WEIGHTED HARDY INEQUALITIES AND THE SIZE OF THE BOUNDARY

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Abstract. We establish necessary and sufficient conditions for a domain $\Omega \subset \mathbb{R}^n$ to admit the (p,β)-Hardy inequality $\int_\Omega |u|^p d_\Omega{}^{\beta-p} \leq C \int_\Omega |\nabla u|^p d_\Omega{}^\beta$, where $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$ and $u \in C_0^\infty(\Omega)$. Our necessary conditions show that a certain dichotomy holds, even locally, for the dimension of the complement Ω^c when Ω admits a Hardy inequality, whereas our sufficient conditions can be applied in numerous situations where at least a part of the boundary $\partial\Omega$ is "thin", contrary to previously known conditions where $\partial\Omega$ or Ω^c was always assumed to be "thick" in a uniform way. There is also a nice interplay between these different conditions that we try to point out by giving various examples.

1. Introduction

We consider in this paper the weighted Hardy inequality

$$\int_\Omega |u(x)|^p \, d_\Omega(x)^{\beta-p} \, dx \leq C \int_\Omega |\nabla u(x)|^p \, d_\Omega(x)^\beta \, dx \,, \tag{1}$$

where $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. We say that a domain $\Omega \subset \mathbb{R}^n$ admits the (p,β)-Hardy inequality, if there exists a constant $C_0 = C_0(\Omega, p, \beta) > 0$ such that (1) holds for all $u \in C_0^\infty(\Omega)$. These inequalities originate from the one-dimensional considerations of Hardy et. al., see [6] and references therein.

The purpose of this paper is to continue the study of the relations between the (p,β)-Hardy inequality and the size and geometry of the boundary $\partial\Omega$ and the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$. To begin with, let us record the following general case of a dichotomy result that we establish for the dimension of the complement when the domain admits the (p,β)-Hardy inequality. In the unweighted case $\beta = 0$ the corresponding result was obtained by Koskela and Zhong in [11].

Theorem 1.1. *Let $1 < p < \infty$ and $\beta \neq p$, and suppose that a domain $\Omega \subset \mathbb{R}^n$ admits the (p,β)-Hardy inequality. Then there exists $\delta > 0$, depending only on the given data, such that either (i) $\dim_{\mathcal{H}}(\Omega^c) > n - p + \beta + \delta$ or (ii) $\dim_{\mathcal{A}}(\Omega^c) < n - p + \beta - \delta$.*

Here $\dim_{\mathcal{A}}(E)$ is a concept of dimension, introduced by Aikawa (cf. [2]), which is never less than the (upper) Minkowski dimension $\overline{\dim}_{\mathcal{M}}(E)$, and there exists sets with $\overline{\dim}_{\mathcal{M}}(E) < \dim_{\mathcal{A}}(E)$. On the other hand, for sufficiently regular E it is even true that $\dim_{\mathcal{A}}(E)$ agrees with the Hausdorff dimension $\dim_{\mathcal{H}}(E)$. See Section 2.2 for more details. The requirement $\beta \neq p$ is really needed in Theorem 1.1, as an example of Section 6.1 shows.

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Moreover, we prove that a dichotomy as in Theorem 1.1 also holds locally. Roughly speaking, this means that if a domain Ω admits the (p,β)-Hardy inequality and B is a ball, then $B\cap\Omega^c$ is either "thick" (case (i)) or "thin" (case (ii)), with an actual gap between the two possibilities. Notice however that these dichotomy results are not very informative if Ω^c has interior points; but then again, if this is not the case, we actually obtain the dichotomy for the dimension of the boundary $\partial\Omega$.

Now, in order to make such results meaningful, one should ask for examples and sufficient conditions on Hardy inequalities under these two distinct cases. Indeed, the case (i), with the complement of Ω thick in a uniform way, is rather well-accomplished. For instance, by Ancona [3] ($p = 2$), Lewis [15], and Wannebo [23], a domain Ω admits the p-Hardy inequality (that is, (1) with $\beta = 0$) if Ω^c is uniformly p-fat, i.e. satisfies a uniform p-capacity density condition. See also [5], [8], and [12] for related results on pointwise p-Hardy inequalities. On the other hand, by a result of Nečas [19], a domain with a Lipschitz-boundary admits the (p,β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < p-1$. A more general sufficient condition for (p,β)-Hardy inequalities, one that can in some sense be considered as an extension of both the results of Ancona-Lewis-Wannebo and Nečas, was established in [10]. The requirement there was that the boundary of Ω is both uniformly thick, in the sense of Hausdorff contents, and accessible from the points inside the domain, in the sense of John curves; both conditions are valid for example for Lipschitz domains, but also for much more general domains, e.g. for domains with a "good" fractal-type boundary, of which the basic example is a von Koch snowflake domain.

On the other hand, the thickness of the boundary is by no means necessary for a domain to admit Hardy inequalities — apart from the cases of the n-Hardy inequality for domains in $\mathbb{R}^n$ (see Ancona [3] ($n = 2$) and Lewis [15]) and pointwise Hardy inequalities (see [12] and [14]). For example, as noted by Lewis [15], the boundary of $B(0,1)\setminus\{0\}$ is not p-fat at the origin for $1 < p < n$, but still $B(0,1)\setminus\{0\}$ admits the p-Hardy inequality for these p. The same holds true for $\mathbb{R}^n\setminus\{0\}$, so this gives the easiest example where the case (ii) of Theorem 1.1 applies.

Besides such easy examples, there exists but a few known instances of phenomena related to the cases where a part, or the whole, of the boundary of Ω is thin. The following result, however, is implicitly contained in [11].

Theorem 1.2. *Let $\Omega\subset\mathbb{R}^n$ be a domain, and let $1 < p < \infty$. If $\dim_{\mathcal{A}}(\partial\Omega) < n-p$, then Ω admits the p-Hardy inequality.*

The proof of Theorem 1.2, as explained at the end of [11], follows from a quasiadditivity result for Riesz capacities [2, Corollary 7.1.2] (where the assumption $\dim_{\mathcal{A}}(\partial\Omega) < n-p$ is needed), and a general capacity-type characterization for certain inequalities [18, Thm. 2.3.3]. In this paper, we come up with the following result, related to Theorem 1.2.

Theorem 1.3. *Let $\Omega\subset\mathbb{R}^n$ be an unbounded John domain, and let $1 < p < \infty$ and $\beta\in\mathbb{R}$. If $\dim_{\mathcal{A}}(\partial\Omega) < n-p+\beta$, then Ω admits the (p,β)-Hardy inequality.*

Moreover, we show by examples (cf. Section 6) that, contrary to the unweighted case of Theorem 1.2, the accessibility condition, i.e. Ω being John, can not be removed from the assumptions in the weighted case of Theorem 1.3. See Section 4 for the precise definitions and the proof of Theorem 1.3, as well as for more general results for unbounded domains.

There still remains the case where the complement of the domain Ω has *both* thick and thin parts. According to our best knowledge, there have not been any general considerations to deal with such cases, perhaps apart from the mostly unpublished works of Wannebo, cf. [24]. In this paper, we try to take a step into this direction by giving a new (even in the unweighted case $\beta = 0$) sufficient condition for (p, β)-Hardy inequalities, which can be applied for instance in the following situation: Suppose that the boundary of a domain $\Omega \subset \mathbb{R}^n$ has two parts: (i) a thick part $\partial\Omega_1$, which satisfies a uniform Hausdorff content density condition with an exponent $\lambda > n-p+\beta$, and an additional accessibility condition, whence a pointwise (p, β)-Hardy inequality holds at the points $x \in \Omega$ relatively close to this part (cf. [10]); and (ii) a thin part $\partial\Omega_2$, with $\dim_{\mathcal{A}}(\partial\Omega_2) < n-p+\beta$. We then require in our condition, loosely speaking, that there exists $x_0 \in \Omega$, relatively close to the thick part $\partial\Omega_1$, such that for each point $x \in \Omega$ close to the small part $\partial\Omega_2$ we can find a curve γ_x joining x to x_0 in such a way that γ_x never gets too close to the boundary $\partial\Omega$. We are then able to conclude that Ω admits the (p, β)-Hardy inequality. Such sufficient conditions are discussed in detail in Sections 3 and 4, for bounded and unbounded domains, respectively. The main ingredients in the proofs of the both cases are a rather standard chaining argument using the Poincaré inequality on cubes, and estimates for the shadows of Whitney cubes with respect to John-curves; here the fact that $\dim_{\mathcal{A}}$ is small is crucial.

An essential difference between bounded and unbounded domains, as far as weighted Hardy inequalities are concerned, is that the latter (may) allow us to consider weight exponents $\beta \geq p$, which in the bounded case are never relevant. This difference is actually present already in the early results of Hardy et. al. (cf. [6, Thm. 330]), where the weighted (p, β)-Hardy inequality was proved in the *unbounded domain* $\Omega = (0, \infty) \subset \mathbb{R}$ for all $1 < p < \infty$ and $\beta \neq p-1$, but, for example, in the interval $(0, 1)$ the (p, β)-Hardy inequality only holds for $\beta < p - 1$. Our results for unbounded domains can now be considered as generalizations of the case $\beta > p - 1$ of these one-dimensional inequalities to higher dimensions.

In conclusion, even if a full (geometrical) characterization of domains admitting the (p, β)-Hardy inequality is beyond our reach, the necessary conditions for the dimension of the complement in Theorem 1.1 (as well as the local version, given in Theorem 5.3), and the sufficient conditions mentioned above complement each other in quite a nice way, and, together with the results from [10], offer in many cases a chance to completely determine the values of p and β for which *an explicitly given* domain $\Omega \subset \mathbb{R}^n$ admits the (p, β)-Hardy inequality. See Section 6, and also [10], for some examples.

The outline of the paper is as follows: In Section 2 we first introduce some notation and basic tools used in the rest of the paper, and then consider in detail many of the things already mentioned in this Introduction. Namely, we recall the relevant concepts of dimension, give some more information about Hardy inequalities, consider John domains and the corresponding shadows of Whitney cubes, and prove some preliminary lemmas on these subjects. As mentioned above, we establish our sufficient conditions for Hardy inequalities in Sections 3 and 4, for bounded and unbounded domains, respectively. Section 4 also includes some auxiliary results for unbounded domains. The dichotomy for the dimension of the complement is discussed in Section 5. Finally, in Section 6, we give a number of examples which show that, in general, the assumptions in our results are not only for technical reasons, but are needed because they really reflect the behavior of Hardy inequalities. In addition, these examples hopefully give some insight of the possible cases in which our conditions can be applied.

