Quasiconformal mappings and singularity of boundary distortion

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Abstract

We extend a well-known theorem by Jones and Makarov [8] on the singularity of boundary distortion of planar conformal mappings. We use a different technique to recover the previous result and, moreover, generalize the result for quasiconformal mappings of the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$, $n \geq 2$. We also establish an estimate on the Hausdorff (gauge) dimension of the boundary of the image domain outside an exceptional set of given size on the sphere $\partial\mathbb{B}^n$. Furthermore, we show that this estimate is essentially sharp.

1 Introduction

Let $f : \mathbb{D} \to \Omega$ be a conformal mapping of the unit disk $\mathbb{D}$ onto a domain $\Omega \subset \mathbb{C}$. Recall that, by a classical theorem of Beurling [12, p. 215], the boundary function of f is defined in terms of angular limits everywhere on $\partial\mathbb{D}$ except for a set of zero logarithmic capacity. Some time ago, Jones and Makarov [8] established the following remarkable result considering the singularity of boundary distortion of f. They write $f^*\Lambda_\varphi \perp m_2$, if f maps the whole unit circle except a set of zero Λ_φ-measure onto a set of zero area. Here Λ_φ denotes the Hausdorff measure on $\partial\mathbb{D}$ associated to a weight function φ, see below for the definition of this measure.

Theorem A ([8]). *Let φ be a weight function satisfying $\varphi(2r) \leq C\varphi(r)$, $r > 0$. Then the relation*

$$f^*\Lambda_\varphi \perp m_2$$

holds for every univalent function f if and only if

$$\int_0 \Big|\frac{\log \varphi(t)}{\log t}\Big|^2 \frac{dt}{t} = \infty. \tag{1.1}$$

In this note we extend the above result for quasiconformal mappings of the unit ball $\mathbb{B}^n$, $n \geq 2$, of Euclidean space. By definition, a homeomorphism $f : \mathbb{B}^n \to \Omega \subset \mathbb{R}^n$ is K-quasiconformal if $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{B}^n; \Omega)$ and the inequality

$$|Df(x)|^n \leq K J_f(x) \tag{1.2}$$

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holds for almost every $x \in \mathbb{B}^n$. Here $|Df(x)|$ stands for the operator norm of the differential matrix of f at the point x, while $J_f(x)$ denotes the determinant of $Df(x)$. Recall that, by the analog of Beurling's theorem, the boundary mapping of f is defined in terms of radial limits everywhere on $\partial\mathbb{B}^n$ except for a set of zero conformal (n)-capacity, see e.g. [4, Theorem 4.4]. In this setting we establish the following theorem.

Theorem 1.1. *Let $\varphi(t)$ be a weight function satisfying the technical conditions* (1.5), (1.6) *and* (1.7) *below and denote $u = \varphi^{-1}$. Then the relation*

$$f^*\Lambda_\varphi \perp m_n \tag{1.3}$$

holds for every quasiconformal mapping $f : \mathbb{B}^n \to \Omega \subset \mathbb{R}^n$ if and only if

$$\int_0 \left(\frac{u(t)}{u'(t)}\right)^{n-1} \frac{dt}{t^n} = \infty. \tag{1.4}$$

Note that, in the case $n = 2$, the condition (1.4) is equivalent to the condition (1.1), see [10, Remark 5.3]. Thus, in the planar case, we recover the result of Jones and Makarov. Our assumptions on the weight function φ are described more precisely in the following. We will assume that for all sufficiently small $t > 0$ the function $\varphi(t)$ is an increasing and differentiable function, which satisfies $\varphi(0) = 0$, $\varphi(2t) \le \beta\varphi(t)$, and

$$\frac{\varphi'(t)t\log t}{\varphi(t)\log\varphi(t)} \text{ is non-increasing or non-decreasing,} \tag{1.5}$$

and

$$\frac{u(t)}{tu'(t)} \text{ is non-decreasing,} \tag{1.6}$$

and

$$\log\frac{1}{u(t^2)} \le \beta\log\frac{1}{u(t)} \tag{1.7}$$

for $u = \varphi^{-1}$ with some constant $\beta > 1$. Note that these technical assumptions are harmless in the sense that they are satisfied in all interesting situations, see e.g. Remark 1.3 below. It is the condition (1.4) that is interesting in Theorem 1.1. Let us point out that some conditions on the regularity of φ are assumed also in the proof of Theorem A, see [8, p. 447-448].

