WHICH MEASURES ARE PROJECTIONS OF PURELY UNRECTIFIABLE ONE-DIMENSIONAL HAUSDORFF MEASURES

MARIANNA CSÖRNYEI AND VILLE SUOMALA

ABSTRACT. We give a necessary and sufficient condition for a measure μ on the real line to be an orthogonal projection of $\mathcal{H}^1|_A$ for some purely 1-unrectifiable planar set A.

1. INTRODUCTION

Let $A \subset \mathbb{R}^2$ be a purely 1-unrectifiable Borel set with $0 < \mathcal{H}^1(A) < \infty$. The well-known projection results of Besicovitch and Marstrand (see e.g. in [Fa] or [Ma]) tell us that for almost all $t \in \mathbb{R}$ the orthogonal projection of $\mathcal{H}^1|_A$ to the line $\ell_t = \{(x, tx) : x \in \mathbb{R}\}$ is singular with respect to the Lebesgue measure on ℓ_t and moreover has dimension 1. These results, however, do not tell anything about one particular projection. In this paper we answer the following question of D. Preiss: for which measures μ on the real line is there a purely 1-unrectifiable Borel set $A \subset \mathbb{R}^2$ such that $\mu = \operatorname{proj} \mathcal{H}^1|_A$? Here $\mathcal{H}^1|_A$ is the one dimensional Hausdorff measure restricted to the set A. By proj we always mean the orthogonal projection $\operatorname{proj}\colon \mathbb{R}^2 \to \mathbb{R}$ onto the x-axis, $\operatorname{proj}(x, y) = x$ and if ν is a measure on $\mathbb{R}^2$ we define the projected measure $\operatorname{proj} \nu$ by defining $\operatorname{proj} \nu(A) = \nu(\operatorname{proj}^{-1}(A))$ for all Borel sets $A \subset \mathbb{R}$.

Since any purely 1-unrectifiable planar set A intersects all vertical lines in a set of zero $\mathcal{H}^1$ measure it follows that if $\mu = \operatorname{proj} \mathcal{H}^1|_A$ then

$$\mu\{x\} = 0 \text{ for all } x \in \mathbb{R}, \tag{1.1}$$

that is, μ has no point masses. Moreover, for any $A \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(A) < \infty$ the convex density of A is one for $\mathcal{H}^1$-almost all $x \in A$, see [Fa, Theorem 2.3]. This implies that $\mu = \operatorname{proj} \mathcal{H}^1|_A$ must satisfy

$$\limsup_{r \downarrow 0} \mu[x - r, x + r]/(2r) \geq 1 \text{ for } \mu\text{-almost all } x \in \mathbb{R}. \tag{1.2}$$

Recall that (1.2) always holds for singular measures. So the condition (1.2) tells that the absolutely continuous part of μ must have density at least one almost everywhere. Our main result, Theorem 1.1 below shows that the necessary conditions (1.1) and (1.2) for μ are also sufficient.

Date: November 2007.

2000 *Mathematics Subject Classification.* 28A78, 28A80.

Theorem 1.1. *Suppose that μ is a locally finite measure on the real line satisfying* (1.1) *and* (1.2)*. Then there is a purely* 1*-unrectifiable Borel set $A \subset \mathbb{R}^2$ for which $\mu = \operatorname{proj} \mathcal{H}^1|_A$.*

To prove Theorem 1.1 we divide μ into its singular and absolutely continuous parts, and handle these separately. The singular part will be considered in §2 and the absolutely continuous case is dealt with in §3. Though we are mainly interested in projections of $\mathcal{H}^1|_A$ for purely unrectifiable sets A, our methods may be used to construct also other fractal-type measures ν on $\mathbb{R}^2$ for which $\operatorname{proj} \nu = \mu$ for a given measure μ defined on $\mathbb{R}$, see Remark 2.4.

We end this introduction with some notation. We follow the usual convention according to which a measure on $\mathbb{R}^n$ always means a non-negative Borel regular outer measure defined on all subsets of $\mathbb{R}^n$. By a singular measure we mean a measure defined on $\mathbb{R}$ that is singular with respect to the Lebesgue measure $\mathcal{L}$. This is equivalent to saying that $\lim_{r\downarrow 0} \mu[x-r, x+r]/(2r) = \infty$ for μ-almost all $x \in \mathbb{R}$. If ν and μ are finite measures on some $\mathbb{R}^n$ we denote $\nu \leq \mu$ if $\nu(A) \leq \mu(A)$ for all sets $A \subset \mathbb{R}^n$. In this case we may also consider the measure $\mu - \nu$ given by $(\mu - \nu)(A) = \mu(A) - \nu(A)$ for Borel sets $A \subset \mathbb{R}^n$.

2. The singular case

We begin with some notation needed in this section. If $k \in \mathbb{N}$, we call the collection of closed squares

$$\mathcal{Q}_k = \left\{ Q_{i,j} = \left[\frac{j-1}{k}, \frac{j}{k}\right] \times \left[\frac{i-1}{k}, \frac{i}{k}\right] \subset \mathbb{R}^2 \,:\, 1 \leq i,j \leq k \right\}$$

a *k-grid*. A collection of grid squares $\mathcal{Q} \subset \mathcal{Q}_k$ is called *porous* if it does not contain two neighbouring squares, that is $Q \cap Q' = \emptyset$ whenever $Q, Q' \in \mathcal{Q}$ and $Q \neq Q'$.