2. Preliminaries

2.1. **Notation and basic definitions.** The considerations of this paper take place in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$. The open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted $B(x,r)$, and the corresponding closed ball is $\overline{B}(x,r)$. If $B = B(x,r)$ is a ball and $L > 0$, we denote $LB = B(x, Lr)$. When $A \subset \mathbb{R}^n$, ∂A is the boundary and $\overline{A} = A \cup \partial A$ the closure of A, and the complement of A is $A^c = \mathbb{R}^n \setminus A$. The diameter of A is $\operatorname{diam}(A)$, and $|A|$ denotes the n-dimensional Lebesgue measure of A. If $0 < |A| < \infty$ and $f \in L^1(A)$, we denote $⨍_A f\,dx = \frac{1}{|A|}\int_A f\,dx$. Also, $\chi_A \colon \mathbb{R}^n \to \{0,1\}$ is the characteristic function of A. The support of a function $u \colon \Omega \to \mathbb{R}$, $\operatorname{spt}(u)$, is the closure of the set where u is non-zero.

The Euclidean distance between two points, or a point and a set, is denoted $d(\cdot,\cdot)$. When $\Omega \subsetneq \mathbb{R}^n$ is a domain, i.e. an open and connected set, and $x \in \Omega$, we also use notation $d_\Omega(x) = d(x, \partial\Omega)$. In the rest of the paper we always assume that $\Omega \subsetneq \mathbb{R}^n$, so that $\partial\Omega \neq \emptyset$. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted $|x|$.

We let $C > 0$ denote various positive constants which may vary from expression to expression. If a and b are some quantities such that $a \leq Cb$, we write $a \lesssim b$, and if $a \lesssim b$ and $b \lesssim a$, then $a \approx b$. If F is a finite set, then $\#F$ denotes the cardinality of F.

Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$, be a proper subdomain. Then $\mathcal{W} = \mathcal{W}(\Omega)$ denotes a *Whitney decomposition* of Ω, i.e. a collection of closed cubes $Q \subset \Omega$ with pairwise disjoint interiors and having edges parallel to the coordinate axes, such that $\Omega = \bigcup_{Q \in \mathcal{W}} Q$. Furthermore, the diameters of $Q \in \mathcal{W}$ are in the set $\{2^{-j} : j \in \mathbb{Z}\}$ and satisfy the condition

$$\operatorname{diam}(Q) \leq d(Q, \partial\Omega) \leq 4\operatorname{diam}(Q).$$

We refer to [21] for the existence and further properties of Whitney decompositions. For $j \in \mathbb{Z}$ we define

$$\mathcal{W}_j = \{Q \in \mathcal{W} : \operatorname{diam}(Q) = 2^{-j}\}.$$

Also, if $\Omega' \subset \Omega$, we denote

$$\mathcal{W}|_{\Omega'} = \{Q \in \mathcal{W}(\Omega) : Q \cap \Omega' \neq \emptyset\}.$$

When Q is a cube, c_Q denotes the center of Q.

The usual *restricted Hardy-Littlewood maximal function* of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$M_R f(x) = \sup_{0<r<R} \fint_{B(x,r)} |f(y)|\, dy\,,$$

where $0 < R \le \infty$ may depend on x. The well-known maximal theorem of Hardy, Littlewood and Wiener (see e.g. [21]) states that if $1 < p < \infty$, we have $||M_R f||_p \le C(n,p)||f||_p$ for all $0 < R \le \infty$.

When $1 < q < \infty$, we define $M_{R,q} f = \big(M_R |f|^q\big)^{1/q}$. It follows from the maximal theorem that $M_{R,q}$ is bounded on L^p for each $q < p < \infty$.

2.2. **Concepts of dimension.** As we already mentioned, it has turned out on numerous occasions that the size and dimension of $\partial\Omega$ and Ω^c have a close connection to Hardy inequalities. Let us now recall some relevant ways to measure the dimension of a set.

The λ-Hausdorff content of a set $E \subset \mathbb{R}^n$ is

$$\mathcal{H}^\lambda_\infty(E) = \inf\left\{ \sum_{i=1}^\infty r_i^\lambda : E \subset \bigcup_{i=1}^\infty B(z_i, r_i)\right\},$$

where $z_i \in E$ and $r_i > 0$, and the Hausdorff dimension of $E \subset \mathbb{R}^n$ is

$$\dim_{\mathcal{H}}(E) = \inf\big\{\lambda > 0 : \mathcal{H}^\lambda_\infty(E) = 0\big\}.$$

When $E \subset \mathbb{R}^n$ is a compact set and $r > 0$, we denote

$$\mathcal{M}^\lambda_r(E) = \inf\left\{ N r^\lambda : E \subset \bigcup_{i=1}^N B(z_i, r),\ z_i \in E\right\}.$$

The lower and upper Minkowski dimension of E are then defined to be

$$\underline{\dim}_{\mathcal{M}}(E) = \inf\big\{\lambda > 0 : \liminf_{r\to\infty} \mathcal{M}^\lambda_r(E) = 0\big\}$$

and

$$\overline{\dim}_{\mathcal{M}}(E) = \inf\big\{\lambda > 0 : \limsup_{r\to 0} \mathcal{M}^\lambda_r(E) = 0\big\},$$

respectively. Notice that always $\dim_{\mathcal{H}}(E) \le \underline{\dim}_{\mathcal{M}}(E) \le \overline{\dim}_{\mathcal{M}}(E)$, where all inequalities can be strict; cf. [17, Ch. 5]. But if $\underline{\dim}_{\mathcal{M}}(E) = \overline{\dim}_{\mathcal{M}}(E)$, we simply write $\dim_{\mathcal{M}}(E) = \overline{\dim}_{\mathcal{M}}(E)$.

We need yet another notion of dimension, introduced by Aikawa (cf. [1], [2]); a similar concept also appears in the works of Wannebo, see [24]. When $E \subset \mathbb{R}^n$ is a closed set with an empty interior, we let $G(E)$ denote the set of those $s > 0$ for which there exists a constant $C_s > 0$ such that

$$\int_{B(x,r)} d(y,E)^{s-n}\, dy \le C_s r^s \tag{2}$$

for every $x \in E$ and all $r > 0$. Then the Aikawa dimension of E is defined by $\dim_{\mathcal{A}}(E) = \inf G(E)$. If a set E has a non-empty interior, we set $\dim_{\mathcal{A}}(E) = n$. Thus always $\dim_{\mathcal{A}}(E) \le n$ for $E \subset \mathbb{R}^n$.

It follows from (2) and [11, Lemma 2.6] that $\overline{\dim}_{\mathcal{M}}(E) \le \dim_{\mathcal{A}}(E)$ for every compact set $E \subset \mathbb{R}^n$, but also here the inequality can be strict. For example, if $E = \{(j^{-1}, 0, \ldots, 0) : j \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}^n$, then $\dim_{\mathcal{H}}(E) = 0$, $\dim_{\mathcal{M}}(E) = 1/2$, and $\dim_{\mathcal{A}}(E) = 1$; see Section 6.2 for the justification of this last fact. However, for many sufficiently regular sets all of the dimensions considered above agree; the next lemma gives quite a general condition for such an equivalence.

Lemma 2.1. *Assume that $E \subset \mathbb{R}^n$ is Ahlfors α-regular, i.e., there exist a Borel regular measure μ on E and a constant $c > 0$ such that*

$$c^{-1} r^\alpha \le \mu(B(w,r)) \le c r^\alpha$$

for every $w \in E$ and all $0 < r < \operatorname{diam}(E)$. Then $\dim_{\mathcal{A}}(E) = \dim_{\mathcal{H}}(E) = \alpha$.

Proof. We may clearly assume that $\alpha < n$. As $\alpha = \dim_{\mathcal{H}}(E) \le \dim_{\mathcal{A}}(E)$ (cf. e.g. [17]), it suffices to prove the opposite inequality. To this end, let $\alpha < s < n$, and fix $w \in E$ and $0 < R < \operatorname{diam}(E)$. For $0 < r < 2R$ we denote $A_r = \{x \in B(w,2R) : d(x,E) < r\}$. Using the standard $5r$-covering lemma, we obtain a finite collection of pairwise disjoint balls $B_i = B(w_i, r)$, where $w_i \in E \cap B(w,2R)$ and $i = 1, \ldots, N_r$, so that $A_r \subset \bigcup_i 5B_i$. But then we also have that $B(w,2R) \cap E \subset \bigcup_i 5B_i \cap E$, and so the α-regularity of E yields that

$$c^{-1} N_r r^\alpha \le N_r \mu(B_i) = \mu\Big(\bigcup_i B_i\Big) \le \mu\big(B(w,4R)\big) \le c 4^\alpha R^\alpha,$$

giving the estimate $N_r \le C(c,\alpha)(R/r)^\alpha$ for each $0 < r < 2R$. Now

$$|A_r| \le N_r (5r)^n \le C(c,\alpha,n) R^\alpha r^{n-\alpha}, \tag{3}$$

and so an integration over the level sets of the distance function (here $\lambda_0 = (2R)^{s-n}$) and a use of (3) leads us to

$$\begin{aligned}
\int_{B(w,R)} d(x,E)^{s-n}\,dx &\le \int_{\lambda_0}^\infty \big|\{x \in B(w,2R) : d(x,E)^{s-n} > \lambda\}\big|\,d\lambda \\
&\le C \int_{\lambda_0}^\infty \big|A_{\lambda^{1/(s-n)}}\big|\,d\lambda \\
&\le C R^\alpha \int_{\lambda_0}^\infty \lambda^{(n-\alpha)/(s-n)}\,d\lambda \\
&\le C R^\alpha \tfrac{n-s}{s-\alpha} \lambda_0{}^{(s-\alpha)/(s-n)} \\
&= C R^\alpha R^{s-\alpha} = C R^s,
\end{aligned}$$

where $C = C(c,\alpha,n,s) > 0$. This shows that $\dim_{\mathcal{A}}(E) \le s$, and the claim follows. $\square$

Standard examples of (Ahlfors) α-regular sets include compact Lipschitz submanifolds of $\mathbb{R}^n$ and self-similar fractals satisfying the open set condition; see for instance [4] or [17] and references therein for more information. We also note that the measure μ above is always comparable to the α-dimensional Hausdorff measure restricted to the α-regular set E.