We also extend the above result by replacing the n-dimensional Lebesgue measure m_n in (1.3) with a measure Λ_ψ, where the gauge function ψ depends on φ. Recall that the *generalized Hausdorff measure* Λ_φ (or simply *φ-measure*) is defined by

$$\Lambda_\varphi(E) = \lim_{r\to 0}\left(\inf\left\{\sum\varphi(\operatorname{diam} B_i) : E \subset \bigcup B_i,\ \operatorname{diam}(B_i) \le r\right\}\right),$$

where the dimension gauge function φ is required to be continuous and increasing with $\varphi(0) = 0$. In particular, if $\varphi(t) = t^\alpha$ with some $\alpha > 0$, then

Λ_φ is the usual *α-dimensional Hausdorff measure* denoted also by H^α. See [3] or [14] for more information on the generalized Hausdorff measure.

Our main result is the following theorem. We write $f^*\Lambda_\varphi \perp \Lambda_\psi$, if f maps the whole unit sphere except a set of zero φ-measure to a set of zero ψ-measure.

Theorem 1.2. *Let φ be a weight function such that $u = \varphi^{-1}$ satisfies the condition* (1.4) *in addition to the technical properties* (1.5)*,* (1.6) *and* (1.7)*. Then the relation*

$$f^*\Lambda_\varphi \perp \Lambda_\psi \tag{1.8}$$

holds for every K-quasiconformal mapping $f : \mathbb{B}^n \to \Omega \subset \mathbb{R}^n$ if there are positive constants r_0 and C_1 so that

$$\psi(r) \le C_1 r^n \exp\Big(C_2 \int_{[r,r_0]} \Big(\frac{u(t)}{u'(t)}\Big)^{n-1} \frac{dt}{t^n}\Big) \tag{1.9}$$

for all $r < r_0$. Here $C_2 = C_2(n, K, \beta) > 0$.

We will also show in Section 3 that Theorem 1.2 is sharp in the following sense. Suppose that φ satisfies the conditions of Theorem 1.2. Then there is an open set Ω, a constant $\tilde{C} > 0$, and a quasiconformal mapping $f : \mathbb{B}^n \to \Omega$ so that, for any set $E \subset \partial\mathbb{B}^n$ with $\Lambda_\varphi(E) = 0$, we have $\Lambda_\psi(f(\partial\mathbb{B}^n \setminus E)) > 0$ with a dimension gauge ψ satisfying

$$\psi(r) \le r^n \exp\Big(\tilde{C} \int_{[r,r_0]} \Big(\frac{u(t)}{u'(t)}\Big)^{n-1} \frac{dt}{t^n}\Big)$$

for all sufficiently small $r > 0$, provided that Λ_ψ is absolutely continuous with respect to H^1, which is the interesting case.

Let us close this section with a concrete example of our results. This remark also demonstrates which dimension gauge functions are critical for the condition (1.3) to hold.

Remark 1.3. *Let $s \ge 1$ and let*

$$\varphi(t) = \exp\Big(- \Big(c \log \frac{1}{t}\Big)^{1/s}\Big).$$

Let f be a K-quasiconformal mapping of the unit ball $\mathbb{B}^n$, $n \ge 2$. Then the condition (1.3) *holds if and only if $s \le n/(n-1)$. In the case $s = n/(n-1)$ the condition* (1.8) *holds with a dimension gauge function*

$$\psi(r) = r^n \Big(\log \frac{1}{r}\Big)^C, \tag{1.10}$$

where $C > 0$ depends only on n, K and c. Note that the latter statement is stronger than (1.3)*.*

2 Proof of the main result

In this section we prove Theorem 1.2 and the "if"-part of Theorem 1.1 (observe that the latter immediately follows from Theorem 1.2). Most of the machinery needed for the proof has already been developed in [10], including the following geometric concept which we use as a tool.

