{H}_1$  of points of Euclidean height at least one is *precisely invariant* under  $\Gamma_{\infty}$  (i.e. for all  $\gamma \in \Gamma - \Gamma_{\infty}$ , the horoballs  $\mathcal{H}_1$  and  $\gamma\mathcal{H}_1$  have disjoint interior). When  $\Gamma$  is torsion-free and has finite covolume,  $\mathcal{H}_1$  covers a Margulis neighborhood of a cusp of  $\Gamma \backslash \mathbb{H}_{\mathbb{R}}^n$ . Define  $C_{\infty} = \mathcal{H}_1 \cap \mathcal{C}\Gamma$ , and assume that  $\Gamma_{\infty} \backslash \partial C_{\infty}$  is compact (for instance if  $\Gamma$  is geometrically finite, and in particular if  $\Gamma$  has finite covolume). Let  $\gamma_0$  be a hyperbolic element of  $\Gamma$ , with translation axis  $C_0$ . Let  $\Gamma_0$  be the stabilizer of  $C_0$  in  $\Gamma$ , which contains the cyclic group generated by  $\gamma_0$  as a subgroup of finite index. The quadruple  $\mathcal{D} = (\mathcal{C}\Gamma, \Gamma, \Gamma_0, C_{\infty})$  satisfies the hypotheses of the beginning of the section.



For every  $\gamma$  in  $\Gamma$ , let  $\gamma_{\pm}$  be the fixed points of the hyperbolic element  $\gamma\gamma_0\gamma^{-1}$ . If  $r$  is the double coset of  $\gamma$  in  $\Gamma_{\infty} \backslash \Gamma / \Gamma_0$ , and if  $D(r) = d(C_{\infty}, \gamma C_0) > 0$ , then by an easy computation in hyperbolic geometry

$$D(r) = -\log \frac{1}{2} \|\gamma_+ - \gamma_-\|, \quad (-10-)$$

where  $\|\cdot\|$  is the standard Euclidean norm on  $\mathbb{R}^{n-1}$ .

Let  $\mathcal{R}_{\Gamma_0}$  be the set of fixed points of the conjugates of  $\gamma_0$ , endowed with its Fréchet filter. For every  $\alpha \in \mathcal{R}_{\Gamma_0}$ , let  $\alpha^*$  be the other endpoint of the translation axis of a conjugate of  $\gamma_0$  containing  $\alpha$  at infinity. The distance-like map  $d_{C_{\infty}}$  coincides with the Hamenstädt distance  $d_{\infty, \partial \mathcal{H}_1}$  on the limit set  $\Lambda\Gamma$ . In  $\mathbb{R}^{n-1} = \partial_{\infty}\mathbb{H}_{\mathbb{R}}^n - \{\infty\}$ , the Hamenstädt distance  $d_{\infty, \partial \mathcal{H}_1}$  coincides with the Euclidean metric (see for instance [HP2]). For every  $\xi$  in  $\Lambda_c\Gamma - \mathcal{R}_{\Gamma_0}$  and  $\alpha \in \mathcal{R}_{\Gamma_0}$ , define  $l_{\alpha} + i\theta_{\alpha}$  to be the complex distance between the oriented geodesic lines from  $\infty$  to  $\xi$  and from  $\alpha^*$  to  $\alpha$ . Note that  $l_{\alpha^*} = l_{\alpha}$  and  $\theta_{\alpha^*} = \theta_{\alpha} + \pi$ . Then, we have by Equation (-10-), by Equation (-3-) and by Lemma 2.2, respectively,

$$c(\Gamma_{\infty}\xi) = \liminf_{\alpha \in \mathcal{R}_{\Gamma_0}} 2 \frac{\|\xi - \alpha\|}{\|\alpha - \alpha^*\|} = \liminf_{\alpha \in \mathcal{R}_{\Gamma_0}} 2 e^{-[\infty, \alpha, \xi, \alpha^*]} = \liminf_{\alpha \in \mathcal{R}_{\Gamma_0}} \cosh l_{\alpha} - \cos \theta_{\alpha}. \quad (-11-)$$

Furthermore, by definition,  $\text{Sp}(\mathcal{D}) = \{c(\xi) : \xi \in \Gamma_{\infty} \backslash (\Lambda_c\Gamma - \mathcal{R}_{\Gamma_0})\}$ .

### Example 2: Spiraling around a closed geodesic in a complex hyperbolic manifold

Let  $n \geq 2$ . The elements of  $\mathbb{C}^{n-1}$  are identified with their coordinate column vectors and for every  $w, w'$  in  $\mathbb{C}^{n-1}$ , we denote by  $w^*w'$  their standard Hermitian product, where  $w^*$  is the conjugate transpose of  $w$ , and  $|w|^2 = w^*w$ .

Let  $\mathbb{H}_{\mathbb{C}}^n$  be the Siegel domain model of the complex hyperbolic  $n$ -space. Its underlying manifold is

$$\mathbb{H}_{\mathbb{C}}^n = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : 2 \text{Re } w_0 - |w|^2 > 0\}.$$

The complex hyperbolic distance  $d_{\mathbb{H}_{\mathbb{C}}^n}$  is defined by the Riemannian metric

$$ds_{\mathbb{C}}^2 = \frac{4}{(2 \text{Re } w_0 - |w|^2)^2} ((dw_0 - dw^* w)((\overline{dw_0} - w^* dw) + (2 \text{Re } w_0 - |w|^2) dw^* dw)$$

(see for instance [Gol, Sect. 4.1]). The complex hyperbolic space has constant holomorphic sectional curvature  $-1$ , hence its real sectional curvatures are bounded between  $-1$  and

$-\frac{1}{4}$ . If we want to consider  $\mathbb{H}_{\mathbb{C}}^n$  as a CAT(-1) space, we will use the distance  $d'_{\mathbb{H}_{\mathbb{C}}^n} = \frac{1}{2}d_{\mathbb{H}_{\mathbb{C}}^n}$ . The boundary at infinity of  $\mathbb{H}_{\mathbb{C}}^n$  is

$$\partial_{\infty}\mathbb{H}_{\mathbb{C}}^n = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : 2 \operatorname{Re} w_0 - |w|^2 = 0\} \cup \{\infty\}.$$

The horoballs centered at  $\infty$  in  $\mathbb{H}_{\mathbb{C}}^n$  are the subspaces

$$\mathcal{H}_s = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : 2 \operatorname{Re} w_0 - |w|^2 \geq s\},$$

for  $s > 0$ . The submanifold  $\{(w_0, w) \in \mathbb{H}_{\mathbb{C}}^n : w = 0\}$ , with the induced Riemannian metric, is the right halfplane model of the real hyperbolic plane with constant curvature  $-1$ , and it is totally geodesic in  $\mathbb{H}_{\mathbb{C}}^n$ . Hence the map  $c_0 : \mathbb{R} \rightarrow \mathbb{H}_{\mathbb{C}}^n$  defined by  $c_0 : t \mapsto (e^{-t}, 0)$  is a unit speed geodesic line for  $d_{\mathbb{H}_{\mathbb{C}}^n}$ , starting from  $\infty$ , ending at  $(0, 0) \in \partial_{\infty}\mathbb{H}_{\mathbb{C}}^n$  and meeting the horosphere  $\partial\mathcal{H}_2$  at time  $t = 0$ . In particular, the distance between two horospheres centered at  $\infty$  is

$$d_{\mathbb{H}_{\mathbb{C}}^n}(\partial\mathcal{H}_s, \partial\mathcal{H}_{s'}) = |\log(s'/s)|. \quad (-12-)$$

Let  $G_{\mathcal{H}_2}$  be the group of isometries of  $\mathbb{H}_{\mathbb{C}}^n$  preserving (globally)  $\mathcal{H}_2$ . The *Cygan distance*  $d_{\text{Cyg}}$  (see for instance [Gol, page 160]) is the unique distance on  $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^n - \{\infty\}$  invariant under  $G_{\mathcal{H}_2}$  such that

$$d_{\text{Cyg}}((w_0, w), (0, 0)) = \sqrt{2|w_0|}.$$

Similarly, we introduced in [PP2, Lem. 6.1] the *modified Cygan distance*  $d'_{\text{Cyg}}$ , as the unique distance on  $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^n - \{\infty\}$  invariant under  $G_{\mathcal{H}_2}$  such that

$$d'_{\text{Cyg}}((w_0, w), (0, 0)) = \sqrt{2|w_0| + |w|^2}.$$

Let  $\Gamma$  be a discrete subgroup of isometries of  $\mathbb{H}_{\mathbb{C}}^n$  with finite covolume. Assume that  $\infty$  is a parabolic fixed point, whose stabilizer in  $\Gamma$  we denote by  $\Gamma_{\infty}$ , such that the horoball  $\mathcal{H}_2$  is precisely invariant under  $\Gamma_{\infty}$ . Let  $\gamma_0$  be a hyperbolic element of  $\Gamma$ , with translation axis  $C_0$ . Let  $\Gamma_0$  be the stabilizer of  $C_0$  in  $\Gamma$ . The quadruple  $\mathcal{D} = (\mathbb{H}_{\mathbb{C}}^n, \Gamma, \Gamma_0, \mathcal{H}_2)$  satisfies the hypotheses of the beginning of the section.

In the following result, we compute the associated map  $D : \Gamma_{\infty} \backslash \Gamma / \Gamma_0 \rightarrow \mathbb{R}$  where  $D([\gamma]) = d'_{\mathbb{H}_{\mathbb{C}}^n}(\mathcal{H}_2, \gamma C_0)$ .

**Lemma 3.4** *If  $[\gamma] \in \Gamma_{\infty} \backslash \Gamma / \Gamma_0$  and  $D([\gamma]) > 0$  then, with  $\gamma_{\pm}$  the fixed points of the hyperbolic element  $\gamma\gamma_0\gamma^{-1}$ , we have*

$$D([\gamma]) = -\log \frac{1}{2} \frac{d_{\text{Cyg}}(\gamma_-, \gamma_+)^2}{d'_{\text{Cyg}}(\gamma_-, \gamma_+)}.$$

**Proof.** Identify  $\mathbb{H}_{\mathbb{C}}^n \cup \partial_{\infty}\mathbb{H}_{\mathbb{C}}^n$  with its image in the projective space  $\mathbb{P}_n(\mathbb{C})$  by  $(w_0, w) \mapsto [w_0 : w : 1]$  and  $\infty \mapsto [1 : 0 : 0]$ .

Let  $(w_0, w) \in \partial_{\infty}\mathbb{H}_{\mathbb{C}}^n$  such that  $w_0 \neq 0$ . The projective action of  $g = \begin{pmatrix} 1 & 0 & 0 \\ \frac{w}{w_0} & \text{Id} & 0 \\ \frac{1}{w_0} & \frac{w^*}{\overline{w_0}} & 1 \end{pmatrix}$

is an isometry of (the image of) the Siegel domain, fixing the point  $(0, 0)$  of  $\partial_{\infty}\mathbb{H}_{\mathbb{C}}^n$ , and mapping  $\infty$  to  $(w_0, w)$ , hence sending the geodesic line between  $(0, 0)$  and  $\infty$  to the one between  $(0, 0)$  and  $(w_0, w)$ . Therefore the map  $\gamma_{w_0, w} : \mathbb{R} \rightarrow \mathbb{H}_{\mathbb{C}}^n$  defined by

$$\gamma_{w_0, w}(t) = \left( \frac{w_0}{1 + w_0 e^t}, \frac{w}{1 + w_0 e^t} \right)$$

is a geodesic line with endpoints  $(w_0, w)$  and  $(0, 0)$  in  $\partial_\infty \mathbb{H}_\mathbb{C}^n$ . The point  $\gamma_{w_0, w}(t)$  belongs to the horosphere  $\mathcal{H}_{s(t)}$ , where

$$s(t) = 2\operatorname{Re}\left(\frac{w_0}{1 + w_0 e^t}\right) - \left|\frac{w}{1 + w_0 e^t}\right|^2 = \frac{2\operatorname{Re}(w_0(1 + \bar{w}_0 e^t)) - |w|^2}{|1 + e^t w_0|^2} = \frac{2e^t |w_0|^2}{|1 + e^t w_0|^2}.$$

If  $w_0 = r e^{i\phi}$  (in polar coordinates) and if  $T = e^t$ , then

$$s(t) = \frac{2Tr^2}{T^2 r^2 + 2Tr \cos \phi + 1}.$$

The map  $t \mapsto s(t)$  reaches its maximum at  $T = 1/r$ , that is at  $t = -\log |w_0|$ , and its maximum value is

$$s = \frac{|w_0|}{1 + \frac{\operatorname{Re} w_0}{|w_0|}} = \frac{|w_0|^2}{|w_0| + |w|^2/2} = \frac{1}{2} \frac{d_{\operatorname{Cyg}}(0, (w_0, w))^4}{d'_{\operatorname{Cyg}}(0, (w_0, w))^2}.$$

The result then follows from Equation (-12-), since  $\mathcal{H}_2$  and  $\gamma C_0$  are disjoint if and only if  $s < 2$ .  $\square$

In  $\mathbb{H}_\mathbb{C}^n$  with its  $\operatorname{CAT}(-1)$  distance  $d'_{\mathbb{H}_\mathbb{C}^n}$ , the distance-like map  $d_{\mathcal{H}_2}$  coincides (as seen in Section 2) with the Hamenstädt distance  $d_{\infty, \partial \mathcal{H}_2}$ . Recall (see [HP3, Prop. 3.12]) that

$$d_{\infty, \partial \mathcal{H}_2} = \frac{1}{\sqrt{2}} d_{\operatorname{Cyg}}. \quad (-13-)$$

Let  $\mathcal{R}_{\Gamma_0}$  be the set of fixed points of the conjugates of  $\gamma_0$ , endowed with its Fréchet filter. For every  $\alpha \in \mathcal{R}_{\Gamma_0}$ , let  $\alpha^*$  be the other endpoint of the translation axis of a conjugate of  $\gamma_0$  containing  $\alpha$  at infinity. We therefore have

$$\operatorname{Sp}(\mathcal{D}) = \left\{ c(\Gamma_\infty \xi) = \liminf_{\alpha \in \mathcal{R}_{\Gamma_0}} \sqrt{2} \frac{d'_{\operatorname{Cyg}}(\alpha, \alpha^*) d_{\operatorname{Cyg}}(\xi, \alpha)}{d_{\operatorname{Cyg}}(\alpha, \alpha^*)^2} : \xi \in \Lambda_c \Gamma - \mathcal{R}_{\Gamma_0} \right\}. \quad (-14-)$$

## 4 The basic properties of the approximation spectra

Let  $\mathcal{D} = (X, \Gamma, \Gamma_0, C_\infty)$  be a quadruple of data as defined in Section 3. In this section, we study the upper bound of the approximation spectrum  $\operatorname{Sp}(\mathcal{D}) \subset \mathbb{R}$  of  $\mathcal{D}$ , and we give a closedness result for  $\operatorname{Sp}(\mathcal{D})$ .

### 4.1 The nontriviality of the approximation spectra

A map  $f : [0, +\infty[ \rightarrow ]0, +\infty[$  is called *slowly varying* if it is measurable and if there exist constants  $B > 0$  and  $A \geq 1$  such that for every  $x, y$  in  $\mathbb{R}_+$ , if  $|x - y| \leq B$ , then  $f(y) \leq A f(x)$ . Recall that this implies that  $f$  is locally bounded, hence it is locally integrable; also, if  $\log f$  is Lipschitz, then  $f$  is slowly varying.

Let  $\epsilon$  be a positive real number, and let  $f, g : [0, +\infty[ \rightarrow ]0, +\infty[$ . A geodesic ray or line  $\rho$  in  $X$  will be called  $(\epsilon, g)$ -*Liouville* if there exist a sequence  $(t_n)_{n \in \mathbb{N}}$  of positive times converging to  $+\infty$  and a sequence  $(\gamma_n)_{n \in \mathbb{N}}$  of elements of  $\Gamma$  such that  $\rho(t)$  belongs to  $\mathcal{N}_\epsilon(\gamma_n C_0)$  for every  $t$  in  $[t_n, t_n + g(t_n)]$ . A geodesic ray or line  $\rho$  in  $X$  such that  $\rho(+\infty) \notin \partial_\infty C_\infty$  will be called *f-well approximated* if there exist infinitely many  $\gamma$  in  $\Gamma/\Gamma_0$  such that

$$\tilde{d}_\infty(\rho(+\infty), \gamma \Lambda \Gamma_0) \leq f(D([\gamma])) e^{-D([\gamma])}.$$

The following result is proved in [HP5, Lemma 4.8].

**Lemma 4.1** *Let  $f : [0, +\infty[ \rightarrow ]0, 1[$  be slowly varying, and let  $g : t \mapsto -\log f(t)$ . Let  $\epsilon > 0$ . There exists  $c = c(\epsilon, f) > 0$  such that for every geodesic ray or line  $\rho$  in  $X$  such that  $\rho(+\infty) \notin \partial_\infty C_\infty \cup \bigcup_{\gamma \in \Gamma} \gamma \partial_\infty C_0$ , if  $\rho$  is  $(\epsilon, g)$ -Liouville, then  $\rho$  is  $(cf)$ -well approximated, and conversely, if  $\rho$  is  $(\frac{1}{c}f)$ -well approximated, then  $\rho$  is  $(\epsilon, g)$ -Liouville.  $\square$*

Our first result says in particular that  $\{0\} \not\subset \text{Sp}(\mathcal{D})$ . We refer to Section 5 for much stronger results for particular cases of  $\mathcal{D}$ .

**Proposition 4.2** *The approximation spectrum of  $\mathcal{D}$  contains 0 as a nonisolated point, and hence the Hurwitz constant of  $\mathcal{D}$  is positive.*

The following consequence, amongst other similar ones, follows from Equation (-9-).

**Corollary 4.3** *Let  $M$  be a nonelementary complete connected Riemannian manifold with sectional curvature at most  $-1$  and dimension at least 2. Let  $A_0$  be a closed geodesic in  $M$ , and let  $A_\infty$  be a closed codimension 0 submanifold of  $M$  with smooth compact locally convex boundary, disjoint from  $A_0$ . Then the spiraling spectrum  $\text{Sp}_{A_\infty, A_0}(M)$  around  $A_0$  contains 0 as a nonisolated point.  $\square$*

**Proof of Proposition 4.2.** Let us first prove that there exists an element  $\gamma$  in  $\Gamma - \Gamma_0$  such that  $d(C_0, \gamma C_0)$  and  $d(C_\infty, \gamma C_0)$  are both as big as we need.

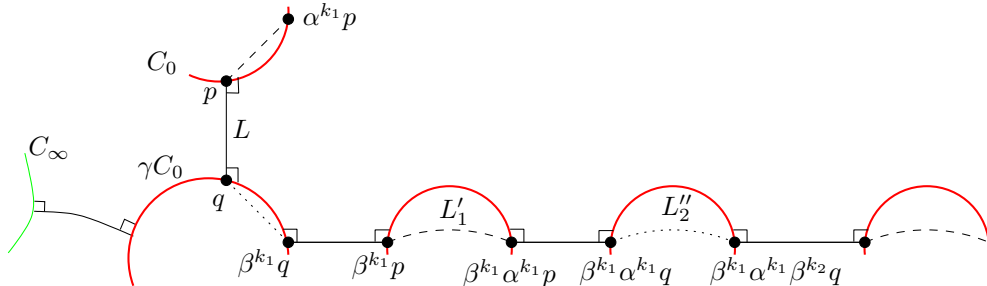
By the lemmata 3.1 and 3.2, there exists a nontrivial double class  $[\gamma_0] \in \Gamma_\infty \backslash \Gamma / \Gamma_0$  such that  $d(C_\infty, \gamma_0 C_0)$  is big. Since  $\Gamma_0$  contains a hyperbolic element, there exists a hyperbolic element  $\gamma_1$  in  $\Gamma$  whose attractive fixed point  $(\gamma_1)_+$  belongs to  $\gamma_0 \partial_\infty C_0$ , and in particular is not in  $\partial_\infty C_\infty$ . As  $\gamma_0 \notin \Gamma_0$  and  $\Gamma_0$  is almost malnormal, we have  $(\gamma_1)_+ \notin \partial_\infty C_0$  by Lemma 2.1 (2). Since  $(\gamma_1)_+ \notin \partial_\infty C_0 \cup \partial_\infty C_\infty$ , then for  $n$  big enough,  $\gamma = \gamma_1^n$  is an element in  $\Gamma - \Gamma_0$  such that  $d(C_0, \gamma C_0)$  and  $d(C_\infty, \gamma C_0)$  are both big enough.

