Adams inequality on metric measure spaces

Tero Mäkäläinen

Abstract

In this paper, we prove the Adams inequality in complete metric spaces supporting a Poincaré inequality with a doubling measure. We also prove the trace inequalities for the Riesz potentials.

1 Introduction

In the Euclidean spaces we have the following Adams inequality, see e.g. [Ad], [Ma], [Tu] or [Zi]:

Theorem 1.1. *Let ν be a Radon measure on $\mathbb{R}^n$ and let $1 \le p < q < \infty$ with $p < n$. Suppose that there is a constant M such that for all balls $B(x,r) \subset \mathbb{R}^n$,*

$$\nu(B(x,r)) \le Mr^{\alpha},$$

where $\alpha = q(n-p)/p$. Then

$$(1) \qquad \left(\int_{\mathbb{R}^n} |u|^q d\nu \right)^{1/q} \le CM^{1/q} \left(\int_{\mathbb{R}^n} |\nabla u|^p dx \right)^{1/p},$$

for all $u \in C_0^{\infty}(\mathbb{R}^n)$, where $C = C(p,q,n) > 0$.

In potential theory, this type of inequalities arise from investigation of imbeddings $\mathcal{G} : L^p(\mu) \to L^q(\nu)$, where $\mathcal{G}$ is a potential, [AH]. These imbeddings are often referred to as trace inequalities. In the Euclidean setting a necessary and sufficient condition for trace type theorems is proven, see [AH, Chapter 7.2]. The sharp result is a growth condition for the measure ν involving Riesz capacity of a ball. See also [Zi, Chapter 4] for more discussions on the Adams type inequality.

For Sobolev functions, inequality (1) is an extension of the Sobolev inequality, since if ν is n-dimensional Lebesgue measure, then $q = p^* = np/(n-p)$.

In this paper, we study the Adams type inequality and trace inequalities for Riesz potential on metric measure spaces. Before we state our main results, we discuss the standard assumptions on the spaces and the background of analysis on metric measure spaces.

2000 *Mathematics Subject Classification.* 46E35, 31C15, 26D10.

Key words and phrases. Trace inequality, Riesz potential, metric space, Sobolev function, the Poincaré inequality.

The author was partially supported by the Academy of Finland, project 207288.

The results in this paper are formulated for Lipschitz functions. In a metric space (X, d), a function $u : X \to \mathbb{R}$ is said to be *Lipschitz continuous*, denoted by $u \in \mathrm{Lip}(X)$, if for some constant $L > 0$

$$|u(x) - u(y)| \leq L d(x, y),$$

for every $x, y \in X$. We also use the notation $u \in \mathrm{Lip}_0(X)$ when the function u has compact support. For a Lipschitz function $u : X \to \mathbb{R}$, we define

$$\mathrm{Lip}\, u(x) := \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}.$$

We require the following standard conditions on the mass and on the geometry of the metric space. First, we assume that the space is equipped with a doubling measure. A measure μ is *doubling* if balls have positive and finite measure and there exists a constant $C_d \geq 1$ such that for all balls $B(x, r)$ in X,

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)).$$

Note that the doubling measure μ has a density lower bound [He, pp. 103-104]: There exist constants c, $s > 0$ that depend only on the doubling constant of μ, such that

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq c \left(\frac{r}{R} \right)^s, \tag{2}$$

whenever $r < R$, $x \in X$ and $y \in B(x, R)$. Usually we consider s to be the natural dimension of the space X, and in this paper we assume that $s > 1$. We call such spaces doubling spaces, or spaces of homogeneous type.

Second, we assume that the space admits a Poincaré inequality:

Definition 1.2. A metric measure space (X, d, μ) is said to admit a *weak* $(1, p)$*-Poincaré (or weak p-Poincaré) inequality*, $1 \leq p < \infty$, with constants $C_p > 0$ and $\tau \geq 1$, if

$$⨍_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C_p r \left(⨍_{B(x,\tau r)} (\mathrm{Lip}\, u)^p \, d\mu \right)^{1/p} \tag{3}$$

for all balls $B(x, r) \subset X$ and for every Lipschitz function $u : X \to \mathbb{R}$. Here barred integrals mean integral averages and u_B is the average value of u over B.