Definition 2.1. *Let $E \subset \mathbb{R}^n$ be a compact set. Let $\alpha : (0,1) \to (0,1)$ be a continuous function such that*

$$\frac{\alpha(t)}{t} \text{ is a non-decreasing function} \tag{2.1}$$

and let $\lambda : \mathbb{N} \to \mathbb{N}$ be a function. Let $\mathcal{Q}$ be a collection of pairwise disjoint cubes $Q_i \subset \mathbb{R}^n \setminus E$. We define for each such collection $\mathcal{Q}$ and for every $k \in \mathbb{N}$ a function

$$\chi_k^{\mathcal{Q}}(x) = \begin{cases} 1, \text{ if one can find cubes } Q_i^k(x) \in \mathcal{Q},\ i = 1, ..., \lambda(k), \\ \text{such that } Q_i^k(x) \subset A_k(x) \text{ and } \operatorname{diam}(Q_i^k(x)) \geq \alpha(2^{-k}) \text{ for all } i; \\ 0, \text{ otherwise.} \end{cases}$$

Here $A_k(x) = \{y \in \mathbb{R}^n : 2^{-k} < |x - y| < 2^{-k+1}\}$. Let

$$S_j^{\mathcal{Q}}(x) = \sum_{k=1}^{j} \chi_k^{\mathcal{Q}}(x).$$

We say that a set E is weakly mean porous with parameters α and λ, if there exists a collection $\mathcal{Q}$ as above and an integer $j_0 \in \mathbb{N}$ such that

$$\frac{S_j^{\mathcal{Q}}(x)}{j} > \frac{1}{2} \tag{2.2}$$

for all $x \in E$ and for all $j \geq j_0$.

Let us remark that the concepts of porosity and mean porosity are well-known tools in geometric analysis, see e.g. [9]. In the proof of our main result, we will apply the following sharp estimate on the Hausdorff (gauge) dimension of weakly mean porous sets established in [10, Corollary 3.5].

Lemma 2.2 ([10])**.** *Let $E \subset \mathbb{R}^n$ be a weakly mean porous set with parameters α and λ such that*

$$\frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n} \text{ is a non-increasing function of } k \tag{2.3}$$

and

$$\sum_{k=j_0}^{\infty} \frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n} = \infty. \tag{2.4}$$

Then $m_n(E) = 0$ *and, moreover, there is a positive constant* $C(n)$ *such that* $\Lambda_h(E) = 0$ *for each premeasure* h*, which satisfies*

$$h(2^{-j}) \le M 2^{-jn} \exp\Big(C(n) \sum_{k=j_0}^{j} \frac{\lambda(k)\alpha(2^{-k})^n}{(2^{-k})^n} \Big)$$

for all $j > j_0$ *with some positive constant* M*.*

Throughout the proofs we denote by C positive constants depending only on the given data n, K and β. These constants may vary from expression to expression as usual.

The *average derivative* a_f of a quasiconformal mapping f is defined by

$$a_f(x) = \Big(\frac{1}{m_n(B_x)} \int_{B_x} J_f \, dm_n \Big)^{1/n},$$

where $x \in \mathbb{B}^n$ and $B_x = B(x, \frac{1}{2}(1 - |x|))$. Recall that a_f satisfies a Harnack inequality

$$1/C \le \frac{a_f(z)}{a_f(y)} \le C \tag{2.5}$$

for any points z, y belonging to some Whitney-ball $B_x \subset \mathbb{B}^n$, see e.g. [4]. Also note that

$$1/C \int_Q a_f^n \, dm_n \le \int_Q |Df|^n dm_n \le C \int_Q a_f^n \, dm_n \tag{2.6}$$

for all cubes Q in a Whitney decomposition $\mathcal{W}$ of $\mathbb{B}^n$, see e.g. [2, Theorem 3.4]. A Whitney decomposition of $\mathbb{B}^n$ refers to a collection of closed dyadic cubes $Q \subset \mathbb{B}^n$ with pairwise disjoint interiors such that

$$\bigcup_{Q \in \mathcal{W}} Q = \mathbb{B}^n$$

and that $\operatorname{diam}(Q) \le \operatorname{dist}(Q, \partial\mathbb{B}^n) \le 4 \operatorname{diam}(Q)$. See [15] for the existence of such a decomposition.