Now, let  $\gamma$  be as above. Let  $[p, q]$  be the shortest segment between  $C_0$  and  $\gamma C_0$ , with  $p \in C_0$ . Let  $\alpha$  be a hyperbolic element in  $\Gamma_0$  with big translation length, and  $\beta = \gamma \alpha \gamma^{-1}$ . Let  $(k_n)_{n \in \mathbb{N}}$  be a sequence of positive integers. In particular,  $L = d(p, q)$ ,  $L'_n = d(p, \alpha^{k_n} p)$  and  $L''_n = d(q, \beta^{k_n} q)$  are big (independently of  $(k_n)_{n \in \mathbb{N}}$ ).

For every  $n$  in  $\mathbb{N}$ , define  $\gamma_n = \beta^{k_1} \alpha^{k_1} \beta^{k_2} \alpha^{k_2} \dots \beta^{k_n} \alpha^{k_n}$ , so that  $\gamma_0 = \text{id}$  and  $\gamma_1 = \beta^{k_1} \alpha^{k_1}$ . Consider the piecewise geodesic ray which is geodesic between the consecutive points

$$p, q, \beta^{k_1} q, \beta^{k_1} p, \gamma_1 p, \dots, \gamma_n p, \gamma_n q, \gamma_n \beta^{k_{n+1}} q, \gamma_n \beta^{k_{n+1}} p, \gamma_{n+1} p, \dots$$

Then if  $A = \min\{L, L'_n, L''_n : n \in \mathbb{N}\}$  is big enough, as the comparison angles at the above points between the incoming and outgoing segments are at least  $\pi/2$  by convexity, this piecewise geodesic ray is quasi-geodesic.



Hence, it stays at bounded distance (depending only on  $A$ ) from a geodesic ray  $\rho^*$  starting from  $p$ . Note that, by convexity, the segments  $[\gamma_n q, \gamma_n \beta^{k_n+1} q]$  and  $[\gamma_n \beta^{k_n+1} p, \gamma_{n+1} p]$  (which are long if the  $L''_n, L'_n$  are big) are contained in images under  $\Gamma$  of  $C_0$ . The point at infinity  $\xi$  of  $\rho^*$  is in particular a conical limit point (since  $\Gamma_0$  is convex-cocompact, there are points in one orbit under  $\Gamma$  that accumulate to  $\xi$  while staying at bounded distance from  $\rho^*$ ). Up to taking the translation length of  $\alpha$ , and hence  $A$ , big enough, the point  $\xi$  belongs neither to  $\partial_\infty C_\infty$ , nor to any  $\gamma' \partial_\infty C_0$  for  $\gamma' \in \Gamma$  (otherwise, two copies of  $C_0$  would be close for a too long time, contradicting Lemma 2.1 (4)). Hence the approximation constant  $c(\Gamma_\infty \xi)$  is well defined.

In order to apply Lemma 4.1, we fix  $\epsilon > 0$ . If the sequence  $(k_n)_{n \in \mathbb{N}}$  tends to  $+\infty$ , then the geodesic ray  $\rho^*$  spends longer and longer time in the images by  $\Gamma$  of the  $\epsilon$ -neighbourhood of  $C_0$ . Thus,  $c(\Gamma_\infty \xi)$  is equal to 0, by Lemma 4.1.

To prove that 0 is not isolated, take the sequence  $(k_n)_{n \in \mathbb{N}}$  to be constant, with  $k_1$  big compared with  $\kappa(\epsilon)$ ,  $L$  and the bounded distance between  $\rho^*$  and the above quasi-geodesic. In particular,  $\rho^*$  is  $(\epsilon, g)$ -Liouville for  $g$  a constant map, having a big value if  $k_1$  is big. By Lemma 2.1 (4), since  $\rho^*$  spends intervals of time of only bounded length outside  $\Gamma \mathcal{N}_\epsilon C_0$ , the geodesic ray  $\rho^*$  is not  $(\epsilon, g')$ -Liouville for  $g' > g$  a big enough constant map.

By Lemma 4.1, this implies that the approximation constant of (the image modulo  $\Gamma_\infty$  of)  $\xi$  is positive, and small if  $k_1$  is big.  $\square$

**Remark.** Let us notice here that the approximation constants are generically equal to 0, hence that the non vanishing of an approximation constant is a quite rare behaviour. We will make this explicit only in a particular case.

Assume that  $X$  is a Riemannian manifold and  $C_0$  a geodesic line. For every  $v \in \Gamma \backslash T^1 X$ , let  $\xi_v \in \Gamma_\infty \backslash \partial_\infty X$  be the (orbit by  $\Gamma_\infty$  of the) endpoint of a geodesic line in  $X$  whose tangent vector at the origin maps to  $v$  by the quotient by  $\Gamma$ . (Several choices are possible, but they will give the same approximation constant.) Let  $\mu$  be a (finite, positive, Borel) measure on  $\Gamma \backslash T^1 X$  invariant and ergodic under the quotient geodesic flow  $(\phi_t)_{t \in \mathbb{R}}$ . Assume that the support of  $\mu$  contains the orbit under  $\Gamma$  of the lift of  $C_0$  to  $T^1 X$  by its unit tangent vector, and that the (measurable) subset of unit vectors  $v$  such that  $\xi_v \in \Gamma_\infty \backslash ((\partial_\infty X - \Lambda_c \Gamma) \cup \partial_\infty C_\infty)$  has measure 0. For instance, this is true if  $\Gamma$  has finite covolume,  $\mu$  is the Liouville measure and  $C_\infty$  is a precisely invariant horoball, or if  $\Gamma$  is cocompact and  $\mu$  is the maximal entropy measure, and  $C_\infty$  is the translation axis of a hyperbolic element. The ergodicity assumption implies that  $\{\phi_t v\}_{t \in \mathbb{R}^+}$  is dense in the support of  $\mu$  for almost every  $v$ . Recall that if two unit tangent vectors are very close, then the geodesic lines they define are close for a long time. Hence for  $\mu$ -almost every  $v$ , we have  $\xi_v \in \Gamma_\infty \backslash (\Lambda_c \Gamma - \partial_\infty C_\infty)$  and  $c(\xi_v) = 0$ .

## 4.2 The boundedness of the approximation spectra

If  $\Gamma$  is geometrically finite (see for instance [Bow]), then there exists a  $\Gamma$ -equivariant family  $\mathcal{H}$  of horoballs centered at the parabolic fixed points of  $\Gamma$ , with pairwise disjoint interiors. There are many possible choices for such an  $\mathcal{H}$  (though only one maximal one if  $\Gamma$  has only one orbit of parabolic fixed points). In the computations of Section 4.4, we will choose natural ones. We call  $X_0 = \mathcal{C}\Gamma - \bigcup \mathcal{H}$  the *thick part* of  $\mathcal{C}\Gamma$ . Clearly,  $X_0$  is  $\Gamma$ -invariant, and  $\Gamma$  acts isometrically on it. We call  $\Gamma \backslash X_0$  the *convex core* of  $\Gamma \backslash X$ .

The next result gives a sufficient condition for the Hurwitz constant of  $\mathcal{D}$  to be finite.

In particular, this condition is satisfied when  $X$  is a Riemannian manifold and  $\Gamma$  has finite covolume. Recall that the Hurwitz constant of  $\mathcal{D}$  is

$$K_{\mathcal{D}} = \sup \text{Sp}(\mathcal{D}) \in [0, +\infty] .$$

**Theorem 4.4** *If  $\Gamma$  is geometrically finite, then  $\text{Sp}(\mathcal{D})$  is bounded, hence  $0 < K_{\mathcal{D}} < \infty$ .*

The following consequence, amongst other similar ones, follows from Equation (-9-).

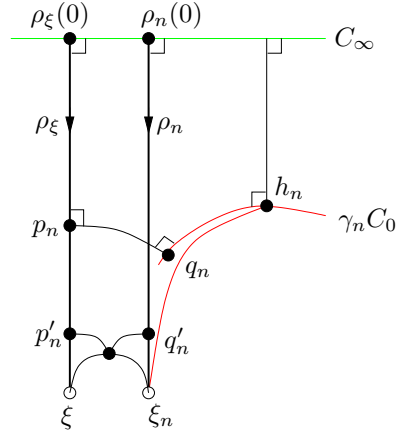
**Corollary 4.5** *Let  $M$  be a geometrically finite complete connected Riemannian manifold with sectional curvature at most  $-1$  and dimension at least 2. Let  $A_0$  be a closed geodesic in  $M$ , and let  $A_\infty$  be a closed codimension 0 submanifold of  $M$  with smooth compact locally convex boundary, disjoint from  $A_0$ . Then the spiraling spectrum  $\text{Sp}_{A_\infty, A_0}(M)$  around  $A_0$  is bounded.  $\square$*

**Proof of Theorem 4.4.** Let  $\mathcal{H}$  be as above. Since  $\Lambda\Gamma_0$  contains at least two points,  $C_0$  is not contained in any element of  $\mathcal{H}$ , hence  $C_0$  intersects  $X_0$ . Since  $\Gamma$  is geometrically finite, the diameter  $\Delta$  of the quotient metric space  $\Gamma \backslash X_0$  is finite. For every  $\xi \in \Lambda_c \Gamma - \bigcup_{\gamma \in \Gamma} \gamma \partial_\infty C_0$ , let  $\rho_\xi$  be a geodesic ray starting from the closest point to  $\xi$  on  $C_\infty$  and converging to  $\xi$ . As  $\xi$  is a conical limit point, there exists a sequence of positive times  $(t_n)_{n \in \mathbb{N}}$  converging to  $+\infty$  such that  $\rho_\xi(t_n) \in X_0$  for every  $n$ . Hence, there exists a sequence of elements  $(\gamma_n)_{n \in \mathbb{N}}$  such that for every  $n$  in  $\mathbb{N}$ ,

$$d(\rho_\xi(t_n), \gamma_n C_0) \leq \Delta .$$

For  $n$  big enough, the distance between the convex subsets  $C_\infty$  and  $\gamma_n C_0$  is big: By Lemma 3.1, there are only a finite number of double cosets  $[\gamma_n] \in \Gamma_\infty \backslash \Gamma / \Gamma_0$  for which  $D([\gamma_n]) = d(C_\infty, \gamma_n C_0) \leq T$  for any  $T > 0$ . As  $X$  is  $\text{CAT}(-1)$ , there is some  $\epsilon_0 >$  (note that this is not a “small  $\epsilon_0$ ”) such that the interval  $\rho_\xi([0, t_n])$  is contained in  $\mathcal{N}_{\epsilon_0} \gamma_n C_0$  for all  $n$ , which contradicts Lemma 2.1 (4) as  $t_n \rightarrow +\infty$ .

We denote by  $[p_n, q_n]$  the shortest segment between  $\rho_\xi$  and  $\gamma_n C_0$ , with  $p_n$  in  $\rho_\xi$  (which exists since  $\xi \notin \bigcup_{n \in \mathbb{N}} \gamma_n \partial_\infty C_0$ ). In particular,  $d(p_n, q_n) \leq \Delta$ . Let  $h_n$  be the point of  $\gamma_n C_0$  the closest to  $C_\infty$ . Since  $\Gamma_0$  is convex-cocompact, there exists a constant  $c_1 \geq 0$  (depending only on  $\Gamma_0$ ) such that the distance between  $q_n$  and a geodesic ray starting from  $h_n$  and staying inside  $\gamma_n C_0$  is at most  $c_1$ , for every  $n$  in  $\mathbb{N}$ . Denote by  $\xi_n$  the point at infinity of this geodesic ray, which does not belong to  $\partial_\infty C_\infty$ . Let  $\rho_n$  be the geodesic ray starting from the closest point to  $\xi_n$  in  $C_\infty$  and converging to  $\xi_n$ .



By the distance formulae of the hyperbolic comparison quadrilateral with vertices corresponding to  $\xi_n, \rho_n(0), h_n$  and to the closest point of  $h_n$  on  $C_\infty$ , with one comparison angle 0 and the three others at least  $\pi/2$  (see [Bea, 7.17]), if  $n$  (and hence  $d(h_n, C_\infty)$ ) is big enough, then the distance between  $\rho_n(0)$  and the closest point to  $h_n$  on  $C_\infty$  is at most 1. Therefore  $h_n$  is at distance at most a universal constant  $c_2$  from  $\rho_n$ . By convexity,  $q_n$  is hence at distance at most  $c_1 + c_2$  from a point  $h'_n$  of  $\rho_n$ , so that

$$d(p_n, \rho_n) \leq d(p_n, h'_n) \leq d(p_n, q_n) + d(q_n, h'_n) \leq \Delta + c_1 + c_2 .$$



In particular,  $d(\rho_n(0), h'_n)$  tends to  $+\infty$  as  $n \rightarrow +\infty$ . By the distance formulae of the hyperbolic comparison quadrilateral with vertices corresponding to  $\rho_\xi(0), \rho_n(0), h'_n, p_n$  (see [Bea, 7.17]), with two comparison angles at least  $\pi/2$  and the length of the opposite segment, that is  $d(p_n, h'_n)$ , bounded by a constant, if  $n$  is big enough, then  $\rho_\xi(0)$  is at distance at most 1 from  $\rho_n(0)$ .

Let  $r_n$  be the class of  $\gamma_n$  in  $\Gamma_\infty \backslash \Gamma / \Gamma_0$ , so that  $D(r_n) = d(h_n, C_\infty)$ . Using the triangle inequality, we have

$$d(p_n, \rho_\xi(0)) \geq d(q_n, C_\infty) - d(q_n, p_n) \geq d(\gamma_n C_0, C_\infty) - d(q_n, p_n) = D(r_n) - \Delta .$$

Since  $d(\rho_\xi(0), \rho_n(0)) \leq 1$ , there exists a universal constant  $c_3 \geq 0$  such that some point on the geodesic line between  $\xi$  and  $\xi_n$  is at distance at most  $c_3$  from both a point  $p'_n$  of  $\rho_\xi$  and a point  $q'_n$  of  $\rho_n$ . As  $p_n$  is at bounded distance from  $\rho_n$ , there exists (by a geometric argument of geodesic triangles) a constant  $c_4 \geq 0$  (which depends only on  $\Delta$ ) such that

$$d(p'_n, \rho_\xi(0)) \geq d(p_n, \rho_\xi(0)) - c_4 .$$

By the definition of the distance-like map  $d_{C_\infty}$  and by the triangle inequality, we have

$$d_{C_\infty}(\xi, \xi_n) \leq e^{-\frac{1}{2}(d(\rho_\xi(0), p'_n) - c_3 + d(\rho_n(0), q'_n) - c_3)} .$$

By the triangle inequality and by the previous inequalities, we hence have

$$d_{C_\infty}(\xi, \xi_n) \leq e^{-d(\rho_\xi(0), p'_n) + c_3 + \frac{d(p'_n, q'_n)}{2} + \frac{1}{2}} \leq e^{\Delta + 2c_3 + c_4 + \frac{1}{2}} e^{-D(r_n)} .$$

Hence the approximation constant  $c(\xi)$  of  $\xi$ , for every  $\xi$  in  $\Gamma_\infty \backslash (\Lambda_c \Gamma - (\partial_\infty C_\infty \cup \bigcup_{\gamma \in \Gamma} \gamma \Lambda \Gamma_0))$ , is at most  $e^{\Delta + 2c_3 + c_4 + \frac{1}{2}}$ , which proves the result.  $\square$

**Remark.** Note that this upper bound depends only on  $(X, \Gamma, C_\infty)$ , but not on  $C_0$ .

In special cases, the proof of Theorem 4.4 may be improved to give a simple explicit constant.

**Proposition 4.6** *If  $\Gamma$  is geometrically finite, if  $\Delta$  is the diameter of the quotient by  $\Gamma$  of the thick part of  $\mathcal{C}\Gamma$ , if  $C_\infty$  is a point in  $X$  or an horoball in  $X$ , and if  $C_0$  is a geodesic line, then*

$$K_\emptyset \leq (1 + \sqrt{2})e^\Delta .$$

The following consequence follows from Equation (-9-).

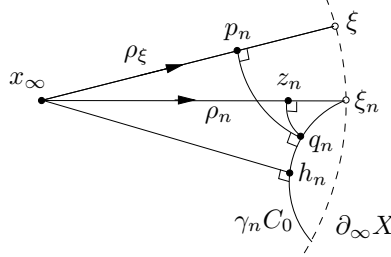
**Corollary 4.7** *Let  $M$  be a geometrically finite complete connected Riemannian manifold with sectional curvature at most  $-1$  and dimension at least 2, and let  $\Delta$  be the diameter of the convex core of  $M$ . Let  $A_0$  be a closed geodesic in  $M$ , and let  $A_\infty$  be either a ball or a Margulis neighbourhood of a cusp of  $M$ . Then the spiraling spectrum  $\text{Sp}_{A_\infty, A_0}(M)$  around  $A_0$  is contained in  $[0, (1 + \sqrt{2})e^\Delta]$ .  $\square$*

**Proof of Proposition 4.6.** Assume first that  $C_\infty = \{x_\infty\}$  with  $x_\infty \in X$ . For every  $\xi$  belonging to  $\Lambda_c \Gamma - (\partial_\infty C_\infty \cup \bigcup_{\gamma \in \Gamma} \gamma \partial_\infty C_0)$ , we define  $\rho_\xi, \gamma_n, p_n, q_n, h_n$  as in the beginning of the proof of Theorem 4.4, so that  $\rho_\xi(0) = x_\infty$ . Let  $\xi_n$  be the endpoint of the geodesic line  $C_0$  such that  $q_n \in [h_n, \xi_n[$ . Let  $\rho_n$  be the geodesic ray from  $x_\infty$  to  $\xi_n$ . Let  $r_n =$

$[\gamma_n] \in \Gamma_\infty \setminus \Gamma / \Gamma_0$ , so that  $d(x_\infty, q_n) \geq D(r_n)$ . Let  $z_n$  be the closest point on  $\rho_n$  to  $q_n$ , which satisfies

$$d(z_n, q_n) \leq \delta ,$$

with  $\delta = \log(1 + \sqrt{2})$ , by looking at the comparison triangle of the geodesic triangle with vertices  $x_\infty, \xi_n, h_n$  (see the picture below).



By the triangle inequality, for all  $t$  big enough,

$$\begin{aligned} & d(\rho_n(t), x_\infty) + d(\rho_\xi(t), x_\infty) - d(\rho_n(t), \rho_\xi(t)) \\ & \geq (d(\rho_n(t), z_n) + d(z_n, x_\infty)) + (d(\rho_\xi(t), p_n) + d(p_n, x_\infty)) - \\ & \quad (d(\rho_n(t), z_n) + d(z_n, p_n) + d(p_n, \rho_\xi(t))) \\ & = d(x_\infty, p_n) + d(x_\infty, z_n) - d(p_n, z_n) \\ & \geq d(x_\infty, q_n) - d(q_n, p_n) + d(x_\infty, q_n) - d(q_n, z_n) - (d(p_n, q_n) + d(q_n, z_n)) \\ & \geq 2D(r_n) - 2\Delta - 2\delta . \end{aligned}$$

Hence,

$$d_{x_0}(\xi, \xi_n) = \lim_{t \rightarrow +\infty} e^{-\frac{1}{2}(d(\rho_n(t), x_\infty) + d(\rho_\xi(t), x_\infty) - d(\rho_n(t), \rho_\xi(t)))} \leq e^{-D(r_n) + \Delta + \delta},$$

which proves that  $c(\Gamma_\infty \xi) \leq e^{\Delta + \delta}$ .

If  $C_\infty$  is a horoball with point at infinity  $\xi_\infty$ , the proof is similar, by replacing the geodesic rays starting from  $x_\infty$  by geodesic lines starting from  $\xi_\infty$  and meeting  $\partial_\infty C_\infty$  at time 0, and using the fact that  $d(\rho_\xi(0), \rho_n(0))$  tends to 0 as  $n \rightarrow +\infty$ .  $\square$

Theorem 1.1 in the introduction follows from Proposition 4.6, by Equation (-9-).

### 4.3 On the closedness of the approximation spectra

In this subsection, we prove that in the constant curvature manifold case, the spiraling spectrum around a closed geodesic is closed.