The basis for our constructions is the following combinatorial lemma that enables us to find relatively good approximations for the set A using k-adic squares when k is so large that for "most" intervals $I_j = [\frac{j-1}{k}, \frac{j}{k}]$, $1 \leq j \leq k$, we have $1/k \ll \mu(I_j) \ll 1$.

Lemma 2.1. *Let $\mathcal{Q} \subset \mathcal{Q}_k$ be an arbitrary collection of grid squares containing at most one square from each row $1 \leq i \leq k$. Denote by l_j the number of squares that $\mathcal{Q}$ contains from the jth column. If $l_j < k/18$ for all $1 \leq j \leq k$, then there is a porous collection $\mathcal{Q}' \subset \mathcal{Q}_k$ of grid squares that contains at most one square from each row and exactly l_j squares from column j for all $1 \leq j \leq k$.*

Proof. If $Q \in \mathcal{Q}_k$, we denote by $N(Q)$ the union of Q and its neighbouring squares. Assume that $Q_{i_0,j_0} \in \mathcal{Q}$ has a neighbouring square in the collection $\mathcal{Q}$.

We define three index sets

$$I = \{1 \le i \le k \,:\, Q \subset N(Q_{i,j_0}) \text{ for some } Q \in \mathcal{Q}\},$$
$$J = \{1 \le j \le k \,:\, Q \subset N(Q_{i_0,j}) \text{ for some } Q \in \mathcal{Q}\},$$
$$I' = \{1 \le i \le k \,:\, Q_{i,j} \in \mathcal{Q} \text{ for some } j \in J\}.$$

Then $\#I \le 3(l_{j_0-1} + l_{j_0} + l_{j_0+1}) < k/2$ and also $\#J \le 9$ since $\mathcal{Q}$ contains at most one square from each row $i_0 - 1, i_0$ and $i_0 + 1$. Moreover $\#I' < \frac{k}{18}\#J \le k/2$.

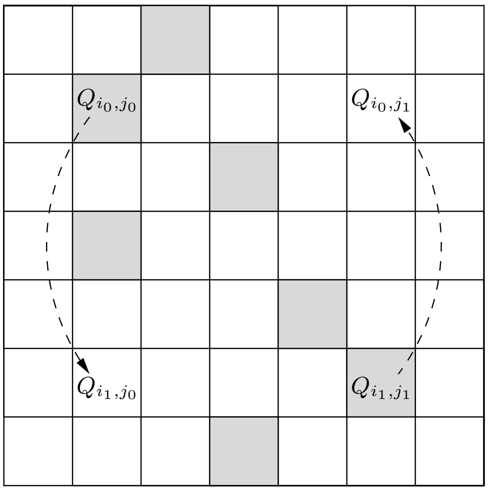

FIGURE 1. Constructing $\mathcal{Q}'$ from $\mathcal{Q}$. The neighbours of Q_{i_1,j_0} and Q_{i_0,j_1} are not contained in the original collection $\mathcal{Q}$.

We now choose an index $i_1 \in \{1, \ldots, k\} \setminus (I \cup I')$ and replace the square Q_{i_0,j_0} in $\mathcal{Q}$ by Q_{i_1,j_0}. If $Q_{i_1,j_1} \in \mathcal{Q}$ for some j_1 we also replace the square Q_{i_1,j_1} in $\mathcal{Q}$ by Q_{i_0,j_1}, see Figure 1. These replacements do not affect the good properties of $\mathcal{Q}$, it still contains at most one square from each row and exactly l_j squares from the jth column. But if the original collection had m squares with some neighbours in $\mathcal{Q}$, the modified collection has at most $m - 1$ squares with neighbours in $\mathcal{Q}$.

It is clear that by repeating the above process enough many times, we are finally led to some porous collection $\mathcal{Q}'$ with the desired properties. □

The following lemma is merely a restatement of the singularity of μ. We give the details for convenience.

Lemma 2.2. *Suppose that μ is a singular measure on $[0,1]$ with no point masses. For $k \in \mathbb{N}$ and $1 \le j \le k$, we denote $m_j = \mu[\frac{j-1}{k}, \frac{j}{k}]$ and let l_j be the greatest integer for which $l_j \le k m_j$. Then $\lim_{k\to\infty} \sum_{j=1}^{k} l_j/k = \mu[0,1]$.*

Proof. Let $\varepsilon > 0$ and $M = 5\mu[0,1]/\varepsilon$. Since μ is singular, we have $\lim_{r\downarrow 0} \mu[x - r, x + r]/(2r) = \infty$ for μ-almost all $x \in [0,1]$ and choosing $k_0 \in \mathbb{N}$ large enough, we have $\mu(A_k) < \varepsilon$ for all $k \geq k_0$ where

$$A_k = \left\{x \in [0,1] \,:\, \mu\left[x - 1/k, x + 1/k\right] < M^2/k\right\}.$$

