DISCONTINUOUS SOLUTIONS OF LINEAR, DEGENERATE ELLIPTIC EQUATIONS

XIAO ZHONG

ABSTRACT. We give examples of discontinuous solutions of linear, degenerate elliptic equations with divergence structure. These solve positively conjectures of De Giorgi.

1. INTRODUCTION

We consider the second order, linear, elliptic equation with divergence structure

$$\operatorname{div}(\mathbb{A}(x)\nabla u(x)) = 0, \tag{1}$$

where $\mathbb{A}(x) = [a_{ij}(x)]_{i,j=1,2,\ldots,n}$ is a symmetric matrix with measurable coefficients, defined in a domain $\Omega \subset \mathbb{R}^n, n \geq 2$. We assume the following ellipticity condition

$$\lambda(x)|\xi|^2 \leqslant \langle \mathbb{A}(x)\xi, \xi\rangle \leqslant \Lambda(x)|\xi|^2 \tag{2}$$

for all $\xi \in \mathbb{R}^2$ and for almost every $x \in \Omega$. Here $\lambda(x)$ and $\Lambda(x)$ are measurable functions, finite and positive for almost every $x \in \Omega$. Equation (1) is *uniformly elliptic* if $\Lambda(x)/\lambda(x)$ is essentially bounded in Ω, and *strictly ellptic* if $\lambda(x) \geq \lambda_0$ for a positive constant λ_0, in the sense of Gilbarg and Trudinger [11].

If equation (1) is uniformly and strictly elliptic, i.e., $\lambda(x)^{-1}$ and $\Lambda(x)$ are essentially bounded, it is well known that the weak solutions are Hölder continuous. In the planar case, the study goes back to the work of Morrey [15, 16], see [26], [14] and [22] for the study of the best Hölder continuity exponent. In higher dimensions ($\mathbb{R}^n, n \geqslant 3$), Hölder continuity of solutions was settled in the late 1950's by De Giorgi [5] and Nash [20]. Hölder continuity also follows from the Harnack inequality, due to Moser [17, 18]. We refer to [13], [23], [24] for the quasilinear elliptic equations.

On the contrary, when equation (1) is degenerate, i.e., $\lambda(x)^{-1}$ is unbounded, or when equation (1) is singular, i.e., $\Lambda(x)$ is unbounded, finding the optimal conditions on $\lambda(x)$ and $\Lambda(x)$ which guarantee the continuity of weak solutions seems to be far from settled. For the existing approaches, we refer to [19], [7], [4] and [8], where *local* conditions on $\lambda(x), \Lambda(x)$ are imposed, for example, that equation (1) is uniformly elliptic and $\lambda(x)$ is a Muckenhaupt A_2 weight. We also refer to [1], [12], [9], [2] and [3], where the structure assumptions on the whole matrix $\mathbb{A}$ are made, for example, that the operator is the sum of squares of vector fields satisfying Hörmander's rank condition.

In 1995, De Giorgi gave a talk in Lecce, Italy, and discussed the natural question: are size assumptions on $\lambda(x)^{-1}$ and $\Lambda(x)$ sufficient to guarantee the continuity of

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weak solutions? He raised the following conjectures on the continuity of weak solutions of equation (1). The first one concerns the singular case in higher dimensions.

Conjecture 1. [De Giorgi [6]] Let $n \geq 3$. Suppose that $\mathbb{A}(x)$ satisfies (2) with $\lambda(x) = 1$ and with $\Lambda(x)$ satisfying

$$\int_\Omega \exp(\Lambda(x))\,dx < \infty. \tag{3}$$

Then all weak solutions of equation (1) are continuous in Ω.

The second one concerns the degenerate case in higher dimensions.

Conjecture 2. [De Giorgi [6]] Let $n \geq 3$. Suppose that $\mathbb{A}(x)$ satisfies (2) with $\Lambda(x) = 1$ and with $\lambda(x)$ satisfying

$$\int_\Omega \exp(\lambda(x)^{-1})\,dx < \infty. \tag{4}$$

Then all weak solutions of equation (1) are continuous in Ω.

The third one concerns the case, both singular and degenerate, in higher dimensions.

Conjecture 3. [De Giorgi [6]] Let $n \geq 3$. Suppose that $\mathbb{A}(x)$ satisfies (2) with $\Lambda(x) = \lambda(x)^{-1}$ satisfying

$$\int_\Omega \exp(\Lambda(x)^2)\,dx < \infty. \tag{5}$$

Then all weak solutions of equation (1) are continuous in Ω.

The fourth one concerns the planar case, $n = 2$, which is different from the higher dimensional cases.

Conjecture 4. [De Giorgi [6]] Let $n = 2$. Suppose that $\mathbb{A}(x)$ satisfies (2) with $\Lambda(x) = 1$ and with $\lambda(x)$ satisfying

$$\int_\Omega \exp(\sqrt{\lambda(x)^{-1}})\,dx < \infty. \tag{6}$$

Then all weak solutions of equation (1) are continuous in Ω.