Let us also recall that the *quasihyperbolic distance* $k_\Omega(x_1, x_2)$ between two points x_1, x_2 in a domain $\Omega \subsetneq \mathbb{R}^n$ is defined as the infimum of

$$\int_\gamma \frac{ds}{\operatorname{dist}(x, \partial\Omega)}$$

over all rectifiable curves joining x_1 to x_2 in Ω.

Before the proof of our main result, we prove an additional technical property for the weight function φ.

Lemma 2.3. *Let $\varphi(t)$ be a weight function satisfying the conditions* (1.5), (1.6), (1.7). *Then there exists a constant $C = C(n, \beta) > 0$ such that*

$$\int_{[0,r]} \frac{\varphi(t)^{1/n}}{t} dt \leq \varphi(r)^C$$

for all sufficiently small $r > 0$.

Proof. Let us write $\varphi(t) = \exp(-\alpha(t))$. We show first that

$$\log\log\frac{1}{r} = o(\alpha(r)) \qquad \text{as } r \to 0. \tag{2.7}$$

Suppose that this assertion fails. Then there is a constant $c > 0$ such that for an arbitrarily small $r > 0$ we have that

$$\varphi(r) \geq \exp(-c\log\log\frac{1}{r}) = (\log\frac{1}{r})^{-c}.$$

This implies that

$$\varphi^{-1}(s) \leq \exp(-s^{-1/c})$$

or equivalently

$$\log\frac{1}{\varphi^{-1}(s)} \geq s^{-1/c}$$

for some arbitrarily small $s > 0$. This, however, contradicts the condition (1.7). Thus we have proven (2.7).

On the other hand, it follows from the assumptions with the help of Gronwall's lemma [17, p. 436] that, for all sufficiently small $t > 0$,

$$\varepsilon \leq \left|\frac{\alpha'(t)t\log t}{\alpha(t)}\right| \leq \frac{1}{\varepsilon}$$

with some $\varepsilon > 0$ depending only on β (cf. the proof of [10, Remark 5.3]). By combining this with (2.7) we obtain

$$\begin{aligned}\frac{\varphi(t)^{1/n}}{t} = \frac{\exp(-\frac{1}{n}\alpha(t))}{t} &\leq \frac{\exp(-\frac{1}{2n}\alpha(t) + \log(\alpha(t)) - \log\log\frac{1}{t})}{t} \\ &= \frac{\alpha(t)}{t\log\frac{1}{t}}\exp(-\frac{1}{2n}\alpha(t)) \\ &\leq C|\alpha'(t)|\exp(-\frac{1}{2n}\alpha(t))\end{aligned}$$

for all sufficiently small $t > 0$. The claim follows.

Proof of Theorem 1.2. Let E_∞ consist of those points $\xi \in \partial\mathbb{B}^n$ for which $\int_{[0,\xi]} a_f(x)\,|dx| = \infty$. Then $\mathrm{cap}_n(E_\infty) = 0$ by [4, Theorem 4.4] and thus, by Lemma 2.3, $\Lambda_\varphi(E_\infty) = 0$, see e.g. [13, p. 120] or [11, Remark 1.3]. Let us then write

$$G_j = \{\xi \in \partial\mathbb{B}^n \setminus E_\infty : \int_{[0,\xi]} a_f(x)\,|dx| \leq j\}$$

for $j \in \mathbb{N}$, and let S_j consist of the union of Stolz cones at G_j, i.e.,