We now fix $k \geq k_0$ and choose a collection $J_i = (x_i - 1/k, x_i + 1/k)$, $i = 1, \dots N$, of non-overlapping intervals such that $x_i \in [0,1] \setminus A_k$ for all i and $[0,1] \setminus A_k \subset \bigcup_{i=1}^N 2J_i = \bigcup_{i=1}^N (x_i - 2/k, x_i + 2/k)$. Letting $I_j = [\frac{j-1}{k}, \frac{j}{k}]$ for $1 \leq j \leq k$, we define

$$B_k = \bigcup_{\mu(I_j) < M/k} I_j.$$

Since any of the intervals $2J_i$ can intersect at most 5 of the intervals I_j we have $\mu(B_k \cap 2J_i) \leq 5M/k \leq 5\mu(J_i)/M$ for all i and consequently

$$\begin{aligned}\mu(B_k) = \mu(A_k) + \mu(B_k \setminus A_k) &\leq \varepsilon + \sum_i \mu(B_k \cap 2J_i)\\ &\leq \varepsilon + \tfrac{5}{M}\sum_i \mu(J_i) \leq \varepsilon + 5\mu[0,1]/M = 2\varepsilon.\end{aligned}$$

If $m_j = \mu(I_j) \geq M/k$ we have $l_j/k \geq (1 - 1/M)\mu(I_j)$. Thus

$$\begin{aligned}\sum_{j=1}^k l_j/k \geq \sum_{\mu(I_j)\geq M/k} l_j/k &\geq \sum_{\mu(I_j)\geq M/k} (1-1/M)\mu(I_j) = (1-1/M)\mu([0,1]\setminus B_k)\\ &\geq (1-1/M)\mu[0,1] - \mu(B_k) \geq \mu[0,1] - 3\varepsilon\end{aligned}$$

for all $k \geq k_0$. Letting $\varepsilon \downarrow 0$ we have the claim. □

Our next step towards proving Theorem 1.1 is the following lemma.

Lemma 2.3. *Let μ be a finite and singular measure on $[0,1]$ with no point masses and let $\delta > 0$. Then there is a purely 1-unrectifiable Borel set $A \subset [0,1] \times [0,\delta]$ such that $\mathcal{H}^1(A) > \mu[0,1]/2$ and $\operatorname{proj}\mathcal{H}^1|_A \leq \sqrt{2}\mu$.*

Proof. We first note that we may assume without loss of generality that $\mu[0,1] \leq \delta = 1$. Indeed, in the general case, we may first choose $k \in \mathbb{N}$ so large that $k > 1/\delta$ and $m_j = \mu[\frac{j-1}{k}, \frac{j}{k}] < \delta/2$ for all $1 \leq j \leq k$ and denote by l_j the greatest integer for which $l_j \leq m_j k$. Then we can write $\mu|_{[\frac{j-1}{k}, \frac{j}{k}]} = \sum_{i=1}^{l_j+1} \mu_{i,j}$ so that $\mu_{i,j}[\frac{j-1}{k}, \frac{j}{k}] \leq 1/k$ for each i, j, and, using a rescaled version of the statement, we can find purely unrectifiable sets $A_{i,j} \subset [\frac{j-1}{k}, \frac{j}{k}] \times [\frac{i-1}{k}, \frac{i}{k}]$ so that $\mathcal{H}^1(A_{i,j}) \geq \mu_{i,j}[\frac{j-1}{k}, \frac{j}{k}]/2$ and $\operatorname{proj}\mathcal{H}^1|_{A_{i,j}} \leq \sqrt{2}\mu_{i,j}$. Since the squares $[\frac{j-1}{k}, \frac{j}{k}] \times [\frac{i-1}{k}, \frac{i}{k}]$ are non-overlapping and they are inside $[0,1] \times [0, \frac{l_j+1}{k}]$ where $\frac{l_j+1}{k} \leq \max(1, 2l_j)/k < \delta$, we can take $A = \bigcup_{i,j} A_{i,j} \subset [0,1] \times [0,\delta]$.

Let $\mu[0,1] \leq \delta = 1$. We construct a set $A \subset [0,1] \times [0,1]$ by iterative use of Lemma 2.1. First choose numbers $\varepsilon_s > 0$ for $s \in \mathbb{N}$ such that $\sum_{s=1}^{\infty} \varepsilon_s \leq \mu[0,1]/2$.

Step 1: Define $m_j = m_{j,k} = \mu[\frac{j-1}{k}, \frac{j}{k}]$ for all $j, k \in \mathbb{N}$, $j \leq k$ and let $l_j = l_{j,k}$ be the greatest integer satisfying $l_j \leq m_j k$. By Lemma 2.2 we may choose $k = k_1$ large enough so that

$$\sum_{j=1}^{k} l_j/k > \mu[0,1] - \varepsilon_1. \tag{2.1}$$

Increasing k if necessary, we may assume that $l_j < k/18$ for all $1 \leq j \leq k$ since μ contains no point masses. Using Lemma 2.1, we may choose a porous collection of k_1-grid squares $\mathcal{R}^1 = \{Q^1_{i,j}\} \subset \mathcal{Q}_{k_1}$ that contains exactly l_j squares from the jth column and at most one square from each row. Let $A_1 = \bigcup_{i,j} Q^1_{i,j}$ be the union of all these squares.