Conjecture 1, Conjecture 2, and Conjecture 3 are still open. As far as we know, the best known result is due to Trudinger [25], which is far from the conjectures. In my opinion, one needs new ideas to deal with these challenging problems. Concerning Conjecture 4, Onninen and the author [21] recently proved that all weak solutions of equation (1) are continuous under the assumption that

$$\int_\Omega \exp(\alpha\sqrt{\lambda(x)^{-1}})\,dx < \infty$$

for some constant $\alpha > 1$.

Another interesting part of [6] is that De Giorgi also conjectured that his conjectures above are sharp; the integrability conditions (3), (4) and (6) are optimal to guarantee the continuity of weak solutions. For example, in Conjecture 1, one can not replace (3) by the following weaker one

$$\int_\Omega \exp(\alpha\Lambda(x)^{1-\delta})\,dx < \infty \tag{7}$$

for some $\delta > 0$ and any $\alpha > 0$. De Giorgi conjectured that one can construct a function $\Lambda(x)$ satisfying the integrability condition (7) such that equation (1) satisfying (2) with $\lambda(x) = 1$ and with this $\Lambda(x)$ has discontinuous weak solutions.

In [6], De Giorgi even gave hints to construct such counter examples to show the sharpness of the above conjectures. He made the following conjectures. Let $\Omega = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : |x| < 1/e\}$ and A, B be subsets of Ω

$$A = \{x \in \Omega : 2|x_n| > |x|\}, \quad B = \{x \in \Omega : 2|x_n| < |x|\}.$$

The first conjecture shows the sharpness of Conjecture 1.

Conjecture 5. [De Giorgi [6]] Let $n \geq 3$. For any $\varepsilon > 0$, define a function τ_1 in Ω as follows

$$\tau_1(x) = \begin{cases} |\log|x||^{1+\varepsilon} & \text{if } x \in A; \\ 1 & \text{if } x \in B. \end{cases}$$

Then equation (1) with $\mathbb{A}(x) = \tau_1(x)I$ has a weak solution, discontinuous at the origin.

The second one shows the sharpness of Conjecture 2.

Conjecture 6. [De Giorgi [6]] Let $n \geq 3$. For any $\varepsilon > 0$, define a function τ_2 in Ω as follows

$$\tau_2(x) = \begin{cases} 1 & \text{if } x \in A; \\ |\log|x||^{-(1+\varepsilon)} & \text{if } x \in B. \end{cases}$$

Then equation (1) with $\mathbb{A}(x) = \tau_2(x)I$ has a weak solution, discontinuous at the origin.

The third one concerns the planar case, and shows the sharpness of Conjecture 4.

Conjecture 7. [De Giorgi [6]] Let $n = 2$. For any $\varepsilon > 0$, define a function τ_3 in Ω as follows

$$\tau_3(x) = \begin{cases} 1 & \text{if } x \in A; \\ |\log|x||^{-(2+\varepsilon)} & \text{if } x \in B. \end{cases}$$

Then equation (1) with $\mathbb{A}(x) = \tau_3(x)I$ has a weak solution, discontinuous at the origin.

In this paper, we show that Conjecture 5, Conjecture 6 and Conjecture 7 are true.

THEOREM 1.1. *Conjecture 5 is true.*

THEOREM 1.2. *Conjecture 6 is true.*

THEOREM 1.3. *Conjecture 7 is true.*

Theorem 1.1 deals with the singular case in higher dimensions. It is essentially due to Franchi, Serapioni and Serra Cassano [10, Theorem 1]. They showed that equation (1) with $\mathbb{A}(x) = \tilde{\tau}_1(x)I$ has a discontinuous weak solution, where the function $\tilde{\tau}_1(x)$ is not exactly the function τ_1 in Conjecture 5, but comparable to it.

For the degenerate case, Theorem 1.2 and Theorem 1.3 are both new. To my knowledge, the best known result in this direction is the following one, due to

Franchi, Serapioni and Serra Cassano [10, Theorem 1]: Let $x = (x_1, x_2) \in B(0,1) \subset \mathbb{R}^2$ and define

$$\tilde{\tau}_3(x) = \begin{cases} (\varepsilon|x_2|)^\gamma & \text{if } \eta|x_1| \le |x_2|; \\ (\eta|x_1|)^\delta & \text{if } \eta|x_1| > |x_2|, \end{cases}$$

where $0 < \delta < \gamma < 1/2$, $0 < \varepsilon < 1/2$ and $0 < \eta^2 < (\gamma+\delta)/2$. Then equation (1) with $\mathbb{A}(x) = \tilde{\tau}_3(x)I$ has a discontinuous weak solution. As we can see, the function $\tilde{\tau}_3(x)$ goes to zero much faster than the function $\tau_3(x)$ in Conjecture 7, as x approaches to the origin. So, this example is far from Conjecture 7.