There are also different definitions for the Poincaré inequality on a metric measure space. However many definitions coincide when the space is complete and supports a doubling Borel regular measure, see discussion in [KZ, Chapter 1.2] and references therein. The Poincaré inequality forces the space to be sufficiently regular in a geometric sense.

Recently there has been progress in the theory of Sobolev spaces in general metric measure spaces, see for instance [Ch], [HaK], [Ha], [HeK], [KKM], [KoM], [Sh] and references therein. In [Sh], Shanmugalingam contructs a Sobolev type space on metric spaces, which yields the same space studied by Cheeger in [Ch]. When the metric space satisfies our general assumptions, the Sobolev type spaces introduced by Hajłasz [Ha] also coincide with the spaces mentioned above.

If the metric space is equipped with a doubling measure and it supports a Poincaré inequality, then Lipschitz functions are dense in the space of Sobolev functions on the metric measure space, see [Sh2]. Therefore the results in this paper can also be applied to the Sobolev functions.

When these standard assumptions on the space and on the measure hold, the space has nice geometric properties and allows us to conduct deep analysis of such a space, and recently such analysis was done in many areas of studies. For instance, in [HeK], [KoM] quasiconformal mappings in metric spaces are studied. Also some results of Euclidean potential theory can be generalized to metric spaces, see [KM1], [KM2], [KS] and [Sh2]. Thanks to Cheeger's definition of partial derivatives [Ch], it is even possible to study partial differential equations on such spaces, see [BBS] and [BMS]. In [HaK], the Sobolev inequality is shown to be true in this setting.

The aim of this paper is to show that the Adams-type inequality also holds on metric spaces under some general assumptions. In this paper we prove a trace inequality for a Riesz potential (Theorem 4.1). Similar results for other similar potentials can be found in [EKM, Theorem 6.2.1].

We could not obtain the sharp results as in [AH] for general metric measure spaces, since our measure is not assumed to be (Ahlfors) Q-regular and hence we do not have connection between the measure and the capacity. When the measure μ is Q-regular, the results are achieved easily from proofs in the Euclidean case, basically by replacing the Lebesgue measure with the measure μ. In this paper the difficulty comes from the fact that only the lower bound for the measure μ is needed, see (2), without any upper bound.

Similar problems as in this paper are studied also in [KK], with a different approach.

The case $p = 1$ needs a special treatment as usual. We prove the following global Adams inequality in the case $p = 1$.

Theorem 1.3. *Let (X, d, μ) be a complete metric measure space such that it admits a weak $(1,1)$-Poincaré inequality and μ is a doubling Radon measure. Let ν be a Radon measure on X. Suppose that there are $M \geq 0$ and $q \geq 1$, such that for all balls $B(x,r) \subset X$ of radius $r < \operatorname{diam} X$, it holds*

$$\frac{\nu(B(x,r))}{\mu(B(x,r))^q} \leq M r^{-q}.$$

Then

$$\left(\int_X |u|^q d\nu \right)^{1/q} \leq C M^{1/q} \int_X \operatorname{Lip} u \; d\mu,$$

for all $u \in \operatorname{Lip}_0(X)$, where the constant $C > 0$ depends only on q, s, the doubling constant and the constants in the Poincaré inequality.

Two key elements in the proof of Theorem 1.3 are isoperimetric inequality and co-area formula (Theorem 2.3), which are already studied by Ambrosio [Am] and Miranda [Mi]. We follow the approach in [Zi, Lemma 4.9.1], which can be easily generalized to our setting by using Lemma 3.1.

For the case $p > 1$, we have the following theorem.

Theorem 1.4. *Let (X, d, μ) be a complete metric measure space such that it admits a weak $(1, t)$-Poincaré inequality for some $1 \le t < p$, and μ is a doubling Radon measure. Suppose that ν is a Radon measure on X, satisfying*

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \le M r^{\alpha} \qquad \text{with} \qquad \alpha = \frac{sq}{p} - s - \frac{q}{t},$$

for all balls $B(x, r) \subset X$ of radius $r < \operatorname{diam} X$, where $1 < p < q < \infty$, $p/t < s$ and M is a positive constant. Here s is from (2). If $u \in \operatorname{Lip}_0(B_0)$ for some ball $B_0 = B(x_0, r_0) \subset X$, for which $r_0 < \operatorname{diam} X/10$, we have

$$\left(\int_{B_0} |u|^q d\nu\right)^{1/q} \le C\mu(B_0)^{1/q-1/p}\, r_0^{\frac{t-1}{t}+\frac{s}{p}-\frac{s}{q}} M^{1/q} \left(\int_{B_0} (\operatorname{Lip} u)^p d\mu\right)^{1/p},$$

where $C = C(p, q, s, t, C_d, C_p, \tau) > 0$.