**Theorem 4.8** *Let  $\mathcal{D} = (X, \Gamma, \Gamma_0, C_\infty)$  be a quadruple of data such that  $X$  is the real hyperbolic  $n$ -space,  $\Gamma$  is geometrically finite,  $C_0 = \mathcal{C}\Gamma_0$  is a geodesic line, and  $C_\infty$  is a horoball. Then  $\text{Sp}(\mathcal{D})$  is equal to the closure in  $\mathbb{R}$  of the set of the approximation constants of the (orbit under  $\Gamma_\infty$  of the) hyperbolic fixed points of  $\Gamma$  (by elements not conjugated to elements of  $\Gamma_0$ ).*

In particular, the approximation spectrum  $\text{Sp}(\mathcal{D})$  is closed, the Hurwitz constant of  $\mathcal{D}$  is the maximum of  $\text{Sp}(\mathcal{D})$ , and the approximation constants of the (orbit under  $\Gamma_\infty$  of the) hyperbolic fixed points of  $\Gamma$  are dense in  $\text{Sp}(\mathcal{D})$ .

**Remark.** The result is still true if  $C_0$  is any totally geodesic subspace of dimension at least 1 and at most  $n - 1$ ; the adaptation of the proof below is left to the reader.

The following consequence follows from Equation (-9-).

**Corollary 4.9** *Let  $M$  be a geometrically finite complete connected Riemannian manifold with constant sectional curvature  $-1$  and dimension at least 2. Let  $A_0$  be a closed geodesic in  $M$ , and let  $A_\infty$  be a Margulis neighbourhood of a cusp of  $M$ . Then the spiraling spectrum  $\text{Sp}_{A_\infty, A_0}(M)$  around  $A_0$  is closed, and equals the closure of the set of spiraling constants of the geodesic lines spiraling around closed geodesics distinct from  $A_0$ .  $\square$*

**Proof of Theorem 4.8.** Let  $\mathcal{L}_0$  be the set of images under  $\Gamma$  of the two oriented geodesic defined by  $C_0$ . Let  $\tilde{v}$  be an element of  $T^1X$ , and  $\tilde{x}$  be its base point. For every  $C$  in  $\mathcal{L}_0$ , define  $p_C$  to be the point of  $C$  the closest to  $\tilde{x}$ , which depends continuously on  $\tilde{v}$ ; define  $\theta_C$  to be the angle at  $p_C$  between the parallel transport along  $[\tilde{x}, p_C]$  of  $\tilde{v}$  and  $C$ , which depends continuously on  $\tilde{v}$ . Let

$$\tilde{f}(\tilde{v}) = \inf_{C \in \mathcal{L}_0} \cosh d(\tilde{x}, C) - \cos \theta_C .$$

Since  $\mathcal{L}_0$  is locally finite and is preserved by  $\Gamma$ , the lower bound defining  $\tilde{f}$  is locally a minimum over a finite set. Thus, the map  $\tilde{f} : T^1X \rightarrow \mathbb{R}$  is continuous and invariant under  $\Gamma$ , and it defines a continuous map  $f : \Gamma \backslash T^1X \rightarrow \mathbb{R}$ . As the image of  $C_0$  in  $\Gamma \backslash X$  is compact and  $\Gamma \backslash X$  is a proper metric space, the distance to this image is a proper map on  $\Gamma \backslash X$ . Hence  $f$ , which is at least  $\cosh d(\tilde{x}, C) - 1 \geq 0$ , is proper. Let  $(\phi^t : T^1X \rightarrow T^1X)_{t \in \mathbb{R}}$  be the geodesic flow of  $X$ , and denote again by  $(\phi^t)_{t \in \mathbb{R}}$  its quotient flow under  $\Gamma$ .

We will use the following result of F. Maucourant [Mau, Theo. 2 (2)], whose main tool is Anosov's closing lemma (and which builds on a partial result of [HP2]). The result extends to our orbifold case.

**Theorem 4.10** *Let  $V$  be a complete Riemannian manifold with sectional curvature at most  $-1$ , let  $(\phi^t)_{t \in \mathbb{R}}$  be its geodesic flow, and let  $J_0$  be the subset of  $T^1V$  of periodic unit vectors. If  $f : T^1V \rightarrow \mathbb{R}$  is a proper continuous map, then*

$$\mathbb{R} \cap \{ \liminf_{t \rightarrow +\infty} f(\phi^t v) : v \in T^1V \} = \overline{\{ \min_{t \in \mathbb{R}} f(\phi^t v) : v \in J_0 \}} . \quad \square$$

Assume that  $X$  is the upper halfspace model of  $\mathbb{H}_{\mathbb{R}}^n$ , and that  $C_\infty$  is centered at  $\infty$ . By the general assumptions on the data  $\mathcal{D}$ , we are in the situation of Example 1 of Section 3. Let  $\xi$  be the endpoint of the geodesic line defined by  $\tilde{v}$ , and note that the geodesic line defined by  $\tilde{v}$  is asymptotic to the geodesic line from  $\infty$  to  $\xi$ . There are three cases to consider:

- (1) If  $\xi \in \Lambda_c \Gamma - \mathcal{R}_{\Gamma_0}$ , then it follows from the definition of  $\tilde{f}$  and from Equation (-11-) that  $\liminf_{t \rightarrow +\infty} \tilde{f}(\phi^t \tilde{v}) = c(\Gamma_\infty \xi)$ .
- (2) If  $\xi \in \mathcal{R}_{\Gamma_0}$ , then  $\liminf_{t \rightarrow +\infty} \tilde{f}(\phi^t \tilde{v}) = 0$ , and we have already seen (in Proposition 4.2) that 0 belongs to  $\text{Sp}(\mathcal{D})$ .

- (3) If  $\xi \in \partial_\infty X - \Lambda_c$ , then since  $\Gamma$  is geometrically finite, either  $\xi$  does not belong to the limit set, or  $\xi$  is a parabolic fixed point. In both cases, since  $f$  is proper, we have  $\liminf_{t \rightarrow +\infty} f(\phi^t \tilde{v}) = +\infty$ , which is not in  $\mathbb{R}$ .

These observations imply that  $\mathbb{R} \cap \{ \liminf_{t \rightarrow +\infty} f(\phi^t v) : v \in \Gamma \backslash T^1 X \}$  is contained in  $\text{Sp}(\mathcal{D}) = \{c(\Gamma_\infty \xi) : \xi \in \Lambda_c \Gamma - \mathcal{R}_{\Gamma_0}\}$ .

By considering  $\tilde{v}$  a vertical unit tangent vector ending at a given  $\xi \in \Lambda_c \Gamma - \mathcal{R}_{\Gamma_0}$ , the opposite inclusion also holds. If  $J'_0$  is the subset of vectors in  $J_0$  that are not the image in  $\Gamma \backslash X$  of unit vector tangents to  $C_0$ , then the set  $A$  of the approximation constants of the (orbits under  $\Gamma_\infty$  of the) points of  $\partial_\infty X$  fixed by hyperbolic elements not conjugated to elements of  $\Gamma_0$ , is equal to  $\{ \inf_{t \in \mathbb{R}} f(\phi^t v) : v \in J'_0 \}$ . Furthermore, the approximation constant of the (orbit under  $\Gamma_\infty$  of a) point of  $\partial_\infty X$  fixed by a hyperbolic element conjugated to an element of  $\Gamma_0$ , is equal to 0. Hence, by Theorem 4.10, we have  $\text{Sp}(\mathcal{D}) = \overline{A} \cup \{0\}$ . Since 0 is not isolated in  $\text{Sp}(\mathcal{D})$  (see Theorem 4.2), this implies that  $\text{Sp}(\mathcal{D}) = \overline{A}$ . This proves Theorem 4.8.  $\square$

#### 4.4 Some upper bounds on the approximation spectra

We give estimates of the Hurwitz constants of data  $\mathcal{D} = (X, \Gamma, \Gamma_0, C_\infty)$  in a number of arithmetically defined cases. The estimates are not likely to be very sharp, except for Proposition 4.11. In the following five examples,  $X$  is  $\mathbb{H}_{\mathbb{R}}^2$ ,  $\mathbb{H}_{\mathbb{R}}^2$ ,  $\mathbb{H}_{\mathbb{R}}^3$ ,  $\mathbb{H}_{\mathbb{R}}^5$  or  $\mathbb{H}_{\mathbb{C}}^2$  respectively. These dimensions are chosen with number theoretical applications in mind, see Section 6. In the first four examples, the convex set  $C_\infty$  is the horoball centered at infinity consisting of the points with Euclidean height at least 1 in the upper half space model of  $\mathbb{H}_{\mathbb{R}}^n$ . The group  $\Gamma$  is specified in each example, and the subgroup  $\Gamma_0$  is the stabilizer in  $\Gamma$  of any geodesic in  $X$  whose quotient in  $\Gamma \backslash X$  is compact.

The following classical fact is used repeatedly in the examples: For every  $\alpha \leq \pi/2$ , the distance  $\ell$  in  $\mathbb{H}_{\mathbb{R}}^2$  between the points of angle  $\alpha$  and of angle  $\pi/2$  with respect to the real line  $\partial_\infty \mathbb{H}_{\mathbb{R}}^2 - \{\infty\}$  on any Euclidean circle centered at a point in  $\partial_\infty \mathbb{H}_{\mathbb{R}}^2 - \{\infty\}$  is

$$\ell = \log \cot \frac{\alpha}{2} = -\log \frac{\sqrt{1 + \tan^2 \alpha} - 1}{\tan \alpha} . \quad (-15-)$$

This equation is used to compute the distances of points in isometrically embedded copies of  $\mathbb{H}_{\mathbb{R}}^2$  in  $\mathbb{H}_{\mathbb{R}}^n$  and  $\mathbb{H}_{\mathbb{C}}^2$ .

(1) Let  $\Gamma = \text{PSL}_2(\mathbb{Z})$ . It is well known that the hyperbolic triangle  $F$  in  $\mathbb{H}_{\mathbb{R}}^2$  with vertices at  $\infty$  and  $\pm e^{i\frac{\pi}{3}}$  is a fundamental polygon for  $\Gamma = \text{PSL}_2(\mathbb{Z})$ . The horoball  $C_\infty$  covers the maximal Margulis neighbourhood  $U$  of the (only) cusp of  $M = \Gamma \backslash \mathbb{H}_{\mathbb{R}}^2$ . The compact set  $K = \overline{F - C_\infty}$  covers the complement of  $U$  in  $M$ . The symmetries imply that the diameter  $\Delta$  of  $M - U$  equals the distance between the cone points of  $M$  with angles  $\pi$  and  $2\pi/3$ . By Equation (-15-),

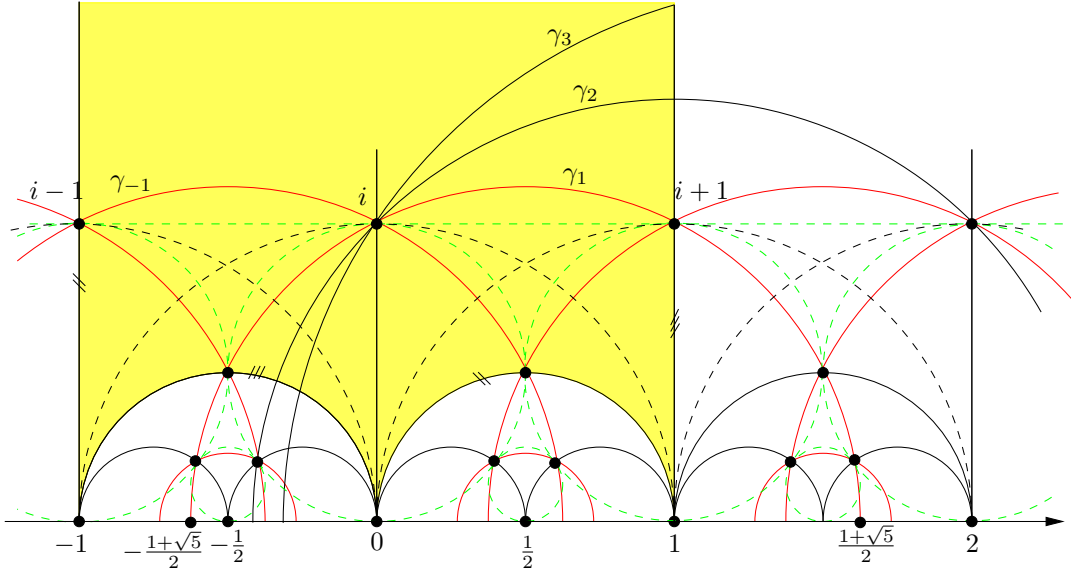
$$\Delta = d(i, e^{i\frac{\pi}{3}}) = \frac{1}{2} \log 3 \approx 0.55 .$$

By Proposition 4.6, we thus have  $\text{Sp}(\mathcal{D}) \subset [0, (1 + \sqrt{2})\sqrt{3}] \approx [0, 4.19]$ . Notice that this upper bound is uniform amongst the subgroups  $\Gamma_0$ .

Let us give an exact computation of the Hurwitz constant in a particular case. Notice that the second assertion of the result below shows a different behaviour than the classical Lagrange spectrum.

**Proposition 4.11** *Let  $\Gamma_0$  be the cyclic subgroup of  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$  generated by  $\gamma_1 = \pm \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ , and let  $\mathcal{D} = (\mathbb{H}_{\mathbb{R}}^2, \Gamma, \Gamma_0, C_\infty)$ . Then  $K_{\mathcal{D}} = 1 - 1/\sqrt{5}$ , and  $K_{\mathcal{D}}$  is not isolated in the approximation spectrum  $\mathrm{Sp}(\mathcal{D})$ .*

**Proof.** The element  $\gamma_1$  is hyperbolic, and its translation axis  $L_1$  is the geodesic line in  $\mathbb{H}_{\mathbb{R}}^2$  with endpoints at  $(1 \pm \sqrt{5})/2$  (see the picture below). This geodesic line projects to the shortest geodesic segment that connects the cone points of angles  $\pi$  and  $2\pi/3$  on  $\Gamma \backslash \mathbb{H}_{\mathbb{R}}^2$ . The translation length  $\ell_1$  of  $\gamma_1$  satisfies  $2 \cosh(\ell_1/2) = 3$  (see [Bea, page 173]), and the translates of  $L_1$  intersect in pairs at the orbit of  $i$ , and form a net of equilateral triangles as in the figure below. The edges of the triangles have length, by Equation (-15-), equal to  $d(i, i+1) = \mathrm{argcosh}(3/2) = \ell_1/2$ , and the angles  $\theta \in [0, \frac{\pi}{2}]$  of the triangles satisfy  $\cos \theta = 3/5$ . These facts are easily seen by considering the 6-fold cover of  $M$  by the modular torus  $M'$  which is the quotient of  $\mathbb{H}_{\mathbb{R}}^2$  by the commutator subgroup of  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$  (see (2) below). Notice that  $\gamma_1$  and its translates by  $z \mapsto z \pm 1$  are lifts of the three shortest periodic geodesics which intersect at the three Weierstrass points of  $M'$  (see for instance [Sch, Theo. 2]).



Any geodesic line that connects  $\infty$  with  $\xi \in \mathbb{R} - \mathbb{Q}$  intersects infinitely many triangles. Let  $T$  be one of the triangles, and let  $\gamma$  be a geodesic line that intersects the interior of  $T$ . The points of intersection of  $\gamma$  with  $T$  and one of the vertices of  $T$  determine a triangle with angles  $\theta, \phi_1, \phi_2$ . By symmetry and the fact that small triangles are almost Euclidean, it is clear that the supremum over all geodesics intersecting the interior of  $T$  of  $\min(\cos \phi_1, \cos \phi_2)$  equals  $\cos((\pi - \theta)/2) = \sqrt{(1 - \cos \theta)/2} = 1 - 1/\sqrt{5}$ , and this value is not attained, since  $\xi \notin \mathbb{Q}$ . Thus, for any  $\xi \in \mathbb{R} - \mathbb{Q}$ ,  $c(\xi) \leq 1 - 1/\sqrt{5}$ .

A simple calculation shows, for any nonzero integer  $n$ , that  $\gamma_n = \pm \begin{pmatrix} n^2 + 1 & n \\ n & 1 \end{pmatrix} \in \Gamma$  is a hyperbolic element whose axis passes through  $i$  and  $n + i$ . The translation distance of  $\gamma_n$ , which is  $2 \mathrm{argcosh}(n^2/2 + 1)$  by [Bea, page 173], is twice the distance between the points  $i$  and  $n + i$ , by Equation (-15-). Thus, the axis of  $\gamma_n$  intersects any  $\Gamma$ -translate of the axis of  $\gamma_1$  at a  $\Gamma$ -translate of  $i$ , and always at the same angle. As  $n \rightarrow \infty$ , this angle

approaches the angle  $\theta'$  with  $\cos \theta' = 1/\sqrt{5}$  between the axis of  $\gamma_0$  and the imaginary axis at  $i$ . Thus, by Lemma 2.2 and Equation (-11-), the approximation constants of the lines from  $\infty$  to the fixed points of  $\gamma_n$  converge to  $1 - 1/\sqrt{5}$  (while being different). The result follows.  $\square$

(2) Let  $\Gamma$  be the commutator subgroup of  $\mathrm{PSL}_2(\mathbb{Z})$ . It is well known (see for instance [Sch]) that  $\Gamma$  is a torsion-free subgroup of index 6 in  $\mathrm{PSL}_2(\mathbb{Z})$ , and that the quotient  $\Gamma \backslash \mathbb{H}_{\mathbb{R}}^2$  is a punctured torus, called *the modular torus*.

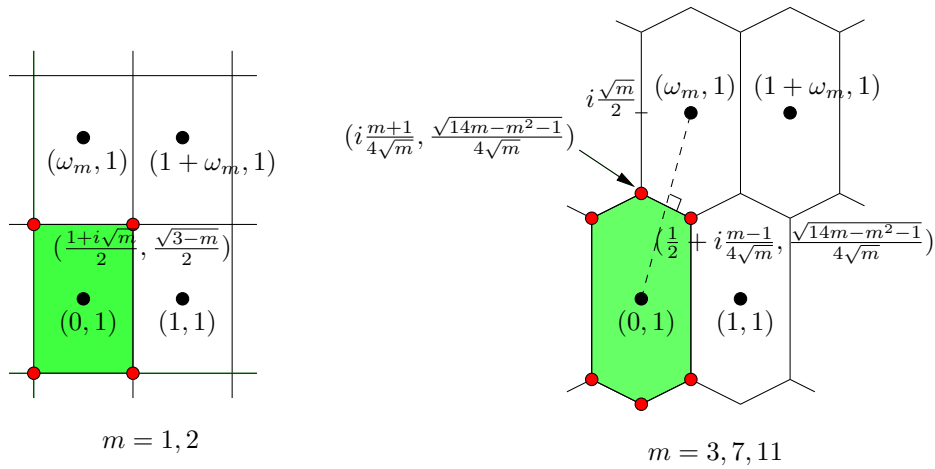
For every  $k \in \mathbb{N}$ , let  $H_k$  be the horosphere centered at  $k$  with Euclidean height one. It is well known (see for instance [Coh, Sch]) that the modular torus is isometric to the quotient of the ideal hyperbolic square  $P$  with vertices  $\infty, -1, 0, 1$  by the gluing of the opposite faces of  $P$ , such that the horoball  $C_\infty$  maps by the two gluings to the horoballs  $H_{-1}$  and  $H_1$ . In particular,  $C_\infty$  covers the maximal Margulis neighbourhood  $U$  of the cusp of  $M = \Gamma \backslash \mathbb{H}_{\mathbb{R}}^2$ .

Let  $T'$  be the closure of the relatively compact component of  $\mathbb{H}_{\mathbb{R}}^2 - (C_\infty \cup H_{-1} \cup H_0)$ . Then the closure of  $M - U$  is the union of the triangles with horocyclic sides  $T'$  and  $T' + 1$ , glued along their vertices. The diameter of  $T'$  for the induced distance of  $\mathbb{H}_{\mathbb{R}}^2$  is, by a convexity argument, equal to  $d(i, i+1) = \operatorname{argcosh}(3/2)$ . Any point of  $T' + 1$  is at distance at most  $d(i, e^{i\frac{\pi}{3}}) = \frac{\log 3}{2}$  (by Equation (-15-)) from one vertex of  $T' + 1$ . Therefore

$$\Delta \leq \frac{\log 3}{2} + \operatorname{argcosh}(3/2).$$

By Proposition 4.6, we thus have  $\operatorname{Sp}(\mathcal{D}) \subset [0, (1 + \sqrt{2})e^\Delta] \subset [0, 10.95]$ .