Step n: Suppose that we are given a collection $\mathcal{R}^{n-1} = \{Q^{n-1}_{i,j}\} \subset \mathcal{Q}_k$ of porous k-grid squares, $k = k_{n-1}$, that contains at most one square from each row and $l_j = l_{j,n-1}$ squares from the jth column such that $l_j/k \leq \mu[\frac{j-1}{k}, \frac{j}{k}]$ and $\sum_{j=1}^{k} l_j/k \geq \mu[0,1] - \sum_{s=1}^{n-1} \varepsilon_s$. We now perform the Step 1 construction inside each of the squares $Q^{n-1}_{i,j}$ replacing $[0,1] \times [0,1]$ by $Q^{n-1}_{i,j}$, μ by $(k_{n-1}\mu[\frac{j-1}{k_{n-1}}, \frac{j}{k_{n-1}}])^{-1}\mu|_{[\frac{j-1}{k_{n-1}}, \frac{j}{k_{n-1}}]}$, and ε_1 by ε_n/k_{n-1}. In particular, we choose $k = k_n$ so large that (2.1) holds for each $Q = Q^{n-1}_{i,j}$ with $\mu[0,1] - \varepsilon_1$ replaced by $1/k_{n-1} - \varepsilon_n/k_{n-1}$. Denoting by $\mathcal{R}^n = \{Q^n_{i,j}\} \subset \mathcal{Q}_{k_n}$ the collection of the squares obtained inside all the squares $Q \in \mathcal{R}^{n-1}$ we define $A_n = \bigcup_{i,j} Q^n_{i,j}$. It is clear that $\mathcal{R}^n$ has the same properties as $\mathcal{R}^{n-1}$. Namely, it is porous, contains at most one square from each row and $l_j = l_{j,n}$ squares from the jth column such that $l_{j,n}/k_n \leq \mu[\frac{j-1}{k_n}, \frac{j}{k_n}]$ for all $1 \leq j \leq k_n$, and moreover $\sum_{j=1}^{k_n} l_{j,n}/k_n \geq \mu[0,1] - \sum_{s=1}^{n} \varepsilon_s$. We finally define $A = \bigcap_n A_n$.

It remains to show that A is purely 1-unrectifiable and that it has the desired properties $\mathcal{H}^1(A) > \mu[0,1]/2$ and $\operatorname{proj} \mathcal{H}^1|_A \leq \sqrt{2}\mu$. We start from the pure unrectifiability of A. Suppose that $\Gamma \subset \mathbb{R}^2$ is a C^1-curve. Since the collections $\mathcal{R}^n$ are porous for all $n \in \mathbb{N}$ it follows that the set $\Gamma \cap A$ has no density points, i.e. points $x \in \Gamma \cap A$ for which $\lim_{r \downarrow 0} \mathcal{H}^1(\Gamma \cap A)/(2r) = 1$. This implies that $\mathcal{H}^1(\Gamma \cap A) = 0$ and thus A is purely 1-unrectifiable.

Recall that $\mathcal{R}^n$ contains at most one square from each row, hence A contains at most one point on each, except possibly for countably many, horizontal line. Let proj_2 denote the projection to the y-axis $(x, y) \to y$, and let ν be the measure defined by $\nu(B) = \mathcal{H}^1(\operatorname{proj}_2(A \cap B))$. Since projection cannot increase the $\mathcal{H}^1$ measure, it is clear that $\nu \leq \mathcal{H}^1|_A$. It is also easy to see that $\nu \geq \frac{1}{\sqrt{2}}\mathcal{H}^1|_A$: indeed, since $\mathcal{R}^n$ contains at most one square from each row, hence for $k = k_n$ and for each interval $I = (\frac{i-1}{k}, \frac{i}{k})$, $A \cap (\operatorname{proj}_2^{-1} I)$ can be covered by a square of side length $1/k$, i.e. of diameter $\sqrt{2}/k$. Therefore it is enough to show that $\nu(A) \geq \mu[0,1]/2$ and $\operatorname{proj} \nu \leq \mu$.