To prove the theorems above, we follow the idea of De Giorgi [6] and the idea of Franchi, Serapioni and Serra Cassano [10]. We construct a positive, continuous weak subsolution v in the upper half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, such that it has a zero trace on the hyperplane $\{x \in \mathbb{R}^n : x_n = 0\}$ and that

$$\lim_{x_n \to 0^+} v(0, x_n) > 0.$$

Then we reflect v to the lower half space such that it is odd with respect to x_n. Now solve the Dirichlet problem

$$\operatorname{div}(\mathbb{A}(x)\nabla u(x)) = 0 \quad \text{in } B(0,1),$$

such that u coincide with v on $\partial B(0,1)$ in the sense of Sobolev. Then u is the desired discontinuous weak solution. So, the essential point is to build up a weak subsolution v with the above properties. Actually, in [6], De Giorgi proposed that we can construct such a weak subsolution for Conjecture 5 satisfying

$$v(x) = \frac{x_n}{|x_n|}\left(1 + b|\log|x||^{-\varepsilon}\right), \qquad \text{when } 2|x_n| = |x|,$$

where $b > 0$ is a suitable constant. In our example, the function v that we constructed in the proof of Theorem 1.1 satisfies the above condition. The proofs of the theorems are given in Section 2.

As mentioned by De Giorgi in [6], solving the conjectures above will raise new questions. Indeed, for example, in the plane, there is still a gap between Theorem 1.3 and the result mentioned above by Onninen and the author in [21]. Theorem 1.3 only shows that the square root in the integrability condition (6) in Conjecture 4 is sharp. One may naturally ask the following question: is there a function $\lambda(x)$ satisfying

$$\int_\Omega \exp(\beta\sqrt{\lambda(x)^{-1}})\, dx < \infty,$$

for a positive constant β, such that equation (1) satisfying (2) with $\Lambda(x) = 1$ and with this $\lambda(x)$ has a discontinuous weak solution? Of course, we require that $\beta \le 1$, since, otherwise, all weak solutions are continuous. My guess is that the answer to above question is positive if $\beta < 1$. We may also ask similar questions in higher dimensions.

Finally, we give the definition of weak solutions of equation (1). Following Trudinger [25], we define the scalar products

$$\begin{aligned} \mathcal{A}(u,v) &= \int_\Omega \langle \mathbb{A}(x)\nabla u, \nabla v\rangle\, dx; \\ \mathcal{A}_1(u,v) &= \int_\Omega \langle \mathbb{A}(x)\nabla u, \nabla v\rangle + \Lambda(x)uv\, dx \end{aligned} \tag{8}$$

on the spaces $C_0^1(\Omega), C^1(\Omega)$, respectively. Here $\mathbb{A}(x)$ is the symmetric matrix satisfying (2). The weighted Sobolev spaces, $H_0^1(\mathcal{A},\Omega)$, $H^1(\mathcal{A},\Omega)$ are then defined as the completions of $C_0^1(\Omega)$, $C^1(\Omega)$ under $\mathcal{A}$, $\mathcal{A}_1$, respectively. $H_0^1(\mathcal{A},\Omega)$ and $H^1(\mathcal{A},\Omega)$ are Hilbert spaces. We can now define a *weak solution* (*weak subsolution*, *weak supersolution*) of equation (1) as a function u in $H^1(\mathcal{A},\Omega)$ satisfying

$$\int_\Omega \langle \mathbb{A}(x)\nabla u, \nabla\phi\rangle \, dx = 0 \quad (\le 0,\ \ge 0) \tag{9}$$

for all non-negative functions $\phi \in H_0^1(\mathcal{A},\Omega)$. So, the weak solutions here are variational ones. We refer to [25] for the detailed discussions on the properties of Sobolev spaces $H_0^1(\mathcal{A},\Omega)$ and $H^1(\mathcal{A},\Omega)$ and on the existence and properties of the weak solutions.

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2. Proofs

We first consider the planar case and prove Theorem 1.3.

Proof of Theorem 1.3. Let $n = 2$. We use the polar coordinates $(r,\theta), r \ge 0, 0 \le \theta < 2\pi$, where $r = |x|, x_1 = r\cos\theta, x_2 = r\sin\theta$ for a point $x = (x_1,x_2) \in \mathbb{R}^2$. Let r_0 be a small number, chosen later. Define the sets

$$A = \{x \in B(0,r_0) : 2|x_2| > |x|\}, \quad B = \{x \in B(0,r_0) : 2|x_2| < |x|\}.$$

We will define a function in the upper half plane $\mathbb{R}^2_+ = \{(r,\theta) : r > 0, 0 < \theta < \pi\}$. Note that

$$\begin{aligned} A \cap \mathbb{R}^2_+ &= \{(r,\theta) : 0 < r < r_0, \pi/6 < \theta < 5\pi/6\}, \\ B \cap \mathbb{R}^2_+ &= \{(r,\theta) : 0 < r < r_0, 0 < \theta < \pi/6, \text{ and } 5\pi/6 < \theta < \pi\}. \end{aligned}$$