In a forthcoming paper the author applies the Adams inequality to study p-harmonic functions on metric measure spaces and to characterize removable sets for Hölder continuous Cheeger p-harmonic functions.

The proof splits into two steps. First, we prove the inequality

$$|u|^p \le C I_{1,B}((\operatorname{Lip} u)^p),$$

where $I_{1,B}$ is a generalization of the Riesz potential, see Theorem 3.2 and Remark 3.3. Second part is to apply the Adams inequality for the Riesz potential, also called as the Fractional Integration Theorem, which states that

$$I_{1,B} : L^p(B, \mu) \longrightarrow L^q(B, \nu)$$

is a bounded operator, see Corollary 4.2.

In Theorem 1.4, we assume that weak $(1, t)$-Poincaré inequality holds for some $1 \le t < p$. This better Poincaré inequality follows from the weak $(1, p)$-Poincaré inequality by the result in [KZ]. The case $p = s$ is not included in the Theorem 1.4 when the weak $(1, 1)$-Poincaré inequality holds. This case is more delicate and is treated in section 6.

The case $p > s$ is not interesting, since the claim follows from [HaK, Theorem 5.1 (3)].

This paper is organized as follows. In section 2 we give the main definitions and some preliminary results. A few key lemmas are proven in section 3. The Adams inequality for Riesz potential is discussed in section 4. Section 5 contains the proofs of Theorem 1.3 and Theorem 1.4. Finally, in section 6 we prove the Adams inequality for borderline case $p = s$.

2 Preliminaries

Throughout the paper we denote by $C > 0$ a constant, whose value may vary between each usage, even in the same line.

The triple (X, d, μ) denotes a complete metric measure space. The equipped measure μ is assumed to be a Radon measure, which means that the measure is Borel regular and the measure of every compact set is finite. We also assume that the measure of every nonempty open set is positive

The ball with center $x \in X$ and radius $r > 0$ is denoted by

$$B(x,r) = \{y \in X : d(y,x) < r\}$$

and we use the notation $\sigma B(x,r) = B(x, \sigma r)$. We write

$$u_A = \frac{1}{\mu(A)} \int_A u \, d\mu = \fint_A u \, d\mu,$$

for a measurable $A \subset X$ and a measurable function $u : X \to [-\infty, \infty]$. The norm of v in $L^p(X,\mu) = L^p(X)$ is denoted by

$$||v||_p = ||v||_{p,\mu} = \left(\int_X |v|^p d\mu \right)^{1/p}.$$

Definition 2.1. The *Riesz potential* of a nonnegative, measurable function f on a metric measure space (X, d, μ) is

$$I_1(f)(x) = \int_X \frac{f(y)d(x,y)}{\mu(B(x, d(x,y)))} d\mu(y),$$

We will also use the notation

$$I_{1,A}(f)(x) = \int_A \frac{f(y)d(x,y)}{\mu(B(x, d(x,y)))} d\mu(y),$$

for a measurable sets $A \subset X$.

For properties of the above natural generalization of the Riesz potential, we refer the reader to [He]. From other sources, the reader may find other generalizations of the Riesz potential to metric spaces. Relations between different definitions depend on regularity assumptions of the measure.

Following [Am] and [Mi], we define the class of sets of finite perimeter on metric measure spaces.

Definition 2.2. Let $E \subset X$ be a Borel set and $A \subset X$ an open set. The *perimeter of E in A* is

$$P(E, A) := \inf \Big\{ \liminf_{h \to \infty} \int\limits_A \operatorname{Lip} u_h \, d\mu : \\ (u_h) \subset \operatorname{Lip}_{\mathrm{loc}}(A), u_h \to \chi_E \text{ in } L^1_{loc}(A) \Big\},$$

where χ_E denotes the characteristic function of E. We say that E has finite perimeter in X if $P(E, X) < \infty$.

Next we give the generalized isoperimetric inequality and co-area formula. For proofs see [Am, Theorem 4.3] and [Mi].

