SMALL POROSITY, DIMENSION AND REGULARITY IN METRIC MEASURE SPACES

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Abstract. Let X be a metric measure space with an s-regular measure μ. We prove that if $A \subset X$ is ϱ-porous, then $\dim_{\mathrm{p}}(A) \leq s - c\varrho^s$ where $\dim_{\mathrm{p}}$ is the packing dimension and c is a positive constant which depends on s and the structure constants of μ. This is an analogue of a well known asymptotically sharp result in Euclidean spaces. We illustrate by an example that the corresponding result is not valid if μ is a doubling measure. However, in the doubling case we find a fixed $N \subset X$ with $\mu(N) = 0$ such that $\dim_{\mathrm{p}}(A) \leq \dim_{\mathrm{p}}(X) - c(\log \frac{1}{\varrho})^{-1}\varrho^t$ for all ϱ-porous sets $A \subset X \setminus N$. Here c and t are constants which depend on the structure constant of μ. Finally, we characterize uniformly porous sets in complete s-regular metric spaces in terms of regular sets by verifying that A is uniformly porous if and only if there is $t < s$ and a t-regular set F such that $A \subset F$.

1. Introduction

The purpose of this paper is twofold: we study dimensional properties of porous sets in s-regular and doubling metric measure spaces and characterize uniformly porous sets in terms of regularity. For definitions we refer to Sections 2 and 3.

In Euclidean spaces dimensional properties of porous sets have been studied extensively, see for example [BS], [JJKS], [KS], [KR], [L], [MV], [M1], [N], [S], [T] and references therein. It is well known that if $A \subset \mathbb{R}^n$ is ϱ-porous, meaning that A contains holes of relative size ϱ at all small scales, then

$$\dim_{\mathrm{p}}(A) \leq n - c\varrho^n \tag{1.1}$$

where $\dim_{\mathrm{p}}$ is the packing dimension and c is a positive constant depending on n only (see [MV, T]). Furthermore, (1.1) is asymptotically sharp as ϱ tends to zero ([KR], [KS, Remark 4.2]). In [DS] and [BHR] it is shown that the dimension of a porous measure in a (globally) s-regular space is smaller than s. In this paper

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we address the question to what extend the quantitative estimate (1.1) is valid in metric measure spaces X. It turns out that the following analogue of (1.1) holds provided that X is equipped with a (locally) s-regular measure μ: if $A \subset X$ is ϱ-porous, then

$$\dim_{\mathrm{p}}(A) \leq s - c\varrho^s \tag{1.2}$$

where c is a positive constant which depends on s and the structure constants a_μ and b_μ of μ (see Theorem 4.8). Note that by [Cu, Theorem 3.16] $\dim_{\mathrm{p}}(X) = s$ provided that X is s-regular. We also show that the dependence on a_μ and b_μ is necessary, that is, unlike in $\mathbb{R}^n$, it is not possible to find c which depends on s only (see Remark 4.13.(3)).

In (1.2) it is not sufficient to assume that μ is doubling: in Example 4.9 we construct a geodesic doubling metric space X having a subset with maximal dimension and porosity. However, in general the failure of the dimension drop is due to a fixed set with μ-measure zero provided that μ is doubling. More precisely, in Theorem 4.10 we show that there exists $N \subset X$ with $\mu(N) = 0$ such that $\dim_{\mathrm{p}}(A) \leq \dim_{\mathrm{p}}(X) - c(\log \frac{1}{\varrho})^{-1}\varrho^t$ for all ϱ-porous $A \subset X \setminus N$. Here t and c are constants which depend on the structure constant c_μ of μ.

As in Euclidean spaces, in complete s-regular metric measure space X uniform porosity is closely related to regularity. We prove that $A \subset X$ is uniformly porous if and only if there are $t < s$ and a t-regular set $F \subset X$ such that $A \subset F$ (see Theorem 5.3). The easier if-part was proven in [BHR], but we give some quantitative estimates on the relations between porosity, t and s.

The paper is organized as follows: In Section 2 we discuss the concept of porosity we are using in metric measure spaces whilst Section 3 is dedicated to measure theoretic preliminaries. Dimension estimates for porous sets are dealt in Section 4. In last section we focus on connections between uniform porosity and regularity.

2. Notation

Let $X = (X, d)$ be a separable metric space and $A \subset X$. For $x \in X$ and $r > 0$, we set

$$\begin{aligned} \mathrm{por}(A, x, r) = \sup\{\varrho \geq 0 :& \text{ there is } y \in X \text{ such that } B(y, \varrho r) \cap A = \emptyset \\ & \text{ and } \varrho r + d(x, y) \leq r\}. \end{aligned} \tag{2.1}$$

Here $B(x, r)$ denotes the closed ball centred at x with radius r. The *porosity of A at a point x* is defined to be

$$\mathrm{por}(A, x) = \liminf_{r \downarrow 0} \mathrm{por}(A, x, r) \tag{2.2}$$

and the *porosity of A* is given by

$$\mathrm{por}(A) = \inf_{x \in A} \mathrm{por}(A, x). \tag{2.3}$$

We call $A \subset X$ *porous* if $\operatorname{por}(A) > 0$, and more precisely, *$\varrho$-porous* provided that $\operatorname{por}(A) > \varrho$. Furthermore, $A \subset X$ is *uniformly (ϱ-)porous* if there exist constants $\varrho > 0$ and $r_p > 0$ such that $\operatorname{por}(A, x, r) > \varrho$ for all $x \in A$ and $0 < r < r_p$.

Remarks 2.1. (1) Even though it would be more accurate to use the term lower porosity for $\operatorname{por}(A, x)$ and $\operatorname{por}(A)$ to distinguish them from upper porosities defined by replacing lim inf by lim sup in (2.2), we keep the terminology shorter. Upper porosities are irrelevant for our purposes; there is no nontrivial upper bound for dimensions of upper porous sets. In fact, there exist sets in $\mathbb{R}^n$ with maximum upper porosity and with Hausdorff dimension n, see [M2, §4.12].

(2) We follow the convention introduced in [MMPZ] to use $\operatorname{por}(A, x, r)$ and $\operatorname{por}(A, x)$ instead of

$$\operatorname{por}^*(A, x, r) = \sup\{\varrho \geq 0 \,:\, B(y, \varrho r) \subset B(x, r) \setminus A \text{ for some } y \in X\}$$

and

$$\operatorname{por}^*(A, x) = \liminf_{r \downarrow 0} \operatorname{por}^*(A, x, r)$$

to guarantee that $0 \leq \operatorname{por}(A, x, r) \leq \frac{1}{2}$ for all $A \subset X$, $x \in A$ and $r > 0$. From the point of view of our results, however, there is no difference between por and por* since we always have $\operatorname{por}(A, x, r) \leq \operatorname{por}^*(A, x, r) \leq 2 \operatorname{por}(A, x, 2r)$, and therefore, $\operatorname{por}(A, x) \leq \operatorname{por}^*(A, x) \leq 2 \operatorname{por}(A, x)$.

(3) To emphasize the underlying metric space, we write $\operatorname{por}^*_{(X,d)}$ instead of por* in what follows. Observe that $0 \leq \operatorname{por}^*_{(\mathbb{R}^n, |\cdot|)}(A) \leq \frac{1}{2}$ for all $A \subset \mathbb{R}^n$, where $|\cdot|$ denotes the usual Euclidean metric. This is not necessarily true in general metric spaces. Indeed, choosing $0 < \varepsilon < 1$, we have $\operatorname{por}^*_{(\mathbb{R}^n, |\cdot|^\varepsilon)}(A) = \operatorname{por}^*_{(\mathbb{R}^n, |\cdot|)}(A)^\varepsilon$ for every $A \subset \mathbb{R}^n$. Hence, for example, $\operatorname{por}^*_{(\mathbb{R}^n, |\cdot|^\varepsilon)}(\{x\}) = (\frac{1}{2})^\varepsilon$ for every $x \in \mathbb{R}^n$. In the following remark, we show that $*$-porosity may be exactly one.

(4) We work in $\mathbb{R}^2$ with the polar coordinates. Define

$$X = \{(lq, 2\pi q) : 0 \leq l \leq 1 \text{ and } q \in \mathbb{Q} \cap [0, 1)\}$$

and equip X with the path metric. We claim that $\operatorname{por}^*\big(\{(0, 0)\}\big) = 1$. Let $0 < r < 1$. For each $i \in \mathbb{N}$ choose $q_i \in \mathbb{Q} \cap [0, r]$ such that $\sup\{q_i : i \in \mathbb{N}\} = r$. It follows immediately that for every $i \in \mathbb{N}$ and $\varepsilon > 0$

$$B\big((q_i, 2\pi q_i), q_i - \varepsilon\big) \subset B\big((0, 0), r\big) \setminus \{(0, 0)\},$$

that is, $\operatorname{por}^*\big(\{(0, 0)\}, (0, 0), r\big) \geq (q_i - \varepsilon)/r$. Hence $\operatorname{por}^*\big(\{(0, 0)\}, (0, 0), r\big) = 1$ for every $0 < r < 1$ and the claim is proved.

(5) The following simple but extremely useful fact will be frequently needed: If $\operatorname{por}(A) > \varrho$, then $A = \cup_{k \in \mathbb{N}} A_k$ where

$$A_k = \{x \in A \,:\, \operatorname{por}(A, x, r) > \varrho \text{ for all } 0 < r < 1/k\}.$$

Given any $\varepsilon > 0$, we may, using the separability, choose A_{kj} such that $A_k = \cup_{j \in \mathbb{N}} A_{kj}$ and $\operatorname{diam}(A_{kj}) < \varepsilon$ for all k and j. (Here diam is the diameter of a set.)

3. Measure theory in metric spaces

This section contains some basic facts of measure and dimension theory in metric spaces that will be needed later. Recall that X is a separable metric space. By a measure we always mean a Borel regular outer measure defined on all subsets of X, see [M2, Definition 1.1]. We say that μ is σ-finite if $X = \cup_{k\in\mathbb{N}} A_k$ where $\mu(A_k) < \infty$ for each k.