$$S_j = \bigcup_{\xi \in G_j} \bigcup_{0 \le t < 1} B(t\xi, \frac{1}{2}(1-t)).$$

Then S_j is open and $\operatorname{diam} f(S_j) \le C \sup_{\xi \in G_j} \{\int_{[0,\xi]} a_f(x) \, |dx|\} \le Cj$, where the first inequality is implied by the Harnack inequality (2.5), [4, Remark 3.3], and the fact that the discrete length of a curve (as defined in [7]) is always at least the Euclidean distance between the end points. Thus we have that $m_n(f(S_j)) < \infty$. It follows by the inequalities (1.2) and (2.6) that the function $u_j : \mathbb{B}^n \to (0, \infty)$,

$$u_j(x) = \begin{cases} a_f(x)^n & \text{for } x \in S_j; \\ 0 & \text{otherwise} \end{cases}$$

belongs to $L^1(\mathbb{B}^n)$. This implies with the help of Besicovitch's covering theorem that there is a set $E_j \subset G_j$ with $\Lambda_\varphi(E_j) = 0$ so that

$$\int_{B(\xi,r) \cap \mathbb{B}^n} u_j \, dm_n = o(\varphi(r)) \text{ as } r \to 0$$

for all $\xi \in G_j \setminus E_j$ (cf. [18, p. 118]). In particular, for all $\xi \in G_j \setminus E_j$,

$$\int_{B_{t\xi}} a_f^n \, dm_n = \int_{B_{t\xi}} u_j \, dm_n \le \int_{B(\xi, \frac{3}{2}(1-t))} u_j \, dm_n = o(\varphi(1-t)) \text{ as } t \to 1$$

and hence, by the Harnack inequality (2.5),

$$a_f(t\xi)^n \le C(1-t)^{-n} \int_{B_{t\xi}} a_f^n \, dm_n = o\Big(\frac{\varphi(1-t)}{(1-t)^n}\Big) \text{ as } t \to 1. \tag{2.8}$$

Let us define $E = E_\infty \cup \bigcup_j E_j$. Then $\Lambda_\varphi(E) = 0$ by the subadditivity of φ-measure. We claim that $f(\partial\mathbb{B}^n \setminus E)$ is of zero ψ-measure. Since $\partial\mathbb{B}^n \setminus E \subset \bigcup_j (G_j \setminus E_j)$, it suffices to show that, for all $j \in \mathbb{N}$, $f(G_j \setminus E_j)$ is of zero ψ-measure.

For each $m \in \mathbb{N}$ we define a set $F_j^m \subset G_j \setminus E_j$ by

$$F_j^m = \{\xi \in G_j \setminus E_j : a_f(t\xi) \le \frac{\varphi(1-t)^{1/n}}{1-t} \text{ for all } t \ge 1 - 2^{-m}\},$$

whence $\bigcup_m F_j^m = G_j \setminus E_j$ by the inequality (2.8). Moreover, $F_j^1 \subset F_j^2 \subset ...$, and thus, by the subadditivity of the Hausdorff measure, it suffices to show that $f(F_j^m)$ is of zero ψ-measure for an arbitrarily large integer m in order to prove the theorem.

Because of Lemma 2.2 it now suffices to prove that the set $f(F_j^m) \subset \partial\Omega$ is weakly mean porous with parameters $C\alpha$ and $C\lambda$, where (for small t)

$$\alpha(t) = cu(t)/u'(t) \text{ and } \lambda(k) \ge 2^{-k}/\alpha(2^{-k}) \tag{2.9}$$

and $c > 0$ depends only on n, K and β.

Let $j, m \in \mathbb{N}$ and let $\xi \in F_j^m$. Then the radial limit $f(\xi)$ exists by [4, Remark 4.5] and, by the Harnack inequality (2.5) and Lemma 2.3, we obtain

$$\begin{aligned} |f(t\xi) - f(\xi)| &\leq \sum_{Q \in \mathcal{W}: Q \cap [t\xi, \xi] \neq \emptyset} \operatorname{diam} f(Q) \leq C \int_{[t\xi,\xi]} a_f(x)\, |dx| \\ &\leq C \int_{[t,1]} \frac{\varphi(1-s)^{1/n}}{1-s} ds \leq \varphi(1-t)^C \end{aligned} \tag{2.10}$$

for all $t \geq 1 - 2^{-m}$, provided that m was chosen large enough above. Here we denoted by $\mathcal{W}$ a Whitney decomposition of $\mathbb{B}^n$.