The first inequality follows immediately from

$$\nu(A) = \mathcal{H}^1(\operatorname{proj}_2 A) = \lim \mathcal{H}^1(\operatorname{proj}_2 A_n)$$

and

$$\mathcal{H}^1(\operatorname{proj}_2 A_n) = \sum_{j=1}^{k_n} l_{j,n}/k_n \geq \mu[0,1] - \sum_{s=1}^{n} \varepsilon_s \geq \mu[0,1]/2.$$

The second inequality follows from the fact that for each $k = k_n$, above each interval $J = (\frac{j-1}{k}, \frac{j}{k})$, the set A is covered by l_j squares of $\mathcal{R}^n$ of side length $1/k$, hence $\nu(A \cap \operatorname{proj}^{-1}(J)) \leq l_j/k \leq m_j = \mu(J)$. □

To prove Theorem 1.1 for singular μ we still have to show how to find a purely unrectifiable $A \subset \mathbb{R}^2$ such that the measures $\operatorname{proj} \mathcal{H}^1|_A$ and μ are the same. An immediate corollary of Lemma 2.3 is that for any singular measure μ on $[0,1]$ with no point masses and for any $\delta > 0$ there is a purely unrectifiable $A \subset [0,1] \times [0,\delta]$ for which $\mathcal{H}^1(A) \geq 2^{-3/2}\mu[0,1]$ and $\mathcal{H}^1|_A \leq \mu$.

Proof of Theorem 1.1 when μ is singular. Without loss of generality we can assume that μ is supported on $[0,1]$. First we choose a purely unrectifiable set $A_1 \subset [0,1] \times [0,1/2]$ so that $\mathcal{H}^1(A_1) \geq 2^{-3/2}\mu[0,1]$ and $\operatorname{proj} \mathcal{H}^1|_{A_1} \leq \mu$. Then consider $\mu_2 = \mu - \operatorname{proj} \mathcal{H}^1|_{A_1}$ and choose a purely unrectifiable $A_2 \subset [0,1] \times [1/2, 3/4]$ for which $\mathcal{H}^1(A_2) \geq 2^{-3/2}\mu_2[0,1]$ and $\operatorname{proj} \mathcal{H}^1|_{A_2} \leq \mu_2$. Proceeding in this manner we get purely unrectifiable sets $A_n \subset [0,1] \times [(1-2^{-n+1})l, (1-2^{-n})]$ and corresponding measures μ_n so that $\mathcal{H}^1(A_n) \geq 2^{-3/2}\mu_n[0,1]$, $\operatorname{proj} \mathcal{H}^1|_{A_n} \leq \mu_n$ and $\mu_{n+1} = \mu_n - \operatorname{proj} \mathcal{H}^1|_{A_n}$. Then clearly $\mu_{n+1}[0,1] \leq (1-2^{-3/2})\mu_n[0,1]$ for each n, in particular, $\mu_n[0,1] \to 0$. Since $\mu = \sum_{i=1}^{n} \operatorname{proj} \mathcal{H}^1|_{A_i} + \mu_{n+1}$, this shows $\mu = \sum_{i=1}^{\infty} \operatorname{proj} \mathcal{H}^1|_{A_i}$. Since the sets A_i are purely unrectifiable and they are contained in pairwise non-overlapping rectangles, for $A = \bigcup_{i=1}^{\infty} A_i$, $\mu = \sum_{i=1}^{\infty} \operatorname{proj} \mathcal{H}^1|_{A_i} = \operatorname{proj} \mathcal{H}^1|_A$. □

Remark 2.4. The method presented above may be used to construct also other fractal-type measures ν on $\mathbb{R}^2$ such that $\operatorname{proj} \nu = \mu$ for a given locally finite Borel regular measure μ. At least the following statements may be obtained:

(1) If $0 < s < 1$ and $\lim_{r \downarrow 0} \mu[x-r, x+r]/(2r)^s = \infty$ for μ almost all $x \in \mathbb{R}$, then there is a Borel set $A \subset \mathbb{R}^2$ such that $\mu = \operatorname{proj} \mathcal{H}^s|_A$.
(2) If $s > 1$ and $\limsup_{r \downarrow 0} \mu[x-r, x+r]/(2r)^{s-1} < \infty$ for μ almost all $x \in \mathbb{R}$, then there is a Borel set $A \subset \mathbb{R}^2$ such that $\mu = \operatorname{proj} \mathcal{H}^s|_A$.

To prove (1) one uses the following simple observation in place of Lemma 2.3 (the notation is as in Lemma 2.3): If Q is a collection of k-grid squares such that $\sum_{j=1}^{k} l_j \leq k^s$, then there is a collection $\mathcal{Q}'$ containing exactly l_j squares from the jth column such that $\#\{Q \in \mathcal{Q}' : B \cap Q \neq \emptyset\} \leq Ck^s \operatorname{diam}(B)^s$ for all balls $B \subset \mathbb{R}^2$ such that $\frac{1}{k} \leq \operatorname{diam}(B) \leq 1$. To prove (2) we observe that a similar statement holds true if $s > 1$ and $\sum_{j=j_0}^{j_1} l_j \leq Ck(j_1 - j_0)^{s-1}$ for all

$1 \leq j_0 \leq j_1 \leq k$. This is seen just by distributing the l_j squares in the jth column evenly along the rows $1 \leq i \leq k$.