The separability assumption is natural given our interest in dimension estimates since the Hausdorff dimension of a non-separable metric space X is infinite and usually one can find porous sets $A \subset X$ that are non-separable. Moreover, no σ-finite doubling measures exist in non-separable spaces.

We denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure defined on X. As in [M2, §5.3], we define for a bounded set $A \subset X$, $\lambda \geq 0$ and $r > 0$

$$M^\lambda(A,r) = \inf\{kr^\lambda : A \subset \bigcup_{i=1}^{k} B(x_i,r) \text{ for some } x_i \in X, k \in \mathbb{N}\}$$

with the interpretation $\inf\emptyset = \infty$. The (upper) Minkowski dimension of a bounded set A is

$$\dim_{\mathrm{M}}(A) = \inf\{\lambda : \limsup_{r\downarrow 0} M^\lambda(A,r) < \infty\}.$$

The packing dimension of $A \subset X$ is given by

$$\dim_{\mathrm{p}}(A) = \inf\bigl\{\sup_i \dim_{\mathrm{M}}(A_i) \,:\, A_i \text{ is bounded and } A \subset \bigcup_{i=1}^{\infty} A_i\bigr\}.$$

Alternatively, the packing dimension may be defined in terms of the (radius based) packing measures $\mathcal{P}^s$ (see Cutler [Cu, §3.1] for the definition) by the identity (here $\sup\emptyset = 0$)

$$\dim_{\mathrm{p}}(A) = \sup\{s \geq 0 \,:\, \mathcal{P}^s(A) > 0\},$$

see [Cu, Theorem 3.11]. Since $\mathcal{H}^\lambda(A) \leq \liminf_{r\downarrow 0} M^\lambda(A,r)$ for all bounded sets $A \subset X$, we immediately get $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{p}}(A) \leq \dim_{\mathrm{M}}(A)$, where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. It is also easy to see that $\dim_{\mathrm{p}}(X) < \infty$ whenever X carries a doubling measure, consult [Cu, Theorem 3.16].

Let $s > 0$. A measure μ on X is *s-regular on a set* $A \subset X$ if there are constants $0 < a_\mu \leq b_\mu$ and $r_\mu > 0$ such that

$$a_\mu r^s \leq \mu\bigl(B(x,r)\bigr) \leq b_\mu r^s \tag{3.1}$$

for all $x \in A$ and $0 < r < r_\mu$. A set $A \subset X$ is *s-regular* if there is a measure μ which is s-regular on A and $\mu(X \setminus A) = 0$. In particular, a metric space X is *s-regular* if there is a measure μ which is s-regular on X.

A measure μ on X is called *doubling* if there are constants $c_\mu \geq 1$ and $r_\mu > 0$ such that

$$0 < \mu\big(B(x,2r)\big) \leq c_\mu \mu\big(B(x,r)\big) < \infty \tag{3.2}$$

for every $x \in X$ and $0 < r < r_\mu$. When we deal with c_μ we always assume that it is the smallest constant that satisfies (3.2) with a given r_μ. A metric space is *doubling* if there exists a constant $N \in \mathbb{N}$ such that for each $r > 0$, every closed ball with radius $2r$ can be covered by a family of at most N closed balls of radius r. Notice that an s-regular measure on X is doubling, and moreover, by [LS], every complete doubling metric space carries a doubling measure. We use the convention that $c(\mu)$ always denotes a constant depending only on c_μ (or a_μ, b_μ and s) if μ is doubling (or s-regular).

Often in the literature it is assumed that (3.1) and (3.2) are valid for all $0 < r \leq \operatorname{diam}(X)$, that is, μ is globally s-regular or doubling (see for example [BHR]). However, for our purposes this is not needed by Remark 2.1.(5). The following example shows that it is not always possible to choose $r_\mu = \operatorname{diam}(X)$.

Example 3.1. Equip $X = [0,1] \times \mathbb{N} \subset \mathbb{R}^2$ with the metric d defined by

$$d\big((x_1,y_1),(x_2,y_2)\big) = \begin{cases} |x_1 - x_2|, & y_1 = y_2, \\ 1, & y_1 \neq y_2. \end{cases}$$

Let $\mu = \mathcal{H}^1$ be the length measure on X. Now $r \leq \mu(B(x,r)) \leq 2r$ whenever $0 < r < 1$, but $\mu(B(x,1)) = \infty$ for all $x \in X$.

It is straightforward to see that all doubling measures are σ-finite, in particular this is true for s-regular measures.

An easy exercise leads to the following lemma:

Lemma 3.2. *Suppose that μ is a doubling measure on X. For all $x \in X$, $0 < r < r_\mu$ and $0 < \alpha < 1$ we have*

$$\mu\big(B(x,\alpha r)\big) \geq c_\mu^{\lfloor \frac{\log \alpha}{\log 2} \rfloor} \mu\big(B(x,r)\big) \tag{3.3}$$

where $\lfloor a \rfloor$ is the greatest integer p satisfying $p \leq a$. Moreover, if μ is s-regular, then

$$\mu\big(B(x,\alpha r)\big) \geq \frac{a_\mu}{b_\mu} \alpha^s \mu\big(B(x,r)\big) \tag{3.4}$$

for all $x \in X$, $0 < r < r_\mu$ and $0 < \alpha < 1$.

Let μ be an s-regular measure on X. For all $\lambda \geq 0$ and $r > 0$ we define

$$M_\mu^\lambda(A,r) = \frac{\mu\big(A(r)\big)}{r^{s-\lambda}},$$

where

$$A(r) = \{x \in X : \operatorname{dist}(x,A) < r\}$$

is the open r-neighbourhood of A and $\operatorname{dist}(x,A) = \inf\{d(x,a) : a \in A\}$ is the distance of x from A. The following easy lemma shows how $M_\mu^\lambda(A,r)$ can be utilized to calculate $\dim_{\mathrm{M}}(A)$. We give a detailed proof for the convenience of the reader.

Lemma 3.3. *Suppose that μ is an s-regular measure on X. Let $A \subset X$ and $\lambda \geq 0$. Then*

$$2^{-s} b_\mu^{-1} M_\mu^\lambda(A,r) \leq M^\lambda(A,r) \leq 5^s a_\mu^{-1} M_\mu^\lambda(A,r)$$

whenever $0 < r < \frac{r_\mu}{2}$.

Proof. Fix $0 < r < r_\mu$ and $\lambda \geq 0$. For the right hand side inequality, we may assume that $\mu\big(A(r)\big) < \infty$. Attaching to each $x \in A$ a ball $B(x, \frac{1}{5}r)$, we find, using the $5r$-covering theorem, an index set I and points $x_i \in A$, $i \in I$, such that

$$A \subset \bigcup_{i \in I} B(x_i, r)$$

and $B(x_i, \frac{1}{5}r) \cap B(x_j, \frac{1}{5}r) = \emptyset$ for $i \neq j$. Let $\#I$ be the number of elements in I. Since

$$\mu\big(A(r)\big) \geq \mu\Big(\bigcup_{i \in I} B(x_i, \tfrac{1}{5}r)\Big) = \sum_{i \in I} \mu\big(B(x_i, \tfrac{1}{5}r)\big) \geq \#I a_\mu 5^{-s} r^s,$$

it follows that $\#I < \infty$. This in turn implies that

$$M_\mu^\lambda(A,r) = \frac{\mu\big(A(r)\big)}{r^{s-\lambda}} \geq a_\mu 5^{-s} \#I r^\lambda \geq a_\mu 5^{-s} M^\lambda(A,r).$$

For the left hand side inequality, we may assume that $M^\lambda(A,r) < \infty$. Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in X$ such that

$$A \subset \bigcup_{i=1}^k B(x_i, r) \quad \text{and} \quad M^\lambda(A,r) \geq k r^\lambda - \varepsilon.$$

Since now

$$\mu\big(A(r)\big) \leq \mu\Big(\bigcup_{i=1}^k B(x_i, 2r)\Big) \leq \sum_{i=1}^k \mu\big(B(x_i, 2r)\big) \leq k b_\mu (2r)^s,$$

we get

$$M_\mu^\lambda(A,r) = \frac{\mu\big(A(r)\big)}{r^{s-\lambda}} \leq 2^s b_\mu k r^\lambda \leq 2^s b_\mu \big(M^\lambda(A,r) + \varepsilon\big).$$

The proof is finished by letting $\varepsilon \downarrow 0$. $\square$

The next observation shows that any porous set on a space carrying a doubling measure must have zero measure. Note that the proposition (with its simple proof) is easily seen to hold for upper porous sets as well.