We define a function in $B(0,r_0) \cap \mathbb{R}^2_+$ as

$$v(r,\theta) = \begin{cases} \left(1 + \left(\log\frac{1}{r}\right)^{-\beta}\right)\varphi(\theta) & \text{if } 0 < \theta < \pi/6; \\ \left(1 + \left(\log\frac{1}{r}\right)^{-\beta}\right)\varphi(\pi-\theta) & \text{if } 5\pi/6 < \theta < \pi; \\ \left(1 + \left(\log\frac{1}{r}\right)^{-\beta}\right)\varphi(\pi/6) + \left(\log\frac{1}{r}\right)^{-(2+\gamma)}\psi(\theta) & \text{if } \pi/6 < \theta < 5\pi/6. \end{cases}$$

Here β and γ are positive constants. We require that

$$0 < \beta < \gamma < \varepsilon, \tag{10}$$

where $\varepsilon > 0$ is the constant from the function $\tau_3(x)$ in Conjecture 7. Recall that $\tau_3(x)$ reads in polar coordinates as

$$\tau_3(r,\theta) = \begin{cases} \left(\log\frac{1}{r}\right)^{-(2+\varepsilon)} & \text{if } 0 < \theta < \pi/6 \text{ and if } 5\pi/6 < \theta < \pi; \\ 1 & \text{if } \pi/6 < \theta < 5\pi/6, \end{cases}$$

and that it is even with repect to x_2. (10) is required in order to show that the function v is a subsolution in $\mathbb{R}^2_+$, as we will see soon. This is the reason that we assume that ε is positive. In other words, equation (1) has to degenerate fast enough so that it has discontinuous weak solutions.

In the definition of the function v, the function φ, defined on $[0, \pi/6]$, is a smooth function with $\varphi(0) = 0, \varphi(\pi/6) > 0$. We require that it is positive, increasing and strictly convex. For example, $\varphi(\theta) = \theta^2$ will do. The function ψ, defined on $[\pi/6, 5\pi/6]$, is a smooth function with $\psi(\pi/6) = \psi(5\pi/6) = 0$. We require that it is positive and that $\psi'(\pi/6) > 0, \psi'(5\pi/6) < 0$. For example, $\psi(\theta) = \sin(\frac{3}{2}(\theta - \pi/6))$ will do.

While v is well defined in the upper half plane, we extend it to the whole plane such that it is odd with respect to x_2, that is, $v(x_1, x_2) = -v(x_1, -x_2)$. Note that v is continuous outside the origin and that v vanishes on the x_1 axis, except the origin.

First, we claim that $v \in H^1(\mathcal{A}, B(0, r_0))$. Indeed, informally, we can check that the following integral is finite,

$$\int_{B(0,r_0)} \tau_3(x)|\nabla v|^2\, dx = \int_0^{r_0} \int_0^{2\pi} \tau_3(r, \theta) \left((\partial_r v)^2 + \frac{(\partial_\theta v)^2}{r^2} \right) r\, d\theta\, dr < \infty.$$

This is easy to check. A formal proof involves a simple approximation. We omit the details.

Next, we claim that v is a weak subsolution of the equation

$$\operatorname{div}(\tau_3(x)\nabla u) = 0,$$

in $B(0, r_0) \cap \mathbb{R}^2_+$. Indeed, note that the function v and τ_3 are piecewise smooth in $A \cap \mathbb{R}^2_+$ and in $B \cap \mathbb{R}^2_+$. By an easy argument of integration by parts, we only need to check that

i) $\operatorname{div}(\tau_3(x)\nabla v) \geq 0$, pointwise in $A \cap \mathbb{R}^2_+$ and in $B \cap \mathbb{R}^2_+$;

ii) on the boundary $\{(r, \theta_0) : 0 < r < r_0, \theta_0 = \pi/6, \ \text{ or } \theta_0 = 5\pi/6\}$, the following transmission condition is satisfied

$$\lim_{\theta \to \theta_0^-} \tau_3(r, \theta)\partial_\theta v(r, \theta) \leq \lim_{\theta \to \theta_0^+} \tau_3(r, \theta)\partial_\theta v(r, \theta). \tag{11}$$

We prove i) by direct computation as follows. Write the equation in polar coordinates

$$\frac{1}{\tau_3}\operatorname{div}(\tau_3 \nabla v) = \begin{cases} \partial_{rr} v + r^{-1}\partial_r v + r^{-2}\partial_{\theta\theta} v + (2 + \varepsilon)(r \log \frac{1}{r})^{-1}\partial_r v & \text{in } B \cap \mathbb{R}^2_+; \\ \partial_{rr} v + r^{-1}\partial_r v + r^{-2}\partial_{\theta\theta} v & \text{in } A \cap \mathbb{R}^2_+. \end{cases}$$

In $B \cap \mathbb{R}^2_+$, if $0 < \theta < \pi/6$, we have

$$\begin{aligned}
\partial_r v &= \beta r^{-1}\big(\log \frac{1}{r}\big)^{-(1+\beta)}\varphi(\theta); \\
\partial_{rr} v &= \beta(1 + \beta)r^{-2}\big(\log \frac{1}{r}\big)^{-(2+\beta)}\varphi(\theta) - \beta r^{-2}\big(\log \frac{1}{r}\big)^{-(1+\beta)}\varphi(\theta); \\
\partial_{\theta\theta} v &= \left(1 + \big(\log \frac{1}{r}\big)^{-\beta}\right) \varphi''(\theta).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{\tau_3}\operatorname{div}(\tau_3 \nabla v) =& \beta(3 + \beta + \varepsilon)r^{-2}\big(\log \frac{1}{r}\big)^{-(2+\beta)}\varphi(\theta) \\
&+ r^{-2}\left(1 + \big(\log \frac{1}{r}\big)^{-\beta}\right) \varphi''(\theta) \\
=& \varphi''(\theta)r^{-2} + o(r^{-2}) > 0,
\end{aligned}$$

if $r < r_0$ and r_0 is small enough. The case $5\pi/6 < \theta < \pi$ is similar to the above one.