Let us write $\gamma = f([0, \xi))$. We choose an integer $i_0 \geq m$ so large that $2^{-i_0+1} \leq \operatorname{dist}(f(0), \partial f(\mathbb{B}^n))$, and we define for all integers $k \geq i_0$ a function

$$\chi_k(f(\xi)) = \begin{cases} 1, & \text{if } k_\Omega(f(t_a\xi), f(t_b\xi)) \leq C \frac{2^{-k}}{\alpha(2^{-k})} \\ 0, & \text{otherwise,} \end{cases}$$

where $f(t_a\xi)$ and $f(t_b\xi)$ are the last entry point along $f([0,\xi])$ into $A_k(f(\xi))$ and the first exit point after $f(t_a\xi)$ along $f([0,\xi])$ from $A_k(f(\xi))$, respectively, and k_Ω is the quasihyperbolic metric in Ω. We also define for all integers $i \geq i_0$ a function

$$S_i(f(\xi)) = \sum_{k=i_0}^{i} \chi_k(f(\xi)).$$

We then claim that

$$\frac{S_i(f(\xi))}{i} > \frac{1}{2} \tag{2.11}$$

for all sufficiently large $i \in \mathbb{N}$ provided that c in (2.9) is chosen small enough.

Let us consider an annulus $A_k(f(\xi))$ such that $\chi_k(f(\xi)) = 0$. The quasihyperbolic distance k_Ω between the points $f(t_a\xi)$ and $f(t_b\xi)$ is at least $C2^{-k}/\alpha(2^{-k})$. Due to the quasi-invariance of the quasihyperbolic metric under quasiconformal mappings [5, p. 62], we then have that the quasihyperbolic distance $k_{\mathbb{B}^n}(t_a\xi, t_b\xi)$ is at least $C2^{-k}/\alpha(2^{-k})$ (with C still depending only on n, K and β).

Consider the largest $t < 1$ with

$$|f(t\xi) - f(\xi)| = 2^{-i}.$$

It follows from (2.10) that $2^{-i} \leq \varphi(1-t)^C$ and hence, by (1.7),

$$\log \frac{1}{1-t} \leq \log\left(\frac{1}{u(2^{-i/C})}\right) \leq C \log\left(\frac{1}{u(2^{-i})}\right). \tag{2.12}$$

On the other hand, we have by the observations above that

$$\log \frac{1}{1-t} = k_{\mathbb{B}^n}(0, t\xi) \geq \sum k_{\mathbb{B}^n}(t_a\xi, t_b\xi) \geq \sum C\frac{2^{-k}}{\alpha(2^{-k})}, \tag{2.13}$$

where the summation is over all $i_0 \leq k \leq i$ with $\chi_k(f(\xi)) = 0$.

Suppose that the assertion (2.11) fails for some large integer i. Then, by combining (2.12) and (2.13), we arrive at (the summation indices follow from the assumption (1.6))

$$\begin{aligned}\log\Big(\frac{1}{u(2^{-i})}\Big) &\geq C\sum_{k=i_0}^{i/2} \frac{2^{-k}u'(2^{-k})}{cu(2^{-k})} \\ &\geq C\frac{1}{c}\Big(\log\Big(\frac{1}{u(2^{-i/2})}\Big) - \log\Big(\frac{1}{u(2^{-i_0})}\Big)\Big).\end{aligned}$$

But this inequality is a contradiction with property (1.7) if we choose i large enough and c small enough (depending on β). This proves (2.11).

For the final arguments of the proof we define a collection $\mathcal{Q}$ of disjoint cubes in the domain Ω in the following way. Let $\mathcal{W}$ be a Whitney decomposition of Ω. Then let $\mathcal{Q}$ consist of the interiors of all the cubes in the Whitney decompositions of the cubes $Q \in \mathcal{W}$. We claim that

$$\chi_k^{\mathcal{Q}}(f(\xi)) \geq \chi_k(f(\xi)). \tag{2.14}$$

for all $k \geq i_0$ with parameters $C\alpha$ and $C\lambda$ (defined above).