3. The absolutely continuous case

In this section we prove Theorem 1.1 for absolutely continuous μ. Let us begin with some preparations. For $\lambda > 0$ we define similitudes $f_i^\lambda \colon \mathbb{R}^2 \to \mathbb{R}^2$ for $i = 1, 2, 3$ by the formulae $f_1^\lambda(x, y) = \frac{1}{3}(x, y) + (0, 0)$, $f_2^\lambda(x, y) = \frac{1}{3}(x, y) + (\frac{1}{3}, \lambda\frac{2}{3})$, and $f_3^\lambda(x, y) = \frac{1}{3}(x, y) + (\frac{2}{3}, \lambda\frac{1}{3})$. Let $C_\lambda \subset [0, 1] \times [0, \lambda]$ be the self similar set induced by the similitudes f_i^λ, see Figure 2.

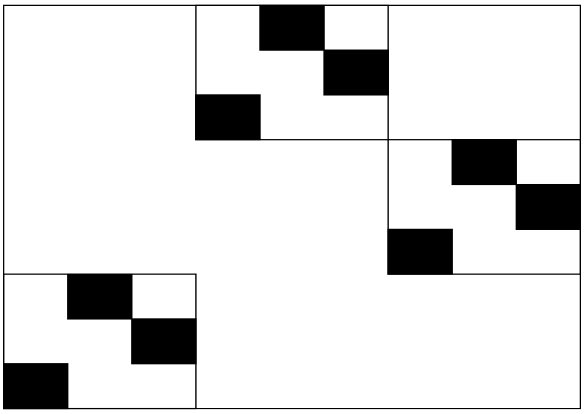

Figure 2. The set C_λ.

Define $h(\lambda) = \mathcal{H}^1(C_\lambda)$. Since the projection of C_λ to the y-axis has length λ we have $h(\lambda) \geq \lambda$, in particular $\lim_{\lambda\to\infty} h(\lambda) = \infty$. It is also easy to see that $\lim_{\lambda\downarrow 0} h(\lambda) = 1$. For all $0 < \lambda_0, \lambda_1 < \infty$ the set C_{λ_1} is obtained from C_{λ_0} by the vertical stretching/flattening $(x, y) \mapsto (x, \frac{\lambda_1}{\lambda_0} y)$ and we observe that h is continuous and non-decreasing. It is also useful to note that if ν is the natural probability measure on C_λ, then $\operatorname{proj}\nu = \mathcal{L}|_{[0,1]}$ and since $\mathcal{H}^1|_{C_\lambda} = h(\lambda)\nu$ we see that $\operatorname{proj}\mathcal{H}^1|_{C_\lambda} = h(\lambda)\mathcal{L}$.

For any $0 < \lambda < \infty$ we define an operation $\mathcal{O}_\lambda$ on all rectangles $R = (x, y) + [0, l_x] \times [0, l_y] \subset \mathbb{R}^2$ for which $l_y \geq \lambda l_x$ by the formula

$$\mathcal{O}_\lambda(R) = (x, y) + l_x \left(\bigcup_{i=1}^{3} f_i^\lambda([0, 1] \times [0, \lambda]) \right).$$

Recall that then $\mathcal{O}_\lambda(R) \subset R$. We now define the increasing function $g \colon (1, \infty) \to (0, \infty)$ by $g(t) = \max h^{-1}(\{t\})$ for all $t > 1$ (If h is one to one we can simply take $g = h^{-1}$ and then g is continuous but we do not know if this is the case).

Proof of Theorem 1.1 when μ is absolutely continuous. We assume that $\operatorname{spt}\mu \subset [0, 1]$ and identify μ with its density $\mu \colon [0, 1] \to [0, \infty)$ when convenient. For simplicity, we assume that μ is continuous and that $\mu^{-1}\{t\}$ has measure zero for all $t \geq 1$. The general case reduces to this as discussed at the end of the proof.

The purely unrectifiable set A is now constructed in the following manner. Let $t_{\max} = \max_{x\in[0,1]} \mu(x)$ and $A_0 = [0,1] \times [0, g(t_{\max})]$. Suppose that $A_k = \bigcup_{j=1}^{3^k} R_j^k$ has been defined where $R_j^k = [(j-1)3^{-k}, j3^{-k}] \times J_j^k$ for all $1 \leq j \leq 3^k$ and $3^k \ell(J_j^k) \geq g(t_j)$ where $t_j = \max_{x\in[(j-1)3^{-k}, j3^{-k}]} \mu(x)$. We then define

$$A_{k+1} = \bigcup_{j=1}^{3^k} \mathcal{O}_{g(t_j)}(R_j^k)$$

and finally $A = \bigcap_k A_k$. Then A is purely 1-unrectifiable which can be seen by looking at the set $A_t = A \cap \operatorname{proj}^{-1}(\mu^{-1}(t,\infty))$ for a fixed $t > 1$: The set $\mu^{-1}(t,\infty) \subset [0,1]$ is an open set and if $I \subset \mu^{-1}(t,\infty)$ is a triadic interval of length 3^{-j}, the set $A \cap \operatorname{proj}^{-1}(I)$ consists of three distinct parts so that the distance between any two of them is at least $\min\{\frac{1}{9}, \frac{g(t)}{3}\}3^{-j}$. It follows as in the proof of Lemma 2.3 that no C^1-curve Γ can intersect A_t in a set of positive measure. Since $A_t \subset A$ for all $t > 1$ and $\mathcal{H}^1(A_t) \to \mathcal{H}^1(A_1)$ as $t \to 1$ it follows that A is purely 1-unrectifiable. Recall that we assumed that the level sets of μ, in particular $\mu^{-1}\{1\}$, have measure zero.