(3) Let  $m$  be a positive squarefree integer, and let  $\Gamma$  be the *Bianchi group*  $\mathrm{PSL}_2(\mathcal{O}_{-m})$ , where  $\mathcal{O}_{-m}$  is the ring of integers of  $\mathbb{Q}(i\sqrt{m})$ . All Bianchi groups contain the transformation  $z \mapsto z + 1$ , and thus, the interior of the horoball  $C_\infty$  is precisely invariant, by Shimizu's Lemma. Since  $\iota : z \mapsto -\frac{1}{z}$  also belongs to  $\Gamma$  and since the horoballs  $C_\infty$  and  $\iota C_\infty$  are tangent, the horoball  $C_\infty$  covers the maximal Margulis neighbourhood of the cusp of  $M = \Gamma \backslash \mathbb{H}_{\mathbb{R}}^3$ . Fundamental domains for the Bianchi groups have been determined in [Bia, Swa, EGM] and we will use the tables of [Hat, page 346].



In the above picture,  $\omega_m$  is equal to  $i\sqrt{m}$  if  $m \equiv 1, 2 \pmod{4}$  and is equal to  $\frac{1+i\sqrt{m}}{2}$  if  $m \equiv 3 \pmod{4}$ . The shaded area represents the vertical projection to  $\mathbb{C}$  of the Ford

fundamental domain at infinity  $F_m$ . The couples are the coordinates in  $\mathbb{H}_{\mathbb{R}}^3 \subset \mathbb{C} \times \mathbb{R}$  of the finite vertices of the polyhedron  $F_m$  projecting to these points, and of the center of the (unique) compact codimension 1 face of  $F_m$ .

**Cases  $m = 1, 2$  :** A Ford fundamental domain of  $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m}) = \mathrm{PSL}_2(\mathbb{Z}[i\sqrt{m}])$  is given by the polyhedron  $F_m$  with five vertices, one at  $\infty$  and four finite ones at  $(\pm\frac{1}{2} \pm \frac{\sqrt{m}}{2}i, \frac{\sqrt{3-m}}{2})$ . The diameter  $\Delta_m$  of the image of  $K_m = \overline{F_m - C_\infty}$  in  $\Gamma \backslash \mathbb{H}_{\mathbb{R}}^3$  satisfies, by the symmetries and Equation (-15-),

$$\Delta_m \leq d\left(\left(\frac{1}{2} + \frac{\sqrt{m}}{2}i, \frac{\sqrt{3-m}}{2}\right), (0, 1)\right) = \log \frac{\sqrt{3-m}}{2 - \sqrt{1+m}},$$

which is  $\log(1 + \sqrt{2})$  if  $m = 1$  and  $\log(2 + \sqrt{3})$  if  $m = 2$ . Now, as in (1) above,  $\mathrm{Sp}(\mathcal{D})$  is contained in  $[0, (1 + \sqrt{2})^2] \approx [0, 5.83]$  if  $m = 1$  and  $[0, (1 + \sqrt{2})(2 + \sqrt{3})] \approx [0, 9.01]$  if  $m = 2$ .

**Cases  $m = 3, 7, 11$  :** A Ford fundamental domain of  $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m}) = \mathrm{PSL}_2(\mathbb{Z}[\frac{1+i\sqrt{m}}{2}])$  is given by the polyhedron  $F_m$  with seven vertices at  $\infty$ ,  $(\pm i\frac{m+1}{4\sqrt{m}}, \frac{\sqrt{14m-m^2-1}}{4\sqrt{m}})$  as well as  $(\pm\frac{1}{2} \pm i\frac{m-1}{4\sqrt{m}}, \frac{\sqrt{14m-m^2-1}}{4\sqrt{m}})$ . The diameter  $\Delta_m$  of the image of  $K_m = \overline{F_m - C_\infty}$  in  $\Gamma \backslash \mathbb{H}_{\mathbb{R}}^3$  satisfies, by the symmetries and Equation (-15-),

$$\Delta_m \leq d\left(\left(i\frac{m+1}{4\sqrt{m}}, \frac{\sqrt{14m-m^2-1}}{4\sqrt{m}}\right), (0, 1)\right) = \log \frac{\sqrt{14m-m^2-1}}{4\sqrt{m}-m-1}.$$

Now, as in (1) above,  $\mathrm{Sp}(\mathcal{D})$  is contained in  $[0, (1 + \sqrt{2})\frac{\sqrt{14m-m^2-1}}{4\sqrt{m}-m-1}]$  which is for instance approximately  $[0, 4.664]$  if  $m = 3$ .

(4) By the classification of the hyperbolic Coxeter simplices (see for instance [VS]), there exists one, called  $F$  thereafter, whose Coxeter diagram is



Up to isometry of  $\mathbb{H}_{\mathbb{R}}^5 = \mathbb{R}^4 \times ]0, +\infty[$ , we may assume that its ideal vertex is at infinity, and that the opposite face lies on the Euclidean unit sphere centered at 0.

Let  $\Gamma = \Gamma_5$  be the group of isometries of  $\mathbb{H}_{\mathbb{R}}^5$  generated by the reflexions on the codimension-one faces of  $F$ . The one-cusped orbifold  $\Gamma_5 \backslash \mathbb{H}_{\mathbb{R}}^5$  is the minimal volume cusped hyperbolic orbifold of dimension 5, see [Hil].

The horoball  $C_\infty$  is the maximal precisely invariant horoball centered at  $\infty$ , see [Hil, Prop. 5]. It is easy to see (see for instance [Hil, page 216]) that the vertical projection of  $F$  in  $\mathbb{R}^4$  (which is a Euclidean Coxeter simplex with Coxeter diagram  $\circ \text{---} \circ \text{---} \overset{4}{\circ} \text{---} \circ$ ), has diameter  $1/\sqrt{2}$ . Thus the diameter of  $\overline{F - C_\infty}$  is at most  $2d_{\mathbb{H}_{\mathbb{R}}^5}(i, \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = 2\log(1 + \sqrt{2})$ , and  $\mathrm{Sp}(\mathcal{D})$  is contained in  $[0, (1 + \sqrt{2})^3] \approx [0, 14.08]$ .

Let  $\mathbb{H}$  be the skew field of Hamilton's quaternions. The Hurwitz ring  $\mathcal{O}'$  consists of all quaternions in  $\mathbb{H}$  of the form  $\frac{1}{2}(a_0 + a_1i + a_2j + a_3k)$  such that the coefficients  $a_0, a_1, a_2, a_3 \in \mathbb{Z}$  have equal parity. The *Hurwitz modular group*  $\Gamma = \mathrm{PSL}_2(\mathcal{O}')$  (defined using the Dieudonné determinant) is, up to conjugation, a subgroup of index 4 in  $\Gamma_5$ , see [JW]. Since a fundamental domain for  $\Gamma$  can be built as the connected union of four copies of the fundamental domain  $F$  of  $\Gamma_5$ , the approximation spectrum  $\mathrm{Sp}(\mathcal{D})$  is contained in  $[0, (1 + \sqrt{2})^9] \approx [0, 2787]$ , a very rough estimate.

(5) Before giving the fifth and last example, notice that the model of  $\mathbb{H}_{\mathbb{C}}^2$  used therein will differ from the one used in Example 2 of Section 3 to facilitate references to [FP] on which the example is based: We will use the Siegel domain of the complex hyperbolic plane, whose underlying space is, as a subset of the complex projective plane  $\mathbb{P}^2(\mathbb{C})$ , with nonhomogeneous coordinates,

$$\mathbb{H}_{\mathbb{C}}^2 = \{[W_0 : W : 1] \in \mathbb{P}^2(\mathbb{C}) : 2 \operatorname{Re} W_0 + |W|^2 < 0\} .$$

Consider the Hermitian form  $q = Z_0 \bar{Z}_2 + Z_2 \bar{Z}_0 + Z_1 \bar{Z}_1$  on  $\mathbb{C}^3$ , whose signature is  $(1, 2)$ . The *Eisenstein-Picard modular group*  $\Gamma = \operatorname{PU}_q(\mathcal{O}_{-3})$  is the projective unitary group of the form  $q$ , acting projectively on  $\mathbb{H}_{\mathbb{C}}^2$ . Let

$$C_{\infty} = \{[W_0 : W : 1] \in \mathbb{P}^2(\mathbb{C}) : 2 \operatorname{Re} W_0 + |W|^2 \leq -2\} ,$$

which is a horoball centered at  $\infty = [-1 : 0 : 0]$ . Let  $\omega = \frac{-1+i\sqrt{3}}{2}$  and

$$P = \begin{pmatrix} 1 & 1 & \omega \\ 0 & \omega & -\omega \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & \omega \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which define elements of  $\Gamma$ . A fundamental domain  $D$  for  $\Gamma$  is constructed in [FP, Theo. 4.15], as a simplex with one infinite vertex at  $\infty$ , which is the geodesic cone with cone point  $\infty$  over a tetrahedron  $T_0$  with four finite vertices

$$z_0 = [\bar{\omega} : 0 : 1], \quad z_1 = [-1 : -\omega : 1], \quad z_2 = [-1 : 1 : 1], \quad z_3 = [\omega : 0 : 1].$$

By [FP, Prop. 4.6], the images of the interior of  $D$  by  $P, Q, R$  are disjoint from  $D$ , the element  $R$  maps  $T_0$  to itself, and the geodesic cones with vertex  $\infty$  over the four faces of  $T_0$  are paired, by  $PQ^{-1}$  and  $P$ .

The horoball  $C_{\infty}$ , which is invariant by  $P, Q$ , and which meets its image by  $R$  only in  $u_0 = [-1 : 0 : 1] \in T_0$ , is hence the maximal precisely invariant horoball centered at  $\infty$ . As  $D$  is a cone with vertex  $\infty$  and  $T_0$  intersects  $C_{\infty}$  at  $u_0$ , the diameter of the complement of the horoball  $C_{\infty}$  in  $D$  is attained by two points of  $T_0$ . Note that (see [FP], Definition 4.5 and the claim before Proposition 4.6 therein) the faces of  $T_0$  are foliated by geodesic arcs between points of the edges, and that the edges are geodesic arcs. Hence, by convexity of the distance map, the maximal distance between two points of  $T_0$  is attained by a pair of vertices, that is by the maximal length of an edge of the tetrahedron  $T_0$ .

The intersection of  $\mathbb{H}_{\mathbb{C}}^2$  with the complex lines of equations  $W = 0$  and  $W_0 = -1$  are totally geodesic, and are respectively a copy of the constant curvature  $-1$  real hyperbolic left halfplane and disc of radius  $\sqrt{2}$  and center  $0$  in  $\mathbb{C}$  (and  $u_0$  corresponds to the point  $(-1, 0)$  in this left halfplane and to the center of this disc). Hence, for  $i = 0, 3$ , we have  $d_{\mathbb{H}_{\mathbb{C}}^2}(z_i, u_0) = \log(2 + \sqrt{3})$  by Equation (-15-). For  $i = 1, 2$ , we have  $d_{\mathbb{H}_{\mathbb{C}}^2}(z_i, u_0) = \log(3 + 2\sqrt{2})$ . Thus for  $0 \leq i, j \leq 3$ , by the triangular inequality, we have

$$d_{\mathbb{H}_{\mathbb{C}}^2}(z_i, z_j) \leq d_{\mathbb{H}_{\mathbb{C}}^2}(z_i, u_0) + d_{\mathbb{H}_{\mathbb{C}}^2}(z_j, u_0) \leq 2 \log(3 + 2\sqrt{2}) .$$

Therefore, in the metric of  $\mathbb{H}_{\mathbb{C}}^2$  with sectional curvature between  $-4$  and  $-1$ , we have the estimate

$$\Delta \leq \max_{0 \leq i, j \leq 3} d'_{\mathbb{H}_{\mathbb{C}}^2}(z_i, z_j) = \log(3 + 2\sqrt{2}) ,$$

and a corresponding estimate on the approximation spectrum

$$\operatorname{Sp}(\mathcal{D}) \subset [0, (1 + \sqrt{2})(3 + 2\sqrt{2})] \approx [0, 14.07] .$$



## 5 Hall rays in approximation spectra

In this section, for some quadruples of data  $\mathcal{D} = (X, \Gamma, \Gamma_0, C_\infty)$ , we will prove that the approximation spectrum  $\text{Sp}(\mathcal{D})$  contains a segment  $[0, c]$  for some  $c > 0$ .

We start by recalling the following result from [PP2]. It says that given a family of almost disjoint neighborhoods of geodesic lines, there exists a geodesic ray or line, with starting point (at infinity in the case of a line) any given point outside these neighborhoods, that has a prescribed penetration in one given neighborhood, and does not penetrate too much in the neighborhoods thereafter.

**Theorem 5.1** [PP2, Theorem 5.9] *For every  $\epsilon > 0$  and  $\delta \geq 0$ , there exists a positive constant  $h'_1$  such that the following holds. Let  $X$  be a complete simply connected Riemannian manifold with sectional curvature at most  $-1$  and dimension at least 3. Let  $(L_n)_{n \in \mathbb{N}}$  be a family of geodesic lines in  $X$ , such that  $\text{diam}(\mathcal{N}_\epsilon L_n \cap \mathcal{N}_\epsilon L_m) \leq \delta$  for all  $n \neq m$  in  $\mathbb{N}$ . For every  $\xi \in (X \cup \partial_\infty X) - (\mathcal{N}_\epsilon L_0 \cup \partial_\infty L_0)$ , let  $f_0 : T_\xi^1 X \rightarrow [0, +\infty[$  be either  $f_0 = \text{ftp}_{L_0}$ , or  $f_0 = \ell_{\mathcal{N}_\epsilon L_0}$  if  $X$  has constant curvature, or  $f_0 = \text{ctp}_{L_0}$  (in which case  $\xi \in \partial_\infty X - \partial_\infty L_0$ ) if the metric spheres for the Hamenstädt distances (on  $\partial_\infty X - \{\xi'\}$  for any  $\xi' \in \partial_\infty X$ ) are topological spheres. Let  $h \geq h'_1$ .*

*Then there exists a geodesic ray or line  $\rho$  starting from  $\xi$  and entering  $\mathcal{N}_\epsilon L_0$  at time 0 with  $f_0(\rho) = h$ , such that  $\ell_{\mathcal{N}_\epsilon L_n}(\rho) \leq h'_1$  for every  $n \neq 0$  such that  $\rho([\delta, +\infty[)$  meets  $\mathcal{N}_\epsilon L_n$ .  $\square$*

Note that the condition on the metric spheres of the Hamenstädt distance being topological spheres is satisfied by all negatively curved symmetric spaces.

The following result is an analog of Theorem 5.13 of [PP2], where we considered cusp excursions. It has as hypothesis the conclusion of the previous theorem. It says that if a given family of almost disjoint neighborhoods of geodesic lines is rich enough, then we can find a geodesic line which has a prescribed upper asymptotic penetration in these neighborhoods.

We first define what we mean precisely by this. Let  $X$  be a proper  $\text{CAT}(-1)$  space and let  $\xi \in X \cup \partial_\infty X$ . Let  $\epsilon > 0$ ,  $\delta, \kappa \geq 0$ . Let  $(C_\alpha)_{\alpha \in \mathcal{A}}$  be a family of convex subsets of  $X$  such that  $\text{diam}(\mathcal{N}_\epsilon C_\alpha \cap \mathcal{N}_\epsilon C_\beta) \leq \delta$  for all  $\alpha \neq \beta$  in  $\mathcal{A}$ . For each  $\alpha \in \mathcal{A}$  such that  $\xi \notin C_\alpha \cup \partial_\infty C_\alpha$ , let  $f_\alpha : T_\xi^1 X \rightarrow [0, +\infty]$  be a map such that  $\|f_\alpha - \ell_{\mathcal{N}_\epsilon C_\alpha}\|_\infty \leq \kappa$ . These assumptions guarantee that for every  $\rho \in T_\xi^1 X$ , the set  $\mathcal{E}_\rho$  of times  $t \geq 0$  such that  $\rho$  enters in some  $C_\alpha$  at time  $t$  with  $f_\alpha(\rho) > \delta + \kappa$  is discrete in  $[0, +\infty[$ , and that such an  $\alpha$  is then unique, denoted by  $\alpha_t$ . Hence  $\mathcal{E}_\rho = (t_i)_{i \in \mathcal{N}}$  for some initial segment  $\mathcal{N}$  in  $\mathbb{N}$ , with  $t_i < t_{i+1}$  for  $i, i+1$  in  $\mathcal{N}$ . With  $a_i(\rho) = f_{\alpha_{t_i}}(\rho)$ , the (finite or infinite) sequence  $(a_i(\rho))_{i \in \mathcal{N}}$  will be called the *penetration sequence* of  $\rho$  with respect to  $(\mathcal{N}_\epsilon C_\alpha, f_\alpha)_{\alpha \in \mathcal{A}}$  (and  $\delta, \kappa$ ). We will be interested in the possible values of  $\limsup_{i \rightarrow +\infty} a_i(\rho)$ , when  $\mathcal{N} = \mathbb{N}$ .

**Theorem 5.2** *Let  $\epsilon > 0$  and  $\delta, \nu, \nu' \geq 0$ . Let  $X$  be a proper  $\text{CAT}(-1)$  space, with  $\partial_\infty X$  infinite, and let  $\xi \in X \cup \partial_\infty X$ . Let  $(L_\alpha)_{\alpha \in \mathcal{A}}$  be a family of geodesic lines in  $X$ , such that  $\text{diam}(\mathcal{N}_\epsilon L_\alpha \cap \mathcal{N}_\epsilon L_\beta) \leq \delta$  for all  $\alpha \neq \beta$  in  $\mathcal{A}$ . For every  $\alpha \in \mathcal{A}$  such that  $\xi \notin C_\alpha \cup \partial_\infty C_\alpha$ , let  $f_\alpha$  be either  $\ell_{\mathcal{N}_\epsilon L_\alpha}$  or  $\text{ftp}_{L_\alpha}$  or  $\text{ctp}_{L_\alpha}$ , and in this last case, assume that  $\xi \in \partial_\infty X$ . Let  $\kappa$  be the upper bound of the  $\|f_\alpha - \ell_{\mathcal{N}_\epsilon L_\alpha}\|_\infty$  for all  $\alpha$  in  $\mathcal{A}$  such that  $\xi \notin C_\alpha \cup \partial_\infty C_\alpha$ . Assume that  $\bigcup_{\alpha \in \mathcal{A}} \partial_\infty L_\alpha$  is dense in  $\partial_\infty X$ . Assume that for every  $h \geq \nu$  and  $\alpha \in \mathcal{A}$  such that  $\xi \notin \mathcal{N}_\epsilon L_\alpha \cup \partial_\infty L_\alpha$ , there exists a geodesic ray or line  $\rho$  starting from  $\xi$  and entering*

$\mathcal{N}_\epsilon L_\alpha$  at time  $t = 0$  with  $f_\alpha(\rho) = h$ , and with  $f_\beta(\rho) \leq \nu'$  for every  $\beta$  in  $\mathcal{A} - \{\alpha\}$  such that  $\rho([\delta, +\infty[)$  meets  $\mathcal{N}_\epsilon L_\beta$ . Let  $(a_i(\rho'))_{n \in \mathcal{N}}$  be the penetration sequence of a geodesic ray or line  $\rho'$  with respect to  $(\mathcal{N}_\epsilon L_\alpha, f_\alpha)_{\alpha \in \mathcal{A}}$  (and  $\delta, \kappa$ ).

Then, there exists  $h_* = h_*(\epsilon, \delta, \kappa, \nu, \nu') > 0$  such that for every  $h \geq h_*$ , there exists a geodesic ray or line  $\rho$  starting from  $\xi$  such that

$$\limsup_{i \rightarrow +\infty} a_i(\rho) = h .$$

**Proof.** We start by recalling two lemmas from [PP2], which explain the relative penetration behaviour of a pair of geodesic lines in the  $\epsilon$ -neighbourhoods of geodesic lines.

