LARGE POROSITY AND DIMENSION OF SETS IN METRIC SPACES

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Abstract. A. Salli proved in [14] an asymptotically sharp dimension estimate for sets in $\mathbb{R}^n$ with large porosity. We generalize this result to a collection of metric spaces. From the metric space we assume, among other properties, that it can be locally mapped into $\mathbb{R}^n$ in a way that allows us to use Euclidean projections. We show that $\mathbb{R}^n$ with any norm satisfies these conditions as well as every Heisenberg type group. We also discuss the necessity of the conditions by examining various metric spaces where the estimates fail.

1. Introduction

Lower-porous sets have holes of certain relative size in all small enough scales. They differ from upper-porous sets, which have holes only in some sequences of scales. For the dimension of lower-porous sets we can find upper bounds that depend on the porosity. These cannot be found for upper-porous sets. This can be seen by constructing a maximally upper-porous set in $\mathbb{R}^n$ that has dimension n (see [12, §4.12]). In this paper we will work only with lower-porosity and therefore every time we speak of porosity we mean lower-porosity.

When one looks at the dimension results for large porosity one notices that there are essentially two different methods utilized to prove them. The use of conical densities is the first one and it was introduced by P. Mattila in [11] where he proved that when a set in $\mathbb{R}^n$ has porosity close to its maximum the dimension of the set cannot be much bigger than $n-1$. This result was later improved by A. Salli in [14]. He used the fact that a uniformly porous set in $\mathbb{R}^n$ lives locally closer and closer to some convex sets when porosity tends to its maximum value. We call this the second method. It gave the dimension estimate

$$\dim_{\mathrm{p}} A \leq n - 1 + \frac{C}{\log(\frac{1}{1-2\varrho})} \tag{1.1}$$

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for ϱ-porous sets $A \subset \mathbb{R}^n$ with a constant C depending only on n. Here $\dim_{\mathrm{p}}$ is the packing dimension.

The idea of using conical densities was reused by A. Käenmäki and V. Suomala in [10] to prove that a k-porous set in $\mathbb{R}^n$ having k-porosity close to $\frac{1}{2}$ must have dimension at most close to $n - k$. By k-porosity we mean that there are holes in k orthogonal directions in reference balls. Again this result was improved using the second method in [8] by E. Järvenpää, M. Järvenpää, A. Käenmäki and V. Suomala.

For porous measures the second method has been used extensively. First time it was used indirectly in [3] where J.-P. Eckmann, E. Järvenpää and M. Järvenpää defined porosity of measures and proved a result similar to (1.1) for doubling measures. They showed that the porosity of a doubling measure μ can be obtained in terms of porosities of sets that have positive μ-measure. Recalling the result of Salli then gives the dimension estimate. E. Järvenpää and M. Järvenpää gave a proof for a similar estimate for the packing dimension of general measures in [5], but the proof actually works only for Hausdorff dimension as explained in [6]. Again the second method was used in the form of first proving that large part of the mass of any porous measure inside a small dyadic cube is near some convex polyhedron.

D. Beliaev and S. Smirnov stated a proof for a similar result for general mean porous measures in $\mathbb{R}^n$ in [2]. In mean porosity we require holes to appear only in some percentage of (for example) dyadic scales. They tried to estimate mean porous measures by mean porous sets, which is in fact not possible as is shown by an example in [1]. Although the proof for mean porous measures was flawed, the paper generalized the result of Salli to mean porous sets.

The estimate for the packing dimension of mean porous measures was finally proven by D. Beliaev, E. Järvenpää, M. Järvenpää, A. Käenmäki, T. Rajala, S. Smirnov and V. Suomala in [1]. Also here the second method was used. The first method was recently used also for porous measures by A. Käenmäki and V. Suomala in [9].

The purpose of this paper is to indicate that the second method can be used in more general settings to estimate the dimension of porous sets in certain metric spaces. We prove that the estimate of Salli holds in normed vector spaces and Heisenberg type groups. The idea in the proof is to use Euclidean projections to a set of directions to move a cover of a porous set to hyperplanes of $\mathbb{R}^n$.

In Section 2 we introduce the notion of porosity and state our theorem and some of its corollaries. Section 3 will deal with porosity in normed vector spaces and Section 4 in Heisenberg type groups. In Section 5 we prove our main theorem and in the last section, Section 6, we give examples illustrating that the dimension results for large porosity do not generalize to geodesic regular metric spaces nor to bi-Lipschitz images of $\mathbb{R}^n$.

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2. Porosity in Metric Spaces

Let (X,d) be a metric space. First we note that $B_{(X,d)}(x,r)$ is a closed ball in X centred at x with radius r. If we are using only one metric d in our space, we may also write $B_X(x,r)$. By S^{n-1} we mean the unit sphere in $\mathbb{R}^n$. Let us now go through some relevant definitions. Following the convention introduced in [13], we define for a set $A \subset X$, a point $x \in X$ and a radius $r > 0$

$$\begin{aligned} \operatorname{por}(A,x,r) = \sup\{\varrho \geq 0 :& \text{there is } y \in X \text{ such that } B_X(y,\varrho r) \cap A = \emptyset \\ & \text{and } \varrho r + d(x,y) \leq r\}. \end{aligned} \tag{2.1}$$

The *porosity of A at a point x* is defined to be

$$\operatorname{por}(A,x) = \liminf_{r \downarrow 0} \operatorname{por}(A,x,r) \tag{2.2}$$

and the *porosity of A* is given by

$$\operatorname{por}(A) = \inf_{x \in A} \operatorname{por}(A,x). \tag{2.3}$$

We call $A \subset X$ *porous* if $\operatorname{por}(A) > 0$, and more precisely, *$\varrho$-porous* provided that $\operatorname{por}(A) > \varrho$. From (2.1) we see that there can be only ϱ-porous sets with $\varrho < \frac{1}{2}$. We call a set A maximally porous, if $\operatorname{por}(A) = \frac{1}{2}$.