2.3. **On Hardy inequalities.** Let us recall that the pointwise (p,β)-Hardy inequality, for $u \in C_0^\infty$ at $x \in \Omega$, reads as

$$|u(x)| \le C d_\Omega(x)^{1-\frac{\beta}{p}} M_{2d_\Omega(x),q}\big(|\nabla u| d_\Omega{}^{\beta/p}\big)(x), \tag{4}$$

where q is some exponent so that $1 < q < p$. These inequalities were introduced in [10], following the considerations in the unweighted case, conducted by Hajłasz [5] and Kinnunen and Martio [8]. It is easy to see, using the boundedness of M_q on $L^p(\Omega)$, that if the pointwise (p,β)-Hardy inequality (4) holds for a function u at every $x \in \Omega$ with a constant $C_1 > 0$, then u satisfies the usual (p,β)-Hardy inequality with a constant $C = C(C_1, p, q, n) > 0$ (see [10]). However, pointwise Hardy inequalities are strictly stronger than usual Hardy inequalities in the sense that there exist domains which admit the (p,β)-Hardy inequality (for some p and β), but where the corresponding pointwise inequalities fail.

For technical reasons, we introduce the following notation. Let $\Omega' \subset \Omega$. We say that *the pair* (Ω', Ω) *admits* the (p,β)-Hardy inequality if there exists a constant $C > 0$ such that

$$\int_{\Omega'} |u(x)|^p \, d_\Omega(x)^{\beta-p} \, dx \le C \int_\Omega |\nabla u(x)|^p \, d_\Omega(x)^\beta \, dx \tag{5}$$

for all $u \in C_0^\infty(\Omega)$. In particular, if the pointwise (p,β)-Hardy inequality holds at every $x \in \Omega'$ for every $u \in C_0^\infty(\Omega)$ with a constant $C_1 > 0$, then the pair (Ω', Ω) admits the (p,β)-Hardy inequality, with a constant $C = C(C_1, p, \beta, n) > 0$.

Let us record the following lemma which gives a sufficient (and trivially necessary) condition for a pair (Ω', Ω) to admit the (p,β)-Hardy inequality.

Lemma 2.2. *Suppose that there exists $C_1 > 0$ such that*

$$\int_{\Omega'} |u|^p {d_\Omega}^{\beta-p} \le C_1 \left[\int_{\Omega'} |u|^{p-1} |\nabla u| {d_\Omega}^{\beta-p+1} + \int_\Omega |\nabla u|^p {d_\Omega}^\beta \right]$$

for all $u \in C_0^\infty(\Omega)$. Then the pair (Ω', Ω) admits the (p,β)-Hardy inequality.

Proof. An application of a general version of Young's inequality gives

$$\begin{aligned} \int_{\Omega'} |u|^{p-1} |\nabla u| {d_\Omega}^{\beta-p+1} &= \int_{\Omega'} \left(|u|^p \, {d_\Omega}^{\beta-p} \right)^{\frac{p-1}{p}} \left(|\nabla u|^p \, {d_\Omega}^\beta \right)^{\frac{1}{p}} \\ &\le \frac{1}{2C_1} \int_{\Omega'} |u|^p \, {d_\Omega}^{\beta-p} + C(C_1, p) \int_\Omega |\nabla u|^p \, {d_\Omega}^\beta, \end{aligned}$$

and the claim follows. □

2.4. **Chains and shadows of cubes on John domains.** Let us first recall the definition of John domains. Let $c \ge 1$. We say that a domain $\Omega \subset \mathbb{R}^n$ is a *c-John domain*, with center point x_0, if for every $x \in \Omega$ there exists a curve (called a John curve) $\gamma\colon [0, l] \to \Omega$, parametrized by arc length, such that $\gamma(0) = x$, $\gamma(l) = x_0$, and

$$d(\gamma(t), \partial\Omega) \ge \tfrac{1}{c} t \tag{6}$$

for each $t \in [0, l]$. Such domains were first considered by F. John, cf. [7]. Geometrically the John condition (6) means that each point in Ω can be joined to the central point by a "twisted cone". If Ω is a c-John domain with center point x_0, then $\Omega \subset B\big(x_0, c\, d(x_0, \partial\Omega)\big)$, and so Ω is bounded. However, in Section 4 we will need a similar notion for unbounded domains, but we postpone these considerations until that section.

When $\Omega \subset \mathbb{R}^n$ is a domain (not necessarily John), we say that $\Omega' \subset \Omega$ is a *c-John subset* (of Ω) if for each $x \in \Omega'$ there exists a c-John curve $\gamma = \gamma_{x,x_0}$ *of* Ω joining x to a fixed center point $x_0 \subset \Omega'$ *in* Ω', i.e. $\gamma([0, l]) \subset \Omega'$; we emphasize that the distance in (6) is still taken with respect to $\partial\Omega$. For our purposes, it is convenient — and also sufficient — to assume that such an $\Omega' \subset \Omega$ is always a union of Whitney cubes $Q \in \mathcal{W}(\Omega)$, and hence we take this as a standing assumption throughout the paper. In this setting, we use the notation $\mathcal{W}' = \mathcal{W}(\Omega)|_{\Omega'}$.

When $\Omega' \subset \Omega$ is a c-John subset, with center point $x_0 \in \Omega'$, and $x \in \Omega'$, we let $\mathcal{J}_c(x, x_0)$ denote the collection of all c-John curves joining x to x_0 in Ω'. If $Q \in \mathcal{W}'$, we write

$$P(Q) = \big\{\widetilde{Q} \in \mathcal{W}' : \widetilde{Q} \cap \gamma \neq \emptyset \text{ for some } \gamma \in \mathcal{J}_c(x, x_0),\ x \in Q\big\}.$$

The *(John-)shadow* $S(\widetilde{Q})$ of a cube $\widetilde{Q} \in \mathcal{W}'$ is then defined to be

$$S(\widetilde{Q}) = \{Q \in \mathcal{W}' : \widetilde{Q} \in P(Q)\}.$$

We obtain the following easy lemma immediately from the definition of the shadow.

Lemma 2.3. *Let $\Omega' \subset \Omega$ be a c-John subset, and let $Q \in \mathcal{W}'$. Then* $\operatorname{diam} S(Q) \le C \operatorname{diam}(Q)$*, where* $C = C(c) > 0$.

The next Lemma is crucial for our sufficient condition for Hardy inequalities.

Lemma 2.4. *Let $\Omega' \subset \Omega$ be a c-John subset so that $\partial\Omega' \cap \partial\Omega \neq \emptyset$, and assume that*

$$d(x, \partial\Omega' \cap \partial\Omega) \le C_1 d_\Omega(x) \tag{7}$$

for every $x \in \Omega'$. Then, if $\dim_{\mathcal{A}}(\partial\Omega' \cap \partial\Omega) < s < n$, there exists a constant $C = C(C_1, C_s, n, c, s) > 0$ *such that*

$$\sum_{Q \in S(\widetilde{Q})} \operatorname{diam}(Q)^s \le C \operatorname{diam}(\widetilde{Q})^s$$

for every $\widetilde{Q} \in \mathcal{W}'$. Here C_s is the constant from (2).

Proof. Let $\widetilde{Q} \in \mathcal{W}'$, take $w \in \partial\Omega$ such that $d(w, \widetilde{Q}) = d(\widetilde{Q}, \partial\Omega)$, and denote $B = B\big(w, \operatorname{diam}(\widetilde{Q})\big)$. It follows easily from Lemma 2.3 that there exists $C_2 = C_2(n, c) > 0$ such that $S(\widetilde{Q}) \subset C_2 B$. Hence, using the properties of the Whitney cubes, assumption (7), and the definition of the Aikawa dimension, we obtain that

$$\begin{aligned}\sum_{Q \in S(\widetilde{Q})} \operatorname{diam}(Q)^s &\le C \sum_{Q \in S(\widetilde{Q})} \int_Q d_\Omega(x)^{s-n}\, dx \\ &\le C \int_{C_2 B} d(x, \partial\Omega' \cap \partial\Omega)^{s-n}\, dx \le C \operatorname{diam}(\widetilde{Q})^s,\end{aligned}$$

where $C = C(C_1, n, c, s) > 0$. □

3. A sufficient condition for Hardy inequalities

In this section we prove our sufficient condition for weighted Hardy inequalities. To be precise, given a subset $\Omega' \subset \Omega$, we give a sufficient condition *for the pair* (Ω', Ω) to admit the (p, β)-Hardy inequality. We discuss the obvious way how this can be used to obtain the (p, β)-Hardy inequality in the whole domain at the end of this section. Our theorem can in some sense be considered as an extension of Theorem 1.2, altough we only deal here with bounded subsets and need an additional accessibility condition, which by the way can not be removed from the assumptions, see e.g. Example 6.3. However, just as in Theorem 1.2, it is essential here as well that *the part* $\partial\Omega' \cap \partial\Omega$ of the boundary has a small Aikawa dimension; the upper bound is $n - p + \beta$ for the (p, β)-Hardy inequality.

The idea behind our condition is the following: We want to estimate the quantity $|u|^p {d_\Omega}^{p-\beta}$ in each of the Whitney cubes $Q \in \mathcal{W}'$, which are all assumed to be relatively close to the small part $\partial\Omega' \cap \partial\Omega$ of the boundary. We also presume that there exists some fixed cube $Q_0 \in \mathcal{W}'$ where the average $|u_{Q_0}|$ is well controlled by the (weighted) integral of the gradient — this is the meaning of the requirement (8) in Theorem 3.1 below. The existence of such a cube follows if Ω has a thick and accessible boundary part, see the remark after the proof of the theorem. The assumption that Ω' is a c-John subset, with center point $x_0 \in Q_0$, assures that we may estimate the difference between the averages $|u_Q|$ and $|u_{Q_0}|$ by the integral of $|\nabla u|$ along a *nice* chain of cubes from Q to Q_0. Summation over every $Q \in \mathcal{W}'$ then gives us an estimate for $\int_{\Omega'} |u|^p {d_\Omega}^{p-\beta}$ in terms of the integral of $|\nabla u|$ in Ω', but this estimate still involves a sum related to the sizes of the shadows of the cubes in $\mathcal{W}'$. However, as the boundary $\partial\Omega' \cap \partial\Omega$ has a small Aikawa dimension, and we assume that $d_\Omega(x) \approx d(x, \partial\Omega' \cap \partial\Omega)$ for all $x \in \Omega'$, Lemma 2.4 gives us a suitable bound, so that finally everything is controlled by the weighted integral of the gradient, meaning that the Hardy inequality holds for the pair (Ω', Ω).