**Lemma 5.3** [PP2, Lemma 2.5] *Let  $C$  be a convex subset in  $X$ , let  $\epsilon > 0$  and let  $\xi_0 \in (X \cup \partial_\infty X) - (\mathcal{N}_\epsilon C \cup \partial_\infty C)$ . If two geodesic rays or lines  $\rho, \rho'$  which start from  $\xi_0$  intersect  $\mathcal{N}_\epsilon C$ , then the first intersection points  $x, x'$  of  $\rho, \rho'$  respectively with  $\mathcal{N}_\epsilon C$  are at a distance at most  $c'_1(\epsilon) = 2 \operatorname{argsinh}(\coth \epsilon)$ .  $\square$*

**Lemma 5.4** [PP2, Lemmas 2.7 and 2.8] *For every  $\epsilon, \eta > 0$ , there exist (explicit) constants  $c'_2(\epsilon), c'_3(\epsilon) > 0$  and  $c(\epsilon, \eta) > 0$  such that the following holds. Let  $X$  be a CAT(-1) space,  $C$  a convex subset in  $X$ ,  $\xi_0 \in X \cup \partial_\infty X$ , and  $\rho, \rho'$  geodesic rays or lines starting from  $\xi_0$ . If  $\rho$  enters  $\mathcal{N}_\epsilon C$  at a point  $x \in X$  and exits  $\mathcal{N}_\epsilon C$  at a point  $y \in X$  such that  $d(x, y) \geq c(\epsilon, \eta)$  and  $d(y, \rho') \leq \eta$ , then  $\rho'$  enters  $\mathcal{N}_\epsilon C$  at a point  $x' \in X$  such that  $d(x, x') \leq c'_2(\epsilon) d(x, \rho')$  and exits  $\mathcal{N}_\epsilon C$  at a point  $y' \in X \cup \partial_\infty X$  such that*

$$d(y, y') \leq c'_3(\epsilon) d(y, \rho') \text{ or } d(x', y') > d(x, y) . \quad \square$$

Let  $X, (L_\alpha, f_\alpha)_{\alpha \in \mathcal{A}}, \xi, c, c', \kappa$  be as in the statement of Theorem 5.2. Note that by the equations (-4-) and (-5-), we have

$$\kappa \leq 2 c'_1(\epsilon) + 2\epsilon + 4 \log(1 + \sqrt{2}) .$$

In particular,  $\kappa$  is finite. We start the proof of this theorem by defining the constants that will be used therein. Let

$$c_* = \kappa + \max \{ \kappa, c'_1(\epsilon)(1 + c'_2(\epsilon)) + \delta, c(\epsilon, \delta + c'_1(\epsilon)), c'_1(\epsilon)(c'_2(\epsilon) + c'_3(\epsilon) + 2\delta) + \nu' + \kappa \} ,$$

where the constants  $c'_1(\cdot), c'_2(\cdot)$  and  $c(\cdot, \cdot)$  are defined in the lemmas 5.3 and 5.4. Note that  $c_* > \max\{\kappa + 2\delta, \nu'\}$ , since  $c'_1(\epsilon) \geq 1$  for all  $\epsilon > 0$ . Let

$$h_* = h_*(\epsilon, \delta, \kappa, \nu, \nu') = \max\{c_*, \nu\} . \quad (-16-)$$

Let  $h \geq h_*$ , and let  $\alpha_0 \in \mathcal{A}$  be such that  $\xi \notin \mathcal{N}_\epsilon L_{\alpha_0} \cup \partial_\infty L_{\alpha_0}$ . The existence of such an index follows from the assumptions: Indeed, as  $\partial_\infty X$  is (Hausdorff and) infinite, and by the density of  $\bigcup_{\alpha \in \mathcal{A}} \partial_\infty L_\alpha$ , the set  $\mathcal{A}$  is infinite; note that  $\partial_\infty L_\alpha \cap \partial_\infty L_\beta$  is empty if  $\alpha \neq \beta$ , otherwise, as geodesic rays converging to the same point at infinity become exponentially close, we would have  $\operatorname{diam}(\mathcal{N}_\epsilon L_\alpha \cap \mathcal{N}_\epsilon L_\beta) = +\infty$ ; hence  $\xi$  belongs to at most one  $\partial_\infty L_\alpha$  if  $\xi \in \partial_\infty X$ ; if  $\xi \in X$ , then  $\xi$  belongs to at most finitely many  $\mathcal{N}_\epsilon L_\alpha$  for  $\alpha \in \mathcal{A}$ , as  $X$  is proper and  $\operatorname{diam}(\mathcal{N}_\epsilon L_\alpha \cap \mathcal{N}_\epsilon L_\beta) \leq \delta$  if  $\alpha \neq \beta$ .

As  $h \geq h_* \geq \nu$ , there exists, by the assumptions of Theorem 5.2, a geodesic ray or line  $\rho_0$  starting from  $\xi$ , entering  $\mathcal{N}_\epsilon L_{\alpha_0}$  at time  $t = 0$ , such that  $f_{\alpha_0}(\rho_0) = h$ , and  $f_\alpha(\rho_0) \leq \nu'$  for every  $\alpha \neq \alpha_0$  such that  $\rho_0(]0, +\infty[)$  meets  $\mathcal{N}_\epsilon L_\alpha$ .

If a geodesic ray or line  $\rho$  starting from  $\xi$  meets  $\mathcal{N}_\epsilon L_\alpha$ , let  $t_\alpha^-(\rho)$  and  $t_\alpha^+(\rho)$  be the entrance and exit times.

We construct, by induction, sequences  $(\rho_k)_{k \in \mathbb{N}}$  of geodesic rays or lines starting from  $\xi$ ,  $(\alpha_k)_{k \in \mathbb{N}}$  of elements of  $\mathcal{A}$ , and  $(t_k)_{k \in \mathbb{N} - \{0\}}$  of elements in  $[0, +\infty[$  converging to  $+\infty$ , such that for every  $k \in \mathbb{N}$ ,

- (1)  $\rho_k$  enters the interior of  $\mathcal{N}_\epsilon L_{\alpha_0}$  at time 0, with  $d(\rho_k(0), \rho_{k-1}(0)) \leq \frac{1}{2^k}$  if  $k \geq 1$ ;
- (2)  $\rho_k$  enters  $\mathcal{N}_\epsilon L_{\alpha_k}$  and  $f_{\alpha_k}(\rho_k) = h$ ;
- (3) if  $0 \leq j \leq k-1$ , then  $\rho_k(]0, +\infty[)$  enters the interior of  $\mathcal{N}_\epsilon L_{\alpha_j}$  before entering  $\mathcal{N}_\epsilon L_{\alpha_k}$  with  $t_{\alpha_j}^-(\rho_k) < t_k = t_{\alpha_{k-1}}^+(\rho_k)$ ;
- (4) if  $k \geq 1$ , then for every  $\alpha$  such that  $\rho_k(]0, +\infty[)$  meets  $\mathcal{N}_\epsilon L_\alpha$ , we have
  - $|f_\alpha(\rho_k) - f_\alpha(\rho_{k-1})| < \frac{1}{2^k}$  if  $t_\alpha^-(\rho_k) < t_k$ ,
  - $f_\alpha(\rho_k) \leq c_*$  if  $\alpha \neq \alpha_k$  and  $t_k \leq t_\alpha^-(\rho_k) \leq t_{\alpha_k}^-(\rho_k) + \delta$ ,
  - $f_\alpha(\rho_k) \leq \nu'$  if  $t_\alpha^-(\rho_k) > t_{\alpha_k}^-(\rho_k) + \delta$ .

Let us first prove that the existence of such sequences implies Theorem 5.2. By the assertion (1), the sequence  $(\rho_k(0))_{k \in \mathbb{N}}$  stays at bounded distance from  $\rho_0(0)$ , by a geometric series argument. Hence as  $X$  is proper, up to extracting a subsequence, the sequence  $(\rho_k)_{k \in \mathbb{N}}$  converges to a geodesic ray or line  $\rho_\infty$  starting from  $\xi$ , entering in  $\mathcal{N}_\epsilon L_{\alpha_0}$  at time  $t = 0$ , by the continuity of the entering point in the interior of the  $\epsilon$ -neighborhood of a convex subset of  $X$  (see for instance [PP2, Lemma 3.1]). Let us prove that

$$\limsup_{i \rightarrow +\infty} a_i(\rho_\infty) = h .$$

The lower bound  $\limsup_{i \rightarrow +\infty} a_i(\rho_\infty) \geq h$  is immediate by a semicontinuity argument. Indeed, for every  $k > i$  in  $\mathbb{N}$ , we have by the assertions (2), (3) and (4),

$$|f_{\alpha_i}(\rho_k) - h| = |f_{\alpha_i}(\rho_k) - f_{\alpha_i}(\rho_i)| \leq \sum_{j=i}^{k-1} |f_{\alpha_i}(\rho_{j+1}) - f_{\alpha_i}(\rho_j)| \leq \sum_{j=i}^{k-1} \frac{1}{2^{j+1}} \leq \frac{1}{2^i} .$$

Hence by the continuity of  $f_{\alpha_i}$  (see Section 2), we have the inequality  $f_{\alpha_i}(\rho_\infty) \geq h - \frac{1}{2^i}$ , whose right side converges to  $h$  as  $i$  tends to  $+\infty$ , which proves the lower bound, by the definition of  $\kappa$  and of the penetration sequence, as  $h \geq h_* \geq c_* > \delta + \kappa$ .

To prove the upper bound  $\limsup_{i \rightarrow +\infty} a_i(\rho_\infty) \leq h$ , assume by absurd that there exists  $\eta > 0$  such that for every  $\lambda > 0$ , there exists  $\alpha = \alpha(\lambda) \in \mathcal{A}$  such that  $\rho_\infty$  enters  $\mathcal{N}_\epsilon L_\alpha$  with  $f_\alpha(\rho_\infty) \geq h + \eta$  and  $t_\alpha^-(\rho_\infty) > \lambda + 2c'_1(\epsilon)$ , where  $c'_1(\epsilon)$  has been defined in Lemma 5.3. Take

$$\lambda_0 = \max \left\{ t_{i+1} : \frac{1}{2^i} \geq \frac{\eta}{2} \right\}$$

and  $\alpha = \alpha(\lambda_0)$ .

By continuity of  $f_\alpha$ , if  $k$  is big enough, we have  $f_\alpha(\rho_k) \geq h + \frac{\eta}{2} > h_*$ . In particular,  $\alpha \neq \alpha_k$  by the assertion (2). Since

$$h_* \geq c_* \geq \kappa \geq |f_\alpha(\rho_k) - \ell_{\mathcal{N}_\epsilon L_\alpha}(\rho_k)|,$$

the geodesic  $\rho_k$  meets  $\mathcal{N}_\epsilon L_\alpha$ . The entry time of  $\rho_k$  in  $\mathcal{N}_\epsilon L_\alpha$  is positive, as  $d(\rho_k(0), \rho_\infty(0)) \leq c'_1(\epsilon)$  and the entrance points of  $\rho_k$  and  $\rho_\infty$  in  $\mathcal{N}_\epsilon L_\alpha$  are at distance at most  $c'_1(\epsilon)$ , both by Lemma 5.3, and as the entrance time of  $\rho_\infty$  in  $\mathcal{N}_\epsilon L_\alpha$  is bigger than  $2c'_1(\epsilon)$ . Hence, since  $\nu' \leq c_* \leq h_*$  by the definitions of  $c_*$  and of  $h_*$ , we have  $t_\alpha^-(\rho_k) < t_k$ , otherwise, by the assertion (4),  $f_\alpha(\rho_k) \leq \max\{c_*, \nu'\} \leq h_*$ , a contradiction. Let  $i \leq k-1$  be the minimum element of  $\mathbb{N}$  such that for  $j = i, \dots, k-1$ , the geodesic  $\rho_{j+1}$  meets  $\mathcal{N}_\epsilon L_\alpha$  at a positive time with  $t_\alpha^-(\rho_{j+1}) < t_{j+1}$ . By the triangular inequality, we have

$$|t_\alpha^-(\rho_{i+1}) - t_\alpha^-(\rho_\infty)| \leq d(\rho_{i+1}(t_\alpha^-(\rho_{i+1})), \rho_\infty(t_\alpha^-(\rho_\infty))) + d(\rho_{i+1}(0), \rho_\infty(0)) \leq 2c'_1(\epsilon),$$

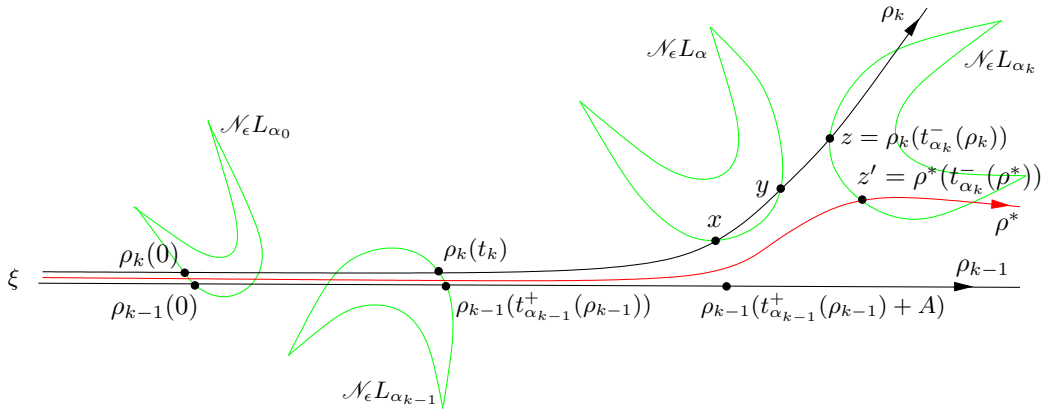
by applying twice Lemma 5.3. Hence, by the definition of  $i$  and of  $\alpha$ ,

$$t_{i+1} > t_\alpha^-(\rho_{i+1}) \geq t_\alpha^-(\rho_\infty) - 2c'_1(\epsilon) > \lambda_0 + 2c'_1(\epsilon) - 2c'_1(\epsilon) = \lambda_0.$$

By the definition of  $\lambda_0$ , we hence have  $\frac{1}{2^i} < \frac{\eta}{2}$ . By the definition of  $i$  and by the assertion (4), we have

$$\begin{aligned} f_\alpha(\rho_i) &= f_\alpha(\rho_k) + \sum_{j=i}^{k-1} (f_\alpha(\rho_j) - f_\alpha(\rho_{j+1})) \geq h + \frac{\eta}{2} - \sum_{j=i}^{k-1} \frac{1}{2^{j+1}} \\ &\geq h + \frac{\eta}{2} - \frac{1}{2^i} \geq h \geq h_*, \end{aligned}$$

and in particular by the same argument as for  $\rho_k$  above,  $\rho_i$  enters  $\mathcal{N}_\epsilon L_\alpha$  at a positive time and  $t_\alpha^-(\rho_i) < t_i$ . This contradicts the minimality of  $i$ . This completes the proof of Theorem 5.2, assuming the existence of the sequences with properties (1)–(4).



Let us now construct the sequences  $(\rho_k)_{k \in \mathbb{N}}$ ,  $(\alpha_k)_{k \in \mathbb{N}}$  and  $(t_k)_{k \in \mathbb{N} - \{0\}}$ . We already have defined  $\rho_0$  and  $\alpha_0$ , and they satisfy the properties (1)–(4). Let  $k \geq 1$ , and assume that  $\rho_{k-1}$ ,  $\alpha_{k-1}$ , as well as  $t_{k-1}$  if  $k \geq 2$ , have been constructed.

For every small  $\mu > 0$  and big  $A > 0$  (to be precised later on), consider the set  $\mathcal{E}' = \mathcal{E}'_k$  of couples  $(\alpha, \rho)$  where  $\alpha \in \mathcal{A}$  and  $\rho$  is a geodesic ray or line, starting from  $\xi$ , entering

$\mathcal{N}_\epsilon L_{\alpha_0}$  at time  $t = 0$ , which is  $\mu$ -close to  $\rho_{k-1}$  on  $[0, t_{\alpha_{k-1}}^+(\rho_{k-1}) + A]$ , and which enters  $\mathcal{N}_\epsilon L_\alpha$  with  $t_\alpha^-(\rho) \geq t_{\alpha_{k-1}}^+(\rho_{k-1}) + \frac{A}{2}$ . This set  $\mathcal{E}'$  is not empty, as  $\bigcup_{\alpha \in \mathcal{A}} \partial_\infty L_\alpha$  is dense in  $\partial_\infty X$ , and as the assumption that  $\text{diam}(\mathcal{N}_\epsilon L_\alpha \cap \mathcal{N}_\epsilon L_\beta) \leq \delta$  for all  $\alpha \neq \beta$  in  $\mathcal{A}$  implies that a compact subset of  $X$  meets only finitely many  $\mathcal{N}_\epsilon L_\alpha$  for  $\alpha \in \mathcal{A}$  (we may even find such a couple  $(\alpha, \rho)$  with  $\rho(+\infty) \in \partial_\infty L_\alpha$ , which implies that a subray of  $\rho$  is contained in  $\mathcal{N}_\epsilon L_\alpha$ ). Let  $(\alpha_k, \rho^*)$  be an element of  $\mathcal{E}'$  with  $t_{\alpha_k}^-(\rho^*)$  minimal, which exists since the family  $(\mathcal{N}_\epsilon L_\alpha)_{\alpha \in \mathcal{A}}$  is locally finite and by a continuity argument. Note that  $\xi \notin \mathcal{N}_\epsilon L_{\alpha_k} \cup \partial_\infty L_{\alpha_k}$ , as  $t_{\alpha_k}^+(\rho^*) > 0$ .

By the last hypothesis of Theorem 5.2, let  $\rho_k$  be a geodesic ray or line starting from  $\xi$  with  $f_{\alpha_k}(\rho_k) = h$  and  $f_\alpha(\rho_k) \leq \nu'$  for every  $\alpha \in \mathcal{A}$  such that  $\rho_k(]t_{\alpha_k}^-(\rho_k) + \delta, +\infty[)$  enters  $H_\alpha$ . In particular, this proves the assertion (2) at rank  $k$ , since

$$h \geq h_* \geq c_* > \kappa \geq |f_{\alpha_k}(\rho_k) - \ell_{\mathcal{N}_\epsilon L_{\alpha_k}}(\rho_k)|. \quad (-17-)$$

Let  $z = \rho_k(t_{\alpha_k}^-(\rho_k))$  and  $z' = \rho^*(t_{\alpha_k}^-(\rho^*))$  be the entering points of  $\rho_k$  and  $\rho^*$  in  $\mathcal{N}_\epsilon L_{\alpha_k}$ . By Lemma 5.3, we have  $d(z, z') \leq c'_1(\epsilon)$ . Hence, by hyperbolicity and as  $(\alpha_k, \rho^*)$  is in  $\mathcal{E}'$ , if  $A$  is big enough, then  $\rho_k$  is  $(2\mu)$ -close to  $\rho_{k-1}$  between  $\xi$  and  $\rho_{k-1}(t_{\alpha_{k-1}}^+(\rho_{k-1}) + 1)$ . In particular, if  $\mu$  is small enough, and using properties (1) and (3) at rank  $k-1$ , we have

- $\rho_k$  enters the interior of  $\mathcal{N}_\epsilon L_{\alpha_0}$ , at a time that we may assume to be 0, with  $d(\rho_k(0), \rho_{k-1}(0)) \leq \frac{1}{2^k}$  (this proves the assertion (1) at rank  $k$ );
- for  $0 \leq j \leq k-1$ , as  $\rho_{k-1}$  meets the interior of  $\mathcal{N}_\epsilon L_{\alpha_j}$  at a time strictly between 0 and  $t_{\alpha_{k-1}}^+(\rho_{k-1})$ , by the inductive assertions (3) if  $k \neq 1$  and  $j \leq k-2$ , or (1) if  $k=1$  or (2) if  $j=k-1$  (by Equation (-17-) where  $k$  has been replaced by  $k-1$ ), the geodesic ray  $\rho_k$  also meets the interior of  $\mathcal{N}_\epsilon L_{\alpha_j}$  at a time strictly between 0 and  $t_{\alpha_{k-1}}^+(\rho_{k-1})$ ; this allows, in particular, to define  $t_k = t_{\alpha_{k-1}}^+(\rho_k)$ , and proves the assertion (3) at rank  $k$ ;
- for every  $\alpha$  such that  $\rho_k(]0, +\infty[)$  meets  $\mathcal{N}_\epsilon L_\alpha$  and  $t_\alpha^-(\rho_k) < t_k$ , we may assume, by the continuity of  $f_\alpha$ , up to taking  $\mu$  small enough, that  $|f_\alpha(\rho_k) - f_\alpha(\rho_{k-1})| < \frac{1}{2^k}$ .