As in [12, §5.3], we define for a bounded set $A \subset X$, $\lambda \geq 0$ and $r > 0$

$$M^\lambda(A,r) = \inf\{kr^\lambda : A \subset \bigcup_{i=1}^{k} B_X(x_i,r) \text{ for some } x_i \in X \text{ and } k \in \mathbb{N}\}$$

with the interpretation $\inf \emptyset = \infty$. The (upper) Minkowski dimension of a bounded set A is

$$\dim_{\mathrm{M}}(A) = \inf\{\lambda : \limsup_{r \downarrow 0} M^\lambda(A,r) < \infty\}.$$

The packing dimension of $A \subset X$ is given by

$$\dim_{\mathrm{p}}(A) = \inf\bigl\{\sup_i \dim_{\mathrm{M}}(A_i) \ : \ A_i \text{ is bounded and } A \subset \bigcup_{i=1}^{\infty} A_i\bigr\}.$$

We use the notation $\mathcal{H}^d$ for the d-dimensional Hausdorff measure and $\dim_{\mathrm{H}}$ for the Hausdorff dimension, see [12] for the definitions. Recall that for all sets $A \subset X$ we have

$$\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{p}}(A).$$

Next we fix some notation in $\mathbb{R}^n$. We denote the convex hull of $E \subset \mathbb{R}^n$ by $\operatorname{conv}(E)$ and the boundary of E by $\partial(E)$. Let $x \in \mathbb{R}^n$, $v \in S^{n-1}$ and $\alpha \in]0,\pi[$. With these parameters we define a cone

$$C(x,v,\alpha) = \{y \in \mathbb{R}^n \mid d_E(y, L(x,v)) \leq \sin(\alpha) d_E(x,y)\},$$

where

$$L(x,v) = \{x + tv \in \mathbb{R}^n \mid t \in [0,\infty[\}$$

and d_E is the Euclidean metric. The orthogonal complement of E is denoted by $E^\perp$ and the Euclidean inner product between vectors $x, y \in \mathbb{R}^n$ by $(x|y)$.

Let (X,d) be a metric space. The following definition gives the maximum amount of disjoint balls of radius R in X such that the centres of the balls can be mapped for fixed $y \in \mathbb{R}^n$ and $R > 0$ into $B_{\mathbb{R}^n}(y,R)$ with a map $f : Y \to \mathbb{R}^n$, where $Y \subset X$. Define for every $R > 0$ and $y \in \mathbb{R}^n$

$$\begin{aligned} N(R,y,f) = \max\Big\{ m \mid x_1,\ldots,x_m \in Y \text{ such that } f(x_i) \in B_{\mathbb{R}^n}(y,R) \\ \text{and } B_X(x_i,R) \cap B_X(x_j,R) = \emptyset \text{ for } i \neq j \Big\}. \end{aligned}$$

Next we state our main theorem. After that the assumptions of the theorem are motivated by corollaries and the role of each assumption is clarified in a remark. More examples satisfying the assumptions will be given in the last section of the paper. There the dimension estimates derived from the Theorem 2.1 are not of the type (1.1).

Theorem 2.1. *Let (X,d) be a separable metric space. Assume that there are constants r_0, R_i, R_o, c, $t > 0$, $0 < s \leq 1$ and $n \in \mathbb{N}$ so that every $x \in X$ and $0 < r < r_0$ have the following properties: If $y, z \in B_X(x,r_0)$ and $d_X(y,z) = r$, then for every $\epsilon \in]0,1[$*

$$B_X(z,(1-\epsilon)r) \cap B_X(y, c\epsilon^s r) \neq \emptyset. \tag{2.4}$$

There exists an injective map $f_{x,r} : B_X(x,4r) \to \mathbb{R}^n$ so that for all $0 < R < r$ and $y \in B_X(x,2r)$

$$B_{\mathbb{R}^n}(f_{x,r}(y), R_i r) \cap f_{x,r}(B_X(x,4r)) \subset f_{x,r}(B_X(y,r)), \tag{2.5}$$

$$f_{x,r}(B_X(y,R)) \subset B_{\mathbb{R}^n}(f_{x,r}(y), R_o R) \tag{2.6}$$

and

$$\begin{aligned} \operatorname{conv}\big(f_{x,r}(B_X(y,r)) \cup B_{\mathbb{R}^n}(f_{x,r}(y), R_i r)\big) \cap f_{x,r}(B_X(x,4r)) \\ = f_{x,r}(B_X(y,r)). \end{aligned} \tag{2.7}$$

Assume for every $y \in \mathbb{R}^n$ and $0 < R < r$

$$N(R,y,f_{x,r}) \leq c\Big(\frac{r}{R}\Big)^{t-n}. \tag{2.8}$$

Then for any ϱ-porous subset $A \subset X$ we have

$$\dim_{\mathrm{p}} A \leq t - 1 + \frac{C}{\log(\frac{1}{1-2\varrho})}, \tag{2.9}$$

where the constant C depends on n, R_i, R_o, s, t and c.

Remark 2.2. Assuming separability is natural when we want to get dimension estimates. Assumption (2.4) guarantees that the porous set lives in a suitable neighbourhood of the holes. Assumptions (2.5), (2.6) and (2.7) allow us to use Euclidean projections when finding a cover for the porous set.

The first inclusion (2.5) says that there is a Euclidean ball with radius $R_i r$ inside an image of a ball of radius r. We take the intersection with the whole image here to allow the maps $f_{x,r}$ to have, for example, holes inside their images. To estimate to the other direction we assume (2.6), which says that the images of small balls are included in a slightly bigger Euclidean balls.

According to the equality (2.7), the images of balls of radius r are relatively convex with respect to the whole image. Moreover, taking the union with the Euclidean ball of radius $R_i r$ guarantees the existence of big enough cones inside the images of the balls, see inclusion (5.1). Here we again have the intersection with the whole image for the same reason as in (2.5). Growth bound (2.8) gives an estimate on the relative change of the number of balls needed for a cover when we move from $\mathbb{R}^n$ to X.

As the first corollary we have a generalization of the estimate (1.1) to normed vector spaces.

Corollary 2.3. *Let $||\cdot||$ be a norm in $\mathbb{R}^n$. Then for every ϱ-porous subset $A \subset \mathbb{R}^n$ we have a dimension estimate*

$$\dim_{\mathrm{p}} A \leq n - 1 + \frac{C}{\log(\frac{1}{1-2\varrho})},$$

where the constant C depends only on n.

The second corollary shows that with the functions $f_{x,r}$ in Theorem 2.1 we can prove estimate (1.1) in $\mathbb{R}^n$ with modified group structures. In particular, we prove the estimate in Heisenberg groups.

Corollary 2.4. *Let $G = \mathbb{R}^n \times \mathbb{R}^m$ be a Heisenberg type group with S as its bilinear form. Then for every ϱ-porous subset $A \subset G$ we have*

$$\dim_{\mathrm{p}} A \leq n + 2m - 1 + \frac{C}{\log(\frac{1}{1-2\varrho})},$$

where the constant C depends only on n, m and S.