After this informal introduction, let us now state the actual theorem and give the exact proof.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $1 < p < \infty$, $\beta \in \mathbb{R}$. Assume that $\Omega' \subset \Omega$ is a c-John subset with center point $x_0 \in \Omega'$, satisfying the following conditions: $\partial\Omega' \cap \partial\Omega \neq \emptyset$ and $d(x, \partial\Omega' \cap \partial\Omega) \leq C_1 d_\Omega(x)$ for every $x \in \Omega'$, $\dim_{\mathcal{A}}(\partial\Omega' \cap \partial\Omega) < n - p + \beta$, and $x_0 \in Q_0 \in \mathcal{W}'$, where Q_0 is such that*

$$|u_{Q_0}|^p \leq C \operatorname{diam}(Q_0)^{p-\beta-n} \int_\Omega |\nabla u(y)|^p \, d_\Omega(y)^\beta \, dy \tag{8}$$

holds whenever $u \in C_0^\infty(\Omega)$, with a constant $C > 0$ independent of u. Then the pair (Ω', Ω) admits the (p, β)-Hardy inequality.

Proof. Let $u \in C_0^\infty(\Omega)$ and denote $v = |u|^p$. Then $v \in W_0^{1,1}(\Omega)$ and $|\nabla v| \leq p|u|^{p-1}|\nabla u|$ for a.e. $x \in \Omega$. When $Q \in \mathcal{W}'$, there exists a c-John curve γ_Q joining c_Q to x_0 in Ω'. We let $\mathcal{C}(\gamma_Q) = \{\widetilde{Q} \in \mathcal{W}' : \widetilde{Q} \cap \gamma_Q \neq \emptyset\}$ denote the corresponding chain of Whitney cubes, so that $\mathcal{C}(\gamma_Q) \subset P(Q)$. An application of the 1-Poincaré inequality for the chain $\mathcal{C}(\gamma_Q)$, as in [20, Lemma 8], leads to

$$|v_Q - v_{Q_0}| \le C \sum_{\widetilde{Q} \in P(Q)} \operatorname{diam}(\widetilde{Q}) \fint_{\widetilde{Q}} |\nabla v(y)|\, dy\,, \tag{9}$$

where the constant $C = C(n) > 0$ is independent of Q. Since $|u(x)|^p \le |v(x) - v_Q| + |v_Q - v_{Q_0}| + |v_{Q_0}|$ and $d_\Omega(x) \approx \operatorname{diam}(Q)$ for every $x \in Q$, it follows from (9) that

$$\begin{aligned} \int_{\Omega'} |u|^p {d_\Omega}^{\beta-p} \le C \Bigg(& \sum_{Q \in \mathcal{W}'} \operatorname{diam}(Q)^{\beta-p} \int_Q |v - v_Q| \\ & + \sum_{Q \in \mathcal{W}'} \operatorname{diam}(Q)^{\beta-p+n} \sum_{\widetilde{Q} \in P(Q)} \operatorname{diam}(\widetilde{Q}) \fint_{\widetilde{Q}} |\nabla v| \\ & + \sum_{Q \in \mathcal{W}'} \operatorname{diam}(Q)^{\beta-p+n} |v_{Q_0}| \Bigg). \end{aligned} \tag{10}$$

By the 1-Poincaré inequality, the first sum in (10) is no more than

$$C(n) \sum_{Q \in \mathcal{W}'} \operatorname{diam}(Q)^{\beta-p+1} \int_Q |\nabla v| \le C(n,p) \int_{\Omega'} |u|^{p-1} |\nabla u| {d_\Omega}^{\beta-p+1}. \tag{11}$$

We change the order of summation in the second term of (10), and then use the conclusion of Lemma 2.4,

$$\sum_{Q \in S(\widetilde{Q})} \operatorname{diam}(Q)^{n-p+\beta} \le C \operatorname{diam}(\widetilde{Q})^{n-p+\beta} \qquad \text{for all } \widetilde{Q} \in \mathcal{W}', \tag{12}$$

to obtain

$$\begin{aligned} & \sum_{Q \in \mathcal{W}'} \operatorname{diam}(Q)^{\beta-p+n} \sum_{\widetilde{Q} \in P(Q)} \operatorname{diam}(\widetilde{Q}) \fint_{\widetilde{Q}} |\nabla v| \\ & \quad \le \sum_{\widetilde{Q} \in \mathcal{W}'} \operatorname{diam}(\widetilde{Q})^{1-n} \int_{\widetilde{Q}} |\nabla v| \sum_{Q \in S(\widetilde{Q})} \operatorname{diam}(Q)^{n-p+\beta} \\ & \quad \le C \sum_{\widetilde{Q} \in \mathcal{W}'} \operatorname{diam}(\widetilde{Q})^{\beta-p+1} \int_{\widetilde{Q}} |\nabla v| \\ & \quad \le C \int_{\Omega'} |u|^{p-1} |\nabla u| {d_\Omega}^{\beta-p+1}, \end{aligned} \tag{13}$$

where $C = C(p, n, \beta, \lambda) > 0$. Recall here that we may always assume that Ω' is a union of Whitney cubes, and hence $\Omega' = \bigcup \{\widetilde{Q} \in \mathcal{W}'\}$.

Finally, to estimate the last term in (10), we use again (12), the p-Poincaré inequality, and the assumption (8), and conclude that

$$\begin{aligned}\sum_{Q\in\mathcal{W}'} \operatorname{diam}(Q)^{n-p+\beta}|v_{Q_0}| &\le \operatorname{diam}(Q_0)^{n-p+\beta}\fint_{Q_0}|u|^p \\ &\le C\operatorname{diam}(Q_0)^{\beta-p}\int_{Q_0}|u-u_{Q_0}|^p+\operatorname{diam}(Q_0)^{n-p+\beta}|u_{Q_0}|^p \\ &\le C\operatorname{diam}(Q_0)^{\beta-p}\operatorname{diam}(Q_0)^p\int_{Q_0}|\nabla u|^p+\int_\Omega|\nabla u|^p d_\Omega{}^\beta \\ &\le C\int_\Omega|\nabla u|^p d_\Omega{}^\beta.\end{aligned} \tag{14}$$

The (p,β)-Hardy inequality for the pair (Ω',Ω) follows now by combining the estimates from (10), (11), (13), and (14), since then

$$\int_{\Omega'}|u|^p d_\Omega{}^{\beta-p}\le C\bigg[\int_{\Omega'}|u|^{p-1}|\nabla u|d_\Omega{}^{\beta-p+1}+\int_\Omega|\nabla u|^p d_\Omega{}^\beta\bigg],$$

and Lemma 2.2 yields the claim. □

Remark. Condition (8) in the assumptions of the previous theorem is closely related to pointwise Hardy inequalities. In particular, it follows from the estimates of [10] that if the *visual boundary* near Q_0 is thick enough (especially, the boundary must have Hausdorff dimension strictly greater than $n-p+\beta$), then (8) holds. See [10] for precise statements.

Let us end this section with a few words about the use of Theorem 3.1 in practice. When $1<p<\infty$ and $\beta\in\mathbb{R}$, and a domain $\Omega\subset\mathbb{R}^n$ is given, one should first analyze the size and geometry of the boundary, and then try to divide Ω into finitely many pairwise disjoint subsets $\Omega_1,\dots,\Omega_K$ in such a way that, for each $k=1,\dots,K$, either the pointwise (p,β)-Hardy holds for every $x\in\Omega_k$, or Ω_k satisfies the assumptions of Theorem 3.1 (or, in the unbounded case, those of Theorem 4.3 from the next section). If this can be done, then we conclude that each pair (Ω_k,Ω) admits the (p,β)-Hardy, and thus it is clear that Ω admits the (p,β)-Hardy inequality.

4. Unbounded domains

In order to extend the condition of Theorem 3.1 for unbounded domains we need to make the notions of Section 2.4 applicable to the unweighted case. Especially, we need to specify what we mean by unbounded John domains. Here we follow Väisälä [22], and say that an unbounded $\Omega\subset\mathbb{R}^n$ is a c-John domain if each pair of points $x_1,x_2\in\Omega$ can be joined by a curve $\gamma=\gamma_{x_1,x_2}\colon[0,l]\to\Omega$, parametrized by arc length, so that $d(\gamma(t),\partial\Omega)\ge\frac{1}{c}\min\{t,l-t\}$ for all $t\in[0,l]$. It turns out that for bounded domains this definition would be equivalent to the definition that we used in Section 2.4, but possibly with different constants; cf. [22, Thm. 3.6].

Nevertheless, for our purposes it is very useful to have a fixed central point x_0 in a John domain, since we may use such a x_0 as the end-point in the chains, as in the proof of Theorem 3.1. We would like to use a similar idea in the unbounded case as well, but the definition above lacks the notion of a central point. Hence the following lemma, a slightly modified version of [22, Thm. 4.6], which allows us to "exhaust" an unbounded c-John domain by bounded c'-John subsets (in the sense of Section 2.4), turns out to be very useful.

Lemma 4.1. *Let $\Omega \subsetneq \mathbb{R}^n$ be an unbounded c-John domain, and let $w \in \partial\Omega$. Then $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$, where each Ω_k is a c'-John subset of Ω, with $c' = c'(c,n) > 0$ independent of k, and moreover, $\overline{B}(w,k) \cap \Omega \subset \Omega_k$ for each $k \in \mathbb{N}$.*

The proof of Lemma 4.1 follows easily from the first part of the proof of [22, Thm. 4.6] and [22, Lemma 4.3].

Inspired by this result, we say that an unbounded subset $\Omega' \subset \Omega$ is an unbounded c-John subset of Ω if $\Omega' = \bigcup_{k=1}^{\infty} \Omega'_k$, where Ω'_k is a bounded c-John subset of Ω and $\overline{B}(w,k) \cap \Omega' \subset \Omega'_k$ for each $k \in \mathbb{N}$, where $w \in \partial\Omega$ is some fixed boundary point. We are then able to use the results from the bounded case for Ω'_k, as we shall do in the proof of Theorem 4.3.