In $A \cap \mathbb{R}^2_+$, we have

$$\begin{aligned}
\partial_r v =& \beta r^{-1}\big(\log\frac{1}{r}\big)^{-(1+\beta)}\varphi(\pi/6) + (2+\gamma)r^{-1}\big(\log\frac{1}{r}\big)^{-(3+\gamma)}\psi(\theta);\\
\partial_{rr} v =& \beta(1+\beta)r^{-2}\big(\log\frac{1}{r}\big)^{-(2+\beta)}\varphi(\pi/6) - \beta r^{-2}\big(\log\frac{1}{r}\big)^{-(1+\beta)}\varphi(\pi/6)\\
&+(2+\gamma)(3+\gamma)r^{-2}\big(\log\frac{1}{r}\big)^{-(4+\gamma)}\psi(\theta) - (2+\gamma)r^{-2}\big(\log\frac{1}{r}\big)^{-(3+\gamma)}\psi(\theta);\\
\partial_{\theta\theta} v =& \big(\log\frac{1}{r}\big)^{-(2+\gamma)}\psi''(\theta).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{\tau_3}\mathrm{div}(\tau_3\nabla v) =& \beta(1+\beta)r^{-2}\big(\log\frac{1}{r}\big)^{-(2+\beta)}\varphi(\pi/6)\\
&+(2+\gamma)(3+\gamma)r^{-2}\big(\log\frac{1}{r}\big)^{-(4+\gamma)}\psi(\theta) + r^{-2}\big(\log\frac{1}{r}\big)^{-(2+\gamma)}\psi''(\theta)\\
=& \beta(1+\beta)\varphi(\pi/6)r^{-2}\big(\log\frac{1}{r}\big)^{-(2+\beta)} + o(r^{-2}\big(\log\frac{1}{r}\big)^{-(2+\beta)}) > 0,
\end{aligned}$$

if $r < r_0$ and r_0 is small enough, because of our choice (10) of β and γ, $0 < \beta < \gamma$. This proves i).

To check ii) is easy. When $\theta_0 = \pi/6$, we have

$$\lim_{\theta\to\theta_0^-}\tau_3(r,\theta)\partial_\theta v(r,\theta) = \big(\log\frac{1}{r}\big)^{-(2+\varepsilon)}\left(1 + \big(\log\frac{1}{r}\big)^{-\beta}\right)\varphi'(\pi/6),$$

and

$$\lim_{\theta\to\theta_0^+}\tau_3(r,\theta)\partial_\theta v(r,\theta) = \big(\log\frac{1}{r}\big)^{-(2+\gamma)}\psi'(\pi/6).$$

Hence

$$\lim_{\theta\to\theta_0^-}\tau_3(r,\theta)\partial_\theta v(r,\theta) \le \lim_{\theta\to\theta_0^+}\tau_3(r,\theta)\partial_\theta v(r,\theta),$$

if $r < r_0$ and r_0 is small enough, because of our choice (10) of γ, $0 < \gamma < \varepsilon$. The case $\theta_0 = 5\pi/6$ is similar. This proves ii), and hence the claim.

Finally, solve the Dirichlet problem

$$\begin{cases} \mathrm{div}(\tau_3(x)\nabla u) = 0, & \text{in } B(0,r_0);\\ u - v \in H^1_0(\mathcal{A}, B(0,r_0)). \end{cases} \tag{12}$$

The existence of weak solutions follows from the Riesz representation theorem. We claim that the weak solution u of (12) is discontinuous at the origin. Indeed, first, since the equation is uniformly and strictly elliptic outside the origin, u is continuous outside the origin. Second, since v is odd and τ_3 is even with respect to x_2, $-u(x_1,-x_2)$ is also a weak solution of (12). Thus u is odd with respect to x_2, by the uniqueness of the solution of (12). This together with the continuity of u outside the origin implies that $u(x_1,0) = 0$ if $x_1 \neq 0$. Finally, because $\tau_3(x)^{-1} \in L^1(B(0,r_0))$ and $u \in H^1(\mathcal{A}, B(0,r_0))$, we know that $u \in W^{1,1}(B(0,r_0))$, the unweighted Sobolev space. Thus u has a trace on $B(0,r_0) \cap \{x_2 = 0\}$, being necessarily zero, and so is v. This implies that $u - v \in H^1_0(\mathcal{A}, B(0,r_0) \cap \mathbb{R}^2_+)$. Since u is a weak solution and v

is a weak subsolution in $B(0, r_0) \cap \mathbb{R}^2_+$, the comparison principle implies that $u \geq v$ in $B(0, r_0) \cap \mathbb{R}^2_+$. Thus

$$\lim \inf_{x_2 \to 0^+} u(0, x_2) \geq \lim_{x_2 \to 0^+} v(0, x_2) = \varphi(\pi/6) > 0.$$

Recall that $u(x_1, 0) = 0$ if $x_1 \neq 0$. Hence u is discontinuous at the origin. This finishes the proof of Theorem 1.3. □

With minor changes of the proof above, one can prove the following more general result.