Theorem 2.3. *Let (X, d, μ) be a complete doubling metric measure space, and $E \subset X$ be a set of finite perimeter. Then*

(i) if (X, d, μ) admits a weak $(1,1)$-Poincaré inequality, the following relative isoperimetric inequality holds for all balls $B = B(x,r) \subset X$:

$$\min\{\mu(E \cap B), \mu((X \setminus E) \cap B)\} \le C \left(\frac{r^s}{\mu(B)} \right)^{1/(s-1)} [P(E, \tau B)]^{\frac{s}{s-1}},$$

where $s > 1$ is any exponent satisfying (2).

(ii) for any nonnegative $u \in \mathrm{Lip}_{\mathrm{loc}}(X)$ the co-area formula holds :

$$\int_{-\infty}^{\infty} P(\{u > t\}, B(x,r))\ dt = \int_{B(x,r)} \mathrm{Lip}\, u\ d\mu,$$

for every ball $B(x,r) \subset X$.

Here we state the Marcinkiewicz Interpolation Theorem without proof. For more discussion on the theorem, see [St].

Let (p_0, q_0) and (p_1, q_1) be pairs of numbers such that $1 \le p_i \le q_i < \infty$ for $i = 0, 1$, $p_0 < p_1$, and $q_0 \ne q_1$, and let ν be a Radon measure on X. An subadditive operator T is of *weak-type* (p_i, q_i) if there is a constant C_i such that for all $u \in L^{p_i}(X)$ and $\alpha > 0$,

$$\nu(\{x : |(Tu)(x)| > \alpha\}) \le (\alpha^{-1} C_i ||u||_{p_i})^{q_i}.$$

Theorem 2.4. *(Marcinkiewicz Interpolation Theorem) Suppose an operator T is simultaneously of weak-types (p_0, q_0) and (p_1, q_1). If for some $0 < \theta < 1$*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then T is of strong type (p, q), i.e. for all $u \in L^p(X)$

$$||Tu||_{q,\nu} \le C C_0^{1-\theta} C_1^{\theta} ||u||_p,$$

where $C = C(p_i, q_i, \theta)$, $i = 0, 1$.

3 Several Lemmas

First, to prove Theorem 1.3 we need the following metric space version of boxing inequality. It is also proven in [KKST, Theorem 3.1], but for reader's convenience we give the proof here.

Lemma 3.1. *Let (X, d, μ) be a complete doubling metric measure space supporting a weak $(1,1)$-Poincaré inequality. Let $E \subset X$ be a bounded open set of finite perimeter. Then there exist a constant $C > 0$ and a sequence of balls $B(x_i, r_i)$ with $x_i \in E$ such that*

$$E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i)$$

and

$$\sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \le C P(E, X),$$

where C depends only on the doubling constant and the constants in the Poincaré inequality.

Proof. If $\mu(X) < \infty$, let $x_0 \in E$ and balls $B(x_0, r_i)$, where

$$r_i = \frac{\mu(X)}{P(E, X)} 2^i.$$

Now the lemma holds with the balls $B(x_0, r_i)$ and $C = 1$. So we may assume $\mu(X) = \infty$.

Now for any $x \in E$ we define

$$f(r) = \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))}.$$

Since E is open we can find $r_1 > 0$ such that $f(r_1) = 1$. By the assumption that E is bounded, $f(r) \to 0$ when $r \to \infty$. Let $i_0 = \min\{i : f(2^i r_1) < 1/2\}$, and we get that $f(2^{i_0-1}r_1) \geq 1/2$ and $f(2^{i_0}r_1) < 1/2$.

Now by the doubling property of μ and by the choice of i_0 we obtain

$$\frac{1}{2C_d} \leq \frac{\mu(B(x, 2^{i_0-1}r_1) \cap E)}{C_d \mu(B(x, 2^{i_0-1}r_1))} \leq \frac{\mu(B(x, 2^{i_0}r_1) \cap E)}{\mu(B(x, 2^{i_0}r_1))} < \frac{1}{2}.$$

Set $r_x = 2^{i_0}r_1$. Then

$$\min\{\mu(B(x,r_x) \cap E), \mu(B(x,r_x) \cap (X \setminus E))\} \geq \frac{1}{2C_d}\mu(B(x,r_x)).$$