However, now also the case $\partial\Omega' \cap \partial\Omega = \emptyset$ may be relevant (see for instance the example of Section 6.4), and then the assumption $\dim_{\mathcal{A}}(\partial\Omega' \cap \partial\Omega) < n - p + \beta$ is obviously not the right one for us. In such cases we may instead require that the shadows of the cubes $\widetilde{Q} \in \mathcal{W}'$ satisfy the following condition:

Let $\Omega \subset \mathbb{R}^n$ and let $\Omega' \subset \Omega$ be an unbounded c-John subset so that $\Omega' = \bigcup_{k=1}^{\infty} \Omega'_k$ as above. When $\widetilde{Q} \in \mathcal{W}'$, we let $S_k(\widetilde{Q})$ denote the shadow of $\widetilde{Q}$ with respect to Ω'_k. We say that Ω' satisfies a uniform cube-count condition if there exists a constant $C > 0$ so that

$$\#\{Q \in S_k(\widetilde{Q}) \cap \mathcal{W}_j\} \le C 2^{\lambda j} \operatorname{diam}(\widetilde{Q})^{\lambda} \tag{15}$$

for every $\widetilde{Q} \in \mathcal{W}'$, $j \in \mathbb{Z}$, and $k \in \mathbb{N}$; note here that $j \in \mathbb{Z}$ is negative for large cubes.

If $\widetilde{Q} \in \mathcal{W}'$ and $\tilde{x}$ is is the center of $\widetilde{Q}$, it follows from Lemma 2.3 that there exists $C = C(c) > 0$ so that $S_k(\widetilde{Q}) \subset B\big(\tilde{x}, C \operatorname{diam}(\widetilde{Q})\big)$ for each $k \in \mathbb{N}$. Hence, if

$$\#\{Q \in \mathcal{W}'_j : Q \subset B\big(\tilde{x}, C \operatorname{diam}(\widetilde{Q})\big)\} \le C 2^{\lambda j} \operatorname{diam}(\widetilde{Q})^{\lambda}$$

for each $\widetilde{Q} \in \mathcal{W}'$ and $j \in \mathbb{Z}$, we conclude that (15) holds with a constant independent of k.

The next lemma shows that the conclusion of Lemma 2.4 holds for all $s > \lambda$ if Ω' satisfies the above uniform cube-count condition (15) with the exponent λ.

Lemma 4.2. *Let $\widetilde{Q} \in \mathcal{W}'$ and assume that condition* (15) *holds with an exponent λ and a constant $C_0 > 0$ for every $j \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then, if $s > \lambda$,*

$$\sum_{Q \in S_k(\widetilde{Q})} \operatorname{diam}(Q)^s \le C \operatorname{diam}(\widetilde{Q})^s,$$

where the constant $C = C(C_0, s, \lambda) > 0$ is independent of $\widetilde{Q}$ and k.

Proof. Assume that $\widetilde{Q} \in \mathcal{W}_{j_0}$ and let $k \in \mathbb{N}$. It follows easily from condition (15) that $S_k(\widetilde{Q}) \cap \mathcal{W}_j = \emptyset$ if $j < j_0 - \frac{\log C_0}{\lambda \log 2}$. Let j_1 be the smallest integer for which $S_k(\widetilde{Q})\cap\mathcal{W}_j\neq\emptyset$. Then $2^{-j_1}\le 2^{-j_0+\frac{\log C_0}{\lambda\log 2}}=C_1 2^{-j_0}$, and we calculate

$$\begin{aligned}\sum_{Q\in S_k(\widetilde{Q})}\operatorname{diam}(Q)^s &\le \sum_{j=j_1}^{\infty}\#\{Q\in S_k(\widetilde{Q})\cap\mathcal{W}_j\}\,2^{-js}\\ &\le C_0\sum_{j=j_1}^{\infty}2^{\lambda j}\operatorname{diam}(\widetilde{Q})^{\lambda}\,2^{-js}\le C_0\operatorname{diam}(\widetilde{Q})^{\lambda}\sum_{j=j_1}^{\infty}2^{-j(s-\lambda)}\\ &\le C\operatorname{diam}(\widetilde{Q})^{\lambda}\,2^{-j_1(s-\lambda)}\le C\operatorname{diam}(\widetilde{Q})^{\lambda}\operatorname{diam}(\widetilde{Q})^{s-\lambda}\\ &\le C\operatorname{diam}(\widetilde{Q})^{s},\end{aligned}$$

where $C=C(C_0,s,\lambda)>0$. □

We can now formulate and prove our main result concerning unbounded domains. The condition (16) in the theorem is satisfied (for instance) if $\Omega'\subset\Omega$ is an unbounded c-John subset so that, either, (i) $\partial\Omega'\cap\partial\Omega\neq\emptyset$, $\dim_{\mathcal{A}}(\partial\Omega'\cap\partial\Omega)<n-p+\beta$, and $d(x,\partial\Omega'\cap\partial\Omega)\le C_1 d_\Omega(x)$ for all $x\in\Omega'$ (Lemma 2.4), or (ii) the uniform cube-count condition (15) holds in Ω' (Lemma 4.2). In particular, (16) holds if Ω is an unbounded c-John domain with $\dim_{\mathcal{A}}(\partial\Omega)<n-p+\beta$, and so Theorem 1.3 follows as a special case of Theorem 4.3.

Theorem 4.3. *Let $\Omega\subset\mathbb{R}^n$ be an unbounded domain, and assume that $\Omega'=\bigcup_{k=1}^{\infty}\Omega'_k$ is an unbounded c-John subset of Ω, so that*

$$\sum_{Q\in S_k(\widetilde{Q})}\operatorname{diam}(Q)^{n-p+\beta}\le C\operatorname{diam}(\widetilde{Q})^{n-p+\beta}\tag{16}$$

for all $\widetilde{Q}\in\mathcal{W}'$ and all $k\in\mathbb{N}$, where $S_k(\widetilde{Q})$ is the shadow of $\widetilde{Q}$ with respect to Ω'_k. Then the pair (Ω',Ω) admits the (p,β)-Hardy inequality

Proof. The proof goes along the lines of the proof of Theorem 3.1, but with some modifications that we present here.

We may clearly assume that $0\in\partial\Omega$, and moreover, that 0 is the fixed reference point for the bounded c-John subsets Ω'_k. Let $u\in C_0^\infty(\Omega)$, and take $k\in\mathbb{N}$ such that $\operatorname{spt}(u)\subset B(0,k)$. We then choose $L\in\mathbb{N}$ to be so large that $L/(c+1)\ge 2$, and consider the bounded c-John subset Ω'_{Lk}. Let x_0 be the center of Ω'_{Lk}. Since $\overline{B}(0,Lk)\cap\Omega\subset\Omega'_{Lk}$, there exists $z\in\Omega'_{Lk}$ such that $|z|\ge Lk$. But $\Omega'_{Lk}\subset B(x_0,cd_\Omega(x_0))$, and thus

$$Lk\le|z|\le|z-x_0|+|x_0|\le cd_\Omega(x_0)+|x_0|\le(c+1)|x_0|,$$

giving $|x_0|\ge 2k$ by the choice of L. Let then $Q_0\in\mathcal{W}'$ be such that $x_0\in Q_0$. It follows that

$$2k\le|x_0|\le d(0,Q_0)+\operatorname{diam}(Q_0)\le d(0,Q_0)+d(Q_0,\partial\Omega)\le 2d(0,Q_0),$$

and so $Q_0\cap B(0,k)=\emptyset$, in particular $u(x)=0$ for every $x\in Q_0$.

If now $Q\in\mathcal{W}'$ is such that $Q\cap\operatorname{spt}(u)\neq\emptyset$, we let $\mathcal{C}(\gamma_Q)$ denote a chain of cubes joining Q to Q_0, just as in the proof of Theorem 3.1, whence $\mathcal{C}(\gamma_Q)\subset\Omega'_{Lk}$. It follows that the estimate (10) holds for u in this case as well, with a constant independent of u. But since $v_{Q_0}=0$ for $v=|u|^p$, the term involving this average vanishes from (10), while the other terms in (10) can be treated just as in the proof of Theorem 3.1, especially the assumption (16) for Ω'_{Lk} offers a substitute for (12). Hence we obtain the (p,β)-Hardy inequality for the function u with a constant independent of u, and it follows that the pair (Ω',Ω) admits the (p,β)-Hardy inequality. □

See Section 6.4 for an example of the use of Theorem 4.3.

A special case of unbounded domains is the one where $\partial\Omega$ is bounded. Using the necessary conditions for weighted pointwise Hardy inequalities (cf. [14]), it is easy to see that such a domain can not admit the pointwise (p,β)-Hardy inequality for any $\beta \geq p-n$. This fact follows also from the next observation, which says that in this case the usual $(p,p-n)$-Hardy fails as well. Whether such an Ω admits Hardy inequalities for any $\beta > p-n$ depends on the size and geometry of the boundary of the particular domain. Let us recall here that, by [10], each domain $\Omega \subsetneq \mathbb{R}^n$ admits the pointwise (p,β)-Hardy inequality if $\beta < p-n$.

Proposition 4.4. *Assume that $\Omega \subset \mathbb{R}^n$ is unbounded with $\partial\Omega$ bounded. Then Ω does not admit the $(p,p-n)$-Hardy inequality for any $1<p<\infty$.*

Proof. Take $r>0$ such that $\partial\Omega \subset B = B(0,r)$. For $k \in \mathbb{N}$, $k \geq 4$ define

$$u_k(x) = \begin{cases} r^{-1} d(x,2B), & x \in 3B \\ 1, & x \in kB \setminus 3B \\ (kr)^{-1} d\big(x,\mathbb{R}^n \setminus 2kB\big), & x \in \mathbb{R}^n \setminus kB. \end{cases}$$

Then u_k is a compactly supported Lipschitz function in Ω for each $k \geq 4$. Using polar coordinates it is easy to calculate that

$$\begin{aligned} \int_\Omega |u_k(x)|^p\, d_\Omega(x)^{(p-n)-p}\, dx &\geq \int_{kB\setminus 3B} d_\Omega(x)^{-n}\, dx \\ &\geq C(n) \int_{3r}^{kr} t^{-n}\, t^{n-1}\, dt = C(n)\big(\log(kr) - \log(3r)\big) \xrightarrow{k\to\infty} \infty, \end{aligned}$$

while the corresponding integrals involving the gradients remain uniformly bounded:

$$\begin{aligned} &\int_\Omega |\nabla u_k(x)|^p\, d_\Omega(x)^{p-n}\, dx \\ &\quad = \int_{3B\setminus 2B} r^{-p} d_\Omega(x)^{p-n}\, dx + \int_{2kB\setminus kB} (kr)^{-p} d_\Omega(x)^{p-n}\, dx \\ &\quad \leq C(n) \left(r^{-p} \int_{2r}^{3r} t^{p-1}\, dt + (kr)^{-p} \int_{kr}^{2kr} t^{p-1}\, dt \right) \\ &\quad \leq C(n,p) < \infty. \end{aligned}$$

By approximation it is now clear that Ω does not admit the $(p,p-n)$-Hardy inequality. □

5. The dimension of the complement

Let us now discuss our necessary condition for a domain admitting a Hardy inequality. This condition is given in terms of the (local) dimension of the complement of Ω, and generalizes the dichotomy results of [11] to the weighted case.