We will prove these corollaries in detail in the next two sections of the paper. Note that in the first corollary the constant C depends only on the dimension of the space and not on the norm $||\cdot||$. We prove the following third corollary of the Theorem 2.1 here.

Corollary 2.5. *Let (X,d) be a geodesic metric space. Assume that X is bi-Lipschitz equivalent to $\mathbb{R}^n$ and that the images of balls under the bi-Lipschitz mapping f are convex. Then for all ϱ-porous subsets $A \subset X$ we have*

$$\dim_{\mathrm{p}} A \leq n - 1 + \frac{C}{\log(\frac{1}{1-2\varrho})},$$

where the constant C depends only on n and the bi-Lipschitz constant of f.

Proof. Let us check that the assumptions of the Theorem 2.1 are satisfied. The space (X,d) is clearly separable and because of geodesity the condition (2.4) holds with $c = 1$ and $s = 1$. As $f_{x,r}$ we can take the restrictions of the bi-Lipschitz map f. Let L be the bi-Lipschitz constant of f. Then the assumption (2.5) is satisfied with $R_i = \frac{1}{L}$ and the assumption (2.6) with $R_o = L$. Assuming convexity of the images of the balls in X under f guarantees that the condition (2.7) holds. A simple volume comparison argument gives condition (2.8) with $t = n$ and c depending on n and L. □

3. Porosity in Normed Vector Spaces

Before any investigation is done on porous sets with different norms it is natural to ask if different norms give different porosity on sets. This is indeed the case as easily seen for example by looking at $(\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \subset \mathbb{R}^2$ which is maximally porous in maximum norm, but not in the Euclidean one.

Because in this section we use different norms let us denote the Euclidean one by $||\cdot||_E$. Let then

$$B(x,r) = \{y \in \mathbb{R}^n \,:\, ||y-x|| \leq r\}$$

and

$$B_E(x,r) = \{y \in \mathbb{R}^n \,:\, ||y-x||_E \leq r\}$$

be the closed balls in $\mathbb{R}^n$.

For a given norm $||\cdot||$ in $\mathbb{R}^n$ and a subspace $V \subset \mathbb{R}^n$ we define the outer radius

$$R_{o,||\cdot||}(V) = \min\{R > 0 \,:\, B(0,1) \cap V \subset B_E(0,R)\}$$

and the inner radius

$$R_{i,||\cdot||}(V) = \max\{R > 0 \,:\, B_E(0,R) \cap V \subset B(0,1) \cap V\}.$$

Clearly $0 < R_{i,||\cdot||}(V) \leq R_{o,||\cdot||}(V) < \infty$ and

$$R_{i,||\cdot||}(\mathbb{R}^n)||\cdot|| \leq ||\cdot||_E \leq R_{o,||\cdot||}(\mathbb{R}^n)||\cdot||.$$

These radii have similar nature as the radii R_o and R_i in the assumptions of Theorem 2.1.

Remark 3.1. With every norm $||\cdot||$ in $\mathbb{R}^n$ all the balls are convex: Let $z, y \in B(x,r)$ and $t \in [0,1]$. Then

$$\begin{aligned}||ty + (1-t)z - x|| &\le ||ty - tx|| + ||(1-t)z - (1-t)x|| = \\ &= t||y-x|| + (1-t)||z-x|| \le r.\end{aligned}$$

Proving Corollary 2.3 with a constant depending also on the norm is very easy. The independence of the norm comes from shrinking and stretching the space.

Proof of Corollary 2.3. We construct the function $f_{x,r}$, independently of x and r, so that it shrinks the original norm $||\cdot||$ in $n-1$ orthogonal directions. Let us first choose the directions $u_1, \ldots, u_{n-1}$. Let u_1 be such a vector that $||u_1|| = 1$ and $||u_1||_E = R_{o,||\cdot||}(\mathbb{R}^n)$. Next take $u_2 \in \{u_1\}^\perp$ so that $||u_2|| = 1$ and $||u_2||_E = R_{o,||\cdot||}(\{u_1\}^\perp)$. We continue choosing rest of the vectors inductively, that is, $u_k \in \{u_1, \ldots, u_{k-1}\}^\perp$ so that $||u_k|| = 1$ and $||u_k||_E = R_{o,||\cdot||}(\{u_1, \ldots, u_{k-1}\}^\perp)$ for all $k = 2, \ldots, n-1$.

Next we start modifying the norm in a reversed order. In the u_{n-1}-direction shrink the norm first by $\frac{R_{i,||\cdot||}(\{u_1,\ldots,u_{n-1}\}^\perp)}{R_{o,||\cdot||}(\{u_1,\ldots,u_{n-2}\}^\perp)}$. The first shrinking gives a norm $||\cdot||_1$. By shrinking a norm $||\cdot||$ by a constant t in the direction of v we mean the following: as the result of shrinking we get a norm $||\cdot||_1$, defined as

$$||x||_1 = ||y + \frac{z}{t}||,$$

where $x = y + z$ with $z \in \{\eta v : \eta \in \mathbb{R}\}$ and $y \in \{v\}^\perp$. Next shrink the norm $||\cdot||_1$ in u_{n-2}-direction by $\frac{R_{i,||\cdot||_1}(\{u_1,\ldots,u_{n-2}\}^\perp)}{R_{o,||\cdot||_1}(\{u_1,\ldots,u_{n-3}\}^\perp)}$. This gives a norm $||\cdot||_2$. Continue the procedure and finally shrink the norm $||\cdot||_{n-2}$ in u_1-direction by $\frac{R_{i,||\cdot||_{n-2}}(\{u_1\}^\perp)}{R_{o,||\cdot||_{n-2}}(\mathbb{R}^n)}$. From the convexity of balls it then follows that we end up with a norm $||\cdot||_s$ for which we have

$$R_{i,||\cdot||_s}(\{u_1, \ldots, u_{k-1}\}^\perp) \ge \frac{R_{i,||\cdot||_s}(\{u_1, \ldots, u_k\}^\perp)}{\sqrt{2}}$$

and

$$R_{o,||\cdot||_s}(\{u_1, \ldots, u_{k-1}\}^\perp) \le \sqrt{5} R_{o,||\cdot||_s}(\{u_1, \ldots, u_k\}^\perp)$$

for all $k \in \{1, \ldots, n-1\}$. Look at Figure 1 for this geometric conclusion. Inside the darker area we can have a ball of radius $\frac{1}{\sqrt{2}} R_{i,||\cdot||_s}(\{u_1\}^\perp)$ and, on the other hand, when shrinking the light gray area it must be contained in a ball of radius $\sqrt{5} R_{o,||\cdot||_s}(\{u_1\}^\perp)$. Note that the constants are not sharp. The two inequalities together yield

$$\frac{R_{o,||\cdot||_s}(\{u_1, \ldots, u_{k-1}\}^\perp)}{R_{i,||\cdot||_s}(\{u_1, \ldots, u_{k-1}\}^\perp)} \le \sqrt{10} \frac{R_{o,||\cdot||_s}(\{u_1, \ldots, u_k\}^\perp)}{R_{i,||\cdot||_s}(\{u_1, \ldots, u_k\}^\perp)} \tag{3.1}$$