THEOREM 2.1. *Let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1/e\}$ and let $\alpha_1 \geq 0$ and $\alpha_2 > 1$ be numbers such that $\alpha_1 + \alpha_2 > 2$. Define a function τ_4 in Ω as follows*

$$\tau_4(x) = \begin{cases} |\log |x||^{\alpha_1} & \text{if } 2|x_2| > |x|; \\ |\log |x||^{-\alpha_2} & \text{if } 2|x_2| < |x|. \end{cases}$$

Then equation (1) with $\mathbb{A}(x) = \tau_4(x) I$ has a weak solution, discontinuous at the origin.

In the above theorem, we require that $\alpha_2 > 1$ to ensure that the function v, defined in the proof of Theorem 1.3, belongs to the Sobolev space $H^1(\mathcal{A}, \Omega)$, and that $\alpha_1 + \alpha_2 > 2$ to ensure that the function v is a weak subsolution.

Next, we consider the degenerate case in higher dimensions and prove Theorem 1.2.

Proof of Theorem 1.2. Let $n \geq 3$. For a point $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we let $r, r \geq 0$, be $r = |x|$, and $\theta, 0 \leq \theta \leq \pi$, be the angle between the line segment $[0, x]$ and the x_n axis. let r_0 be a small number, chosen later. Define the sets

$$\begin{aligned} A =& \{x \in B(0, r_0) : 2|x_n| > |x|\} \\ =& \{x \in \mathbb{R}^n : r < r_0, 0 \leq \theta < \pi/3, \quad \text{or } 2\pi/3 < \theta \leq \pi\}; \\ B =& \{x \in B(0, r_0) : 2|x_n| < |x|\} = \{x \in \mathbb{R}^n : r < r_0, \pi/3 < \theta < 2\pi/3\}. \end{aligned}$$

We define a function in $B(0, r_0) \cap \mathbb{R}^n_+$ as

$$v(x) = \begin{cases} \left(1 + \left(\log \frac{1}{r}\right)^{-\beta}\right) \varphi(\theta) & \text{if } \pi/3 < \theta < \pi/2; \\ \left(1 + \left(\log \frac{1}{r}\right)^{-\beta}\right) \varphi(\pi/3) + \left(\log \frac{1}{r}\right)^{-(1+\gamma)} \psi(\theta) & \text{if } 0 \leq \theta < \pi/3. \end{cases}$$

Here β and γ are positive constants such that

$$0 < \beta < \gamma < \varepsilon, \tag{13}$$

where $\varepsilon > 0$ is the constant from the function $\tau_2(x)$ in Conjecture 6. Recall that $\tau_2(x)$ reads as

$$\tau_2(x) = \begin{cases} \left(\log \frac{1}{r}\right)^{-(1+\varepsilon)} & \text{if } \pi/3 < \theta < \pi/2; \\ 1 & \text{if } 0 \leq \theta < \pi/3, \end{cases}$$

and that it is even with respect to x_n. In the definition of v, the function φ, defined on $[\pi/3, \pi/2]$, is a smooth function with $\varphi(\pi/3) > 0, \varphi(\pi/2) = 0$. We require that it is positive, decreasing and strictly convex and that

$$\varphi''(\theta) + (n-2) \frac{\cos \theta}{\sin \theta} \varphi'(\theta) \geq 1, \quad \text{for all } \pi/3 < \theta < \pi/2. \tag{14}$$

For example, $\varphi(\theta) = e^{-\alpha\theta}\int_\theta^{\pi/2}(\pi/2 - t)e^{\alpha t}\,dt$ with $\alpha = \sqrt{3}(n-2)$ will do. The function ψ, defined on $[0, \pi/3]$, is a smooth function with $\psi(\pi/3) = 0$. We require that it is positive and that $\psi'(\pi/3) < 0$. For example, $\psi(\theta) = \cos(3\theta/2)$ will do.

We then extend the function v to the whole space $\mathbb{R}^n$ such that it is odd with respect to x_n. Since the functions v and τ_2 depend only on r and θ, we will write $v(x) = v(r, \theta)$ and $\tau_2(x) = \tau_2(r, \theta)$. Note that v is continuous outside the origin and v vanishes on the hyperplane $\theta = \pi/2$. The function v defined as above for the higher dimensional cases is similar to the function in the proof of Theorem 1.3 for the planar case. The only difference is that we have different exponents, $-(2+\gamma)$ and $-(1+\gamma)$, in the functions involving ψ.