Let us consider a "good" annulus, i.e., $A_k(f(\xi))$ with $\chi_k(f(\xi)) = 1$. Then

$$k_\Omega(f(t_a\xi), f(t_b\xi)) \leq C\frac{2^{-k}}{\alpha(2^{-k})},$$

which means geometrically that there are at most $C2^{-k}/\alpha(2^{-k})$ Whitney cubes $Q \in \mathcal{W}$ intersecting the quasihyperbolic geodesic joining $f(t_a\xi)$ and $f(t_b\xi)$. But since the length of this curve is at least the width of the annulus $A_k(f(\xi))$ or 2^{-k}, it follows that some of these Whitney cubes must have a large diameter. Indeed, one finds at least $C2^{-k}/\alpha(2^{-k})$ cubes $Q \in \mathcal{Q}$ so that $Q \subset A_k(f(\xi))$ and the diameter of each cube is at least $C\alpha(2^{-k})$. This observation follows by easy geometric arguments and elementary calculations, which we leave to the reader (cf. [10, Lemma 4.6]). Let us point out that the second Whitney decomposition $\mathcal{Q}$ is needed here to ensure that there are enough cubes in $A_k(f(\xi))$ also in the case that the geodesic intersects only, say, one large Whitney cube from the first decomposition $\mathcal{W}$.

In conclusion, we have shown (2.14) and (2.11), which together imply that the set $f(F_j^m)$ is weakly mean porous. The desired estimate on the ψ-measure of $f(F_j^m)$ (and thus for the ψ-measure of $f(\partial\mathbb{B}^n \setminus E)$) then follows from Lemma 2.2, and thus the proof is complete.

3 Sharpness of the results

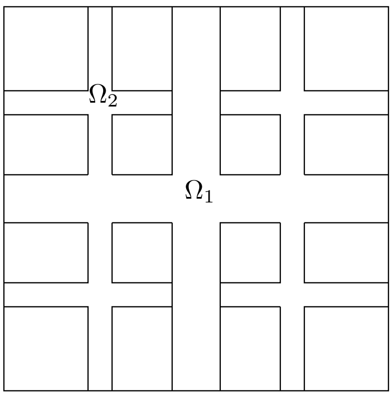
In this section we show the sharpness of Theorem 1.2 in the interesting case that Λ_φ is absolutely continuous wit respect to H^1. At the same time we show the "only if"-part of Theorem 1.1. The example domain that we consider here was constructed also in [10] for a different purpose. This example turns out to be the critical one also for the results of this paper. We give an outline for the construction in the following. See [10, Section 7.2] for the full details.

Given a gauge function φ satisfying (1.5), (1.6) and (1.7) we choose an increasing function α so that

$$\alpha(2^{-k}) = cu(2^{-k})/u'(2^{-k})$$

for all $k \in \mathbb{N}$, where the constant $c > 0$ is to be determined below. We assume (by (1.6)) that $\alpha(t) \leq t/16$ for all $t > 0$. Moreover, we assume that $\alpha(2^{-k})$ is dyadic for all k.

Starting with the square $Q_1 = \{(x, y) \in \mathbb{R}^2 : |x| < 2^{-1} \text{ and } |y| < 2^{-1}\}$ define Ω_1 as the intersection of Q_1 and the open $\alpha(2^{-1})$-neighborhood of the coordinate axes in Q_1. Then subdivide Q_1 into 4 dyadic squares Q_2^i, $i \in \{1, 2, 3, 4\}$, and define Ω_2 as the union of the intersections of $Q_1 \setminus \overline{\Omega_1}$ and the open $\alpha(2^{-2})$-neighborhoods of the centered coordinate axes of each square Q_2^i. Then attach each component of Ω_2 to Ω_1 in the way shown in the picture below.