Proposition 3.4. *Suppose that μ is a doubling measure on X. If $A \subset X$ is porous then $\mu(A) = 0$.*

Proof. By Remark 2.1.(5), we may assume that A is uniformly ϱ-porous for some $\varrho > 0$. Furthermore, we may assume that A is closed since the closure of a uniformly porous set is uniformly porous. Assume on the contrary that $\mu(A) > 0$. Using the density theorem [H, Theorem 1.8], choose $x \in A$ for which

$$\lim_{r \downarrow 0} \frac{\mu\big(A \cap B(x,r)\big)}{\mu\big(B(x,r)\big)} = 1. \tag{3.5}$$

Since A is uniformly ϱ-porous we find $0 < r_p < r_\mu$ such that for all $0 < r < r_p$ there exists $y \in X$ for which

$$B(y, \varrho r) \subset B(x,r) \setminus A.$$

Hence by Lemma 3.2, we get for any $0 < r < r_p$

$$\frac{\mu\big(B(x,r) \setminus A\big)}{\mu\big(B(x,r)\big)} \geq \frac{\mu\big(B(y,\varrho r)\big)}{\mu\big(B(x,r)\big)} \geq c_\mu^{\lfloor \frac{\log \varrho}{\log 2} \rfloor - 1} > 0,$$

contrary to (3.5). □

4. Dimension estimates for porous sets

It is well known that in $\mathbb{R}^n$

$$\dim_{\mathrm{p}}(A) \leq n - c\varrho^n \tag{4.1}$$

for any ϱ-porous set $A \subset \mathbb{R}^n$. Here c is a positive constant depending on n. In particular, $\dim_{\mathrm{p}}(A) < n$ for all porous sets $A \subset \mathbb{R}^n$. In this section we discuss whether these estimates are valid in s-regular metric measure spaces and, more generally, on spaces that carry a doubling measure. In [DS, Lemma 5.8] it is stated that in the s-regular case $\dim_{\mathrm{p}}(A) \leq s - \eta$, where η depends on porosity and the constants of the s-regular measure. The proof is based on dyadic cubes whose side lengths are powers of ϱ. Thus (as in $\mathbb{R}^n$) this argument will give that $\eta = c(\log \frac{1}{\varrho})^{-1}\varrho^s$. In [BHR, Lemma 3.12] a different method is used to show that even the Assouad dimension of a porous subset of a globally s-regular space is less than s. (Recall that the Assouad dimension is always at least the packing dimension.) Pushing this argument further we will show that the factor $(\log \frac{1}{\varrho})^{-1}$ is not needed in the s-regular case. As a tool we need the following generalization of (2.1)-(2.3) which is a modification of the mean ε-porosity from [KR].

Definition 4.1. Let $0 < \varrho \leq 1$, $D > 1$, $0 < p \leq 1$ and $n_0, k_0 \in \mathbb{N}$. For all $k \in \mathbb{N}$ and $x \in X$, we denote by $A_k(x)$ the annulus

$$A_k(x) = \{y \in X \,:\, D^{-k} < d(x,y) \leq D^{-k+1}\}.$$

Furthermore, for $A \subset X$ define

$$\psi_k(x) = \begin{cases} 1 & \text{if } A_k(x) \text{ contains } y \text{ with } \operatorname{dist}(y, A) > \varrho d(y, x) \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$S_{k_0,n}(x) = \sum_{k=k_0+1}^{k_0+n} \psi_k(x).$$

The set $A \subset X$ is $(\varrho, D, p, n_0, k_0)$-mean porous if $S_{k_0,n}(x) \geq pn$ for all $x \in A$ and $n \geq n_0$.

Lemma 4.3 generalizes the arguments of [MV, Lemma 2.8], [KR, Theorem 2.1] and [BHR, Lemma 3.12] to our setting. For the purpose of proving it, we state an auxiliary result that can be found from [H, Exercise 2.10]. If $B = B(x, r)$ is a ball in X and $R > 0$, we denote by RB the ball $B(x, Rr)$. Moreover, we use the notation χ_B for the characteristic function of B.

Lemma 4.2. *Suppose that μ is a globally doubling measure on X, that is, $r_\mu = \operatorname{diam}(X)$. Let $\mathcal{B}$ be a countable family of balls in X and let $\{a_B\}_{B\in\mathcal{B}}$ be a collection of non-negative real numbers. Then for all $R \geq 1$ and $1 \leq q < \infty$*

$$|| \sum_{B\in\mathcal{B}} a_B \chi_{RB} ||_{L^q(X)} \leq C_1 R^t q || \sum_{B\in\mathcal{B}} a_B \chi_B ||_{L^q(X)},$$

where $t = \frac{\log c_\mu}{\log 2}$ and C_1 depends on c_μ only. Moreover, if μ is s-regular, then for all $R \geq 1$ and $1 \leq q < \infty$

$$|| \sum_{B\in\mathcal{B}} a_B \chi_{RB} ||_{L^q(X)} \leq C_2 R^s q || \sum_{B\in\mathcal{B}} a_B \chi_B ||_{L^q(X)}$$

where C_2 depends on a_μ, b_μ and s.

Proof. A straightforward calculation gives the claim in the case $q = 1$ whilst the case $1 < q < \infty$ follows from the Hölder's inequality, the duality of L^q-spaces, the maximal function theorem [H, Theorem 2.2], [H, (2.6)] and Lemma 3.2. □

Lemma 4.3. *Suppose that μ is a doubling measure on X. Let $x_0 \in X$ and $0 < r_0 < \frac{r_\mu}{2}$. If $A \subset B(x_0, r_0)$ is $(\varrho, D, p, n_0, k_0)$-mean porous, then*

$$\mu(A(r)) \leq C(\mu) D^{k_0\delta} \mu(A(2D^{-k_0})) r^\delta \text{ for all } r < D^{-n_0-k_0}$$

where $\delta = c(\mu)(\log D)^{t-1} D^{-3t} p \varrho^t$ and $t = \frac{\log c_\mu}{\log 2}$. Moreover, if μ is s-regular then we may choose $t = s$.

Proof. We assume that μ is a doubling measure on X. Modifying the proof in an obvious way gives the claim in the case that μ is s-regular. Define

$$\widetilde{\mathcal{B}} = \{B(x, r_x) : x \in A(D^{-k_0}) \setminus \overline{A} \text{ and } r_x = \frac{\log D}{20D^2} \operatorname{dist}(x, A)\}.$$

By the $5r$-covering theorem we find a countable pairwise disjoint subfamily $\mathcal{B}$ of $\widetilde{\mathcal{B}}$ such that

$$A(D^{-k_0})\setminus\overline{A}\subset\bigcup_{B\in\mathcal{B}}5B \tag{4.2}$$

and

$$B\subset A(2D^{-k_0}) \text{ for all } B\in\mathcal{B}. \tag{4.3}$$

Letting $j\in\mathbb{N}$ and $x\in A(2^{-j})$, choose $x'\in A$ such that $d(x,x')<2^{-j}$. Assume that there is $k\geq k_0+1$ with $\psi_k(x')=1$. Take $y\in A_k(x')$ such that $\operatorname{dist}(y,A)>\varrho d(y,x')$. Using (4.2) we find $B_k\in\mathcal{B}$ such that $y\in 5B_k$.

We proceed by showing that

$$5B_k\subset A_{k+1}(x')\cup A_k(x')\cup A_{k-1}(x'). \tag{4.4}$$

To verify this, let $B_k=B(z,r_z)$. Noting that

$$\operatorname{dist}(z,A)\leq d(z,y)+\operatorname{dist}(y,A)\leq 5r_z+D^{-k+1}=\frac{\log D}{4D^2}\operatorname{dist}(z,A)+D^{-k+1},$$

we obtain

$$\operatorname{dist}(z,A)\leq D^{-k+1}\big(1-\frac{\log D}{4D^2}\big)^{-1},$$

giving

$$10r_z\leq\frac{\log D}{2D^2}\big(1-\frac{\log D}{4D^2}\big)^{-1}D^{-k+1}\leq\frac{2}{3}\log D D^{-k-1}.$$

Now the width of $A_{k-1}(x')$ is $D^{-k+1}(D-1)>10r_z$ and that of $A_{k+1}(x')$ is $D^{-k-1}(D-1)>10r_z$. Thus, (4.4) follows since $B(z,5r_z)\cap A_k(x')\neq\emptyset$.

Next we conclude that

$$x\in\frac{75D^3}{\varrho\log D}B_k \tag{4.5}$$

under the assumption that $D^{-k}\geq 2^{-j}$. Indeed, since

$$\operatorname{dist}(z,A)\geq\operatorname{dist}(y,A)-d(z,y)>\varrho d(y,x')-5r_z=\varrho d(y,x')-\frac{\log D}{4D^2}\operatorname{dist}(z,A),$$

we get

$$\operatorname{dist}(z,A)\geq\varrho d(y,x')\big(1+\frac{\log D}{4D^2}\big)^{-1}.$$

This in turn implies that

$$r_z\geq\frac{\log D}{20D^2}\varrho d(y,x')\big(1+\frac{\log D}{4D^2}\big)^{-1}\geq\frac{\varrho\log D}{25}D^{-k-2}.$$

Hence

$$D^{-k}\leq\frac{25D^2}{\varrho\log D}r_z,$$

and therefore,

$$\begin{aligned}d(x,z) &\le d(x,x') + d(x',y) + d(y,z) \le 2^{-j} + D^{-k+1} + 5r_z \\ &\le D^{-k} + D^{-k+1} + 5r_z \le \frac{25D^2}{\varrho\log D} r_z + \frac{25D^3}{\varrho\log D} r_z + \frac{25D^3}{\varrho\log D} r_z \le \frac{75D^3}{\varrho\log D} r_z.\end{aligned}$$

This gives (4.5).

Clearly, $D^{-k} \ge 2^{-j}$ provided that $k \le \frac{\log 2}{\log D} j$. Thus if $\psi_k(x') = 1$ for $k_0 + 1 \le k \le \frac{\log 2}{\log D} j$ we find, by (4.5), a ball $B_k \in \mathcal{B}$ such that $x \in \frac{75D^3}{\varrho\log D} B_k$. The fact that A is $(\varrho, D, p, n_0, k_0)$-mean porous gives

$$S_{k_0,n}(x') = \sum_{k=k_0+1}^{k_0+n} \psi_k(x') \ge pn \text{ whenever } n \ge n_0.$$

Letting $j_0 > \frac{\log D}{\log 2}(n_0 + k_0)$, we have for all $j \ge j_0$

$$\#\{k : k_0 + 1 \le k \le \frac{\log 2}{\log D} j \text{ and } \psi_k(x') = 1\} \ge \frac{p}{2}\Big(\frac{\log 2}{\log D} j - k_0\Big).$$

Combining this with (4.4) implies that for all $j \ge j_0$ and $x \in A(2^{-j})$

$$\sum_{B\in\mathcal{B}} \chi_{\frac{75D^3}{\varrho\log D}B}(x) \ge \frac{p}{6}\Big(\frac{\log 2}{\log D} j - k_0\Big). \tag{4.6}$$

Indeed, for at least $\frac{p}{2}(\frac{\log 2}{\log D} j - k_0)$ different k's we find a ball $B_k \in \mathcal{B}$ such that $x \in \frac{75D^3}{\varrho\log D} B_k$. However, because of (4.4) each B_k can be taken into account at most three times.