We may assume that $s > 1$ is an exponent satisfying (2). By the relative isoperimetric inequality (Theorem 2.3(i))

$$\left[\frac{1}{2C_d}\mu(B(x,r_x))\right]^{\frac{s-1}{s}} \leq C\frac{r_x}{\mu(B(x,r_x))^{1/s}}P(E, B(x,\tau r_x)).$$

Thus

$$\frac{\mu(B(x,r_x))}{r_x} \leq CP(E, B(x,\tau r_x)). \tag{4}$$

Next we choose a cover for E: Let $\mathcal{B} = B(x, \tau r_x)$ be the family of all balls such that $x \in E$ and r_x chosen as before. Since E is bounded, r_x are uniformly bounded w.r.t x and by the basic covering theorem ([He, Thm. 1.2]) we obtain a sequence of disjoint balls $B(x_i, \tau r_{x_i}) \in \mathcal{B}$ so that

$$\bigcup_{i=1}^{\infty} B(x_i, 5\tau r_{x_i}) \supset E.$$

Let $B(x_i, r_i) = B(x_i, 5\tau r_{x_i})$. Now by the doubling property of μ and (4)

$$\sum \frac{\mu(B(x_i,r_i))}{r_i} \leq C\sum P(E, B(x_i, \tau r_{x_i})) \leq CP(E,X).$$

□

Second, we prove a lemma, needed in proof of Theorem 1.4. Here we need the chain condition.

We say that X satisfies a *chain condition* if for every $\lambda \geq 1$ there is a constant M such that for each $x \in X$ and all $0 < \rho < R < \operatorname{diam}(X)/4$ there is a sequence of balls $B_0, B_1, B_2, \ldots, B_k$ for some integer k with

1. $\lambda B_0 \subset X \setminus B(x,R)$ and $\lambda B_k \subset B(x,\rho)$,

2. $M^{-1}\operatorname{diam}(\lambda B_i) \le \operatorname{dist}(x, \lambda B_i) \le M \operatorname{diam}(\lambda B_i)$ for $i = 0, 1, 2, \dots, k$,

3. there is a ball $R_i \subset B_i \cap B_{i+1}$, such that $B_i \cup B_{i+1} \subset MR_i$ for $i = 0, 1, 2, \dots, k-1$,

4. no point of X belongs to more than M balls λB_i.

The chain condition above is a bit different from the one stated in [HaK, Ch.6]. With a minor change of the proof in [HaK, Ch.6], we can show that each connected doubling space satisfies the chain condition above. We need to cover each annuli with balls of radii equal to $\varepsilon 2^j \lambda^{-1}$ instead of $\varepsilon 2^j$. Then the argument in [HaK, Ch.6] shows that for a fixed $\sigma > 0$, the balls B_i can be chosen such that $\lambda B_i \subset B(x, (1+\sigma)R)$ for all i.

In the next theorem and remark, we show that if the space admits a weak Poincaré inequality, then we have the following pointwise inequality for Lipschitz continuous functions, see also [He, Thm. 9.5].

Theorem 3.2. *Assume that (X, d, μ) admits a weak $(1,p)$-Poincaré inequality with a doubling Borel measure μ. Let $u \in \operatorname{Lip}(X)$ and fix a ball $B(y,r) \subset X$. Then for each $x \in B(y,r)$ there exists a ball $B(z_x, r/8) \subset B(y, 2r)$ such that*

$$|u(x) - u_{B(z_x, r/8)}|^p \le C r^{p-1} I_{1, B(y,2r)}((\operatorname{Lip} u)^p)(x).$$

Proof. Let $\lambda = \tau$ and $R = r/8$ in the chain condition. Let $x \in B(y,r)$. For any small $\rho > 0$, we have a chain $\{B_i\}_{i=0}^k$, for which

$$\begin{aligned}
|u(x) - u_{B_0}| &\le \sum_{i=0}^{k-1} |u_{B_{i+1}} - u_{B_i}| + |u(x) - u_{B_k}| \\
&\le \sum_{i=0}^{k-1} (|u_{B_{i+1}} - u_{R_i}| + |u_{B_i} - u_{R_i}|) + \rho || \operatorname{Lip} u||_{L^\infty} \\
&\le \sum_{i=0}^{k-1} \left(\fint_{R_i} |u - u_{B_{i+1}}| d\mu + \fint_{R_i} |u - u_{B_i}| d\mu \right) + \rho || \operatorname{Lip} u||_{L^\infty} \\
&\le C \sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}| d\mu + \rho || \operatorname{Lip} u||_{L^\infty} \\
&\le C \sum_{i=0}^{k} r_i \left(\fint_{\tau B_i} (\operatorname{Lip} u)^p d\mu \right)^{1/p} + \rho || \operatorname{Lip} u||_{L^\infty}.
\end{aligned}$$