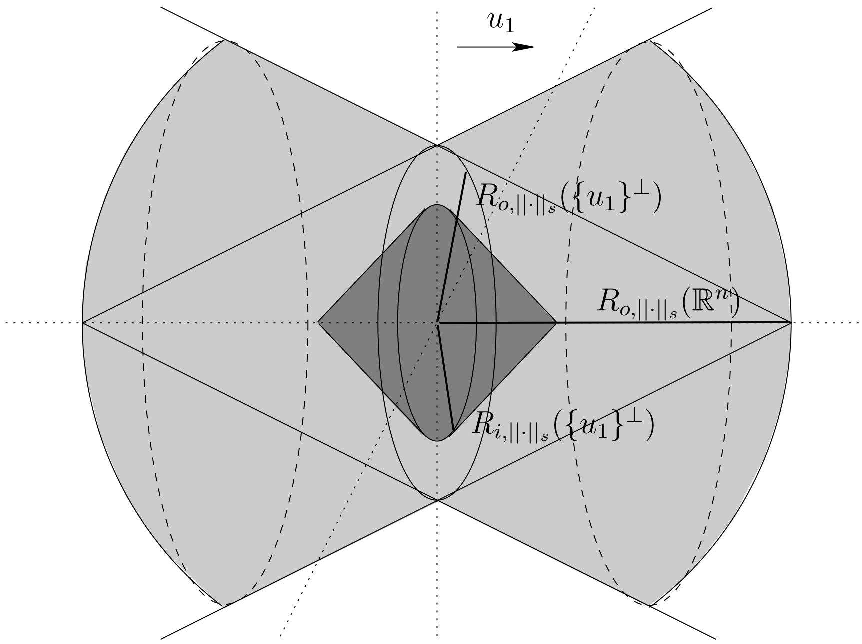

Figure 1: Shrinking in the direction of u_1 (right). By convexity the darker area must be inside the new ball and the original ball must be contained in the lighter area.

for all $k \in \{1, \ldots, n-1\}$. Next observe that

$$R_{i,||\cdot||_s}(\{u_1, \ldots, u_{n-1}\}^\perp) = R_{o,||\cdot||_s}(\{u_1, \ldots, u_{n-1}\}^\perp),$$

since $\{u_1, \ldots, u_{n-1}\}^\perp$ is a line. Finally combine this with (3.1) to get

$$\frac{R_{o,||\cdot||_s}(\mathbb{R}^n)}{R_{i,||\cdot||_s}(\mathbb{R}^n)} \leq (10)^{\frac{n-1}{2}}. \tag{3.2}$$

By shrinking the space the same way we shrank the norm we get an isometry between the original normed space and the new one. In particular, porosity does not change when moving from one space to the other nor does the dimensions of sets.

We choose $f_{x,r}$ to be the identity in the new normed space. Take $R_i = R_{i,||\cdot||_s}(\mathbb{R}^n)$ and $R_o = R_{o,||\cdot||_s}(\mathbb{R}^n)$. The condition (2.4) is satisfied with constants $c = s = 1$ because of the linear structure. By construction the assumptions (2.5) and (2.6) are satisfied. The assumption (2.7) was proven to hold in Remark 3.1. The condition (2.8) is satisfied with $t = n$ and c depending on n, R_o and R_i as seen by a volume comparison principle . By scaling the whole space so that $R_i = 1$ the inequality (3.2) gives an absolute estimate for the constant R_o and hence the constant C depends only on n. $\square$

4. Porosity in Heisenberg Type Groups

We will follow the definitions of [15] for a Heisenberg type group and assume that

$$G = \mathbb{R}^n \times \mathbb{R}^m$$

with a group law

$$(x, y) \circ (x', y') = (x + x', y + y' + S(x, x')),$$

where $S(x, x')$ is a skew-symmetric bilinear function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^m$ with integer coefficients when expressed in the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. With these assumptions G is said to be a Heisenberg type group. We will use one of the natural metrics on the group G which is given by

$$d_G(a, b) = [a^{-1} \circ b]$$

with

$$[c] = \max\{||z||_E, ||t||_E^{\frac{1}{2}}\} \text{ for } c = (z, t) \in G.$$

Clearly the Hausdorff dimension of the space G is $n+2m$. The (first) Heisenberg group is just $\mathbb{H}^1 = \mathbb{R}^2 \times \mathbb{R}^1$ with the bilinear form

$$S((x_1, x_2), (x_1', x_2')) = 2(x_1' x_2 - x_1 x_2').$$

Proof of Corollary 2.4. Recall that

$$d_G((a_1, a_2), (y_1, y_2)) = \max\{||y_1 - a_1||_E, ||y_2 - a_2 - S(a_1, y_1)||_E^{\frac{1}{2}}\}$$

for all $(a_1, a_2), (y_1, y_2) \in G$. Hence a ball centred at $(y_1, y_2) \in G$ looks like a diamond with sides of Euclidean balls. Define a constant

$$C(S) = \max\{||S(b, c)||_E \ : \ ||b||_E = ||c||_E = 1\}$$

and a mapping $f_{0,r} \ : \ B_G(0, 4r) \to \mathbb{R}^{n+m}$ by $f_{0,r}(y_1, y_2) = (y_1, \frac{y_2}{r})$. This mapping stretches the space in the direction where we use the square root metric so that balls with radius r look almost Euclidean. By translating we define $f_{x,r}(y) = f_{0,r}(x^{-1} \circ y)$ for every $x \in G$. Let us now check the assumptions of Theorem 2.1.