We finish the proof by verifying that for all $j \ge j_0$

$$\mu(A(2^{-j})) \le 11 D^{k_0\delta} \mu(A(2D^{-k_0})) 2^{-j\delta} \tag{4.7}$$

where

$$\delta = \frac{\log 2}{18 C_1 75^t} (\log D)^{t-1} D^{-3t} p \varrho^t \text{ and } t = \frac{\log c_\mu}{\log 2}.$$

Our claim easily follows from this. For (4.7) it suffices to show that for all $j \ge j_0$

$$\int_{A(2^{-j})} 2^{\gamma(\varrho\log D)^t (\frac{p}{6} k_0 + \sum_{B\in\mathcal{B}} \chi_{\frac{75D^3}{\varrho\log D}B}(x))}\, d\mu(x) \le 11 D^{k_0\delta} \mu(A(2D^{-k_0})) \tag{4.8}$$

with $\gamma = (3C_1(75D^3)^t)^{-1}$. This is so because from (4.6) we obtain for all $x \in A(2^{-j})$ that

$$2^{\gamma(\varrho\log D)^t \sum_{B\in\mathcal{B}} \chi_{\frac{75D^3}{\varrho\log D}B}(x)} \ge 2^{\gamma(\varrho\log D)^t \frac{p}{6}(\frac{\log 2}{\log D} j - k_0)}.$$

Moreover, combining this with (4.8) gives

$$\begin{aligned}\mu(A(2^{-j})) &= \mu(A(2^{-j}))2^{-j\gamma(\varrho\log D)^t\frac{p}{6}\frac{\log 2}{\log D}}2^{j\gamma(\varrho\log D)^t\frac{p}{6}\frac{\log 2}{\log D}}\\ &= 2^{-j\gamma(\varrho\log D)^t\frac{p}{6}\frac{\log 2}{\log D}}\int_{A(2^{-j})}2^{\gamma(\varrho\log D)^t\frac{p}{6}\frac{\log 2}{\log D}j}\,d\mu\\ &\le 11D^{k_0\delta}\mu(A(2D^{-k_0}))2^{-j\gamma(\varrho\log D)^t\frac{p}{6}\frac{\log 2}{\log D}},\end{aligned}$$

and therefore (4.7) is valid.

To prove (4.8), write

$$u(x)=\gamma(\varrho\log D)^t\sum_{B\in\mathcal{B}}\chi_{\frac{75D^3}{\varrho\log D}B}(x).$$

Next we want to apply Lemma 4.2. Since $A\subset B(x_0,r_0)$ and $r_0<\frac{r_\mu}{2}$, we may assume that μ is globally doubling. (In fact it is enough to assume that the radius of RB is less than r_μ.) Now we obtain from Lemma 4.2 and (4.3) that

$$\begin{aligned}\int_{A(2^{-j})}&2^{u(x)}\,d\mu(x)\le\int_{A(D^{-k_0})}\exp(u(x))\,d\mu(x)\\ &=\sum_{k=0}^{\infty}\int_{A(D^{-k_0})}\frac{u(x)^k}{k!}\,d\mu(x)\\ &\le\mu(A(D^{-k_0}))+\sum_{k=1}^{\infty}\int_{A(D^{-k_0})}\frac{u(x)^k}{k!}\,d\mu(x)\\ &\le\mu(A(D^{-k_0}))+\sum_{k=1}^{\infty}\frac{(\gamma(\varrho\log D)^t)^k}{k!}\int_X\big(\sum_{B\in\mathcal{B}}\chi_{\frac{75D^3}{\varrho\log D}B}(x)\big)^k\,d\mu(x)\\ &\le\mu(A(D^{-k_0}))+\sum_{k=1}^{\infty}\frac{(\gamma(\varrho\log D)^tC_1(75D^3)^tk)^k}{k!(\varrho\log D)^{tk}}\int_X\big(\sum_{B\in\mathcal{B}}\chi_B(x)\big)^k\,d\mu(x)\\ &\le\mu(A(2D^{-k_0}))\big(1+\sum_{k=1}^{\infty}\frac{(\gamma C_1(75D^3)^tk)^k}{k!}\big)\\ &=\mu(A(2D^{-k_0}))\Big(1+\sum_{k=1}^{\infty}\frac{1}{k!}\Big(\frac{k}{3}\Big)^k\Big)\\ &\le 11\mu(A(2D^{-k_0})),\end{aligned}$$

where the last inequality follows since

$$\frac{\frac{1}{(k+1)!}(\frac{k+1}{3})^{k+1}}{\frac{1}{k!}(\frac{k}{3})^k}=\frac{1}{3}\big(1+\frac{1}{k}\big)^k\uparrow\frac{e}{3}$$

as $k \to \infty$. Finally,

$$\begin{aligned}\int_{A(2^{-j})} & 2^{\gamma(\varrho \log D)^t (\frac{p}{6}k_0 + \sum_{B\in\mathcal{B}} \chi_{\frac{75D^3}{\varrho \log D}B}(x))}\, d\mu(x) \\ &= 2^{\gamma(\varrho\log D)^t \frac{p}{6}k_0} \int_{A(2^{-j})} 2^{u(x)}\, d\mu(x) \\ &\le 11 \cdot 2^{\gamma(\varrho\log D)^t \frac{p}{6}k_0} \mu(A(2D^{-k_0})) \\ &= 11 D^{k_0\delta} \mu(A(2D^{-k_0}))\end{aligned}$$

finishing the proof. □

Remark 4.4. (1) Assume that $\operatorname{por}(A,x,r) > \varrho$ for all $x \in A$ and $0 < r < r_p$. Let $D > 1$. Choose $k_0 \in \mathbb{N}$ such that $D^{-k_0} < r_p$. Then there is $y \in X$ such that $B(y, \varrho D^{-k_0}) \subset B(x, D^{-k_0}) \setminus A$ which in turn implies that $y \in A_k(x)$ for some $k_0 + 1 \le k \le k_0 - \lfloor \frac{\log \varrho}{\log D} \rfloor$. Hence, A is $(\varrho, D, -\frac{1}{2}\lfloor \frac{\log\varrho}{\log D}\rfloor^{-1}, -\lfloor\frac{\log\varrho}{\log D}\rfloor, k_0)$-mean porous. Note that it is not possible to obtain mean porosity with $p = 1$ or p close to 1 unless one takes $D = \frac{1}{\varrho}$. The reason for this is that in general metric spaces the annuli $A_k(x)$ may be empty for many k's.

(2) Assume that μ is an s-regular measure on X and $\operatorname{por}(A,x,r) > \varrho$ for all $x \in A$ and $0 < r < r_p$. Let $l = \frac{1}{2}(\frac{a_\mu}{b_\mu})^{\frac{1}{s}}$ and $D = 2(1-l)^{-1} l^{-2}$. Choose $k_0 \in \mathbb{N}$ such that $D^{-k_0} < \min\{r_p, r_\mu\}$. We verify that A is $(\frac{1}{3}l^2\varrho, D, 1, 1, k_0)$-mean porous. Consider $0 < r \le D^{-k_0}$. Since μ is s-regular there exists $y \in B(x, lr) \setminus B(x, l^2 r)$. Assuming that there is $z \in B(y, \frac{1}{2}l^2 r) \cap A$, we find, using uniform porosity of A, $w \in X$ such that $B(w, \frac{1}{2}\varrho l^3 r) \cap A = \emptyset$ and $\frac{1}{2}\varrho l^3 r + d(w,z) \le \frac{1}{2} l^3 r$. This gives

$$d(w,x) \ge d(x,y) - d(y,z) - d(z,w) \ge \frac{1}{2} l^2 (1-l) r = \frac{r}{D}$$

and

$$d(w,x) \le d(x,y) + d(y,z) + d(z,w) < \frac{3}{2} lr < r.$$

Letting $k \ge k_0 + 1$ and choosing $r = D^{-(k-1)}$ in the above inequalities gives $w \in A_k(x)$ for all $k \ge k_0 + 1$. This combined with the fact that

$$\operatorname{dist}(w, A) \ge \frac{1}{2}\varrho l^3 r > \frac{1}{3} l^2 \varrho d(w,x)$$

gives the claim. If $B(y, \frac{1}{2}l^2 r) \cap A = \emptyset$ we may take $w = y$.

In the following two corollaries we verify scaling properties of measures of r-neighbourhoods of bounded uniformly porous sets for small scales r.

Corollary 4.5. *Suppose that μ is a doubling measure on X. Let $x_0 \in X$. There exists $0 < D_1 < 1$ such that if $A \subset B(x_0, r_0)$ is uniformly ϱ-porous and $0 < r_0 <$*

$D_1 \varrho r_p$ *then*

$$\mu(A(r)) \le C(\mu) c_\mu^{-\frac{\log \varrho}{\log 2}} \mu(B(x_0, r_0)) \Big(\frac{r}{r_0}\Big)^\delta \text{ for all } 0 < r < r_0,$$

where $\delta = c(\mu)(\log \frac{1}{\varrho})^{-1} \varrho^t$ *and* $t = \frac{\log c_\mu}{\log 2}$.