Continue the construction by subdividing each square Q_2^i into 4 dyadic squares and defining Ω_3 accordingly. By iterating this process one obtains a simply connected domain Ω (take $\bigcup_j \Omega_j$ and open certain gates as in the picture above to make it simply connected), which satisfies the growth condition

$$k_\Omega(0, x) \leq \phi\Big(\frac{\operatorname{dist}(x, \partial\Omega)}{\operatorname{dist}(0, \partial\Omega)}\Big) + C$$

on the quasihyperbolic metric k_Ω with the function $\phi(t) = \frac{C}{c}\log\frac{1}{u(t)}$. Then by Lemma 2.3, we can use known results from the literature ([6, Theorem 1.2]) to conclude that there exists a (quasi)conformal mapping $f : \mathbb{B}^2 \to \Omega$, which is uniformly continuous with a modulus of continuity $C\varphi(t)$, i.e.,

$$|f(x) - f(y)| \le C\varphi(|x - y|) \tag{3.1}$$

for all $x, y \in \mathbb{B}^2$ provided that the constant c is chosen large enough in the construction above.

Moreover, by standard arguments (cf. [10, Section 7.2] or [9]) involving a construction of a "Frostman measure" on the boundary of Ω and an employment of the Frostman's lemma one observes that $\Lambda_\psi(\partial\Omega) > 0$ with a dimension gauge $\psi(r)$ satisfying

$$\psi(r) \le r^2 \exp\Big(\tilde{C}\int_{[r,r_0]} \frac{u(t)}{u'(t)}\frac{dt}{t^2}\Big)$$

for all $r > 0$ with a sufficiently large constant $\tilde{C} > 0$. In particular, this means that $m_2(\partial\Omega) > 0$ if the integral condition (1.4) fails.

It only remains to show that the ψ-measure of the set $f(\partial\mathbb{B}^2 \setminus E)$ is also positive for any set $E \subset \partial\mathbb{B}^2$ of zero φ-measure. To that end, let $E \subset \partial\mathbb{B}^2$ be an arbitrary set with $\Lambda_\varphi(E) = 0$. This means, by definition, that for any $\varepsilon > 0$ there is a countable collection of balls $B_1, B_2, ...$ with diameters $r_1, r_2, ...$ so that $E \subset \bigcup_i B_i$ and

$$\sum_i \varphi(r_i) < \varepsilon. \tag{3.2}$$

On the other hand, note that the internal diameter of Ω is finite (recall that the internal distance between two points $z, w \in \Omega$ is the infimum of lengths of curves $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1) = w$), and hence the radial limit $f(\xi)$ exists for all points $\xi \in \partial\mathbb{B}^2$. This follows from the Gehring–Hayman theorem, see e.g. [4, Remark 4.5]. Thus $f(E) \subset \partial\Omega$ is well defined, and we observe that $f(E) \subset \bigcup_i f(B_i)$. Furthermore, $\mathrm{diam}(f(B_i)) \le C\varphi(r_i)$ for each ball B_i by (3.1). By combining this with (3.2) we conclude that $H^1(f(E)) = 0$ which implies $\Lambda_\psi(f(E)) = 0$, since we are assuming that Λ_ψ is absolutely continuous with respect to H^1, which as we said above, is the interesting case. It follows that $\Lambda_\psi(f(\partial\mathbb{B}^2 \setminus E)) \ge \Lambda_\psi(\partial\Omega \setminus f(E)) > 0$.

Let us finally point out that the construction of Ω above can be extended to $\mathbb{R}^n$, $n \ge 3$ so that the resulting domain is quasiconformally equivalent to the ball $\mathbb{B}^n$, see [16]. For example, in the case $n = 3$ define Ω_1 in the unit cube Q_1 (of side length 1) by taking the $\alpha(2^{-1})$-neighborhood of the coordinate axes and of the lines $(t, \pm 2^{-2}, 0)$ (we have to include the neighborhoods of these additional lines to make the final set Ω connected). Then subdivide Q_1 into 8 dyadic (open) subcubes Q_2^i and define $\Omega_2 \subset \bigcup Q_2^i \setminus \overline{\Omega}_1$ accordingly.

Iterate this process and, in the end, attach each component of Ω_{j+1} to Ω_j to obtain a connected set Ω. One can then show the sharpness of Theorem 1.2 with similar calculations and conclusions as in the planar case discussed above.

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