Hence, to prove the assertion (4) at rank  $k$ , we consider  $\alpha \in \mathcal{A} - \{\alpha_k\}$  such that  $\rho_k$  meets  $\mathcal{N}_\epsilon L_\alpha$  with  $t_k \leq t_\alpha^-(\rho_k) \leq t_{\alpha_k}^-(\rho_k) + \delta$ , and we prove that  $f_\alpha(\rho_k) \leq c_*$ .

Assume by absurd that  $f_\alpha(\rho_k) > c_*$ . In particular,  $\ell_{\mathcal{N}_\epsilon L_\alpha}(\rho_k) \geq c_* - \kappa > 0$  (by the definition of  $c_*$ ), so that  $\rho_k$  enters  $\mathcal{N}_\epsilon L_\alpha$ . Let  $x = \rho_k(t_\alpha^-(\rho_k))$  be the entering point of  $\rho_k$  in  $\mathcal{N}_\epsilon L_\alpha$ . Note that

$$\ell_{\mathcal{N}_\epsilon L_{\alpha_k}}(\rho_k) \geq f_{\alpha_k}(\rho_k) - \kappa = h - \kappa \geq h_* - \kappa \geq c_* - \kappa,$$

by the definition of  $h_*$ . If  $t_\alpha^-(\rho_k) \geq t_{\alpha_k}^-(\rho_k)$ , then, since

$$\min\{\ell_{\mathcal{N}_\epsilon L_\alpha}(\rho_k), \ell_{\mathcal{N}_\epsilon L_{\alpha_k}}(\rho_k)\} \geq c_* - \kappa > 2\delta$$

by the definition of  $c_*$ , this would imply that the intersection  $\mathcal{N}_\epsilon L_\alpha \cap \mathcal{N}_\epsilon L_{\alpha_k}$  has diameter bigger than  $\delta$ , which contradicts  $\alpha \neq \alpha_k$ . Hence  $t_\alpha^-(\rho_k) < t_{\alpha_k}^-(\rho_k)$  and we have  $t_\alpha^+(\rho_k) \leq t_{\alpha_k}^-(\rho_k) + \delta$ , again since  $\text{diam}(\mathcal{N}_\epsilon L_\alpha \cap \mathcal{N}_\epsilon L_{\alpha_k}) \leq \delta$  and  $\alpha \neq \alpha_k$ . In particular,  $y = \rho_k(t_\alpha^+(\rho_k))$  is a point in  $X$ .

We want to apply Lemma 5.4 with  $\eta = \delta + c'_1(\epsilon)$ ,  $\rho = \rho_k$ ,  $\rho' = \rho^*$ ,  $C = L_\alpha$ . As

$$d(\rho_k(t_{\alpha_k}^-(\rho_k) + \delta), \rho^*) \leq d(z, z') + \delta \leq \eta ,$$

and by convexity, we have  $d(y, \rho^*) \leq \eta$ , and

$$d(x, y) \geq f_\alpha(\rho_k) - \kappa > c_* - \kappa \geq c(\epsilon, \eta) , \quad (-18-)$$

by the definition of  $c_*$ . Note that, in particular, this implies that  $x \in [\xi, z]$ , as  $d(x, y) > c_* - \kappa > \delta$  by the definition of  $c_*$ .

Hence we may indeed apply Lemma 5.4, and the geodesic  $\rho^*$  enters  $\mathcal{N}_\epsilon L_\alpha$  at a point  $x'$  such that

$$d(x, x') \leq c'_2(\epsilon) d(x, \rho^*) \leq c'_2(\epsilon) d(z, \rho^*) \leq c'_2(\epsilon) c'_1(\epsilon) , \quad (-19-)$$

where the middle inequality holds by convexity, and the last one since  $d(z, z') \leq c'_1(\epsilon)$ .

Furthermore, by the alternative at the end of Lemma 5.4 and Equation (-18-), the geodesic  $\rho^*$  exits  $\mathcal{N}_\epsilon L_\alpha$  at a point  $y'$  (possibly at infinity) and

$$d(x', y') > d(x, y) > c_* - \kappa \geq \nu' + \kappa$$

or

$$\begin{aligned} d(x', y') &\geq d(x, y) - d(x, x') - d(y, y') \geq d(x, y) - d(x, x') - c'_3(\epsilon) d(y, \rho^*) \\ &> (c_* - \kappa) - c'_2(\epsilon) c'_1(\epsilon) - c'_3(\epsilon) \eta \geq \nu' + \kappa , \end{aligned}$$

by the definition of  $c_*$ . In both cases,  $d(x', y') > \nu' + \kappa$ .

Let us prove that

$$t_{\alpha_k}^-(\rho^*) > t_{\alpha_k}^-(\rho_k) . \quad (-20-)$$

Otherwise, the point  $z'$  belongs to  $[\xi, x']$ . Hence, with  $x'', z''$  the closest points to  $x', z'$  respectively on  $\rho_k$ , we have  $z'' \in [\xi, x'']$ . Therefore, as closest point maps do not increase distances, we have, using (-19-),

$$d(x, z) \leq d(x, x'') + d(x'', z) \leq d(x, x') + d(z'', z) \leq d(x, x') + d(z', z) \leq (c'_2(\epsilon) + 1) c'_1(\epsilon) .$$

Therefore

$$c_* < f_\alpha(\rho_k) \leq d(x, y) + \kappa \leq d(x, z) + \delta + \kappa \leq c'_1(\epsilon)(c'_2(\epsilon) + 1) + \delta + \kappa ,$$

which contradicts the definition of  $c_*$ .

Now, recall the constants  $\mu > 0$  and  $A \geq 0$  introduced in the definition of  $\rho^*$ . It follows from Equation (-20-), and from the minimality assumption in the definition of  $\alpha_k$ , that we have

$$t_{\alpha_k}^-(\rho^*) < t_{\alpha_{k-1}}^+(\rho_{k-1}) + \frac{A}{2} .$$

Assume that  $\mu$  is small enough and that  $A$  is big enough. Since  $d(x', y') > \nu' + \kappa$ , and as  $\rho^*$  is  $\mu$ -close to  $\rho_{k-1}$  on  $[0, t_{\alpha_{k-1}}^+(\rho_{k-1}) + A]$ , this implies that  $\rho_{k-1}$  enters  $\mathcal{N}_\epsilon L_\alpha$  at a point  $x^\sharp$  close to  $x'$ , and exits at a point  $y^\sharp$  (possibly at infinity) such that  $d(x^\sharp, y^\sharp) > \nu' + \kappa$ . Hence

$$f_\alpha(\rho_{k-1}) \geq d(x^\sharp, y^\sharp) - \kappa > \nu' .$$

This implies that  $t_\alpha^-(\rho_{k-1}) \leq t_{\alpha_{k-1}}^-(\rho_{k-1}) + \delta$ , otherwise we have in particular that  $\alpha \neq \alpha_{k-1}$  and by the assertion (4) at rank  $k-1$  if  $k \geq 2$  or by the construction of  $\rho_0$  if  $k=1$ , we would have  $f_\alpha(\rho_{k-1}) \leq \nu'$ . Hence

$$\begin{aligned} t_{\alpha_{k-1}}^+(\rho_{k-1}) - t_\alpha^-(\rho_{k-1}) &\geq t_{\alpha_{k-1}}^+(\rho_{k-1}) - t_{\alpha_{k-1}}^-(\rho_{k-1}) - \delta = \ell_{\mathcal{N}_\epsilon L_{\alpha_{k-1}}}(\rho_{k-1}) - \delta \\ &\geq f_{\alpha_{k-1}}(\rho_{k-1}) - \kappa - \delta = h - \delta - \kappa \geq h_* - \delta - \kappa \geq c_* - \delta - \kappa \end{aligned}$$

That is,  $\rho_{k-1}$  enters in  $\mathcal{N}_\epsilon L_\alpha$  well before exiting  $\mathcal{N}_\epsilon L_{\alpha_{k-1}}$ , the amount of time being at least the constant  $c_* - \delta - \kappa$  (which is positive by the definition of  $c_*$ ). But as the entering points in  $\mathcal{N}_\epsilon L_\alpha$ , as well as the exiting points out of  $\mathcal{N}_\epsilon L_{\alpha_{k-1}}$ , of the geodesic rays or lines  $\rho_{k-1}$ ,  $\rho_*$  and  $\rho_k$  are very close, this contradicts the fact that  $t_\alpha^-(\rho_k) \geq t_k = t_{\alpha_{k-1}}^+(\rho_k)$ .

This proves the result.  $\square$

**Corollary 5.5** *Let  $X$  be a complete simply connected Riemannian manifold with sectional curvature at most  $-1$  and dimension at least 3, such that the metric spheres for the Hamenstädt distances (on  $\partial_\infty X - \{\xi'\}$  for any  $\xi' \in \partial_\infty X$ ) are topological spheres. Let  $\Gamma$  be a discrete group of isometries of  $X$  with finite covolume, and let  $\gamma_0$  be a hyperbolic element of  $\Gamma$ . Let  $\xi_0 \in \partial_\infty X$  be a parabolic fixed point, and  $H_0$  be a horosphere centered at  $\xi_0$ . For every  $\xi \in \partial_\infty X$  which is not a fixed point of a conjugate of  $\gamma_0$  or a parabolic fixed point, define*

$$c'(\xi) = \liminf \frac{d_{\xi_0, H_0}(\xi, \gamma_-)}{d_{\xi_0, H_0}(\gamma_+, \gamma_-)},$$

where the lower limit is taken over the conjugates  $\gamma$  of  $\gamma_0$  or its inverse, with fixed points  $\gamma_-, \gamma_+$  and  $d_{\xi_0, H_0}(\gamma_+, \gamma_-)$  tending to 0.

Then the subset of  $\mathbb{R}$  consisting of the  $c'(\xi)$  for  $\xi \in \partial_\infty X$  which is neither a fixed point of a conjugate of  $\gamma_0$  nor a parabolic fixed point, contains a segment  $[0, c]$  for some  $c > 0$ .

**Proof.** We will apply Theorem 5.2 with  $(L_\alpha)_{\alpha \in \mathcal{A}}$  the family of translation axes of the conjugates of the element  $\gamma_0$  (where each line appears exactly once), with  $\xi = \xi_0$  and with  $f_\alpha = \mathbf{crp}_{L_\alpha}$  for every  $\alpha$  in  $\mathcal{A}$ . Let  $\kappa = 2c'_1(\epsilon) + 2\epsilon + 4\log(1 + \sqrt{2})$ .

For some positive  $\epsilon, \delta$  this family satisfies the assumption that  $\text{diam}(\mathcal{N}_\epsilon L_\alpha \cap \mathcal{N}_\epsilon L_\beta) \leq \delta$  for all  $\alpha \neq \beta$  in  $\mathcal{A}$ . Otherwise, there would exist a sequence  $(\beta_n)_{n \in \mathbb{N}}$  in  $\Gamma - \Gamma_0$ , where  $\Gamma_0$  is the stabilizer in  $\Gamma$  of the translation axis  $L_0$  of  $\gamma_0$ , such that  $\text{diam}(\mathcal{N}_\epsilon L_\alpha \cap \mathcal{N}_\epsilon L_{\beta_n})$  converges to  $+\infty$ . Up to multiplying  $\beta_n$  on the right and on the left by a power of  $\gamma_0$  or its inverse, the element  $\beta_n$  moves a point of  $L_0$  less than a constant. Hence  $\beta_n$  stays in a compact subset of the isometry group of  $X$ . By discreteness, up to extracting a subsequence,  $\beta_n$  does not depend on  $n$ . But then  $L_0$  and  $\beta_0 L_0$  are two distinct translation axes that meet at least in one point at infinity, which contradicts the discreteness of  $\Gamma$ .

As  $\Gamma$  has finite covolume, the set of fixed points of the conjugates of  $\gamma_0$  is dense in  $\partial_\infty X$ , hence  $\bigcup_{\alpha \in \mathcal{A}} \partial_\infty L_\alpha$  is dense in  $\partial_\infty X$ . The last hypothesis of Theorem 5.2 holds true by Theorem 5.1, with  $\nu = h'_1$  and  $\nu' = h'_1 + \kappa$  (by definition of  $\kappa$ ). By Theorem 5.2, there exists  $h_* > 0$  such that for every  $h \geq h_*$ , there exists a geodesic line  $\rho$  starting from  $\xi_0$  such that  $\limsup_{i \rightarrow +\infty} a_i(\rho) = h$  where  $a_i(\rho)$  is the penetration sequence of  $\rho$  with respect to  $(\mathcal{N}_\epsilon L_\alpha, f_\alpha)_{\alpha \in \mathcal{A}}$  (and  $\delta, \kappa$ ).

For every  $i$  in  $\mathbb{N}$ , let  $\alpha_i \in \mathcal{A}$  be the unique element such that  $a_i(\rho) = \mathbf{crp}_{L_{\alpha_i}}(\rho)$ . For every  $\alpha \in \mathcal{A}$  such that  $\rho$  meets  $\mathcal{N}_\epsilon L_\alpha$  at a time big enough with  $\mathbf{crp}_{L_\alpha}(\rho) > 0$ , let  $L_{\alpha, \pm}$  be

the two endpoints of  $L_\alpha$  such that, by Equation (-3-) and the definition of the crossratio penetration map,

$$\mathbf{crp}_{L_\alpha}(\rho) = [\xi_0, L_{\alpha,-}, \rho(+\infty), L_{\alpha,+}] = \log \frac{d_{\xi_0, H_0}(L_{\alpha,+}, L_{\alpha,-})}{d_{\xi_0, H_0}(\rho(+\infty), L_{\alpha,-})}.$$

Only finitely many  $L_\alpha$ 's meet a given compact subset of  $X$ . Thus, a subsequence  $(i_k)_{k \in \mathbb{N}}$  tends to  $+\infty$  if and only if  $d_{\xi_0, H_0}(L_{\alpha_{i_k}, +}, L_{\alpha_{i_k}, -})$  tends to 0. Also note that, by definition of the penetration sequence, if  $\alpha \in \mathcal{A}$  does not belong to  $\{\alpha_i : i \in \mathbb{N}\}$ , then either  $\rho$  does not meet  $\mathcal{N}_\epsilon L_\alpha$  at a positive time, or  $\mathbf{crp}_{L_\alpha}(\rho) \leq \delta + \kappa < h_* \leq h$ , see Equation (-16-).

Finally, if  $(\gamma_k)_{k \in \mathbb{N}}$  is a sequence of conjugates of  $\gamma_0$  or its inverse with fixed points  $\gamma_{k,-}, \gamma_{k,+}$  and with  $d_{\xi_0, H_0}(\gamma_{k,-}, \gamma_{k,+})$  tending to 0, such that the sequence  $\frac{d_{\xi_0, H_0}(\rho(+\infty), \gamma_{k,-})}{d_{\xi_0, H_0}(\gamma_{k,-}, \gamma_{k,+})}$  is bounded from above, then  $\gamma_{k,-}$  tends to  $\rho(+\infty)$ . Hence, if

$$\liminf_{k \rightarrow +\infty} \frac{d_{\xi_0, H_0}(\rho(+\infty), \gamma_{k,-})}{d_{\xi_0, H_0}(\gamma_{k,-}, \gamma_{k,+})} < e^{-h},$$

then for  $k$  big enough, there exists  $\alpha \in \mathcal{A}$  such that  $L_\alpha$  is the translation axis of  $\gamma_k$ , and  $\mathbf{crp}_{L_{\alpha_i}}(\rho) > h \geq h_* > \delta + \kappa$  so that  $\rho$  meets  $\mathcal{N}_\epsilon L_{\alpha_i}$ , at a positive time. In particular  $\alpha$  belongs to  $\{\alpha_i : i \in \mathbb{N}\}$ . Therefore, we have

$$\liminf \frac{d_{\xi_0, H_0}(\rho(+\infty), \gamma_-)}{d_{\xi_0, H_0}(\gamma_+, \gamma_-)} = e^{-h},$$

where the lower limit is taken as in the statement of the corollary. This proves the result, with  $c = e^{-h_*(\epsilon, \delta, \kappa, h'_1, h'_1 + \kappa)}$ .  $\square$

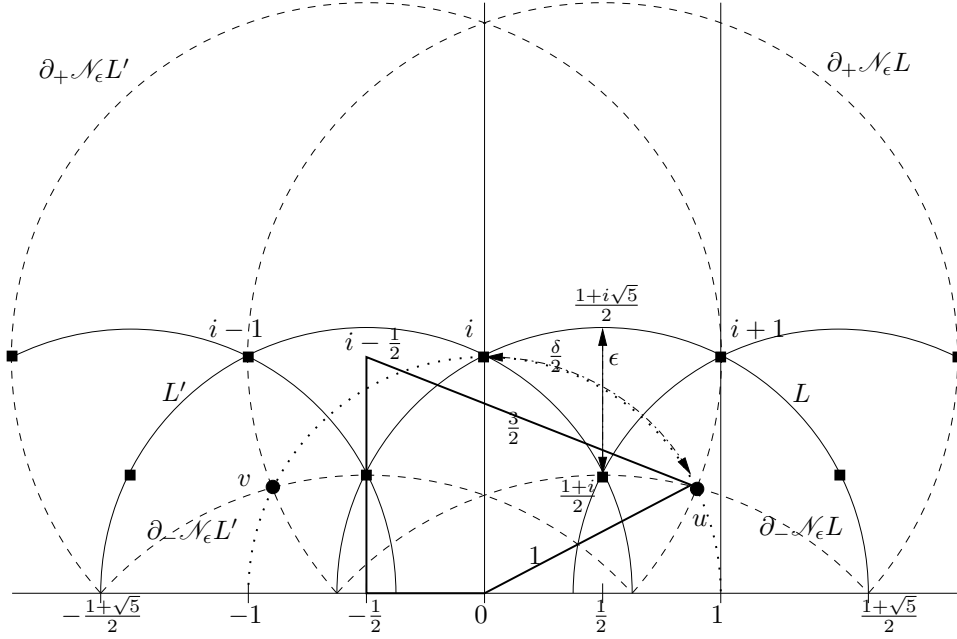
Specializing the above Corollary 5.5 to the particular cases of the real or complex hyperbolic space (see the examples at the end of Section 3), we have the following applications.

**Corollary 5.6** *Let  $n \geq 3$ , let  $\Gamma$  be a discrete group of isometries of  $X = \mathbb{H}_{\mathbb{R}}^n$  with finite covolume, and let  $\Gamma_0$  be the stabilizer in  $\Gamma$  of the translation axis of a hyperbolic element of  $\Gamma$ . Let  $C_\infty$  be a precisely invariant horosphere centered at a parabolic fixed point of  $\Gamma$ , and  $\mathcal{D} = (X, \Gamma, \Gamma_0, C_\infty)$ . Then  $\text{Sp}(\mathcal{D})$  contains a segment  $[0, c]$  for some  $c > 0$ .  $\square$*

By the last equality in the proof of Corollary 5.5, if one wants in particular situations to be able to give an explicit (lower bound on the) constant  $c$  appearing in Corollary 5.6 (which is the same as in Corollary 5.5), one only needs to find explicit  $\epsilon, \delta$  such that  $\text{diam}(\mathcal{N}_\epsilon L_\alpha \cap \mathcal{N}_\epsilon L_\beta) \leq \delta$  for all  $\alpha \neq \beta$  in  $\mathcal{A}$ . We give such a computation in the following remark.