Proof. We may assume that $r_p < \frac{1}{3} r_\mu$. Let $D > 1$ and $D_1 = D^{-2}$. Since $r_0 < D^{-2} \varrho r_p$ we may choose k_0 to be the largest integer k with $\frac{D}{\varrho} r_0 \le D^{-k} < r_p$. By Remark 4.4.(1), A is $(\varrho, D, p, n_0, k_0)$-mean porous for $p = \frac{c_0}{\log \frac{1}{\varrho}}$ where $c_0 > 0$ is a constant depending on D and for $n_0 = -\lfloor \frac{\log \varrho}{\log D} \rfloor$. From Lemma 4.3 we obtain that

$$\mu(A(r)) \le C(\mu) D^{k_0 \delta} \mu(A(2D^{-k_0})) r^\delta \text{ for all } 0 < r < D^{-n_0 - k_0}$$

where $\delta = c(\log \frac{1}{\varrho})^{-1} \varrho^t$ for a constant c that depends on c_μ and D. Note that $\varrho D^{-1-k_0} < D^{-n_0-k_0}$ by the choice of n_0, and therefore, $r_0 < D^{-n_0-k_0}$. The choice of k_0 in turn guarantees that $D^{-k_0} < \frac{D^2}{\varrho} r_0$ giving $A(2D^{-k_0}) \subset B(x_0, 3\frac{D^2}{\varrho} r_0)$. The claim follows by applying the doubling condition and by noting that $D^{k_0 \delta} < r_0^{-\delta} (\frac{\varrho}{D})^\delta < r_0^{-\delta}$. □

Corollary 4.6. *Suppose that* μ *is* s*-regular on* X. *Let* $x_0 \in X$. *There exists a constant* $0 < D_1 < 1$ *such that if* $A \subset B(x_0, r_0)$ *is uniformly* ϱ*-porous and* $0 < r_0 < D_1 r_p$, *then*

$$\mu(A(r)) \le C(\mu) \mu(B(x_0, r_0)) \Big(\frac{r}{r_0}\Big)^\delta \text{ for all } 0 < r < r_0$$

where $\delta = c(\mu) \varrho^s$.

Proof. The proof is similar to that of Corollary 4.5. □

Remark 4.7. Observe that in Corollary 4.5 one can choose D and thus D_1 as close to 1 as one wishes whilst in Corollary 4.6 there is a lower bound for D. (Recall that also in this case $D_1 = D^{-2}$.) This is due to Remark 4.4. However, the result of Corollary 4.6 is stronger since there is no ϱ-dependence on r_0. In both Corollaries the constants $C(\mu)$ and $c(\mu)$ depend on D, the size of annuli determining the mean porosity.

Next we prove the analogue of (4.1) for s-regular metric spaces. As mentioned in the Introduction this is asymptotically sharp as ϱ tends to zero in $\mathbb{R}^n$ and thus also in metric spaces.

Theorem 4.8. *Suppose* μ *is* s*-regular on* X. *If* $A \subset X$ *is* ϱ*-porous, then*

$$\dim_{\mathrm{p}}(A) \le s - c(\mu) \varrho^s.$$

Moreover, if A *is uniformly* ϱ*-porous and* $\operatorname{diam}(A) < r_\mu$, *then*

$$\dim_{\mathrm{M}}(A) \le s - c(\mu) \varrho^s.$$

Proof. By Remark 2.1.(5), A is a countable union of sets A_{ij} with $\operatorname{diam}(A_{ij}) < 1/i < r_\mu$ such that $\operatorname{por}(A, x, r) > \varrho$ for all $x \in A_{ij}$ and $0 < r < 1/i$. Moreover, if A is uniformly porous with $\operatorname{diam}(A) < r_\mu$, then it is a finite union of such sets. Thus it is enough to show that $\dim_{\mathrm{M}}(A_{ij}) \le s - \delta$ for all i and j where $\delta = c(\mu)\varrho^s$. Letting $x \in A_{ij}$ and using Lemma 3.3 and Corollary 4.6 we get for large i

$$\begin{aligned}\limsup_{r\downarrow 0} M^\lambda(A_{ij}, r) &\le 5^s a_\mu^{-1} \limsup_{r\downarrow 0} \frac{\mu\big(A_{ij}(r)\big)}{r^{s-\lambda}} \\ &\le 5^s a_\mu^{-1} C \limsup_{r\downarrow 0} \mu\big(B(x, 1/i)\big) i^\delta r^{\delta+\lambda-s} < \infty\end{aligned}$$

if $\lambda > s - \delta$. Here C is a constant which is independent of r. This gives the claim. □

Theorem 4.8 is not true if we only assume that μ is doubling. An easy example is given by defining $X = (\{0\} \cup_{j\in\mathbb{N}} \{2^{-j}\}) \times [0,1]$ with the metric inherited from $\mathbb{R}^2$ and letting μ be any doubling measure on X. If $N = \{0\} \times [0,1]$, then $\dim_{\mathrm{M}}(N) = \dim_{\mathrm{p}}(N) = 1 = \dim_{\mathrm{p}}(X)$. However, it is easy to see that N is uniformly $\frac{1}{3}$-porous. The following example shows that one can perceive similar behaviour in geodesic metric spaces as well, even for maximally porous sets. Recall that a metric space (X, d) is geodesic if for each pair of points $x, y \in X$ there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$, and the length of γ is equal to $d(x, y)$.

Example 4.9. We give an example of a complete geodesic doubling metric space having a subset with maximal dimension and porosity. The construction is an infinite tree with branches getting smaller and smaller as we go deeper into the tree. The metric is the natural path metric induced by the branches (see Figure 1).

Letting

$$\mathcal{N}_0 = \{\emptyset\}\ ,\ \mathcal{N}_n = \{1,2\}^n \times]0, 2^{-n}] \text{ for all } \mathbb{N} \setminus \{0\} \text{ and } \mathcal{N}_\infty = \{1,2\}^{\mathbb{N}} \times \{0\},$$

define

$$\mathcal{N} = \mathcal{N}_\infty \cup \bigcup_{n=0}^{\infty} \mathcal{N}_n.$$

The metric is given as follows: For $x \in \mathcal{N}$, let $n(x) \in \mathbb{N} \cup \{\infty\}$ so that $x = (x_1, \ldots, x_{n(x)}, x_{n(x)+1}) \in \mathcal{N}_{n(x)}$. We denote the distance of x from the root by

$$l(x) = \begin{cases} 1 - 2^{-(n(x)-1)} + x_{n(x)+1} & \text{, if } 0 < n(x) < \infty \\ 1 & \text{, if } n(x) = \infty \\ 0 & \text{, if } n(x) = 0. \end{cases}$$

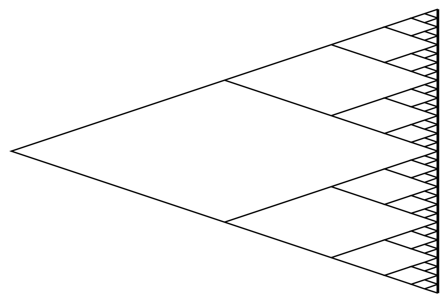

Figure 1: An illustration of space $\mathcal{N}$.

Given $n \in \mathbb{N}$, let

$$x|_n = \begin{cases} (x_1, x_2, \dots, x_n) & \text{, if } n(x) \geq n \\ \emptyset & \text{, if } n(x) < n \end{cases}$$

be the restriction of x. Define the longest common route $\delta(x, y)$ of a point $x = (x_1, x_2, \dots) \in \mathcal{N}$ and $y = (y_1, y_2, \dots) \in \mathcal{N}$ from the root by

$$\delta(x, y) = \sup\{m : x|_m = y|_m \neq \emptyset\}$$

and their longest common part $x \wedge y \in \mathcal{N}_{\delta(x,y)}$ by

$$x \wedge y = \begin{cases} (x|_{\delta(x,y)}, \min\{ \; 2^{-\delta(x,y)}, n(x) - \delta(x, y) + x_{n(x)+1}, & \text{, if } 0 < \delta(x, y) < \infty \\ \qquad\qquad\qquad n(y) - \delta(x, y) + x_{n(y)+1}\}) & \\ \emptyset & \text{, if } \delta(x, y) = 0 \\ x & \text{, if } \delta(x, y) = \infty. \end{cases}$$

With these notations we define the metric $d : \mathcal{N} \times \mathcal{N} \to [0, \infty[$ by

$$d(x, y) = |l(x) - l(x \wedge y)| + |l(y) - l(x \wedge y)|.$$

Figure 1 illustrates the metric space $(\mathcal{N}, d)$ which is obviously geodesic.

We verify that

$$\dim_{\mathrm{M}}(\mathcal{N}) = \dim_{\mathrm{H}}(\mathcal{N}_\infty) = 1 \tag{4.9}$$

and

$$\mathrm{por}(\mathcal{N}_\infty) = \frac{1}{2}. \tag{4.10}$$

Indeed, since $\mathrm{dist}(\mathcal{N}_n, \mathcal{N}_\infty) = 2^{-n}$, the set $\mathcal{N} \setminus \cup_{k=1}^n \mathcal{N}_k$ can be covered by 2^n closed balls centred in $\mathcal{N}_n$ with radii 2^{-n}. On the other hand, $\cup_{k=1}^n \mathcal{N}_k$ can be covered by $n2^n$ balls with radii 2^{-n}, and therefore, $\mathcal{N}$ can be covered by $(n+1)2^n$ such balls. Hence $\dim_{\mathrm{H}}(\mathcal{N}_\infty) \leq \dim_{\mathrm{M}}(\mathcal{N}) \leq 1$. Clearly, $\dim_{\mathrm{H}}(\mathcal{N}_\infty) = 1$ which gives (4.9). For (4.10), take any $x \in \mathcal{N}_\infty$ and $0 < r < 1$. Choose $n \in \mathbb{N}$ such that $2^{-n} \leq r < 2^{-n+1}$. Since $\mathrm{dist}((x|_n, 2^{-n+1} - r), \mathcal{N}_\infty) = r$, we have for all $\varepsilon > 0$

$$B((x|_n, 2^{-n+1} - r), (1 - \varepsilon)r) \subset B(x, 2r) \setminus \mathcal{N}_\infty.$$

This implies (4.10).