To complete the proof we have to show that $\operatorname{proj} \mathcal{H}^1|_A = \mu$. This will be done using the following lemma.

Lemma 3.1. *Let* $1 < t < \infty$*,* $\varepsilon > 0$*, and* $B_{t,\varepsilon} = \mu^{-1}(t, t+\varepsilon)$*. Then* $\frac{1}{c}\mu|_{B_{t,\varepsilon}} \leq (\operatorname{proj} \mathcal{H}^1|_A)|_{B_{t,\varepsilon}} \leq c\mu|_{B_{t,\varepsilon}}$ *where*

$$c = 1 + 54\,(g(t+\varepsilon) - g(t))\,/\min\{1, 3g(t)\}. \tag{3.1}$$

Proof. We begin with a technical remark. Let $E \subset [0,1]$ denote the countable set consisting of the endpoints of all triadic intervals $I \subset [0,1]$. Since A is purely 1-unrectifiable, the measure $\mathcal{H}^1(A)$ does not change if we remove the vertical lines $\operatorname{proj}^{-1}\{x\}$ from the set A for all $x \in E$. This makes the mapping $x \mapsto \operatorname{proj} x$, $A \to [0,1] \setminus E$ one to one. For a given $\lambda > 0$ we do the same for the set C_λ, that is, remove the vertical lines $\operatorname{proj}^{-1}\{x\}$ from C_λ for all $x \in E$. After this we can define a natural bijection between A and C_λ by demanding that $x \mapsto x'$ if and only if $\operatorname{proj}(x') = \operatorname{proj}(x)$.

Since $B_{t,\varepsilon}$ is an open set it is enough to show that $\frac{1}{c}\mu(I) \leq (\operatorname{proj} \mathcal{H}^1|_A)(I) \leq c\mu(I)$ for any triadic interval $I \subset B_{t,\varepsilon}$ and by scaling this reduces to showing that $\frac{1}{c}\mu[0,1] \leq \mathcal{H}^1(A) \leq c\mu[0,1]$ assuming $B_{t,\varepsilon} = [0,1]$.

Let $x, y \in A$, $x \neq y$ and $x_j, y_j \in \{0,1,2\}$ be such that $\operatorname{proj} x = \sum_{j=1}^\infty x_j 3^{-j}$ and $\operatorname{proj} y = \sum_{j=1}^\infty y_j 3^{-j}$. We define $\lambda_x^j = g(\max_{x\in I_x^j} \mu(x))$ where I_x^j is the unique triadic interval of size 3^{-j} containing x. The numbers λ_y^j are defined in a similar manner. Now $\operatorname{proj}_2 x = \sum_{j=1}^\infty \lambda_x^{j-1} x_j' 3^{-j}$ where the mapping $x_j \mapsto x_j'$ is defined by the rules $0 \mapsto 0, 1 \mapsto 2, 2 \mapsto 1$. Similarly $\operatorname{proj}_2 y = \sum_{j=1}^{\infty} \lambda_y^{j-1} y_j' 3^{-j}$. Recall that proj_2 denotes the orthogonal projection onto the vertical coordinate axis.

Let j_0 be the smallest integer for which $I_x^{j_0} \neq I_y^{j_0}$ and let $x', y' \in C_{g(t)}$ so that $\operatorname{proj} x = \operatorname{proj} x'$ and $\operatorname{proj} y = \operatorname{proj} y'$. Then

$$|x' - y'| \geq \min\{\tfrac{1}{9}, \tfrac{g(t)}{3}\} 3^{-j_0} \tag{3.2}$$

since $\operatorname{dist}(f_i^{g(t)}(C_{g(t)}), f_j^{g(t)}(C_{g(t)})) \geq \min\{\frac{1}{9}, \frac{g(t)}{3}\}$ whenever $i, j \in \{1, 2, 3\}$ and $i \neq j$. Moreover

$$\begin{aligned}
|(x-y) - (x'-y')| &= |\operatorname{proj}_2(x - x') - \operatorname{proj}_2(y - y')| \\
&= \Big| \big(\sum_{j=j_0}^{\infty} (\lambda_x^{j-1} - g(t)) x_j' 3^{-j} \big) - \big(\sum_{j=j_0}^{\infty} (\lambda_y^{j-1} - g(t)) y_j' 3^{-j} \big) \Big| \\
&\leq 4(g(t+\varepsilon) - g(t)) \sum_{j=j_0} 3^{-j} = 6(g(t+\varepsilon) - g(t)) 3^{-j_0}
\end{aligned}$$

since $\lambda_x^j, \lambda_y^j \in (g(t), g(t+\varepsilon))$ for all j. Combined with (3.2) this gives $|x - y| \leq c|x' - y'|$ where c is as in (3.1). Thus the natural bijection between $C_{g(t)}$ and A is c-Lipschitz and we get

$$\mathcal{H}^1(A) \leq c\mathcal{H}^1(C_{g(t)}) = c\, t < c\, \mu[0,1]. \tag{3.3}$$