**Remark 5.7** *In the real hyperbolic upper halfplane  $\mathbb{H}_{\mathbb{R}}^2$ , consider the geodesic line  $L$  with endpoints  $\frac{1+\sqrt{5}}{2}$  and  $\frac{1-\sqrt{5}}{2}$ . Let  $(L_\alpha)_{\alpha \in \mathcal{A}}$  be the family of the images of  $L$  by  $\text{PSL}_2(\mathbb{Z})$ , acting by homographies on  $\mathbb{H}_{\mathbb{R}}^2$ , modulo the (global) stabilizer of  $L$ . Let  $\epsilon = \frac{\log 5}{2}$  and  $\delta = 2 \log(2 + \sqrt{5})$ . Then  $\text{diam}(\mathcal{N}_\epsilon L_\alpha \cap \mathcal{N}_\epsilon L_\beta) \leq \delta$  for all  $\alpha \neq \beta$  in  $\mathcal{A}$ .*





**Proof.** Call *Weierstrass points* the points of  $\mathbb{H}_{\mathbb{R}}^2$  in the image of  $i$  by  $\mathrm{PSL}_2(\mathbb{Z})$ . Note that, by an easy computation, the  $\epsilon$ -neighborhood of  $L$  is the tubular neighborhood of  $L$  of biggest radius so that the Weierstrass points in its interior lie on  $L$ . As seen by considering the fundamental domain for the integer horizontal translations in  $\mathbb{H}_{\mathbb{R}}^2$ , between the geodesic lines with endpoints  $0, \infty$  and  $1, \infty$  respectively, if two images of  $L$  by elements of  $\mathrm{PSL}_2(\mathbb{Z})$  are disjoint, then the interiors of their  $\epsilon$ -neighborhoods do not meet. If two images of  $L$  by elements of  $\mathrm{PSL}_2(\mathbb{Z})$  are distinct but meet, to prove that the diameter of the intersection of their  $\epsilon$ -neighborhoods is at most  $\delta$ , we may assume that these images are  $L$  and the image  $L'$  of  $L$  by the translation by  $-1$ . Recall that the boundary of the  $\epsilon$ -neighborhood of the geodesic line carried by the Euclidean circle of center  $x \in \mathbb{R}$  and Euclidean radius  $r$  is the union of two arcs of circles between  $x+r$  and  $x-r$  that are invariant by symmetries on the vertical line through  $x$ . Since the upper arc of circle  $\partial_+ \mathcal{N}_\epsilon L'$  of  $\partial \mathcal{N}_\epsilon L'$  is tangent to the vertical line through the Weierstrass point  $i+1$ , its center is the point  $(-\frac{1}{2}, 1)$ . Since the geodesic line  $L''$  with endpoints  $-1, 1$  is a bisectrix of  $L$  and  $L'$ , it goes through the intersection of  $\partial_+ \mathcal{N}_\epsilon L'$  and the lower arc of circle  $\partial_- \mathcal{N}_\epsilon L$  of  $\partial \mathcal{N}_\epsilon L$ . By convexity and symmetry arguments, the diameter of  $\mathcal{N}_\epsilon L \cap \mathcal{N}_\epsilon L'$  is hence equal to the distance between the intersection points  $u, v$  of  $L''$  and  $\partial_+ \mathcal{N}_\epsilon L'$ , that is to twice the distance between  $i$  and  $u$ . Up to permuting  $u$  and  $v$ , we may assume that  $\mathrm{Re} u > 0$ . By considering the Euclidean quadrangle with vertices at  $u, i - \frac{1}{2}, -\frac{1}{2}, 0$ , we easily compute that the Euclidean height of  $u$  is  $\frac{1}{\sqrt{5}}$ . Using Formula (-15-), we hence have  $d(i, u) = \log(2 + \sqrt{5})$ , and the result follows.  $\square$

The following result is a consequence Corollary 5.6 and Equation (-9-):

**Corollary 5.8** *Let  $M$  be a geometrically finite complete connected Riemannian manifold of constant sectional curvature  $-1$  and of dimension at least 3. Let  $A_0$  be a closed geodesic in  $M$ , and let  $A_\infty$  be a Margulis neighbourhood of a cusp of  $M$ . Then the spiraling spectrum  $\mathrm{Sp}_{A_\infty, A_0}(M)$  around  $A_0$  contains a segment  $[0, c]$  for some  $c > 0$ .  $\square$*

Using Equation (-13-), we obtain the following result in the complex hyperbolic case.

**Corollary 5.9** *Let  $n \geq 2$ , let  $\Gamma$  be a discrete group of isometries of the Siegel domain model of  $\mathbb{H}_{\mathbb{C}}^n$  with finite covolume, and let  $\gamma_0$  be an hyperbolic element of  $\Gamma$ . Assume that the point  $\infty$  is a parabolic fixed point of  $\Gamma$ . For every  $\xi \in \partial_{\infty}\mathbb{H}_{\mathbb{C}}^n$  which is neither a fixed point of a conjugate of  $\gamma_0$  nor a parabolic fixed point, define*

$$c'(\xi) = \liminf \frac{d_{\text{Cyg}}(\xi, \gamma_-)}{d_{\text{Cyg}}(\gamma_+, \gamma_-)},$$

where the lower limit is taken over the conjugates  $\gamma$  of  $\gamma_0$  or its inverse, with fixed points  $\gamma_-, \gamma_+$  and  $d_{\text{Cyg}}(\gamma_+, \gamma_-)$  tending to 0.

Then the subset of  $\mathbb{R}$  consisting of the  $c'(\xi)$  for  $\xi \in \partial_{\infty}\mathbb{H}_{\mathbb{C}}^n$  which is neither a fixed point of a conjugate of  $\gamma_0$  nor a parabolic fixed point, contains a segment  $[0, c]$  for some  $c > 0$ .

□

## 6 Applications to Diophantine approximation

In this section, we apply the results of Sections 4 and 5 to study Diophantine approximation by quadratic irrational elements in  $\mathbb{R}$ ,  $\mathbb{C}$  and the Heisenberg group.

In order to obtain the Khinchin-type results both in Subsection 6.1 and Subsection 6.2, we will apply the following result, which follows as a particular case from [HP5, Theorem 4.6]. We refer for instance to [HP5] for the general definitions of the critical exponent  $\delta \in [0, +\infty]$  and of the Patterson-Sullivan measure  $\mu_{\xi_{\infty}, H_{\infty}}$  associated to a horosphere  $H_{\infty}$  with point at infinity  $\xi_{\infty}$ , for a nonelementary discrete group of isometries  $\Gamma$  of a complete simply connected Riemannian manifold  $X$  with sectional curvature at most  $-1$ . In this paper, we will only be interested in the particular cases explained after the statement.

**Theorem 6.1** [HP5] *Let  $X$  be a complete simply connected Riemannian manifold with sectional curvature at most  $-1$  and dimension at least 2; let  $\Gamma$  be a discrete group of isometries of  $X$  with finite covolume and critical exponent  $\delta$ ; let  $\gamma_0$  be a hyperbolic element of  $\Gamma$  and  $\mathcal{B}_{\Gamma_0}$  be the set of points in  $\partial_{\infty}X$  fixed by some conjugate of  $\gamma_0$  in  $\Gamma$ ; let  $\xi_{\infty}$  be a parabolic fixed point of  $\Gamma$  and  $H_{\infty}$  be a horosphere centered at  $\xi_{\infty}$ ; and let  $f : [0, +\infty[ \rightarrow ]0, +\infty[$  be a slowly varying map (as defined in Section 4.1).*

*If  $\int_1^{+\infty} f(t)^{\delta} dt$  converges (resp. diverges), then  $\mu_{\xi_{\infty}, H_{\infty}}$ -almost no (resp. every) point of  $\partial_{\infty}X - \{\xi_{\infty}\}$  belongs to infinitely many balls of center  $r$  and radius  $f(D(r))e^{-D(r)}$  for the distance  $d_{\xi_{\infty}, H_{\infty}}$ , where  $r$  ranges over  $\mathcal{B}_{\Gamma_0}$ .* □

In our applications in Section 6.1 (resp. 6.2),  $X$  is the upper halfspace model of  $\mathbb{H}_{\mathbb{R}}^n$  (resp. the Siegel domain model of  $\mathbb{H}_{\mathbb{C}}^n$  as in Example 1 of Section 3) and  $H_{\infty}$  is the horosphere (centered at  $\xi_{\infty} = \infty$ ) of the points at Euclidean height 1 (resp.  $H_{\infty} = \{(w_0, w) \in \mathbb{H}_{\mathbb{C}}^n : 2\text{Re } w_0 - |w|^2 = 2\}$ ). In this situation, since  $\Gamma$  has finite covolume,

- the critical exponent is  $\delta = n - 1$  (resp.  $\delta = 2n$ ),
- the Hamenstädt distance  $d_{\xi_{\infty}, H_{\infty}}$  on  $\partial_{\infty}X - \{\infty\}$  is the Euclidean distance, see [HP2, § 2.1] (resp. a multiple of the Cygan distance, see [HP3, § 3.11]), and
- the measure  $\mu_{\xi_{\infty}, H_{\infty}}$  on  $\partial_{\infty}X - \{\infty\}$  is the Hausdorff measure of  $d_{\xi_{\infty}, H_{\infty}}$ , i.e. is the Lebesgue measure (resp. is in the same measure class as the Hausdorff measure of the Cygan distance).

## 6.1 Approximation in $\mathbb{R}$ and $\mathbb{C}$ by irrational quadratic elements

Let  $K$  be either the field  $\mathbb{Q}$  or an imaginary quadratic extension of  $\mathbb{Q}$ , and correspondingly, let  $\widehat{K}$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $\mathcal{O}_K$  be the ring of integers of  $K$ . Denote by  $K_{\text{quad}}$  the set of irrational quadratic elements in  $\widehat{K}$  over  $K$ . For every  $\alpha \in K_{\text{quad}}$ , let  $\alpha^\sigma$  be its Galois conjugate over  $K$ .

The group  $\text{PGL}_2(\widehat{K})$  acts on  $\mathbb{P}^1(K) = \widehat{K} \cup \{\infty\}$  by homographies, and its subgroup  $\text{PGL}_2(\mathcal{O}_K)$  preserves  $K$  and  $K_{\text{quad}}$ . Note that, for every  $\alpha \in K_{\text{quad}}$  and every  $\gamma \in \text{PGL}_2(\mathcal{O}_K)$ , we have  $(\gamma \cdot \alpha)^\sigma = \gamma \cdot (\alpha^\sigma)$ .

Let us fix a finite index subgroup  $\Gamma$  of  $\text{PSL}_2(\mathcal{O}_K)$ . An orbit of  $\Gamma$  in  $K_{\text{quad}}$  will be called a *congruence class* in  $K_{\text{quad}}$  under  $\Gamma$ . We are interested in Section 6.1 in the approximation of elements of  $\widehat{K}$  by elements in the union of a fixed congruence class and of its Galois conjugate.

For every  $\alpha \in K_{\text{quad}}$ , let

$$\mathcal{E}_{\alpha, \Gamma} = \Gamma \cdot \{\alpha, \alpha^\sigma\},$$

endowed with its Fréchet filter, and let

$$h(\alpha) = \frac{2}{|\alpha - \alpha^\sigma|}.$$

Clearly,  $h(\alpha)$  belongs to  $]0, +\infty[$  (as  $\alpha \neq \alpha^\sigma$ ), and  $h(\alpha^\sigma) = h(\alpha)$ . We will see in the proof of Theorem 6.4 that points  $r \in \mathcal{E}_{\alpha, \Gamma}$  exit every finite subset of  $\mathcal{E}_{\alpha, \Gamma}$  if and only if  $h(r)$  tends to  $+\infty$ . Define the *quadratic Lagrange spectrum* relative to  $(\alpha, \Gamma)$  by

$$\text{Sp}_{\alpha, \Gamma} = \left\{ c_{\alpha, \Gamma}(\xi) = \liminf_{r \in \mathcal{E}_{\alpha, \Gamma}} h(r) |\xi - r| : \xi \in \widehat{K} - (K \cup \mathcal{E}_{\alpha, \Gamma}) \right\}.$$

The following result is very classical, its proof (given for the sake of completeness) was indicated to us by Y. Benoist.

**Lemma 6.2** *Let  $\alpha \in \widehat{K}$ . Then  $\alpha$  is irrational quadratic over  $K$  if and only if there exists a hyperbolic element  $\gamma$  in  $\text{PSL}_2(\mathcal{O}_K)$  having  $\alpha$  as a fixed point, the other one then being  $\alpha^\sigma$ .*

**Proof.** Let  $\delta = \dim_{\mathbb{R}} \widehat{K}$ . If  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is an hyperbolic element in  $\text{SL}_2(\mathcal{O}_K)$ , then its two fixed points (in  $\partial_\infty \mathbb{H}_{\mathbb{R}}^{\delta+1} = \widehat{K} \cup \{\infty\}$ ) are distinct solutions of the quadratic equation  $ax + b = x(cx + d)$  with coefficients in  $\mathcal{O}_K$ , and in particular they are irrational quadratic and Galois conjugated by  $\sigma$ .

Conversely, let  $\alpha \in K_{\text{quad}}$ . We refer for instance to [Bor] for general information on linear algebraic groups. Let

$$T(\widehat{K}) = \{\gamma \in \text{SL}_2(\widehat{K}) : \gamma \cdot \alpha = \alpha, \gamma \cdot \alpha^\sigma = \alpha^\sigma\}.$$

Since  $\alpha$  and  $\alpha^\sigma$  are two distinct points in the boundary of  $\mathbb{H}_{\mathbb{R}}^{\delta+1}$ , the subgroup  $T(\widehat{K})$  is the set of  $\widehat{K}$ -points of an algebraic torus  $T$  in  $\text{SL}_2$ . This torus  $T$  is defined by a set of polynomial equations with coefficients in  $K(\alpha)$ , this set being invariant by the Galois group of  $K(\alpha)$  over  $K$ . Hence  $T$  is defined over  $K$ . Notice that  $T$  does not split over  $K$ , as the eigenvectors of an element of  $T$  in  $\mathbb{A}^2$  are  $(\alpha, 1)$  and  $(\alpha^\sigma, 1)$ , that are not multiples

of an element with coordinates in  $K$ . Hence, by Borel-Harish-Chandra's theorem (see for instance [BHC, Theo. 12.3]), the subgroup  $T(\mathcal{O}_K)$  is a lattice in  $T(\widehat{K})$ , and in particular is not trivial. That is, there exists an element in  $\mathrm{SL}_2(\mathcal{O}_K)$  having  $\alpha$  (and  $\alpha^\sigma$ ) as fixed point. The result follows.  $\square$

**Remark 1.** Let  $K = \mathbb{Q}$  and  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ . Notice that the Golden Ratio  $\phi = \frac{1+\sqrt{5}}{2}$  is in the same congruence class under  $\Gamma$  as its Galois conjugate  $\frac{1-\sqrt{5}}{2} = -1/\phi$ , hence  $\mathcal{E}_{\phi,\Gamma} = \Gamma \cdot \phi$ . But  $\frac{1+\sqrt{3}}{2}$  and  $\frac{1-\sqrt{3}}{2}$ , though they are Galois conjugate, are not in the same congruence class under  $\Gamma$ .

The first assertion of the result below gives an almost necessary and sufficient condition for when  $\mathcal{E}_{\alpha,\Gamma}$  is reduced to one congruence class (see [BPP] for more algebraic characterisations), and implies that in general, the set  $\mathcal{E}_{\alpha,\Gamma}$  consists of two disjoint congruence classes  $\Gamma \cdot \alpha$  and  $\Gamma \cdot \alpha^\sigma$ . The second assertion gives another expression of the approximation constants.

Recall that an irrational real number is quadratic over  $\mathbb{Q}$  if and only if its continued fraction expansion is eventually periodic. For every  $x \in \mathbb{P}^1(\mathbb{R})$ , let  $\Gamma_x$  be the stabilizer of  $x$  in  $\Gamma$ . For every  $\alpha \in \mathbb{Q}_{\mathrm{quad}}$ , endow the infinite set (see Lemma 3.1)  $\Gamma_\infty \backslash \Gamma / \Gamma_\alpha$  with its Fréchet filter. Denote by  $N(\alpha) = \alpha\alpha^\sigma$  the *norm* of an element  $\alpha \in \mathbb{Q}_{\mathrm{quad}}$ . For every element  $\gamma$  in  $\Gamma$ , let  $\gamma = \pm \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$ .

**Proposition 6.3** *Let  $\alpha \in \mathbb{Q}_{\mathrm{quad}}$  and  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ .*

- (1) *Then  $\alpha$  is in the same orbit as  $\alpha^\sigma$  under  $\mathrm{PGL}_2(\mathbb{Z})$  if and only if the period of the continued fraction expansion of  $\alpha$  is palindromic up to cyclic permutation.*
- (2) *For every  $x \in \mathbb{R} - (\mathbb{Q} \cup \mathcal{E}_{\alpha,\Gamma})$ , we have*

$$c_{\alpha,\Gamma}(x) = h(\alpha) \liminf_{(\gamma,\varepsilon) \in (\Gamma_\infty \backslash \Gamma / \Gamma_\alpha) \times \{1,\sigma\}} |N(\alpha c(\gamma) + d(\gamma))| |x - \gamma \cdot \alpha^\varepsilon|.$$

For instance,  $\mathcal{E}_{\phi,\Gamma}$  is the set of real numbers whose continued fraction expansion is eventually constant equal to 1, and

$$\mathrm{Sp}_{\phi,\Gamma} = \left\{ \frac{2}{\sqrt{5}} \liminf_{a,b,c,d \in \mathbb{Z}, ad-bc=1, d^2+dc-c^2 \rightarrow +\infty} |d^2+dc-c^2| \left| x - \frac{a\phi + b}{c\phi + d} \right| : x \in \mathbb{R} - (\mathbb{Q} \cup \mathcal{E}_{\phi,\Gamma}) \right\}.$$

**Proof.** (1) This is well known: As proved for instance in [Per, §23], the periodic part of the continued fraction expansion of  $\alpha^\sigma$  is up to cyclic permutation, obtained by reversing the order of the periodic part of the continued fraction expansion of  $\alpha$ ; furthermore two real numbers are in the same homographic orbit under  $\mathrm{PGL}_2(\mathbb{Z})$  if and only if their continued fraction expansions have equal tails.

(2) An easy computation shows that  $|\gamma \cdot \alpha - \gamma \cdot \alpha^\sigma| = |\alpha - \alpha^\sigma| / N(c(\gamma)\alpha + d(\gamma))$ , hence

$$h(\gamma \cdot \alpha) = N(c(\gamma)\alpha + d(\gamma)) h(\alpha).$$

The result follows.  $\square$

**Remark 2.** Assume again that  $K = \mathbb{Q}$ . The quantity  $h(\alpha)$  behaves in a very different way from the naive height  $H(\alpha)$  of  $\alpha$  (defined in the introduction). Clearly,  $h(\alpha+n) = h(\alpha)$

for every  $n$  in  $\mathbb{N}$ , but  $H(\alpha + n) \rightarrow \infty$  as  $n \rightarrow \infty$ . So that there exists a sequence  $(\alpha_i)_{i \in \mathbb{N}}$  in  $\mathbb{Q}_{\text{quad}}$  such that the ratio  $\frac{h(\alpha_i)}{H(\alpha_i)}$  tends to 0 when  $i \rightarrow +\infty$ . But this ratio cannot tend to  $+\infty$ , as for every  $\alpha \in \mathbb{Q}_{\text{quad}}$ , we have

$$\frac{h(\alpha)}{H(\alpha)} \leq 2.$$

Indeed, let  $aX^2 + bX + c$  be a minimal polynomial of  $\alpha$  over  $\mathbb{Z}$ , so that  $H(\alpha) = \max\{|a|, |b|, |c|\}$ . Then

$$h(\alpha) = \frac{2}{\sqrt{(b/a)^2 - 4(c/a)}} = \frac{2|a|}{\sqrt{b^2 - 4ac}} \leq 2|a| \leq 2H(\alpha),$$

There are many possibilities for the relative behaviour of  $h(\alpha)$  and  $H(\alpha)$ . In particular, there exist sequences  $(\alpha_i)_{i \in \mathbb{N}}$  in  $\mathbb{Q}_{\text{quad}}$  and constants  $c, c', c'' > 0$  such that  $h(\alpha_i)$  is equivalent as  $i$  tends to  $+\infty$  either to  $cH(\alpha_i)^{-\frac{1}{2}}$  (take  $c = 1$  and  $\alpha_i = \sqrt{p_i}$  for  $p_i$  the  $i$ -th prime number), or to  $c'H(\alpha_i)^{\frac{1}{2}}$  (take  $c' = 1$  and  $\alpha_i = 1/\sqrt{p_i}$  for  $p_i$  the  $i$ -th prime number) or to  $c''H(\alpha_i)$  (take  $c'' = 2/\sqrt{5}$  and  $\alpha_i = \frac{3+2i+\sqrt{5}}{1+3i+i^2}$  for every  $i \in \mathbb{N}$ ). This difference is good to bear in mind when comparing our results with for example the results of [DS1, DS2, BT] cited in the Introduction. We refer to [BPP] for a treatment of the algebraic number theory aspects of  $h(\alpha)$ .