As in the proof of Theorem 1.3 for the planar case, we will show the following. First, we claim that $v \in H^1(\mathcal{A}, B(0, r_0))$. This is easy to check. We omit the details.

Second, we claim that v is a weak subsolution of the equation

$$\operatorname{div}(\tau_2(x)\nabla u) = 0,$$

in $B(0, r_0) \cap \mathbb{R}^n_+$. To prove this, we only need to check that

i) $\operatorname{div}(\tau_2(x)\nabla v) \geq 0$, pointwise in $A \cap \mathbb{R}^n_+$ and in $B \cap \mathbb{R}^n_+$;

ii) on the boundary $\theta_0 = \pi/3$, the transimission condition is satisfied

$$\lim_{\theta\to\theta_0^-} \tau_2(r,\theta)\partial_\theta v(r,\theta) \leq \lim_{\theta\to\theta_0^+} \tau_2(r,\theta)\partial_\theta v(r,\theta). \tag{15}$$

We prove i) by direct computation as follows. Since the functions v and τ_2 depend only on r and θ, we may write the equation in the following form

$$\frac{1}{\tau_2}\operatorname{div}(\tau_2\nabla v) = \begin{cases} \partial_{rr}v + (n-1)\frac{\partial_r v}{r} + (n-2)\frac{\cos\theta}{\sin\theta}\frac{\partial_\theta v}{r^2} + \frac{\partial_{\theta\theta}v}{r^2} + (1+\varepsilon)\frac{\partial_r v}{r\log\frac{1}{r}} & \text{in } B; \\ \partial_{rr}v + (n-1)\frac{\partial_r v}{r} + (n-2)\frac{\cos\theta}{\sin\theta}\frac{\partial_\theta v}{r^2} + \frac{\partial_{\theta\theta}v}{r^2} & \text{in } A. \end{cases}$$

In $B \cap \mathbb{R}^n_+$, that is, $r < r_0, \pi/3 < \theta < \pi/2$, we have

$$\begin{aligned}
\partial_r v =& \beta r^{-1}\big(\log\frac{1}{r}\big)^{-(1+\beta)}\varphi(\theta);\\
\partial_\theta v =& \left(1 + \big(\log\frac{1}{r}\big)^{-\beta}\right)\varphi'(\theta);\\
\partial_{rr}v =& \beta(1+\beta)r^{-2}\big(\log\frac{1}{r}\big)^{-(2+\beta)}\varphi(\theta) - \beta r^{-2}\big(\log\frac{1}{r}\big)^{-(1+\beta)}\varphi(\theta);\\
\partial_{\theta\theta}v =& \left(1 + \big(\log\frac{1}{r}\big)^{-\beta}\right)\varphi''(\theta),
\end{aligned}$$

and hence,

$$\begin{aligned}
\frac{1}{\tau_2}\operatorname{div}(\tau_2\nabla v) =& \beta(2+\beta+\varepsilon)r^{-2}\big(\log\frac{1}{r}\big)^{-(2+\beta)}\varphi(\theta)\\
&+ (n-2)\beta r^{-2}\big(\log\frac{1}{r}\big)^{-(1+\beta)}\varphi(\theta)\\
&+ \left(\varphi''(\theta) + (n-2)\frac{\cos\theta}{\sin\theta}\varphi'(\theta)\right)r^{-2}\left(1 + \big(\log\frac{1}{r}\big)^{-\beta}\right)\\
\geq& r^{-2} + o(r^{-2}) > 0,
\end{aligned}$$

if $r < r_0$ and r_0 is small enough. In the last step, we used (14).

In $A \cap \mathbb{R}^n_+$, that is, $r < r_0, 0 \le \theta < \pi/3$, we have

$$\begin{aligned}
\partial_r v =& \beta r^{-1}\big(\log\frac{1}{r}\big)^{-(1+\beta)}\varphi(\pi/3) + (1+\gamma)r^{-1}\big(\log\frac{1}{r}\big)^{-(2+\gamma)}\psi(\theta);\\
\partial_{rr} v =& \beta(1+\beta)r^{-2}\big(\log\frac{1}{r}\big)^{-(2+\beta)}\varphi(\pi/3) - \beta r^{-2}\big(\log\frac{1}{r}\big)^{-(1+\beta)}\varphi(\pi/3)\\
&+ (1+\gamma)(2+\gamma)r^{-2}\big(\log\frac{1}{r}\big)^{-(3+\gamma)}\psi(\theta) - (1+\gamma)r^{-2}\big(\log\frac{1}{r}\big)^{-(2+\gamma)}\psi(\theta);\\
\partial_\theta v =& \big(\log\frac{1}{r}\big)^{-(1+\gamma)}\psi'(\theta);\\
\partial_{\theta\theta} v =& \big(\log\frac{1}{r}\big)^{-(1+\gamma)}\psi''(\theta).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{\tau_2}\mathrm{div}(\tau_2\nabla v) =& \partial_{rr}v + (n-1)\frac{\partial_r v}{r} + (n-2)\frac{\cos\theta}{\sin\theta}\frac{\partial_\theta v}{r^2} + \frac{\partial_{\theta\theta}v}{r^2}\\
=& (n-2)\beta\varphi(\pi/3)r^{-2}\big(\log\frac{1}{r}\big)^{-(1+\beta)} + o\left(r^{-2}\big(\log\frac{1}{r}\big)^{-(1+\beta)}\right) > 0,
\end{aligned}$$

if $r < r_0$ and r_0 is small enough. Here we used (13), $0 < \beta < \gamma$. This proves i).