The assumption (2.4) is satisfied by taking $c = \sqrt{2}$ and $s = \frac{1}{2}$. This is due to the fact that the worst case is in the square root direction of the metric. To prove that (2.5) holds with the constant $R_i = \min\{\frac{1}{2}, \frac{1}{4C(S)}\}$, take $y \in B_G(0, 2r)$ and $z \in B_G(0, 4r)$ so that $d_{\mathbb{R}^{n+m}}(f_{0,r}(y), f_{0,r}(z)) \le R_i r$. Now

$$\begin{aligned} d_G(y, z) &= \max\{||y_1 - z_1||_E, ||y_2 - z_2 - S(y_1, z_1)||_E^{\frac{1}{2}}\} \\ &\le \max\left\{\frac{r}{2}, (||y_2 - z_2||_E + ||y_1||_E ||z_1 - y_1||_E C(S))^{\frac{1}{2}}\right\} \\ &\le \max\left\{\frac{r}{2}, \left(\frac{r^2}{2} + 2r\frac{r}{4C(S)}C(S)\right)^{\frac{1}{2}}\right\} \le r. \end{aligned}$$

Next we show that the assumption (2.6) holds with a constant $R_o = 2(C(S)+1)$. Taking $y \in B_G(0,2r)$ and $z \in B_G(0,4r)$ so that $d_G(y,z) < R < r$, we obtain

$$\begin{aligned} d_{\mathbb{R}^{n+m}}(f_{0,r}(y), f_{0,r}(z)) &\leq ||y_1 - z_1||_E + \frac{1}{r}||y_2 - z_2||_E \\ &\leq R + \frac{1}{r}\Big(R^2 + ||S(y_1,z_1)||_E\Big) \\ &\leq R + \frac{1}{r}\Big(R^2 + ||y_1||_E||y_1 - z_1||_E C(S)\Big) \\ &\leq R + \frac{1}{r}\Big(Rr + 2rRC(S)\Big) = R_o R. \end{aligned}$$

Because of the shape of the balls assumption (2.7) clearly holds. Finally, we confirm that the condition (2.8) holds with $t = 2m+n$ and $c = \frac{\mathcal{H}^{n+m}(B_{\mathbb{R}^{n+m}}(0,1))(R_o+1)^{n+m}}{\mathcal{H}^n(B_{\mathbb{R}^n}(0,1))\mathcal{H}^m(B_{\mathbb{R}^m}(0,1))}$. Take $y \in \mathbb{R}^{n+m}$, $0 < R < r$ and $x_1, \dots, x_k \in G$ so that $B_G(x_i,R) \cap B_G(x_j,R) = \emptyset$ when $i \neq j$ and $f_{0,r}(x_i) \in B_{\mathbb{R}^{n+m}}(y,R)$. Notice that the bilinear form S has no effect to the Euclidean Hausdorff measure of the balls which gives

$$\mathcal{H}^{n+m}(f_{0,r}(B_G(x_i,R))) = \mathcal{H}^n(B_{\mathbb{R}^n}(0,1))\mathcal{H}^m(B_{\mathbb{R}^m}(0,1))\frac{R^{n+2m}}{r^m},$$

and, on the other hand, because (2.6) holds we have

$$f_{0,r}(B_G(x_i,R)) \subset B_{\mathbb{R}^{n+m}}(f_{0,r}(x_i), R_o R) \subset B_{\mathbb{R}^{n+m}}(y,(R_o+1)R).$$

Comparing the volumes we get

$$k \leq \frac{\mathcal{H}^{n+m}(B_{\mathbb{R}^{n+m}}(0,1))(R_o+1)^{n+m}}{\mathcal{H}^n(B_{\mathbb{R}^n}(0,1))\mathcal{H}^m(B_{\mathbb{R}^m}(0,1))}\frac{R^{n+m}r^m}{R^{n+2m}} = c\Big(\frac{r}{R}\Big)^m$$

and the proof is finished. □

5. Proof of the Main Theorem

Before we start proving Theorem 2.1 we introduce one more notation. From the two relative radii R_i and R_o given in Theorem 2.1 we define an angle

$$\alpha = \tan^{-1}\Big(\frac{R_i}{R_o}\Big).$$

From the convexity assumption for the images of the balls (2.7) we see that for every $z \in \operatorname{conv}(f_{x,r}(B_X(y,r)) \setminus \{f_{x,r}(y)\}$

$$C(z, \frac{v}{||v||_E}, \alpha) \cap B_{\mathbb{R}^n}(z, ||v||_E) \cap f_{x,r}(B_X(x,4r)) \subset f_{x,r}(B_X(y,r)), \tag{5.1}$$

where $v = f_{x,r}(y) - z$. See Figure 2 for this conclusion.

The next lemma will deal with the Euclidean projection part of our proof. For similar conclusions, see for example [5, Theorem 2.2] and [1, Lemma 3.4].

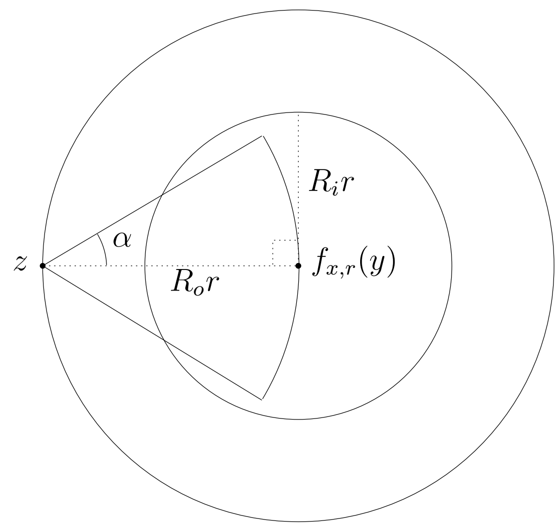

Figure 2: A cone opening with an angle α to the direction of the image of centre of the ball is included in the image of the ball. Here z is chosen to have the maximum distance to $f_{x,r}(y)$ which is the extreme case.

Lemma 5.1. *With the same assumptions as in Theorem 2.1 let $r < r_0$, $x \in X$, $c > 0$, $0 < s \le 1$, $0 < \varrho < \frac{1}{2}$ and $R = \frac{R_i \tan\frac{\alpha}{4}}{R_o} r$. Assume that $\{B_X(x_i, r) \mid i \in I\}$ is a collection of balls with $\{x_i \mid i \in I\} \subset B_X(x, 2r)$. Let*

$$D = \partial\Big(\operatorname{conv}(f_{x,r}(B_X(x,R))) \setminus \bigcup_{i\in I} \operatorname{conv}(f_{x,r}(B_X(x_i,r)))\Big).$$

Then there are at most $c'(1-2\varrho)^{-s(n-1)}$ disjoint Euclidean balls with centres in D and radius $c(1-2\varrho)^s r$, where c' depends only on R_i, R_o, n, c and s.