As the proofs turn out to be a bit technical, let us give a quick overview of how to obtain the result. What we need to prove is that if Ω admits the (p,β)-Hardy inequality and the Hausdorff dimension of $2B\cap\Omega^c$ is small, at most $n-p+\beta$, then actually $\dim_{\mathcal{A}}(B\cap\Omega^c)<n-p+\beta-\delta$ for some $\delta>0$. The idea is to use the Hardy inequality for the function $u(x)=\varphi(x)d_\Omega(x)^{-\alpha}$, where $\varphi(x)=r^{-1}d(x,(2B)^c)$ and $\alpha>0$ is small enough. Then

$$|\nabla u(x)|^p \lesssim |\nabla\varphi(x)|^p d_\Omega(x)^{-\alpha p}+\alpha^p|u(x)|^p d_\Omega(x)^{-p},$$

and so the integral of $|\nabla u|^p {d_\Omega}^\beta$ can be estimated by two terms, of which the other is α^p times the left-hand side of the Hardy inequality for u. Taking α small enough then gives us the desired estimate with $\delta=\alpha p$.

The above argument is not quite rigorous, as u is not bounded, or even supported in Ω, so there is no (*a priori*) reason why the Hardy inequality should hold for u. In addition, we need to assure that certain integrals of the distance function are finite, but nevertheless, after more careful considerations where suitable truncations and approximations are used with the fact that $\dim_{\mathcal{H}}(2B\cap\Omega^c)\le n-p+\beta$, this same idea can be carried out to prove that the Aikawa dimension of Ω^c is indeed less than $n-p+\beta-\delta$. Let us start with a lemma that allows us to use the Hardy inequality also for some functions which are not supported in Ω. As a consequence we also obtain a useful estimate for a distance integral.

Lemma 5.1. *Let $1<p<\infty$, $0\le\beta\le p$, and assume that Ω admits the (p,β)-Hardy inequality with a constant $C_0>0$. Assume further that $B_0\subset\mathbb{R}^n$ is a ball such that*

$$\mathcal{H}^{n-p+\beta}_\infty(4B_0\cap\Omega^c)=0. \tag{17}$$

Then
(i) the (p,β)-Hardy inequality holds for all $u\in C_0^\infty(\Omega\cup 4B_0)$ with a constant $C_1=C_1(C_0,p)>0$;
(ii) we have for each closed ball $\overline{B}(w,r)\subset 3B_0$, with $w\in\Omega^c$, that

$$\int_B d_\Omega(x)^{\beta-p}\,dx\le C_2r^{n-p+\beta},$$

where $C_2=C_2(C_0,p,n)>0$.

Proof. (i) Let $u\in C_0^\infty(\Omega\cup 4B_0)$. For a fixed $j\in\mathbb{N}$ there exists balls $B_i^j=B(w_i,r_i)$ with $w_i\in\Omega^c$, for $i=1,\dots,N$, so that $4B_0\cap\Omega^c\subset\bigcup_{i=1}^N B_i^j$ and $\sum_{i=1}^N {r_i}^{n-p+\beta}\le ||u||_\infty^{-1}2^{-j}$. Now define cut-off functions ψ_j by $\psi_j(x)=\min_i\{1,{r_i}^{-1}d(x,2B_i^j)\}$, and let $u_j=\psi_j u$. We may assume that $u_j\le u_{j+1}$ for each $j\in\mathbb{N}$. As $d(\operatorname{spt}(u_j),\partial\Omega)>0$, it follows that u_j is, for each $j\in\mathbb{N}$, a Lipschitz function with compact support in Ω, and so the (p,β)-Hardy inequality holds for the functions u_j as well. Since $|\nabla u_j|\le|\nabla\psi_j||u|+|\nabla u|$ a.e. in Ω, we obtain from the Hardy inequality that

$$\int|u_j|^p{d_\Omega}^{\beta-p}\le C_1\left[||u||_\infty\int|\nabla\psi_j|^p{d_\Omega}^\beta+\int|\nabla u|^p{d_\Omega}^\beta\right], \tag{18}$$

with the constant $C_1=2^pC_0$. But now, as $\beta\ge0$,

$$|\nabla\psi_j(x)|^p d_\Omega(x)^\beta\le\sum_{i=1}^N {r_i}^{-p}{r_i}^\beta\chi_{3B_i^j}(x)\qquad\text{for a.e. } x\in\Omega,$$

and thus (18) yields that

$$\begin{aligned}\int |u_j|^p {d_\Omega}^{\beta-p} \leq & C_1 \left[||u||_\infty \sum_{i=1}^{N} {r_i}^{n-p+\beta} + \int |\nabla u|^p {d_\Omega}^{\beta} \right] \\ \leq & C 2^{-j} + C_1 \int |\nabla u|^p {d_\Omega}^{\beta},\end{aligned}$$

where $C = C(C_0, p, n) > 0$. Now $u_j(x) \to u(x)$ for a.e. $x \in \Omega \cup 4B_0$, since (17) with $\beta \leq p$ implies that $|4B_0 \cap \Omega^c| = 0$, and claim (i) follows from the monotone convergence theorem.
(ii) Let $w \in \Omega^c$ and $r > 0$ be such that $\overline{B} = \overline{B}(w, r) \subset 3B_0$, so that $\frac{4}{3}\overline{B} \subset 4B_0$. We define $\varphi(x) = (r/3)^{-1} d\big(x, \mathbb{R}^n \setminus \frac{4}{3}B\big)$. Then φ is a Lipschitz function with a compact support in $\Omega \cup 4B_0$, $\varphi \geq 1$ in B, and $|\nabla \varphi| \leq (r/3)^{-1}$ a.e. in $\frac{4}{3}B$. Thus we may apply part (i) of the lemma to conclude that

$$\int_B {d_\Omega}^{\beta-p} \leq \int_{\frac{4}{3}B} |\varphi|^p {d_\Omega}^{\beta-p} \leq C_1 \int_{\frac{4}{3}B} |\nabla \varphi|^p {d_\Omega}^{\beta} \leq C_2 r^{n-p+\beta},$$

since now $d_\Omega(y) \leq \frac{4}{3}r$ for all $y \in \frac{4}{3}B$. □

The proof of the main result of this section, Theorem 5.3, relies on the following lemma, which is basically a "weighted" version of Lemma 2.4 from [11], altough the proof is quite different.

Lemma 5.2. *Let $1 < p < \infty$, $0 < \beta \leq p$, and assume that Ω admits the (p, β)-Hardy inequality with a constant $C_0 > 0$. Assume further that $B_0 \subset \mathbb{R}^n$ is a ball such that $\mathcal{H}^{n-p+\beta}_\infty(4B_0 \cap \Omega^c) = 0$. Then there exist constants $\delta = \delta(C_0, p, \beta, n) > 0$ and $C = C(C_0, p, \beta, n) > 0$ such that if $w \in \Omega^c$ and $\overline{B}(w, r) \subset 2B_0$, then*

$$\int_{B(w,r)} d_\Omega(x)^{-p+\beta-\delta}\, dx \leq C r^{n-p+\beta-\delta}.$$

In particular, $\dim_{\mathcal{A}}(B_0 \cap \Omega^c) \leq n - p + \beta - \delta$.

Proof. Let $B = B(w, r) \subset 2B_0$, where $w \in \Omega^c$, and let $\alpha > 0$ be small, to be chosen later. Define, for $j \in \mathbb{N}$, $\psi_j(x) = \min\{d_\Omega(x), 2^{-j}\}^{-\alpha}$, and let $\varphi(x) = (r/2)^{-1} d(x, \mathbb{R}^n \setminus \frac{3}{2}B)$. Then the functions $u_j = \varphi \psi_j$ are Lipschitz functions with compact support in $3B_0$, and so, by Lemma 5.1(i), the (p, β)-Hardy inequality holds for u_j with a constant $C_1 = C_1(C_0, p) > 0$. Denote $A_j = \{x \in \frac{3}{2}B : d_\Omega(x) \geq 2^{-j}\}$. Then, for a.e. $x \in \frac{3}{2}B \cap A_j$,

$$|\nabla u_j(x)| \leq |\nabla \varphi(x)| d_\Omega(x)^{-\alpha} + \alpha |\varphi(x)| d_\Omega(x)^{-\alpha-1},$$

and $|\nabla u_j(x)| = 0$ for a.e. $x \in 3B_0 \setminus \big(\frac{3}{2}B \cap A_j\big)$. Hence the (p, β)-Hardy inequality for u_j yields

$$\begin{aligned}&\int_{3B_0} |u_j|^p {d_\Omega}^{\beta-p} \\ &\quad \leq C_1 2^p \Big[\int_{\frac{3}{2}B \cap A_j} |\nabla \varphi|^p {d_\Omega}^{-\alpha p+\beta} + \alpha^p \int_{\frac{3}{2}B \cap A_j} |\varphi|^p {d_\Omega}^{-\alpha p-p+\beta} \Big].\end{aligned} \tag{19}$$

Using the upper bound $|u_j| \leq 3 \cdot 2^{\alpha j}$ and the part (ii) of Lemma 5.1, we infer that

$$\int_{\frac{3}{2}B\cap A_j} |\varphi|^p {d_\Omega}^{-\alpha p-p+\beta} \leq \int_{3B_0} |u_j|^p {d_\Omega}^{\beta-p} \leq C2^{j\alpha p} \int_{3B_0} {d_\Omega}^{\beta-p} < \infty. \tag{20}$$

Let us now require that α is so small that $C_1 2^p \alpha^p \leq 1/2$ and $\alpha p < \beta$. Then, if we move the last term of (19) — which is finite by (20) — to the left-hand side, and then use the first inequality of (20), we obtain that

$$\int_{3B_0} |u_j|^p {d_\Omega}^{\beta-p} \leq 2^{p+1} C_1 \int_{\frac{3}{2}B} |\nabla\varphi|^p {d_\Omega}^{-\alpha p+\beta} \leq Cr^{n-p-\alpha p+\beta} \tag{21}$$

for every $j \in \mathbb{N}$, since $|\nabla\varphi| \leq (r/2)^{-1}$ and ${d_\Omega}^{-\alpha p+\beta} \leq (\frac{3}{2}r)^{-\alpha p+\beta}$ in $\frac{3}{2}B$. As $|3B_0 \cap \Omega^c| = 0$ (since $\beta \leq p$ and (17) holds), it follows that $u_j(x) \to \varphi(x) d_\Omega(x)^{-\alpha}$ for a.e. $x \in 3B_0$. Moreover, the sequence (u_j) is monotone increasing, and so (21) yields that there exists $\delta_0 = \delta_0(C_0, p, n, \beta) > 0$ such that for all $\alpha p \leq \delta_0$ we have

$$\int_B {d_\Omega}^{-p+\beta-\alpha p} \leq \int_{3B_0} \varphi^p {d_\Omega}^{-\alpha p} {d_\Omega}^{\beta-p} \leq Cr^{n-p+\beta-\alpha p} \tag{22}$$

with a constant $C = C(C_0, p, \beta, n) > 0$; notice here that $\varphi \geq 1$ in B.