Note that the space $(\mathcal{N}, d)$ is doubling with a doubling constant 3, that is, every closed ball with radius $2r$ can be covered with 3 closed balls with radius r. In particular, it carries a doubling measure by [LS]. Moreover, while $\cup_{k=0}^{n}\mathcal{N}_k$ is 1-regular for all n and $\mathcal{N}_\infty$ is also 1-regular, the set $\mathcal{N} \setminus \mathcal{N}_\infty$ is not.

By taking a closer look at the above examples one recognizes that the chosen porous sets, denoted here by N, are exceptional in the sense that $\mu(N) = 0$ and $\dim_{\mathrm{p}}(A) < \dim_{\mathrm{p}}(X)$ for all porous sets $A \subset X \setminus N$. This is not a coincidence as indicated by the following result.

Theorem 4.10. *Suppose that μ is a doubling measure on X. Then there is $N \subset X$ such that $\mu(N) = 0$ and*

$$\dim_{\mathrm{p}}(A) \le \dim_{\mathrm{p}}(X) - c(\mu)(\log \tfrac{1}{\varrho})^{-1}\varrho^t$$

for all ϱ-porous sets $A \subset X \setminus N$ where $t = \frac{\log c_\mu}{\log 2}$.

We prove Theorem 4.10 using Corollary 4.5 and the following two lemmas.

Lemma 4.11. *Suppose that μ is a doubling measure on X. Then there is $N \subset X$ with $\mu(N) = 0$ such that for all $s > \dim_{\mathrm{p}}(X)$ we have*

$$\lim_{r\downarrow 0} \frac{\mu(B(x,r))}{r^s} = \infty$$

for every $x \in X \setminus N$.

Proof. Choose a decreasing sequence (s_k) such that $s_k \downarrow \dim_{\mathrm{p}}(X)$ as $k \to \infty$. We claim that for all $k \in \mathbb{N}$ there is $N_k \subset X$ with $\mu(N_k) = 0$ such that

$$\lim_{r\downarrow 0} \frac{\mu(B(x,r))}{r^{s_k}} = \infty \tag{4.11}$$

for all $x \in X \setminus N_k$. Fix $k \in \mathbb{N}$ and suppose to the contrary that there are $0 < D < \infty$ and a Borel set $A \subset X$ such that $\mu(A) > 0$ and $\liminf_{r\downarrow 0} \mu(B(x,r))/r^{s_k} < D$ for all $x \in A$. By [Cu, Theorem 3.16] we get $\mathcal{P}^{s_k}(A) \ge \frac{1}{C(\mu)D}\mu(A) > 0$ which is impossible since $\dim_{\mathrm{p}}(A) \le \dim_{\mathrm{p}}(X) < s_k$. Thus (4.11) is proved. Defining $N = \cup_{k=0}^{\infty} N_k$, verifies the claim. $\square$

The following lemma is a substitute for Lemma 3.3 in metric spaces that carry a doubling measure.

Lemma 4.12. *Suppose that μ is a doubling measure on X. Let $0 < \lambda < \dim_{\mathrm{p}}(X)$ and let $N \subset X$ be as in Lemma 4.11. If $A \subset X \setminus N$ has the property that*

$$\limsup_{r\downarrow 0} \frac{\mu(A(r))}{r^\lambda} < \infty,$$

then $\dim_{\mathrm{p}}(A) \le \dim_{\mathrm{p}}(X) - \lambda$.

Proof. Fix $x_0 \in X$. Let $s > \dim_{\mathrm{p}}(X)$ and define

$$A_{ki} = \{x \in A \cap B(x_0, i) \,:\, \mu(B(x,r)) > r^s \text{ for all } 0 < r < 1/k\}$$

for all k, $i \in \mathbb{N}$. Since $A \subset X \setminus N$ we see from Lemma 4.11 that $A = \cup_{k,i} A_{ki}$. It suffices to show that

$$\dim_{\mathrm{M}}(A_{ki}) \leq s - \lambda \text{ for all } k, i \in \mathbb{N}. \tag{4.12}$$

Let k, $i \in \mathbb{N}$. Choose $r_0 > 0$ and $M < \infty$ such that $\mu(A_{ki}(r)) \leq \mu(A(r)) \leq M r^\lambda$ when $0 < r < r_0$. If $0 < r < \min\{\frac{1}{2} r_0, 1/k\}$ we apply the $5r$-covering theorem to the collection $\{B(x,r) \,:\, x \in A_{ki}\}$ to find a pairwise disjoint subcollection $\{B(x_l, r) \,:\, l \in I\}$ such that

$$A_{ki} \subset \bigcup_{l \in I} B(x_l, 5r). \tag{4.13}$$

Now

$$M 2^\lambda r^\lambda \geq \mu(A_{ki}(2r)) \geq \sum_{l \in I} \mu(B(x_l, r)) \geq \# I r^s.$$

This implies that $\# I \leq M 2^\lambda r^{\lambda - s} < \infty$ which combined with (4.13) gives

$$M^{s-\lambda}(A_{ki}, 5r) \leq M 2^\lambda 5^{s-\lambda} r^{\lambda - s} r^{s-\lambda} \leq M 2^\lambda 5^{s-\lambda}.$$

Now (4.12) follows as $r \downarrow 0$. □

Proof of Theorem 4.10. Let N be as in Lemma 4.11 and let $A \subset X \setminus N$. By Remark 2.1.(5), A is a countable union of sets of the form

$$E = \{x \in A \cap B(x_0, r_0) \,:\, \mathrm{por}(A, x, r) > \varrho \text{ for all } 0 < r < r_0\}$$

where $x_0 \in X$ and $r_0 > 0$. Thus it suffices to show that $\dim_{\mathrm{p}}(E) \leq \dim_{\mathrm{p}}(X) - \delta$ where δ is as in Corollary 4.5. But this follows from Lemma 4.12 since $E \subset X \setminus N$ and $\limsup_{r \downarrow 0} \mu(E(r))/r^\delta < \infty$ by Corollary 4.5. □

Remark 4.13. (1) We do not know if $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}(X) - \delta$ in Theorem 4.10. Of course, this question is relevant only when $\dim_{\mathrm{H}}(X) < \dim_{\mathrm{p}}(X)$.

(2) The essential qualitative difference between Theorems 4.8 and 4.10 is that in the doubling case we have an extra factor $(\log \frac{1}{\varrho})^{-1}$ in δ. This extra factor appears also in $\mathbb{R}^n$ if one makes a simple estimate for the Minkowski dimension using mesh cubes whose side lengths are powers of ϱ. To obtain the optimal upper bound one has to utilize the porosity also at the intermediate scales, that is, for all j's with $\varrho^{k+1} \leq 2^{-j} < \varrho^k$. As pointed out in Remark 4.4.(1), these intermediate scales do not necessarily exist in general metric spaces. So we do not know whether the factor $(\log \frac{1}{\varrho})^{-1}$ is necessary or not in Theorem 4.10 although in Remark 4.4.(1) it is which can be seen by considering a disjoint union of Cantor sets C_{ϱ_i} where ϱ_i tends to zero.

(3) In Theorem 4.8 the constant $c(\mu)$ depends on a_μ, b_μ and s. One may ask whether it is possible that $c(\mu)$ depends on s only. However, this is not the case.

If in Example 4.9 one chooses $\mathcal{N}_n = \{1,2\}\times]0,\lambda^n]$ for $\lambda < \frac{1}{2}$, then the resulting space $\mathcal{N}$ is 1-regular, $\mathrm{por}(\mathcal{N}_\infty) = \frac{1}{2}$ and $\dim_{\mathrm{p}}(\mathcal{N}_\infty) \to 1$ as $\lambda \to \frac{1}{2}$. Note that in this case $\frac{b_\mu}{a_\mu} \to \infty$ and thus $c(\mu) \to 0$ as $\lambda \to \frac{1}{2}$.

5. Uniform porosity and regular sets

In this section we prove that if X is s-regular and complete, then the difference between uniformly porous and regular subsets of X is negligible in the following sense: $A \subset X$ is uniformly porous if and only if there is $0 < t < s$ and a t-regular set $F \subset X$ such that $A \subset F$. In $\mathbb{R}^n$ this is well known, see for example [S], [L, Theorem 5.2], [Ca, Proposition 4.3], [KS, (proof of) Theorem 4.1], and [KV, Example 6.8]. We begin by showing that for each $0 < t < s$ there exists a t-regular set $F \subset X$. This is a consequence of the following lemma which was proven in $\mathbb{R}^n$ in an unpublished Licentiate thesis of Pirjo Saaranen. The proof for complete s-regular spaces is almost the same but it is included here for the sake of completeness. We denote by $\mathrm{spt}(\mu)$ the support of μ, that is, $\mathrm{spt}(\mu)$ is the smallest closed set F with $\mu(X \setminus F) = 0$.

Lemma 5.1. *Assume that X is complete, μ is s-regular on X and $0 < t < s$. Let $z \in X$ and $0 < R < r_\mu$. Then there is a measure ν with $\mathrm{spt}(\nu) \subset B(z, 2R)$ such that $\nu(B(z,2R)) = R^t$ and*

$$a_\nu r^t \le \nu(B(x,r)) \le b_\nu r^t$$

for all $x \in \mathrm{spt}(\nu)$ and $0 < r < R$. Here the constants $a_\nu = a_\nu(s,t,a_\mu,b_\mu)$ and $b_\nu = b_\nu(s,t,a_\mu,b_\mu)$ are independent of z and R.