Proof. First we cover the space $\mathbb{R}^n$ with N cones

$$C_j = C(x, v_j, \frac{\alpha}{4}),$$

where $v_1, \ldots, v_N \in S^{n-1}$ and N depends on α and n. Fix $j = 1, \ldots, N$ and select a subcollection of balls

$$\mathcal{B}_j = \{B_X(x_i, r) \mid i \in I \text{ and } f_{x,r}(x_i) \in C_j\}.$$

For any point

$$y \in D_j = \partial\Big(\bigcup_{B\in\mathcal{B}_j} \operatorname{conv}(f_{x,r}(B))\Big) \cap \operatorname{conv}(f_{x,r}(B_X(x,R))),$$

there exists a ball $B_X(x_i, r) \in \mathcal{B}_j$ so that $y \in \partial(\operatorname{conv}(f_{x,r}(B_X(x_i,r))))$. Because $d_E(f_{x,r}(x_i), y) \ge R_i r$ the angle between v_j and $f_{x,r}(x_i) - y$ is at most $\frac{\alpha}{2}$. This

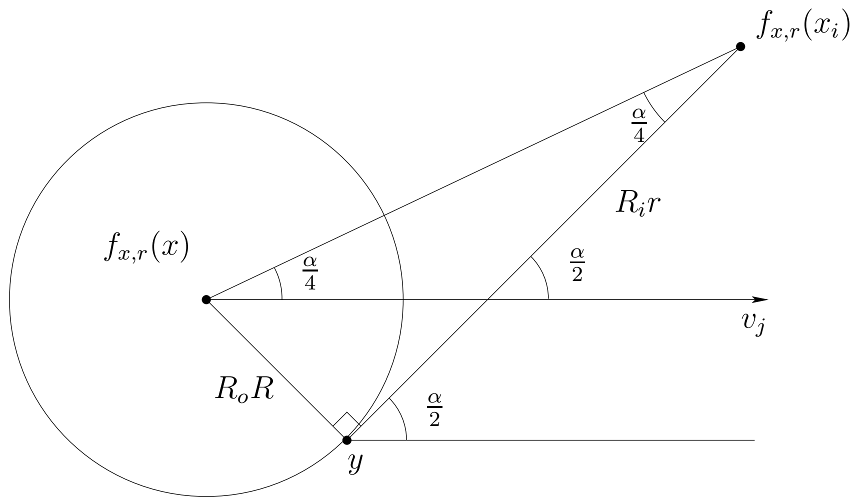

Figure 3: The choice of r and R is based on the worst case scenario.

follows from the choice of r and R. (See Figure 3.) Let $v = f_{x,r}(x_i) - y$. By the inclusion (5.1) we have

$$C(y, \frac{v}{||v||_E}, \alpha) \cap B_{\mathbb{R}^n}(y, ||v||_E) \cap f_{x,r}(B_X(x, 2r)) \subset f_{x,r}(B_X(x_i, r)).$$

These geometric conclusions together give

$$C(y, v_j, \frac{\alpha}{2}) \cap B_{\mathbb{R}^n}(y, R_i r) \cap f_{x,r}(B_X(x, 2r)) \subset f_{x,r}(B_X(x_i, r)).$$

Now that we have cones opening to a fixed direction v_j the projection

$$\mathrm{proj}_j \ : \ D_j \to \{v_j\}^\perp \ : \ x' \mapsto x' - (x'|v_j)v_j$$

satisfies the following inequalities for every $x_1, x_2 \in D_j$

$$d_E(\mathrm{proj}_j(x_1), \mathrm{proj}_j(x_2)) \le d_E(x_1, x_2) \le \Big(\sin(\frac{\alpha}{2})\Big)^{-1} d_E(\mathrm{proj}_j(x_1), \mathrm{proj}_j(x_2)),$$

see Figure 4. Hence it is a bi-Lipschitz map with Lipschitz constant $\big(\sin(\frac{\alpha}{2})\big)^{-1}$.

Take M_j disjoint Euclidean balls $B_{\mathbb{R}^n}(w_i, c(1-2\varrho)^s r)$ with centres $w_i \in D_j$. Because proj_j is $\big(\sin(\frac{\alpha}{2})\big)^{-1}$-bi-Lipschitz the balls $B_{\mathbb{R}^{n-1}}(\mathrm{proj}_j(w_i), \sin(\frac{\alpha}{2})c(1-2\varrho)^s r)$ are also disjoint. On the other hand, by (2.6) they are all centred in

$$B_{\mathbb{R}^{n-1}}(\mathrm{proj}_j(f_{x,r}(x)), R_o R).$$

Hence

$$M_j \le c'\Big(\frac{R_o R}{\sin(\frac{\alpha}{2})c(1-2\varrho)^s r}\Big)^{n-1} = c''(1-2\varrho)^{-s(n-1)},$$

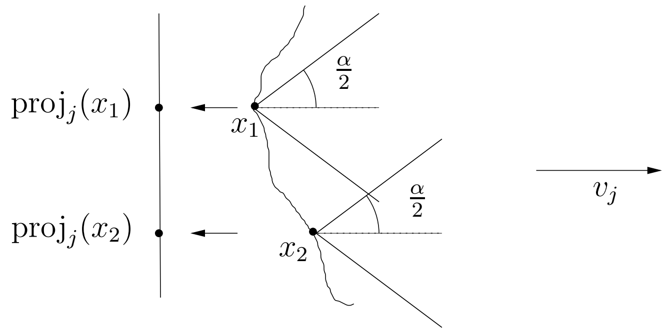

Figure 4: Cones opening to the direction v_j from each point in D_j guarantee the bi-Lipschitzness of the projection.

where c' depends on n and c'' depends on n, c, R_i and R_o. Multiplying this constant by N gives the desired upper bound for the cover of

$$\partial\Big(\bigcup_{i\in I}\mathrm{conv}(f_{x,r}(B_X(x_i,r)))\Big)\cap\mathrm{conv}(f_{x,r}(B_X(x,R))).$$

To finish the proof we cover the set

$$\partial\Big(\mathrm{conv}(f_{x,r}(B_X(x,R)))\Big)$$

with $c'''(1-2\varrho)^{-s(n-1)}$ disjoint Euclidean balls with radius $c(1-2\varrho)^s r$. The existence of such cover follows immediately from the assumption (2.6) and convexity. □