By a similar reasoning we see that $|x'' - y''| \leq c|x - y|$ if $x'', y'' \in C_{g(t+\varepsilon)}$ for which $\operatorname{proj} x = \operatorname{proj} x''$ and $\operatorname{proj} y = \operatorname{proj} y''$. This gives $c\mathcal{H}^1(A) \geq \mathcal{H}^1(C_{g(t+\varepsilon)}) = t + \varepsilon > \mu[0,1]$ and together with (3.3) completes the proof. $\square$

We may now finish the proof of Theorem 1.1. Let $1 < t_0 < t_{max}$, $A_{t_0} = \mu^{-1}(t_0, t_{\max})$, and $\delta > 0$. Since g is nondecreasing we may cover all, except possibly at most countably many, points of (t_0, t_{max}) by pairwise disjoint intervals (t, t') such that $g(t') - g(t) < \delta$. Lemma 3.1 then implies that $\frac{1}{c}\mu|_{A_{t_0}} \leq (\operatorname{proj} \mathcal{H}^1|_A)|_{A_{t_0}} \leq c\mu|_{A_{t_0}}$ where $c = 1 + 54\delta / \min\{1, 3g(t_0)\}$ (recall that $\mu^{-1}\{t\}$ has measure zero for all t). Letting first $\delta \downarrow 0$ and then $t_0 \downarrow 1$ we get $\mu = \operatorname{proj} \mathcal{H}^1|_A$. This proves the Theorem for a continuous μ whose level sets are of measure zero.

For a general μ there are at most countably many values t_n for which $B_n = \mu^{-1}\{t_n\}$ has positive measure and letting $A_n = C_{g(t_n)} \cap \operatorname{proj}^{-1} B_n$ we have $\mu|_{B_n} = \operatorname{proj} \mathcal{H}^1|_{A_n}$. (If $t_n = 1$ we cannot use $C_0 = [0,1] \subset \mathbb{R}^2$ since it is rectifiable, but one easily finds a purely 1-unrectifiable set $A_0 \subset \mathbb{R}^2$ for which $\operatorname{proj} \mathcal{H}^1|_{A_0} = \mathcal{L}$.) Let $B = [0,1] \setminus \bigcup_n B_n$. We now use Lusin's Theorem to find a compact set $F_1 \subset B$ with $\mu(B \setminus F_1) < \frac{1}{2}$ such that $\mu|_{F_1}$ is continuous. Then we extend $\mu|_{F_1}$ to a continuous function $\nu\colon [0,1] \to [1, \infty)$ whose level sets are of measure zero. The above argument now gives us a purely 1-unrectifiable set $A \subset \mathbb{R}^2$ with $\operatorname{proj} \mathcal{H}^1|_A = \nu$ and letting $A^1 = A \cap \operatorname{proj}^{-1}(F_1)$ we have $\operatorname{proj} \mathcal{H}^1|_{A^1} = \mu|_{F_1}$. We continue with the same argument and find a set $F_2 \subset B \setminus F_1$ so that μ is continuous on F_2 and $\mu(B \setminus (F_1 \cup F_2)) < \frac{1}{4}$. Then we define a purely 1-unrectifiable set A^2 such that $\operatorname{proj}\mathcal{H}^1|_{A^2} = \mu|_{F_2}$ and so on. Defining finally A as the union of the sets A_n and A^n we have $\operatorname{proj}\mathcal{H}^1|_A = \mu$. □

Remark 3.2. The construction proving Theorem 1.1 in the absolutely continuous case easily generalises to higher dimensions. Thus, for all absolutely continuous measures μ on $\mathbb{R}^n$ with $\lim_{r\downarrow 0}\mu(B(x,r)/(2r)^n \geq 1$ for μ-almost all x, there is a purely n-unrectifiable Borel set $A \subset \mathbb{R}^{n+1}$ such that $\mu = \operatorname{proj}\mathcal{H}^n|_A$. Here $\operatorname{proj}(x_1,\ldots,x_n,x_{n+1}) = (x_1,\ldots,x_n)$ and $\mathcal{H}^n$ is the non-normalised Hausdorff n-measure. We do not have a characterisation for the singular case in higher dimensions although we conjecture that a singular measure μ on $\mathbb{R}^n$ may be expressed as $\operatorname{proj}\mathcal{H}^n|_A$ for some purely n-unrectifiable $A \subset \mathbb{R}^{n+1}$ if and only if μ itself is purely $(n-1)$-unrectifiable in the sense that $\mu(B) = 0$ for all $(n-1)$-rectifiable sets $B \subset \mathbb{R}^n$.

References

[Fa] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1985.

[Ma] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*, Cambridge University Press, Cambridge, 1995.

MC: Department of Mathematics University College London Gower Street, London WC1E 6BT, UK

E-mail address: mari@math.ucl.ac.uk

VS: Department of Mathematics and Statistics P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland,

E-mail address: visuomal@maths.jyu.fi