To check ii) is easy. We have

$$\lim_{\theta\to\theta_0^+}\tau_2(r,\theta)\partial_\theta v(r,\theta) = \big(\log\frac{1}{r}\big)^{-(1+\varepsilon)}\left(1 + \big(\log\frac{1}{r}\big)^{-\beta}\right)\varphi'(\pi/3),$$

and

$$\lim_{\theta\to\theta_0^-}\tau_2(r,\theta)\partial_\theta v(r,\theta) = \big(\log\frac{1}{r}\big)^{-(1+\gamma)}\psi'(\pi/3).$$

Recall that $\varphi'(\pi/3) < 0, \psi'(\pi/3) < 0$ and that (13) requires $0 < \beta < \gamma < \varepsilon$. Hence,

$$\lim_{\theta\to\theta_0^-}\tau_2(r,\theta)\partial_\theta v(r,\theta) \le \lim_{\theta\to\theta_0^+}\tau_2(r,\theta)\partial_\theta v(r,\theta),$$

if $r < r_0$ and r_0 is small enough. This proves ii), and hence the claim.

Finally, solve the Dirichlet problem

$$\begin{cases} \mathrm{div}(\tau_2(x)\nabla u) = 0, & \text{in } B(0,r_0);\\ u - v \in H^1_0(\mathcal{A}, B(0,r_0)). \end{cases}$$

Then the weak solution u is the desired discontinuous one. The proof is the same as that in the proof of Theorem 1.3. We omit the details. This finishes the proof of Theorem 1.2. □

Finally, we consider the singular case in higher dimensions and prove Theorem 1.1.

Proof of Theorem 1.1. The proof is similar to that of Theorem 1.2 in the degenerate case. We only give the main line of the proof. Let r, θ and the sets A, B be as in the proof of Theorem 1.2. Let v, defined on $B(0,r_0) \cap \mathbb{R}^n_+$, be the same function as in the proof of Theorem 1.2.

$$v(x) = \begin{cases} \left(1 + \big(\log\frac{1}{r}\big)^{-\beta}\right)\varphi(\theta) & \text{if } \pi/3 < \theta < \pi/2;\\ \left(1 + \big(\log\frac{1}{r}\big)^{-\beta}\right)\varphi(\pi/3) + \big(\log\frac{1}{r}\big)^{-(1+\gamma)}\psi(\theta) & \text{if } 0 \le \theta < \pi/3, \end{cases}$$

where $0 < \beta < \gamma < \varepsilon$, and φ and ψ are functions as in the proof of Theorem 1.2. We claim that v is a weak subsolution of the equation

$$\operatorname{div}(\tau_1(x)\nabla u) = 0,$$

in $B(0, r_0) \cap \mathbb{R}^n_+$, where $\tau_1(x)$ is the function defined as in Conjecture 5

$$\tau_1(x) = \begin{cases} 1 & \text{if } x \in B; \\ \left(\log \frac{1}{r}\right)^{1+\varepsilon} & \text{if } x \in A. \end{cases}$$

This can be checked by direct computation as in the proof of Theorem 1.2. We omit the details. Then the weak solution of the equation

$$\begin{cases} \operatorname{div}(\tau_1(x)\nabla u) = 0, & \text{in } B(0, r_0); \\ u - v \in H^1_0(\mathcal{A}, B(0, r_0)). \end{cases}$$

is the desired discontinuous one. This finishes the proof of Theorem 1.1. □

With minor changes of the proof above, one can generalize Theorem 1.2 and Theorem 1.1 to the degenerate and singular case, which shows the sharpness of Conjecture 3.

THEOREM 2.2. *Let* $\Omega = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : |x| < 1/e\}, n \geq 3$ *and let* $\alpha_1 \geq 0$ *and* $\alpha_2 \geq 0$ *be numbers such that* $\alpha_1 + \alpha_2 > 1$. *Define a function* τ_5 *in* Ω *as follows*

$$\tau_5(x) = \begin{cases} |\log |x||^{\alpha_1} & \text{if } 2|x_n| > |x|; \\ |\log |x||^{-\alpha_2} & \text{if } 2|x_n| < |x|. \end{cases}$$

Then equation (1) with $\mathbb{A}(x) = \tau_5(x)I$ *has a weak solution, discontinuous at the origin.*

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Department of Mathematics and Statistics, University of Jyväskylä,, P.O. Box 35 (MaD), Fin-40014, University of Jyväskylä, Finland.

E-mail address: zhong@maths.jyu.fi