It is easy to see from the definition of the Aikawa dimension that, when calculating $\dim_{\mathcal{A}}(B_0 \cap \Omega^c)$, it suffices to consider in (2) only balls $\overline{B}(w, r) \subset 2B_0$ with $w \in B_0 \cap \Omega^c$. Hence we conclude from (22) that $\dim_{\mathcal{A}}(B_0 \cap \Omega^c) \leq n - p + \beta - \delta_0$. $\square$

Remark. If the (q, β)-Hardy inequality holds for all $p_1 < q < p_2$, with a constant C_1, we can take $\delta > 0$ in Lemma 5.2 to be independent of q; more precisely, then $\delta = \delta(p_1, p_2, \beta, n, C_1) > 0$.

We are now ready to prove our main dichotomy result for domains which admit the (p, β)-Hardy inequality. Here we need to assume that $\beta \neq p$, as the result need not hold for the (p, p)-Hardy inequality, see the example of Section 6.1.

Theorem 5.3. *Let $1 < p < \infty$, $\beta \neq p$, and assume that Ω admits the (p, β)-Hardy inequality with a constant $C_0 > 0$. Then there exists $\varepsilon_0 = \varepsilon_0(C_0, p, \beta, n) > 0$ such that for each ball $B_0 \subset \mathbb{R}^n$ either*

$$\dim_{\mathcal{H}}(4B_0 \cap \Omega^c) > n - p + \beta + \varepsilon_0$$

or

$$\dim_{\mathcal{A}}(B_0 \cap \Omega^c) < n - p + \beta - \varepsilon_0.$$

Proof. If $\beta > p$, it is clear that the upper bound for the Aikawa dimension holds with $\varepsilon_0 = (\beta - p)/2$, so we only need to consider the case $\beta < p$. First of all, we may assume that $\beta \geq 1$. Indeed, if this is not the case, we have, by [13, Thm. 3] (essentially Hölder's inequality), that Ω admits the $(p - \beta + 1, 1)$-Hardy inequality with a constant $C_0' = C_0'(C_0, p, \beta) > 0$, and now we may consider this instead of the (p, β)-Hardy inequality.

Using a self-improving property of Hardy inequalities (cf. [13]), we find $\varepsilon_1 = \varepsilon_1(C_0, p, \beta, n) > 0$ and $C_1 = C_1(C_0, p, \beta, n) > 0$ such that Ω admits the

(q,β)-Hardy inequality for all $p-\varepsilon_1 < q \le p$, and the constant in all these inequalities can be taken to be C_1. We require in addition that $\varepsilon_1 \le p-\beta$.

Let then $0<\varepsilon<\varepsilon_1/2$ to be specified later. If

$$\dim_{\mathcal{H}}(4B_0 \cap \Omega^c) > n-p+\beta+\varepsilon,$$

the claim is true. We may hence assume that

$$\dim_{\mathcal{H}}(4B_0 \cap \Omega^c) \le n-p+\beta+\varepsilon,$$

and thus

$$\mathcal{H}^{\lambda}_{\infty}(4B_0 \cap \Omega^c) = 0 \qquad \text{for } \lambda = n-p+\beta+2\varepsilon.$$

As Ω now admits the $(p-2\varepsilon,\beta)$-Hardy inequality and $p-2\varepsilon > p-\varepsilon_1 \ge \beta$, we may use Lemma 5.2 to conclude that there exists $\delta = \delta(C_0,p,\beta,n,\varepsilon_1) > 0$, independent of the particular choice of $\varepsilon < \varepsilon_1/2$ (cf. the remark after Lemma 5.2), such that

$$\dim_{\mathcal{A}}(B_0 \cap \Omega^c) \le n-p+\beta+2\varepsilon-\delta.$$

If we now choose $\varepsilon < \min\{\varepsilon_1/2, \delta/3\}$ the claim follows. □

Theorem 5.3 proves the local dichotomy, but the global dichotomy of Theorem 1.1 follows along the same lines; notice however the subtle difference between the global and local Aikawa dimension, which is present e.g. in the domain of Example 6.5. Nevertheless, if Ω admits the (p,β)-Hardy inequality for $\beta<p$, and if in addition $\dim_{\mathcal{H}}(\Omega^c) \le n-p+\beta+\varepsilon$, we obtain, as above, from Lemma 5.2 that

$$\int_B d_\Omega(x)^{-p+2\varepsilon+\beta-\delta}\,dx \le C r^{n-p+2\varepsilon+\beta-\delta}$$

for *any* ball $B=B(w,r)$ with $w\in\Omega^c$ and $r>0$, where C and δ are independent of B and the particular ε. Choosing ε as in the proof of Theorem 5.3 leads to the global dichotomy result.

6. Examples

6.1. **Dichotomy for the (p,p)-Hardy?** In Theorem 5.3 we assume that a domain $\Omega\subset\mathbb{R}^n$ admits the (p,β)-Hardy inequality, with the requirement that $\beta\ne p$. Here we give an easy example which shows that this restriction is really needed.

Let $\Omega=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : x_n>0\}$. It is then easy to see that Ω admits the (p,β)-Hardy if $1<p<\infty$ and $\beta\ne p-1$; in the case $\beta<p-1$ even the pointwise inequality holds by the results of [10], and in the case $\beta>p-1$ we can apply Theorem 1.3, as Ω is an unbounded John domain with $\dim_{\mathcal{A}}(\partial\Omega)=n-1<n-p+\beta$. In particular, Ω admits the (p,p)-Hardy for each $1<p<\infty$. However, $\dim_{\mathcal{H}}(\Omega^c)=\dim_{\mathcal{A}}(\Omega^c)=n$, so there does not exist $\varepsilon_0>0$ such that $\dim_{\mathcal{H}}(\Omega^c)>n-p+\beta+\varepsilon_0$ or $\dim_{\mathcal{A}}(\Omega^c)<n-p+\beta-\varepsilon_0$ when $\beta=p$, and so the conclusion of Theorem 5.3 indeed fails in this case.

6.2. **Easy sequence.** Let $E = \{(j^{-1}, 0, \dots, 0) : j \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}^n$, and denote $\Omega = \mathbb{R}^n \setminus E$. Let us first show that $\dim_{\mathcal{A}}(E) = 1$; recall that $\dim_{\mathcal{H}}(E) = 0$ and $\dim_{\mathcal{M}}(E) = 1/2$. Since $E \subset [0,1]$ and clearly $\dim_{\mathcal{A}}([0,1]) = 1$, it suffices to show that $\dim_{\mathcal{A}}(E) \geq 1$. To this end, take $x_j = (j^{-1}, 0, \dots, 0)$ and $r_j = (2j^2)^{-1}$, and denote $B_j = B(x_j, r_j)$ for $j \in \mathbb{N}$. Then the balls B_j are pairwise disjoint, and if $R = 1/j_0$ and $0 < s < 1$, we have that

$$\begin{aligned} \int_{B(0,R)} d(x,E)^{s-n}\,dx &\geq \sum_{j=j_0+1}^{\infty} \int_{B_j} d(x,E)^{s-n}\,dx \\ &\approx \sum_{j=j_0+1}^{\infty} \int_0^{r_j} r^{s-1}\,dx \approx \sum_{j=j_0+1}^{\infty} j^{-2s}. \end{aligned} \tag{23}$$

The last sum in (23) diverges if $2s \leq 1$; this is actually an indication of the fact that $\dim_{\mathcal{M}}(E) = 1/2$. Thus, we may assume that $s > 1/2$, whence

$$\sum_{j=j_0+1}^{\infty} j^{-2s} \approx (j_0+1)^{-2s+1} \gtrsim R^s\, R^{s-1}.$$

If we now let $j_0 \to \infty$ (i.e. $R \to 0$), it follows from the above calculations and the fact $s - 1 < 0$ that

$$R^{-s} \int_{B(0,R)} d(x,E)^{s-n}\,dx \longrightarrow \infty.$$

This shows that $s \notin G(E)$ (cf. Section 2.2), and so $\dim_{\mathcal{A}}(E) \geq 1$.

Now, by Theorem 5.3, the domain $\Omega = \mathbb{R}^n \setminus E$ can not admit the (p,β)-Hardy inequality if

$$0 = \dim_{\mathcal{H}}(E) \leq n - p + \beta \leq \dim_{\mathcal{A}}(E) = 1,$$

i.e. if $p - n \leq \beta \leq p - n + 1$. But, as was already mentioned, by the results from [10], every proper subdomain of $\mathbb{R}^n$ admits the (p,β)-Hardy if $\beta < p - n$, and on the other hand it is easy to see that Ω is an unbounded John-domain, and so Theorem 1.3 yields that Ω admits the (p,β)-Hardy if $1 = \dim_{\mathcal{A}}(E) < n - p + \beta$, i.e. $\beta > p - n + 1$. We conclude that Ω admits the (p,β)-Hardy inequality, for $1 < p < \infty$, *if and only if* $\beta < p - n$ or $\beta > p - n + 1$.