Proof. Let $0 < r \le \frac{1}{10}2^{-\frac{1}{t}}R$. Using the $5r$-covering theorem, we choose from the collection $\{B(x, 2\cdot 2^{\frac{1}{t}}r) : x \in B(z,R)\}$ disjoint balls $B(x_i, 2\cdot 2^{\frac{1}{t}}r)$ so that the balls $B(x_i, 10\cdot 2^{\frac{1}{t}}r)$ cover the set $B(z,R)$. We may take $x_1 = z$. Next we estimate the number, say $m(r)$, of balls in this collection. Since by Lemma 3.2

$$\begin{aligned} m(r)b_\mu(2\cdot 2^{\frac{1}{t}}r)^s \ge& \sum_{i=1}^{m(r)} \mu(B(x_i, 2\cdot 2^{\frac{1}{t}}r)) \ge \sum_{i=1}^{m(r)} \frac{a_\mu}{5^s b_\mu}\mu(B(x_i, 10\cdot 2^{\frac{1}{t}}r)) \\ \ge& \frac{a_\mu}{5^s b_\mu}\mu(B(z,R)) \ge \frac{a_\mu^2}{5^s b_\mu}R^s, \end{aligned}$$

we obtain

$$c_1\Big(\frac{R}{r}\Big)^s \le m(r) \text{ for all } 0 < r \le \tfrac{1}{10}2^{-\frac{1}{t}}R, \tag{5.1}$$

where

$$c_1 = \frac{a_\mu^2}{10^s 2^{\frac{s}{t}} b_\mu^2}.$$

Choose $d > 0$ so small that

$$d < \tfrac{1}{10} 2^{-\frac{1}{t}},\ d^{s-t} \leq \frac{c_1}{2} \text{ and } d^t \leq \frac{1}{5}. \tag{5.2}$$

Taking $r = dR$ in the above process gives disjoint balls $B(x_i, 2 \cdot 2^{\frac{1}{t}} dR)$ with $x_i \in B(z, R)$. Moreover, by (5.1) we get the following estimate for the number m of such balls

$$\frac{c_1}{d^s} \leq m. \tag{5.3}$$

Let $M \in \mathbb{N}$ be such that

$$d^{-t} - \frac{1}{2} \leq M < d^{-t} + \frac{1}{2}. \tag{5.4}$$

Because

$$d^{-t} + \frac{1}{2} \leq \frac{1}{2} c_1 d^{-s} + \frac{1}{2} < m$$

by (5.2) and (5.3), we may take M balls from the collection of disjoint balls $\{B(x_i, 2 \cdot 2^{\frac{1}{t}} dR) : i = 1, \ldots, m\}$. Having roughly advanced towards our t-regular measure on this scale by taking suitable balls, we proceed by adjusting the radii of the balls to get exactly the regularity we want.

Fix $d_1 < 1$ so that $d_1^t = M^{-1}$. Inequalities (5.4) and (5.2) combine to give

$$d_1^t \leq \frac{2d^t}{2 - d^t} \leq \frac{10}{9} d^t < 2d^t \tag{5.5}$$

which implies that the balls $B_i = B(x_i, 2d_1 R)$, $i = 1, \ldots, M$ are disjoint. Next we repeat the process by taking $B(x_i, d_1 R)$ as $B(z, R)$. In this manner we get disjoint balls $B(x_{ij}, 2 \cdot 2^{\frac{1}{t}} dd_1 R)$, $j = 1, \ldots, m_{2i}$, where $x_{ij} \in B(x_i, d_1 R)$, $x_{i1} = x_i$ and

$$\frac{c_1}{d^s} \leq m_{2i}.$$

For all $i = 1, \ldots, M$ choose M balls from these collections and adjust the radii to be $d_1^2 R$. Now the balls $B_{ij} = B(x_{ij}, 2d_1^2 R)$ are disjoint and $B_{ij} \subset B_i$ for all $i, j = 1, \ldots, M$, because by (5.5)

$$d_1 R + 2d_1^2 R < d_1 R + 2 \cdot 2^{\frac{1}{t}} d_1 d R = d_1 R(1 + 2 \cdot 2^{\frac{1}{t}} d) < 2d_1 R.$$

We continue this process. At step k we obtain for all sequences $(i_1, \ldots, i_{k-1}) \in \{1, \ldots, M\}^{k-1}$ disjoint balls $B_{i_1 \ldots i_k}$, $i_k = 1, \ldots, M$, with centres in the ball $B(x_{i_1 \ldots i_{k-1}}, d_1^{k-1} R)$ and with radii $2d_1^k R$ such that

$$B_{i_1 \ldots i_k} \subset B_{i_1 \ldots i_{k-1}}.$$

Furthermore, $x_{i_1 \ldots i_{k-1} 1} = x_{i_1 \ldots i_{k-1}}$. Set

$$\tilde{\nu}(B_{i_1 \ldots i_k}) = M^{-k} \tag{5.6}$$

for all $k \in \mathbb{N}$ and $i_1, \dots, i_k \in \{1, \dots, M\}^k$. Since X is complete and s-regular, all closed balls are compact and we can apply the standard mass distribution principle to extend $\tilde{\nu}$ to a probability measure on X satisfying (5.6) with

$$\operatorname{spt}(\tilde{\nu}) = \bigcap_{k=1}^{\infty} \bigcup_{i_1,\dots,i_k} B_{i_1\dots i_k} \subset B(z, 2R).$$

Note that $z \in \operatorname{spt}(\tilde{\nu}) \neq \emptyset$.

Next we prove that $\tilde{\nu}$ is t-regular. Take any $x \in \operatorname{spt}(\tilde{\nu})$ and $0 < r < (1-2d_1)R$. Let $l \in \mathbb{N}$ be such that

$$(1-2d_1)d_1^{l+1}R \leq r < (1-2d_1)d_1^l R. \tag{5.7}$$

By (5.5) and (5.2) we have $d_1 < \frac{1}{10}$. Thus inequalities (5.7) guarantee that

$$B_{i_1\dots i_{l+2}} \subset B(x,r) \subset B_{i_1\dots i_l}$$

for some $(i_1, \dots, i_{l+2}) \in \{1, \dots, M\}^{l+2}$. This in turn implies with (5.6) and (5.7) that

$$\frac{d_1^{2t}}{(1-2d_1)^t R^t} r^t \leq d_1^{(l+2)t} = M^{-(l+2)} \leq \tilde{\nu}(B(x,r)) \leq M^{-l} = d_1^{lt} \leq \frac{1}{(1-2d_1)^t R^t d_1^t} r^t.$$

Finally, defining $\nu = R^t \tilde{\nu}$ gives the desired measure. □

As an immediate consequence we obtain:

Corollary 5.2. *Assume that X is complete and μ is s-regular on X. Then for all $0 < t < s$ there is $F \subset X$ which is t-regular.*

Now we are ready to state the main theorem of this section.

Theorem 5.3. *Suppose that μ is s-regular on X. If $0 < t < s$ and $A \subset X$ is t-regular, then A is uniformly porous. Conversely, if X is complete and $A \subset X$ is uniformly ϱ-porous, then for all $s - c(\mu)\varrho^s < t < s$ there exists a t-regular set $F \subset X$ so that $A \subset F$. Here $c(\mu)$ is as in Corollary 4.6.*

Proof. The first part is proven in [BHR, Lemma 3.12]. We reprove it here to obtain a quantitative estimate which in a sense is optimal (see Remark 5.4). Let ν be a t-regular measure on A. Pick $x \in A$ and $0 < r < \frac{1}{2}\min\{r_\mu, r_\nu\}$, and take $k \in \mathbb{N}$. Attaching to each $z \in B(x,r)$ a ball $B(z, \frac{1}{5}2^{-k}r)$, we find, using the $5r$-covering theorem, a countable index set I_k and points $x_i \in B(x,r)$, $i \in I_k$, such that

$$B(x,r) \subset \bigcup_{i \in I_k} B(x_i, 2^{-k}r)$$

and $B(x_i, \frac{1}{5}2^{-k}r) \cap B(x_j, \frac{1}{5}2^{-k}r) = \emptyset$ for $i \neq j$.

From the s-regularity of μ we get

$$a_\mu r^s \leq \mu\big(B(x,r)\big) \leq \sum_{i \in I_k} \mu\big(B(x_i, 2^{-k}r)\big) \leq \# I_k b_\mu 2^{-ks} r^s,$$

and therefore,

$$\#I_k \geq \frac{a_\mu}{b_\mu} 2^{ks}. \tag{5.8}$$

Taking any set $J_k \subset I_k$ for which $B(x_j, \frac{1}{10}2^{-k}r) \cap A \neq \emptyset$ as $j \in J_k$ and using the fact that ν is t-regular on A, we have

$$\begin{aligned} b_\nu 2^t r^t \geq \nu\big(B(x, 2r)\big) &\geq \sum_{j \in J_k} \nu\big(B(x_j, \tfrac{1}{5}2^{-k}r)\big) \\ &\geq \sum_{j \in J_k} \nu\big(B(z_j, \tfrac{1}{10}2^{-k}r)\big) \geq \#J_k a_\nu (\tfrac{1}{10})^t 2^{-kt} r^t, \end{aligned}$$

where $z_j \in B(x_j, \frac{1}{10}2^{-k}r) \cap A$ as $j \in J_k$. Thus

$$\#J_k \leq \frac{b_\nu}{a_\nu} 20^t 2^{kt}.$$

This upper bound is strictly smaller than the lower bound in (5.8) when

$$k > \frac{\log\big(\frac{b_\mu b_\nu}{a_\mu a_\nu} 20^t\big)}{(s-t)\log 2} =: K(\mu, \nu).$$

Choosing $k > K(\mu, \nu)$, gives $I_k \setminus J_k \neq \emptyset$, and we find $i_0 \in I_k$ such that $B(x_{i_0}, \frac{1}{10}2^{-k}r) \cap A = \emptyset$. It follows that

$$\operatorname{por}^*(A, x, 2r) \geq \tfrac{1}{20} 2^{-k} \tag{5.9}$$

whenever $x \in A$ and $0 < r < \frac{1}{2}\min\{r_\mu, r_\nu\}$. The claim follows from Remark 2.1.(2).