Proof of Theorem 2.1. Because X is separable it can be covered with a countable collection of balls of radius r_0. It is sufficient to estimate the dimension of A in these balls separately as long as the estimate does not depend on the ball. We may therefore assume that $A \subset B_X(x_0, r_0)$ for some $x_0 \in X$. Divide the set A into uniformly porous subsets

$$A_k = \big\{x \in A \mid \mathrm{por}(A,x,r) > \varrho \text{ for all } 0 < r < \frac{1}{k}\big\},$$

where $k \in \mathbb{N}$. We will estimate the Minkowski dimension of A_k. The set A_k can be covered with N_0 balls $B_X(x_i, R)$ with radius

$$R = \min\Big\{\frac{1}{\eta k}, \frac{r_0}{2(1+\eta)}\Big\},$$

where

$$\eta = \frac{R_o}{R_i \tan\frac{\alpha}{4}}.$$

Define $r = \eta R$. Take one covering ball $B_X(x_i, R)$ and form two collections of balls as follows. First define a collection that covers $A_k \cap B_X(x_i, R)$ as

$$\mathcal{B}_C = \{B_X(y, c2^s(1-2\varrho)^s r) \mid y \in A_k \cap B_X(x_i, R)\}$$

and then a collection of holes as

$$\mathcal{B}_H = \{B_X(z_y, \varrho r) \mid B_X(z_y, \varrho r) \cap A_k = \emptyset, y \in A_k \text{ and } \varrho r + d_X(z_y, y) \leq r\}.$$

We may assume that $c2^s(1-2\varrho)^s < 1$. To estimate the number of balls needed to cover $A_k \cap B_X(x_i, R)$ take a maximum subcollection of pairwise disjoint balls from $\mathcal{B}_C$ and estimate from above the number of balls, denoted by N, in this subcollection.

Take a ball $B_X(y, c(1-2\varrho)^s r) \in \mathcal{B}_C$. Then there exists a hole $B_X(z_y, \varrho r) \in \mathcal{B}_H$ so that $B_X(z_y, \varrho r) \cap A_k = \emptyset, y \in A_k$ and $\varrho r + d_X(z_y, y) \leq r$, since A is porous in this scale at point y. For the points y and z_y we have

$$1 - \frac{\varrho r}{d_X(z_y, y)} \leq 2(1-2\varrho),$$

which yields together with the assumption (2.4) that

$$B_X(y, c2^s(1-2\varrho)^s r) \cap B_X(z_y, \varrho r) \neq \emptyset.$$

Therefore with the assumption (2.7) in mind we find for each $B_X(y, c2^s(1-2\varrho)^s r) \in \mathcal{B}_C$ a point

$$y' \in \partial\Big(\operatorname{conv}(f_{(x_i,r)}(B_X(x_i, R))) \setminus \bigcup_{B \in \mathcal{B}_H} \operatorname{conv}(f_{x_i,r}(B))\Big)$$

so that

$$y' \in \operatorname{conv}(f_{x_i,r}(B_X(y, c2^s(1-2\varrho)^s r))).$$

Assumption (2.8) tells us that

$$N(c2^s(1-2\varrho)^s r, y', f_{x_i,r}) \leq c_1(1-2\varrho)^{s(n-t)}.$$

This means that at least

$$Nc_1^{-1}(1-2\varrho)^{s(t-n)} \tag{5.2}$$

balls

$$B_{\mathbb{R}^n}(y', c2^s(1-2\varrho)^s r)$$

are disjoint. By Lemma 5.1 the maximum number of these disjoint balls is

$$c_2(1-2\varrho)^{-s(n-1)}. \tag{5.3}$$

Together (5.2) and (5.3) imply

$$N \leq c_3(1-2\varrho)^{-s(t-1)}.$$

Now that we have an estimate for N we are ready to move to a cover of the set $A_k \cap B_X(x_i, R)$. This is done by tripling the radii of the balls in the disjoint collection. Next take a ball from the new collection and continue covering A_k in it using the same argument. This way we get for every $m \in \mathbb{N}$

$$N_o(c_3(1-2\varrho)^{-s(t-1)})^m$$

balls of radius

$$(3c2^s\eta(1-2\varrho)^s)^m R$$

that cover the set A_k. Now with

$$\lambda = t - 1 + \frac{c_4}{\log(\frac{1}{1-2\varrho})}$$

we have $\lim_{r\to 0} M^\lambda(A_k, r) = 0$ and hence $\dim_{\mathrm{M}}(A_k) \le \lambda$. Because the constant c_4 does not depend on k and x_0 the proof is complete. □

6. Examples where Dimension Estimates Fail

It seems natural to expect that there are very simple groups with natural metrics where the dimension of maximally porous sets may be more than the dimension of the group minus one. The groups introduced by P. Erdös and B. Volkmann in [4] serve as a set of examples. They proved that for each $0 < s < 1$ there is an additive subgroup $G_s \subset \mathbb{R}$ with Hausdorff dimension s.

Example 6.1. The groups G_s constructed by P. Erdös and B. Volkmann are chosen using the following representation of real numbers

$$x = a_1(x) + \sum_{k=2}^{\infty} \frac{a_k(x)}{k!},$$

where $a_i(x) \in \mathbb{Z}$ for all i and $0 \le a_i(x) < i$ for all $i \ge 2$. Define

$$G_s = \{x \in \mathbb{R} \ : \ a_k(x) \le c(x)k^s \text{ or } a_k(x) \ge k - c(x)k^s \text{ for all } k \ge k_0(x)\}.$$

These groups are dense in $\mathbb{R}$ and hence for example $\{0\} \times G_{\frac{1}{2}}$ is $\frac{1}{2}$-porous in $G_{\frac{1}{2}} \times G_{\frac{1}{2}}$, but

$$\dim_{\mathrm{H}}(\{0\} \times G_{\frac{1}{2}}) = \frac{1}{2} > 0 = \dim_{\mathrm{H}}(G_{\frac{1}{2}} \times G_{\frac{1}{2}}) - 1.$$

One immediately sees that the space $G_{\frac{1}{2}} \times G_{\frac{1}{2}}$ satisfies the assumptions of Theorem 2.1 with $f_{x,r}$ being the identity mapping. The problem is that it satisfies the condition (2.8) with a constant $t \ge 2$ and so Theorem 2.1 only gives

$$\dim_{\mathrm{p}}(\{0\} \times G_{\frac{1}{2}}) \le 1.$$

In [7] we proved that the same asymptotic behaviour is true for the dimension of lower-porous subsets of regular spaces as is true in the Euclidean space. The result is that there exists a constant c that depends only on the regularity parameters so that for every ϱ-porous subset A of an s-regular space X the dimension is bounded above by

$$\dim_{\mathrm{p}}(A) \leq s - c\varrho^s.$$

This gives naturally the asymptotic behaviour when porosity goes to zero. The following example shows that an $s-1$ estimate for large porosity can not be true in general s-regular spaces.