6.3. **Square with a twist.** For $1 < \lambda < 2$ we let $K_\lambda \subset \mathbb{R}^2$ denote the usual λ-dimensional von Koch -snowflake curve joining points $(0,0)$ and $(1,0)$ (cf. [9, Section 2]), but reflected so that K_λ is contained in the lower (closed) half-plane $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R},\ y \leq 0\}$. We first consider the bounded domain $\Omega = \Omega(\lambda)$, whose boundary consists of K_λ and the line segments $[0,i]$, $[i, 1+i]$, and $[1, 1+i]$ (in complex notation). Let $1 < p < \infty$. It is easy to see, using the results from [10], that Ω admits the pointwise (p,β)-Hardy inequality for all $\beta < p - 1$, but even the usual $(p, p-1)$-Hardy fails in Ω. Also, if we let Ω_g denote the part of Ω which is below the line segments $[0, (1+i)/2]$ and $[(1+i)/2, 1]$, it follows, by pointwise (p,β)-Hardy inequalities for $x \in \Omega_g$, that the pair (Ω_g, Ω) admits the (p,β)-Hardy for all $\beta < p - 2 + \lambda$. But then again, if $\Omega_b = \Omega \setminus \Omega_g$, we have that $\dim_{\mathcal{A}}(\partial\Omega_b \cap \partial\Omega) = 1$ and Ω_b satisfies clearly the assumptions of Theorem

3.1 when $p-1<\beta<p-2+\lambda$ (take x_0 to be the center of the unit square), and so the pair (Ω_b,Ω) admits the (p,β)-Hardy inequality for these β. Combining the above facts we conclude that Ω admits the (p,β)-Hardy inequality for all $\beta<p-2+\lambda$, with the exception of $\beta=p-1$.

Now let us modify the domain Ω in the following way: For $j\in\mathbb{N}$ we denote $Q_j=1-2^{j-1}+i+2^{-j}I^2$, where $I^2=[0,1]^2$ is the unit square, so that $\{Q_j\}_j$ consists of squares attached to the top of Ω. Let $s\geq 1$. We open a passage of width $\delta_j\approx 2^{-js}$ from Ω to Q_j, for each $j\in\mathbb{N}$, in order to obtain a new domain, denoted Ω^s. Let Ω_g be as above and $\Omega_b^s=\Omega^s\setminus\Omega_g$. Then direct calculations yield that $\dim_{\mathcal{A}}\big(\partial\Omega_b^s\cap\partial\Omega^s\big)=1$. However, it is easy to see that if $s>1$, then the accessibility assumption of Theorem 3.1 fails to hold in Ω_b^s, and, indeed, we show that the (p,β)-Hardy fails to hold in Ω^s whenever $\beta\geq p-1$ if $s>1$. To see this, it suffices to consider Lipschitz functions u_j such that $\operatorname{spt}(u_j)\subset Q_j$ and

$$u_j(x)=\min\big\{1,\max\big\{0,2^{js}d(x,\partial Q_j)-1\big\}\big\}$$

for $x\in Q_j$. Then, for j so large that $2^{-js+1}<2^{-j-1}$, we have

$$\int_{\Omega^s}|u_j|^p d_\Omega{}^{\beta-p}\gtrsim\int_{\frac12 Q_j} d_\Omega{}^{\beta-p}\approx 2^{-j(2-p+\beta)};$$

here $\frac12 Q_j$ is the cube with the same center as Q_j but $\operatorname{diam}\big(\frac12 Q_j\big)=\frac12\operatorname{diam}Q_j$. Now $\big|\operatorname{spt}(|\nabla u_j|)\big|\approx 2^{-js}2^{-j}$, and $|\nabla u_j(x)|^p d_\Omega(x)^\beta\lesssim 2^{jsp}2^{-j\beta}$ for a.e. $x\in\operatorname{spt}(|\nabla u_j|)$, since $\beta>p-1>0$. Thus

$$\int_\Omega|\nabla u_j|^p d_\Omega{}^\beta\lesssim 2^{-js(1-p+\beta)}2^{-j},$$

and because $(1-p+\beta)(1-s)<0$, it follows that

$$\frac{\int_{\Omega^s}|u_j|^p d_\Omega{}^{\beta-p}}{\int_{\Omega^s}|\nabla u_j|^p d_\Omega{}^{\beta}}\gtrsim\frac{2^{-j(2-p+\beta)}}{2^{-js(1-p+\beta)}2^{-j}}=2^{-j(1-p+\beta)(1-s)}\xrightarrow{j\to\infty}\infty.$$

This, together with the easy observation that the $(p,p-1)$-Hardy fails in Ω^s as well, shows that Ω^s, for $s>1$, admits the (p,β)-Hardy inequality if and only if $\beta<p-1$. Notice however that for $s=1$ the assumptions of Theorem 3.1 are satisfied when $p-1<\beta<p-2+\lambda$, and so Ω^1 admits the (p,β)-Hardy also for these β. This shows that the accessibility assumption of Theorem 3.1 is essential for weighted Hardy inequalities.

If we only consider the union of cubes Q_j and open the passages of width $\delta_j\approx 2^{-js}$ just as above, we obtain unbounded domains $\widetilde{\Omega}^s$ with $\dim_{\mathcal{A}}(\widetilde{\Omega}^s)=1$, which fail to admit the (p,β)-Hardy for $\beta\geq p-1$ if $s>1$. This shows that we can not remove the assumption that Ω is John from Theorem 1.3.

6.4. **Snowflake line.** As a positive example of the use of our results for unbounded domains we present the following construction: For $1<\lambda<2$ we let $K_\lambda\subset\mathbb{R}^2$ denote again the usual λ-dimensional von Koch -snowflake curve joining points $(0,0)$ and $(1,0)$. Then we set $\mathcal{K}_\lambda=\bigcup_{j\in\mathbb{Z}}\big(K_\lambda+(j,0)\big)$ and let $\Omega=\Omega(\lambda)$ be the $(0,1)$-component of $\mathbb{R}^2\setminus\mathcal{K}_\lambda$. Now fix $1<p<\infty$. Then, by the results in [10] and [14], Ω admits the pointwise (p,β)-Hardy if and only if $\beta<p-1$. We now claim that the usual (p,β)-Hardy also holds in Ω when (i) $p-1<\beta<p-2+\lambda$ or (ii) $\beta>p-2+\lambda$.

In the case (i) we take $\Omega_1 = \Omega \cap \{(x_1, x_2) : 0 < x_2 \le 2\}$. Then, since $\beta < p - 2 + \lambda$, there exists $C > 0$ such that the pointwise (p,β)-Hardy inequality holds at every $x \in \Omega_1$ with this constant C, and so the pair (Ω_1, Ω) admits the (p,β)-Hardy. But it is also easy to show that $\Omega_2 = \Omega \setminus \Omega_1$ is an unbounded c-John subset of Ω, as defined in Section 4, satisfying the uniform cube-count condition (15) with the exponent $\lambda = 1$, and so Theorem 4.3 yields that the pair (Ω_2, Ω) admits the (p,β)-Hardy for all $\beta > p - 2 + 1 = p - 1$, and claim (i) follows.

On the other hand, in the case (ii) we observe that Ω is an unbounded John domain with $\dim_{\mathcal{A}}(\partial\Omega) = \lambda$. Hence, by Theorem 1.3, we conclude that Ω admits the (p,β)-Hardy whenever $\lambda < 2 - p + \beta$, i.e. $\beta > p - 2 + \lambda$.

Finally, it is quite easy to see that, for a fixed $1 < p < \infty$, both $(p, p-1)$- and $(p, p - n + \lambda)$-Hardy inequalities fail in the domain Ω.

6.5. **Local and global Aikawa dimensions.** Let $E = \{(k, 0, \dots, 0) : k \in \mathbb{Z}\} \subset \mathbb{R}^n$. Then, obviously, $\dim_{\mathcal{H}}(E) = 0$, and also $\dim_{\mathcal{A}}(E \cap B) = 0$ for any ball $B \subset \mathbb{R}^n$, but still $\dim_{\mathcal{A}}(E) = 1$. These facts have at least two implications which are of interest to us: (i) by Theorem 1.1, $\Omega = \mathbb{R}^n \setminus E$ does not admit the (p,β)-Hardy inequality when $p - n \le \beta \le p - n + 1$, and (ii) there is a difference between the local and global Aikawa dimensions.

Let us verify the claims concerning the Aikawa dimension. Let B be a ball of radius $R > 0$ and take $K \in \mathbb{N}$ so that $R \in [K-1, K]$, and let $w \in E$ and $r > 0$ be such that $B(w, r) \subset 2B$; we may in fact assume that $w = 0$. Then, if we denote $B_k = B\big((k, 0, \dots, 0), r\big)$, we have for each $s > 0$ that

$$\begin{aligned}\int_{B(0,r)} d(x,E)^{s-n}\,dx &\le \sum_{|k|\le r} \int_{B_k} d(x,E)^{s-n}\,dx \\ &\le 4K \int_{B(0,r)} t^{s-n}\,dt \le C(R,s,n) r^s,\end{aligned}$$

and so $\dim_{\mathcal{A}}(E \cap B) = 0$.

On the other hand, it is clear that $\dim_{\mathcal{A}}(E) \le 1$. To see that $\dim_{\mathcal{A}}(E) \ge 1$, we denote $C_k = [-k, k] \times B^{n-1}(0,k)$, where $B^{n-1}(0,k) \subset \mathbb{R}^{n-1}$, so that $C_k \subset B(0, k\sqrt{n})$. If $s < 1$, we calculate, using Fubini's theorem, polar coordinates, and the fact that for $(t, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with $|y| \ge 1$ we have $d\big((t,y), E\big) \le 2|y|$, that

$$\begin{aligned}\int_{B(0,k\sqrt{n})} d(x,E)^{s-n}\,dx &\ge \int_{C_k} d(x,E)^{s-n}\,dx \\ &= \int_{-k}^{k} \int_{B^{n-1}(0,k)} d\big((t,y),E\big)^{s-n}\,dy\,dt \gtrsim \int_{-k}^{k}\int_1^k r^{s-n} r^{n-2}\,dr\,dt \\ &\gtrsim k \int_1^k r^{s-2}\,dr \gtrsim k(1 - k^{s-1}) = (k - k^s),\end{aligned}$$

where the constants involved depend only on n and s. Thus

$$(k\sqrt{n})^{-s} \int_{B(0,k\sqrt{n})} d(x,E)^{s-n}\,dx \gtrsim (k^{1-s} - 1) \xrightarrow{k\to\infty} \infty,$$

showing that $\dim_{\mathcal{A}}(E) \ge 1$.

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