**Theorem 6.4** *Let  $K = \mathbb{Q}$  (resp.  $\mathbb{Q}(i\sqrt{m})$  where  $m$  is a squarefree positive integer), let  $\widehat{K} = \mathbb{R}$  (resp.  $\mathbb{C}$ ) and let  $\delta = \dim_{\mathbb{R}} \widehat{K}$ . Let  $\Gamma$  be a finite index subgroup of  $\text{PSL}_2(\mathcal{O}_K)$ .*

- (1) *For every  $\alpha_0 \in K_{\text{quad}}$ , the Lagrange spectrum  $\text{Sp}_{\alpha_0, \Gamma}$  is closed.*
- (2) *There exists  $C \geq 0$  such that for every  $\alpha_0 \in K_{\text{quad}}$ ,*

$$\max \text{Sp}_{\alpha_0, \Gamma} \leq C.$$

- (3) *If  $\widehat{K} = \mathbb{C}$ , then for every  $\alpha_0 \in K_{\text{quad}}$ , there exists  $c > 0$  such that  $\text{Sp}_{\alpha_0, \Gamma}$  contains  $[0, c]$ .*
- (4) *Let  $\alpha_0 \in K_{\text{quad}}$  and  $\varphi : [0, +\infty[ \rightarrow ]0, +\infty[$  be a map such that  $t \mapsto \varphi(e^t)$  is slowly varying. If the integral  $\int_1^{+\infty} \varphi(t)^\delta / t dt$  diverges (resp. converges), then for Lebesgue almost every  $x \in \widehat{K}$ ,*

$$\liminf_{r \in \mathcal{E}_{\alpha_0, \Gamma}} \frac{h(r)}{\varphi(h(r))} |x - r| = 0 \text{ (resp. } = +\infty \text{)}.$$

**Proof.** Consider the data  $\mathcal{D} = (X, \Gamma, \Gamma_0, C_\infty)$ , where  $X$  is the upper halfspace model of  $\mathbb{H}_{\mathbb{R}}^{\delta+1}$ ,  $\Gamma$  is as in the statement,  $\Gamma_0$  is the stabilizer in  $\Gamma$  of the translation axis of a hyperbolic element of  $\Gamma$  one of whose fixed points is  $\alpha_0$ , and  $C_\infty$  is the set of points in  $X$  with Euclidean height at least 1. Note that  $\Gamma_0$  does exist: By Lemma 6.2, the point  $\alpha_0$  is a fixed point of a hyperbolic element of  $\text{PSL}_2(\mathcal{O}_K)$ , hence of a hyperbolic element of  $\Gamma$ , since  $\Gamma$  is a finite index subgroup of  $\text{PSL}_2(\mathcal{O}_K)$ .

The data  $\mathcal{D}$  satisfies the general assumptions of Example 1 of Section 3: Since  $z \mapsto z+1$  belongs to  $\text{PSL}_2(\mathcal{O}_K)$ , the point  $\infty$  is fixed by a parabolic element of  $\text{PSL}_2(\mathcal{O}_K)$ , hence of  $\Gamma$ ; moreover,  $C_\infty$  is precisely invariant under the stabilizer of  $\infty$  in  $\text{PSL}_2(\mathcal{O}_K)$ , hence in  $\Gamma$ ,

by Shimizu's lemma; furthermore, the quotient of  $X$  by  $\mathrm{PSL}_2(\mathcal{O}_K)$ , hence by  $\Gamma$ , has finite volume.

Using the notations of Example 1 of Section 3, we easily check that  $\mathcal{R}_{\Gamma_0} = \mathcal{E}_{\alpha_0, \Gamma}$ . Furthermore, let  $r \in \mathcal{E}_{\alpha_0, \Gamma}$  and let  $\gamma_r \in \Gamma$  be an element such that  $\gamma_r \gamma_0 \gamma_r^{-1}$  fixes  $r$ . Recall that the other fixed point of a hyperbolic element fixing  $r$  is the Galois conjugate of  $r$  over  $K$ , by Lemma 6.2. Hence, by Equation (-10-), we have

$$D([\gamma_r]) = \log h(r) . \quad (-21-)$$

Therefore, by Lemma 3.1, points  $r \in \mathcal{E}_{\alpha_0, \Gamma}$  exit every finite subset of  $\mathcal{E}_{\alpha_0, \Gamma}$  if and only if  $h(r)$  tends to  $+\infty$ . The set of parabolic fixed points of  $\Gamma$  is equal to the set of parabolic fixed point of  $\mathrm{PSL}_2(\mathcal{O}_K)$ , as  $\Gamma$  has finite index, hence it is equal to  $K \cup \{\infty\}$ . Therefore

$$\widehat{K} - (K \cup \mathcal{E}_{\alpha_0, \Gamma}) = \Lambda_c \Gamma - \mathcal{R}_{\Gamma_0} .$$

For every  $\xi$  in this set, we have

$$c(\Gamma_\infty \xi) = \liminf_{r \in \mathcal{E}_{\alpha_0, \Gamma}} h(r) |\xi - r|$$

by Equation (-11-). Hence, the quadratic Lagrange spectrum  $\mathrm{Sp}_{\alpha_0, \Gamma}$  coincides with the approximation spectrum  $\mathrm{Sp}(\mathcal{D})$ .

We can now conclude that the assertions (1), (2) and (3) follow, respectively, from Theorem 4.8, Proposition 4.6, and Corollary 5.6. In particular, in (2) we get an upper bound on  $\mathrm{Sp}_{\alpha_0, \Gamma}$  which depends only on  $\Gamma$ , and not on the (congruence class under  $\Gamma$ ) of  $\alpha_0$ .

To prove the assertion (4), define  $f : t \mapsto \varphi(e^t)$ , which is slowly varying by the assumptions of (4). By an easy change of variable, the integral  $\int_1^{+\infty} f(t)^\delta dt$  diverges if and only if  $\int_1^{+\infty} \varphi(t)^\delta / t dt$  diverges. Hence by Theorem 6.1, by the comments following it, and by Equation (-21-), if  $\int_1^{+\infty} \varphi(t)^\delta / t dt$  diverges, then for almost every (for the Lebesgue measure) point  $\xi$  in  $\widehat{K}$ ,

$$\liminf_{r \in \mathcal{E}_{\alpha_0, \Gamma}} \frac{h(r)}{\varphi(h(r))} |\xi - r| \leq 1 .$$

Replacing  $\varphi$  by  $\frac{1}{k}\varphi$  and letting  $k \in \mathbb{N}$  go to  $+\infty$ , this proves the divergence part of the assertion (4) in Theorem 6.4. The convergence part follows similarly.  $\square$

**Remark.** Replacing in the above proof, as in [PP2, PP1, PP3],  $X$  by  $\mathbb{H}_{\mathbb{R}}^5$ ,  $\delta$  by 4 and  $\Gamma$  by the image in the isometry group of  $X$  of a finite index subgroup of  $\mathrm{SL}_2(\mathcal{O}')$  where  $\mathcal{O}' = \mathbb{Z}(1+i+j+k)/2 + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$  is the Hurwitz order in the usual Hamilton quaternion algebra  $A$  over  $\mathbb{Q}$  (using Dieudonné's determinant), we could get similar Diophantine approximation results of points in  $A(\mathbb{R})$  by points in quadratic extensions of  $A(\mathbb{Q})$ .

To prove Theorem 1.3 of the Introduction, apply Theorem 6.4 with  $K = \mathbb{Q}$  (so that  $\delta = 1$ ),  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ , and  $\varphi : t \mapsto \psi(2/t)/t$ , and notice that  $\log(\varphi(e^t))$  is Lipschitz, and that  $\int_1^{+\infty} \varphi(t)/t dt$  converges if and only if  $\int_1^{+\infty} \psi(t) dt$  converges.

## 6.2 Approximation in the Heisenberg group

Let  $m$  be a squarefree positive integer, let  $K$  be the number field  $\mathbb{Q}(i\sqrt{m})$ , let  $\mathcal{O}_K$  be its ring of integers, and let  $K_{\text{quad}}$  be the set of elements of  $\mathbb{C}$  which are irrational quadratic over  $K$ .

Let  $n \geq 2$ , and let  $(w', w) \mapsto w' \cdot \bar{w} = \sum_{i=1}^{n-1} w'_i \bar{w}_i$  be the usual Hermitian product on  $\mathbb{C}^{n-1}$ . Consider the real Lie group

$$\text{Heis}_{2n-1}(\mathbb{R}) = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : 2 \operatorname{Re} w_0 - w \cdot \bar{w} = 0\}$$

whose law is

$$(w_0, w) \cdot (w'_0, w') = (w_0 + w'_0 + w' \cdot w, w + w').$$

Endow it with the Cygan distance (see for instance [Gol]), which is the unique left-invariant distance such that

$$d_{\text{Cyg}}((w_0, w), (0, 0)) = \sqrt{2|w_0|},$$

as well as with the *Cygan measure* (which is the Hausdorff measure of the Cygan distance). The Lie group  $\text{Heis}_{2n-1}(\mathbb{R})$  is isomorphic to the standard  $(2n-1)$ -dimensional Heisenberg

group, i.e. for  $n = 2$  to the Lie group  $\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$ .

We are interested in the Diophantine approximation of the points of  $\text{Heis}_{2n-1}(\mathbb{R})$  by elements all of whose coordinates are irrational quadratic over  $K$ , i.e. by elements of  $\text{Heis}_{2n-1}(\mathbb{R}) \cap (K_{\text{quad}} \times K_{\text{quad}}^{n-1})$  (or nice subsets of it).

Let us identify  $\text{Heis}_{2n-1}(\mathbb{R})$  with its image in the projective space  $\mathbb{P}_n(\mathbb{C})$  of  $\mathbb{C}^{n+1}$  by the map  $(w_0, w) \mapsto [w_0 : w : 1]$ . Let  $q$  be the Hermitian form of signature  $(1, n)$  on  $\mathbb{C}^{n+1}$  with coordinates  $(z_0, z, z_n)$ , defined by

$$q = -z_0 \bar{z}_n - z_n \bar{z}_0 + z \cdot \bar{z}.$$

The induced action on  $\mathbb{P}_n(\mathbb{C})$  of the special unitary group  $\text{SU}_q$  of  $q$  preserves (the image in  $\mathbb{P}_n(\mathbb{C})$  of)  $\text{Heis}_{2n-1}(\mathbb{R}) \cup \{\infty\}$ . Let  $\text{SU}_q(\mathcal{O}_K)$  be the arithmetic subgroup of  $\text{SU}_q$  which consists of matrices with coefficients in  $\mathcal{O}_K$ . Note that the action of  $\text{SU}_q(\mathcal{O}_K)$  on  $\text{Heis}_{2n-1}(\mathbb{R})$  preserves both the set of rational points

$$\text{Heis}_{2n-1}(\mathbb{Q}) = \text{Heis}_{2n-1}(\mathbb{R}) \cap (K \times K^{n-1})$$

union  $\{\infty\}$ , and the set of irrational quadratic points  $\text{Heis}_{2n-1}(\mathbb{R}) \cap (K_{\text{quad}} \times K_{\text{quad}}^{n-1})$ .

Let

$$\alpha_0 = \frac{i}{2}(\sqrt{m+4} - \sqrt{m}),$$

which is an element of  $K_{\text{quad}}$ , since it is a root of the quadratic polynomial  $X^2 + i\sqrt{m}X + 1$ , whose coefficients are in  $K$  and which is irreducible over  $K$ . (We could have taken many other examples, but  $\alpha_0$  is one of the simplest ones.) Note that  $(\alpha_0, 0) \in \text{Heis}_{2n-1}(\mathbb{R}) \cap (K_{\text{quad}} \times K_{\text{quad}}^{n-1})$ . Let  $\alpha_0^\sigma = \frac{i}{2}(-\sqrt{m+4} - \sqrt{m})$  be the Galois conjugate of  $\alpha_0$  over  $K$ .

Let  $\Gamma$  be a finite index subgroup of  $\text{SU}_q(\mathcal{O}_K)$ , and let  $\mathcal{E}'_{\alpha_0, \Gamma} = \Gamma \cdot \{(\alpha_0, 0), (\alpha_0^\sigma, 0)\}$ . For every  $r \in \mathcal{E}'_{\alpha_0, \Gamma}$ , let  $r^\sigma$  be the componentwise Galois conjugate of  $r$ , and let

$$h'(r) = \frac{1}{d_{\text{Cyg}}(r, r^\sigma)}.$$

Endow  $\mathcal{E}'_{\alpha_0, \Gamma}$  with its Fréchet filter. We will see in the proof below that points  $r \in \mathcal{E}'_{\alpha_0, \Gamma}$  tend to infinity in  $\mathcal{E}'_{\alpha_0, \Gamma}$  if and only if  $h'(r)$  tends to  $+\infty$ .

In order to understand the Diophantine approximation of elements  $\xi$  of  $\text{Heis}_{2n-1}(\mathbb{R})$  by elements in the subset  $\mathcal{E}'_{\alpha_0, \Gamma}$  of the set  $\text{Heis}_{2n-1}(\mathbb{R}) \cap (K_{\text{quad}} \times K_{\text{quad}}^{n-1})$  of irrational quadratic elements of  $\text{Heis}_{2n-1}(\mathbb{R})$ , we introduce the *approximation constant* of  $\xi$ , defined by

$$c'(\xi) = \liminf_{r \in \mathcal{E}'_{\alpha_0, \Gamma}} h'(r) d_{\text{Cyg}}(\xi, r),$$

and the *Heisenberg-Lagrange spectrum*

$$\text{Sp}'_{\alpha_0, \Gamma} = \left\{ c'(\xi) : \xi \in \text{Heis}_{2n+1}(\mathbb{R}) - (\text{Heis}_{2n+1}(\mathbb{Q}) \cup \mathcal{E}'_{\alpha_0, \Gamma}) \right\}$$

**Theorem 6.5** *If  $\Gamma$  is a finite index subgroup of  $\text{SU}_q(\mathcal{O}_K)$ , then*

- (1) *the Heisenberg-Lagrange spectrum  $\text{Sp}'_{\alpha_0, \Gamma}$  is bounded;*
- (2) *there exists  $c > 0$  such that  $\text{Sp}'_{\alpha_0, \Gamma}$  contains an interval  $[0, c[$ ;*
- (3) *let  $\varphi : [0, +\infty[ \rightarrow ]0, +\infty[$  be a map such that  $t \mapsto \varphi(e^t)$  is slowly varying, if the integral  $\int_1^{+\infty} \varphi^{2n}(t)/t dt$  diverges (resp. converges), then for Cygan almost every  $x \in \widehat{K}$ ,*

$$\liminf_{r \in \mathcal{E}'_{\alpha_0, \Gamma}} \frac{h'(r)}{\varphi(h'(r))} d_{\text{Cyg}}(x, r) = 0 \text{ (resp. } = +\infty \text{)}.$$

**Proof.** Let  $X$  be the Siegel domain model of the complex hyperbolic  $n$ -space  $\mathbb{H}_{\mathbb{C}}^n$ , as in Example 2 of Section 3, identified, as well as its boundary at infinity, with its image in  $\mathbb{P}_n(\mathbb{C})$  by  $(w_0, w) \mapsto [w_0 : w : 1]$ . The group  $\Gamma \subset \text{SL}_{n+1}(\mathbb{C})$  acts (with finite kernel) on  $X \subset \mathbb{P}_n(\mathbb{C})$  as a discrete group of isometries of  $X$  with finite covolume, by the restriction of the projective action. Note that  $\infty$  (corresponding to  $[1 : 0 : 0]$ ) is a parabolic fixed point of  $\text{SU}_q(\mathcal{O}_K)$ , hence of  $\Gamma$ . Let

$$\gamma_0 = \begin{pmatrix} m+1 & 0 & -i\sqrt{m} \\ 0 & I & 0 \\ i\sqrt{m} & 0 & 1 \end{pmatrix}.$$

It is easy to check that  $\gamma_0$  is a hyperbolic element of  $\text{SU}_q(\mathcal{O}_K)$  whose fixed points in  $\partial_{\infty} X$  are exactly  $(\alpha_0, 0)$  and  $(\alpha_0^{\sigma}, 0)$ . Since  $\Gamma$  has finite index in  $\text{SU}_q(\mathcal{O}_K)$ , there exists  $k \in \mathbb{N} - \{0\}$  such that  $\gamma_0^k \in \Gamma$ . Let  $\Gamma_0$  be the stabilizer in  $\Gamma$  of the translation axis of  $\gamma_0$ .

The horoball  $\mathcal{H}_2$  is precisely invariant under the stabilizer of  $\infty$  in  $\text{SU}_q(U)$ , hence under  $\Gamma_{\infty}$  in  $\Gamma$ , by the Kamiya-Parker inequality, see for instance [PP2, Lemma 6.4]. Thus, the data  $\mathcal{D} = (X, \Gamma, \Gamma_0, \mathcal{H}_2)$  satisfy the conditions of Example 2 of Section 3. Using the notation of this example,  $\mathcal{E}'_{\alpha_0, \Gamma}$  is the set  $\mathcal{R}_{\Gamma_0}$  of fixed points of the conjugates of  $\gamma_0^k$  in  $\Gamma$ . The set of parabolic fixed points of  $\Gamma$  is equal to set of parabolic fixed points of  $\text{SU}_q(\mathcal{O}_K)$ , which is exactly  $\text{Heis}_{2n-1}(\mathbb{Q}) \cup \{\infty\}$  (see for instance [PP2, Sect. 6.3, Exam. 2]). In particular,

$$\Lambda_c \Gamma - \mathcal{R}_{\Gamma_0} = \text{Heis}_{2n-1}(\mathbb{R}) - (\text{Heis}_{2n-1}(\mathbb{Q}) \cup \mathcal{E}'_{\alpha_0, \Gamma}).$$

Since the Cygan distance and the modified Cygan distance are equivalent (see [PP2, Sect. 6.1]), there exists a constant  $c_1 > 0$  such that for every  $r$  in  $\mathcal{E}'_{\alpha_0, \Gamma}$ , if  $\gamma_r \in \Gamma$  is such that  $r$  is fixed by  $\gamma_r \gamma_0 \gamma_r^{-1}$ , then with  $D([\gamma_r])$  computed in Lemma 3.4, we have

$$|D([\gamma_r]) - \log h'(r)| \leq c_1. \quad (-22-)$$



Hence, it follows from Lemma 3.1 that points  $r \in \mathcal{E}'_{\alpha_0, \Gamma}$  tend to infinity in  $\mathcal{E}'_{\alpha_0, \Gamma}$  if and only if  $h'(r)$  tends to  $+\infty$ .

Again, since the Cygan distance and the modified Cygan distance are equivalent, there exists a constant  $c_2 > 0$  such that for every  $\xi \in \Lambda_c \Gamma - \mathcal{R}_{\Gamma_0}$ , we have

$$\liminf_{r \in \mathcal{E}'_{\alpha_0, \Gamma}} \frac{d_{\text{Cyg}}(\xi, r)}{d_{\text{Cyg}}(r, r^\sigma)} \leq c_2 \liminf_{r \in \mathcal{R}_{\Gamma_0}} \sqrt{2} \frac{d'_{\text{Cyg}}(r, r^\sigma) d_{\text{Cyg}}(\xi, r)}{d_{\text{Cyg}}(r, r^\sigma)^2}.$$

The assertion (1) of Theorem 6.5 then follows from Equation (-14-) and from Proposition 4.6, and the assertion (2) follows from Corollary 5.9.

To prove the assertion (3) of Theorem 6.5, consider the map  $f : t \mapsto \varphi(e^t)$ , which is slowly varying. In particular, by Equation (-22-), there exists a constant  $c_3 > 0$  such that for every  $r \in \mathcal{E}'_{\alpha_0, \Gamma}$ , we have

$$\frac{1}{c_3} \frac{\varphi(h'(r))}{h'(r)} \leq f(D([\gamma_r])) e^{-D([\gamma_r])} \leq c_3 \frac{\varphi(h'(r))}{h'(r)}.$$

Hence the assertion (3) of Theorem 6.5 follows from Theorem 6.1, as in the proof of the assertion (4) in Theorem 6.4.  $\square$

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