Example 6.2. For all $n \in \mathbb{N}$ we define a metric space (S_n, d_n). Here S_n is the attractor of function system

$$f_i : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto \frac{1}{2}x + a_i,$$

where

$$a_i \in \{0, (\frac{1}{2}, 0, \dots, 0), (0, \frac{1}{2}, 0, \dots, 0), \dots, (0, \dots, 0, \frac{1}{2})\}.$$

We define the metric d_n as the path-metric induced by the maximum-metric in $\mathbb{R}^n$. Next we make some observations. The metric space (S_n, d_n) is s-regular, where s is the dimension of the space

$$\dim_{\mathrm{H}}(S_n) = \frac{\log(n+1)}{\log 2}.$$

Secondly by leaving one coordinate out and hence restricting the space (S_n, d_n) we get (S_{n-1}, d_{n-1}). Because of the definition of the metric we get also that

$$\partial B_{S_n}((1, 0, \dots, 0), 1) = \{0\} \times S_{n-1}.$$

It is easy to see that in a geodesic metric space the boundary of any ball is maximally porous. Next we note that when $n \to \infty$

$$\dim_{\mathrm{H}}(S_n) - \dim_{\mathrm{H}}(\partial B_{S_n}((1, 0, \dots, 0), 1)) = \frac{\log(n+1) - \log(n)}{\log 2} \searrow 0.$$

Look at Figure 5 to see what S_3 looks like. Notice that in the picture we have a more symmetric Sierpinski gasket. This is the same space as in the case when we use the path-metric induced by the Euclidean one.

Again by Theorem 2.1 we get trivial bounds for the porous subsets in Example 6.2 using the underlying Euclidean space $\mathbb{R}^n$, but the problem is the same as in Example 6.1. One direction in the Euclidean sense does not have to contribute by one to the dimension of the space.

J. Väisälä has shown in [16] that porosity is qualitatively preserved by quasisymmetric maps, in particular, by bi-Lipschitz maps. Naturally the porosity might decrease when taking a quasisymmetric image of a porous set. Nevertheless we might ask if our previous results can be generalized to quasisymmetric images of $\mathbb{R}^n$. It turns out that this is not true even for bi-Lipschitz images of $\mathbb{R}$ as is shown by the next example.

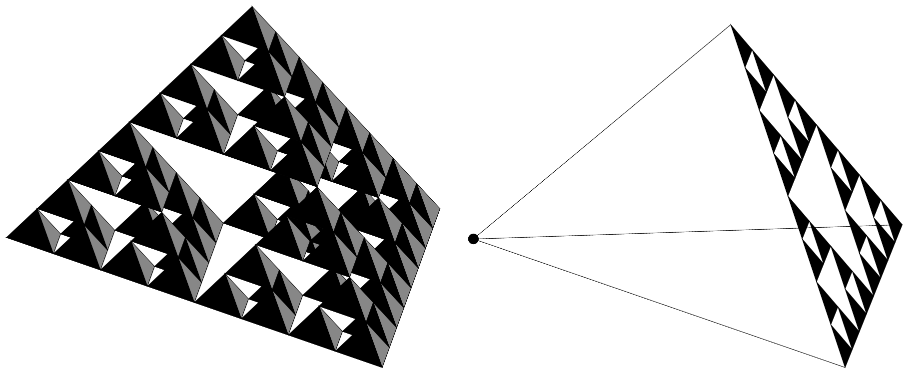

Figure 5: An illustration of space S_3 and S_2 as $\partial B_{S_3}((1,0,\dots,0),1)$.

Example 6.3. Take a $\lambda \in]0, \frac{1}{2}[$ and a Cantor λ-set $C_\lambda \subset \mathbb{R}$ which is the attractor of the function system $\{\lambda x, \lambda x + 1 - \lambda\}$. Look at the graph of a stretched distance function from that set and define the space $X \subset \mathbb{R}^2$ as

$$X = \Big\{(x,y) \in \mathbb{R}^2 \;:\; y = \frac{3-2\lambda}{1-2\lambda} d_E(x, C_\lambda)\Big\}.$$

The metric d in X is given by restricting the maximum metric of $\mathbb{R}^2$ to X. The space (X,d) is now bi-Lipschitz equivalent to $\mathbb{R}$ with bi-Lipschitz constant $\frac{3-2\lambda}{1-2\lambda}$ and the Cantor set in X i.e.

$$C = \Big\{(x,y) \in X \;:\; y = 0\Big\}$$

is maximally porous, but still

$$\dim_{\mathrm{H}}(C) = \dim_{\mathrm{H}}(C_\lambda) = \frac{\log(\frac{1}{2})}{\log(\lambda)} > 0 = 1 - 1.$$

An example of space X with $\lambda = \frac{1}{4}$ is given in Figure 6.

The space of Example 6.3 clearly violates the condition (2.4) in Theorem 2.1. The two previous examples have shown that alone the existence of geodesics in the space or the existence of a bi-Lipschitz map from the space to $\mathbb{R}^n$ is not enough to ensure a dimension result similar to (1.1). On the other hand, these two conditions with an extra assumption on the convexity of balls is sufficient as we proved in the Corollary 2.5. There is still a gap between positive results and negative examples and it remains open, for example, whether or not one can drop the assumption on convexity in Corollary 2.5.

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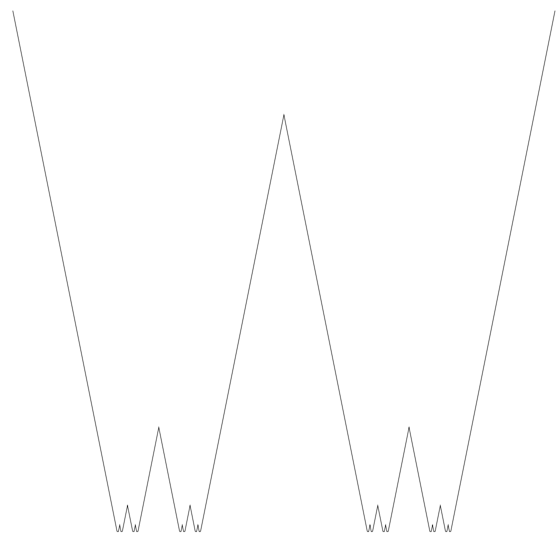

Figure 6: An example of a bi-Lipschitz image of $\mathbb{R}$ where maximally porous sets can have positive dimension.

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