Now we prove the opposite direction. The idea is to use Lemma 5.1 to build a regular measure inside the voids of suitable reference balls. Let $\varrho > 0$ and $r_p > 0$ be such that

$$\operatorname{por}(A, x, r) > \varrho \tag{5.10}$$

for all $x \in A$ and $0 < r < r_p$ Set $\gamma = \frac{\varrho}{5}$ and fix $n_0 \in \mathbb{N}$ such that $\gamma^{n_0} < \min\{D_1 r_p, r_\mu\}$ (see Corollary 4.6). From Corollary 4.6 we see that for all $x \in X$ and $0 < r_0 \leq \gamma^{n_0}$

$$\mu((A \cap B(x, r_0))(r)) \leq C(\mu)\mu(B(x, r_0))\big(\frac{r}{r_0}\big)^\delta \text{ for all } 0 < r < r_0 \tag{5.11}$$

where $\delta = c(\mu)\varrho^s$.

Consider $s - \delta < t < s$. For all $j \in \mathbb{N}$, we choose by means of the $5r$-covering theorem a collection of disjoint balls $\{B(x_{ji}, \gamma^{n_0+j})\}$ so that $x_{ji} \in A$ for all $i \in \mathbb{N}$ and

$$A \subset \bigcup_i B(x_{ji}, 5\gamma^{n_0+j}). \tag{5.12}$$

For each $i \in \mathbb{N}$ we find, using inequality (5.10), $z_{ji} \in B(x_{ji}, \gamma^{n_0+j})$ so that $B(z_{ji}, \varrho\gamma^{n_0+j}) \subset B(x_{ji}, \gamma^{n_0+j}) \setminus A$. Define

$$B_{ji} = B(z_{ji}, \tfrac{\varrho}{5}\gamma^{n_0+j}) = B(z_{ji}, \gamma^{n_0+j+1}).$$

Then obviously $2B_{ji} \subset B(z_{ji}, \varrho\gamma^{n_0+j}) \subset B(x_{ji}, \gamma^{n_0+j}) \setminus A$. Lemma 5.1 implies that for all $i \in \mathbb{N}$ there is a t-regular measure ν_{ji} on $\operatorname{spt}(\nu_{ji}) \subset 2B_{ji}$ with $\nu_{ji}(2B_{ji}) = \gamma^{(n_0+j+1)t}$. Note that the constants $a_{\nu_{ji}}$ and $b_{\nu_{ji}}$ are the same for all j and i, say $a_{\nu_{ji}} = a$ and $b_{\nu_{ji}} = b$. Moreover, it is evident by the properties of ν_{ji} that we may choose $r_{\nu_{ji}} = 3\gamma^{n_0+j}$ by adjusting a and b. We conclude that for all $x \in \operatorname{spt}(\nu_{ji})$ and $0 < r < 3\gamma^{n_0+j}$

$$ar^t \leq \nu_{ji}(B(x,r)) \leq br^t. \tag{5.13}$$

Setting

$$F = \bigcup_{j,i} \operatorname{spt}(\nu_{ji}) \cup A$$

and

$$\nu = \sum_{j,i} \nu_{ji},$$

we clearly have $A \subset F$ and $\nu(X \setminus F) = 0$, and therefore, it suffices to prove that ν is t-regular on F.

We first verify that ν is t-regular on A, that is, there are constants C_1 and C_2 such that for all $x \in A$ and $0 < r < \frac{1}{2}\gamma^{n_0+1}$ we have

$$C_1 r^t \leq \nu(B(x,r)) \leq C_2 r^t. \tag{5.14}$$

For the purpose of proving (5.14), fix $k \in \mathbb{N}$ so that

$$\gamma^{n_0+k+1} \leq r < \gamma^{n_0+k}.$$

From (5.12) it follows easily that $2B_{k+3,i} \subset B(x,r)$ for some i giving

$$\nu(B(x,r)) \geq \nu_{k+3,i}(2B_{k+3,i}) = \gamma^{(n_0+k+4)t} \geq \gamma^{4t} r^t.$$

Thus we may choose $C_1 = \gamma^{4t}$ in (5.14).

For the remaining inequality in (5.14), denote by N_j the number of balls $2B_{ji}$ that intersect $B(x,r)$. Assuming that $j \geq k$, we obtain

$$\bigcup_{2B_{ji} \cap B(x,r) \neq \emptyset} 2B_{ji} \subset (A \cap B(x, \frac{2}{\gamma} r))(\gamma^{n_0+j}). \tag{5.15}$$

Indeed, if $2B_{ji} \cap B(x,r) \neq \emptyset$, then also $B(x_{ji}, \gamma^{n_0+j}) \cap B(x,r) \neq \emptyset$ giving $d(x_{ji}, x) \leq r + \gamma^{n_0+j} \leq r + \frac{1}{\gamma} r \leq \frac{2}{\gamma} r$. Hence (5.15) is valid since $x_{ji} \in A$.

Now (5.15) implies with (5.11) that

$$\begin{aligned} N_j a_\mu 2^s \gamma^{(n_0+j+1)s} &\le \mu\Big(\bigcup_{2B_{ji}\cap B(x,r)\neq\emptyset} 2B_{ji}\Big) \le \mu((A\cap B(x,\frac{2}{\gamma}r))(\gamma^{n_0+j})) \\ &\le C(\mu)\mu(B(x,\frac{2}{\gamma}r))\Big(\frac{\gamma^{n_0+j}}{\frac{2}{\gamma}r}\Big)^\delta \le C(\mu)b_\mu\Big(\frac{2}{\gamma}r\Big)^{s-\delta}\gamma^{(n_0+j)\delta} \\ &\le C(\mu)b_\mu\Big(\frac{2}{\gamma}\Big)^{s-\delta}\gamma^{(n_0+k)s}\gamma^{(j-k)\delta}, \end{aligned}$$

and therefore,

$$N_j \le c_3\gamma^{(k-j)(s-\delta)} \text{ for all } j \ge k, \tag{5.16}$$

where c_3 is a constant which depends on a_μ, b_μ, s and ϱ. Note that $N_j = 0$ provided that $j \le k-1$. This is true because

$$\operatorname{dist}(2B_{ji}, A) \ge 3\gamma^{n_0+j+1} \ge 3\gamma^{n_0+k} > r.$$

From (5.16) we obtain

$$\begin{aligned} \nu(B(x,r)) &= \sum_{j,i}\nu_{ji}(B(x,r)\cap 2B_{ji}) \le \sum_{j=k}^\infty N_j\gamma^{(n_0+j+1)t} \\ &\le c_3\gamma^{t(n_0+1)+k(s-\delta)}\sum_{j=k}^\infty \gamma^{j(\delta+t-s)} \\ &= \frac{c_3}{1-\gamma^{\delta+t-s}}\gamma^{t(n_0+k+1)} \le C_2 r^t \end{aligned}$$

where $C_2 = \frac{c_3}{1-\gamma^{\delta+t-s}}$. This completes the verification of (5.14).

We finish the proof by showing that ν is t-regular on $\operatorname{spt}(\nu_{ji})$ for all j and i. Fix j and i and let $y \in \operatorname{spt}(\nu_{ji})$. We first derive the lower bound in the definition of t-regularity. If $0 < r \le 3\gamma^{n_0+j}$, then $\nu(B(y,r)) \ge \nu_{ji}(B(y,r)) \ge ar^t$ by (5.13). On the other hand, if $3\gamma^{n_0+j} < r < \gamma^{n_0+1}$, we get $B(x_{ji}, \frac{r}{2}) \subset B(y,r)$. Applying (5.14) implies that $\nu(B(y,r)) \ge \nu(B(x_{ji},\frac{r}{2})) \ge C_1 2^{-t}r^t$.

For the upper regularity bound we first assume that $0 < r \le \gamma^{n_0+j+1}$. Now $B(y,r)\cap 2B_{j'i'} = \emptyset$ for all $(j',i') \neq (j,i)$. Indeed, if $j = j'$ this follows from $B(y,r) \subset B(x_{ji},\gamma^{n_0+j})$ and $2B_{ji'} \subset B(x_{ji'},\gamma^{n_0+j})$. In the case $j' < j$ assume that $w \in B(y,r)$. Then $d(w,A) \le d(w,x_{ji}) \le 2\gamma^{n_0+j}$. Further, if $w \in 2B_{j'i'}$, then $d(w,A) \ge \frac{3\varrho}{5}\gamma^{n_0+j'} \ge 3\gamma^{n_0+j}$. The final case $j' > j$ is similar. Thus $\nu(B(y,r)) = \nu_{ji}(B(y,r)) \le br^t$ by (5.13). Finally, supposing that $\gamma^{n_0+j+1} < r < \frac{\varrho}{12}\gamma^{n_0+1}$ gives $B(y,r) \subset B(x_{ji},\frac{6r}{\varrho})$. Hence $\nu(B(y,r)) \le \nu(B(x_{ji},\frac{6r}{\varrho})) \le C_2\big(\frac{6}{\varrho}\big)^t r^t$ by (5.14). □

Remark 5.4. (1) Observe that in (5.9) the porosity ϱ is proportional to $\big(\frac{a_\nu}{b_\nu}\big)^{\frac{1}{s-t}}$. The following simple construction shows that this is sharp. Let $0 < t < 1$ and choose from $[0,1]$ N evenly distributed intervals of length $N^{-\frac{1}{t}}$. Repeat this construction and let ν be the natural measure on the resulting Cantor set A. Then A is t-regular, $\mathrm{por}(A) \approx \frac{1-N^{1-\frac{1}{t}}}{N}$ and $\frac{a_\nu}{b_\nu} = \frac{N^{t-1}}{(1-N^{1-\frac{1}{t}})^t}$. As N tends to infinity, $\mathrm{por}(A) \approx \frac{1}{N} \approx \left(\frac{a_\nu}{b_\nu}\right)^{\frac{1}{1-t}}$. We do not know what is the best asymptotic behaviour as $\frac{a_\nu}{b_\nu}$ is fixed and $t \to s$.

(2) We do not know whether the completeness is needed in Theorem 5.3 although the mass distribution principle is not valid without it.

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