In the second step, we used the Lipschitz continuity of u. Here the number of balls k depends on ρ. We may assume that $\tau B_i \subset B(y, \frac{3}{2} r)$ for all i. Next we choose our ball $B_z := B(z_x, r/8) \subset B(y, 2r)$ where z_x is the center of the ball B_0. We have

$$|u_{B_0} - u_{B_z}| \le C \fint_{B_z \cup B_0} |u - u_{B_z \cup B_0}| d\mu \le C r \left(\fint_{\tau B_z \cup \tau B_0} (\operatorname{Lip} u)^p d\mu \right)^{1/p}.$$

Putting the above two estimates together gives us

$$\begin{aligned}
|u(x) - u_{B_z}| &\le |u(x) - u_{B_0}| + |u_{B_0} - u_{B_z}| \\
&\le C r^{(p-1)/p} \left(\sum_{i=0}^{k} r_i \fint_{\tau B_i} (\operatorname{Lip} u)^p d\mu \right)^{1/p} \\
&\quad + C r^{(p-1)/p} \left(r \fint_{\tau B_z \cup \tau B_0} (\operatorname{Lip} u)^p d\mu \right)^{1/p} + \rho || \operatorname{Lip} u ||_{L^\infty} \\
&\le C r^{(p-1)/p} \left(\int_{B(y,2r)} \frac{(\operatorname{Lip} u(w))^p d(x,w)}{\mu(B(x,d(x,w)))} d\mu(w) \right)^{1/p} \\
&\quad + \rho || \operatorname{Lip} u ||_{L^\infty},
\end{aligned}$$

where the condition 2 of the chain and the finite overlap property of the balls τB_i are needed. The claim follows by letting $\rho \to 0$. $\square$

Note that in Theorem 3.2, it is possible to replace $B(y, 2r)$ by $B(y, (1+\varepsilon)r)$ and $B(z_x, r/8)$ by $B(z_x, \varepsilon r/8)$, where $\varepsilon > 0$ is any small fixed number. Notice also that, in this case the constant depends on ε.

Remark 3.3. If $r < \operatorname{diam} X/10$ and $u = 0$ in $X \setminus B(y,r)$ in Theorem 3.2, we can prove that

$$|u(x)|^p \le C r^{p-1} I_{1,B(y,r)}((\operatorname{Lip} u)^p)(x),$$

for all $x \in B(y,r)$. The proof is similar to that of Theorem 3.2. The only change is that by choosing $R = \frac{5}{2} r$, we find a ball $B(z_x, r) \subset B(y, 5r)$ such that $B(z_x, r) \cap B(y,r) = \emptyset$. In this case, we may assume that $\tau B_i \subset B(y, 4r)$ for all balls B_i in the chain.

The next lemma is due to Muckenhoupt and Wheeden in the setting of Euclidean spaces, see [MW] and [AH, Theorem 3.6.1]. We generalize it to the setting of metric measure spaces.

Lemma 3.4. *Assume that (X, d, μ) is complete and μ is doubling. Let $1 < p < \infty$ and fix a ball $B_0 = B(x_0, r_0) \subset X$ and let ν be any positive Radon measure on B_0. Then*

$$\left(\int_{4B_0} \hat{I}_{3r_0}(\nu)^p d\mu \right)^{1/p} \le C \frac{r_0}{\mu(B_0)^{1/s}} \left(\int_{4B_0} M_1(\nu)^p d\mu \right)^{1/p},$$

where $C = C(s, p, C_d) > 0$ is a constant, M_1 is the fractional maximal operator

$$M_1(\nu)(x) = \sup_{r>0} \left(\frac{\nu(B(x,r))}{\mu(B(x,r))^{1-1/s}} \right)$$

and $\hat{I}_{3r_0}$ is a local Riesz potential

$$\hat{I}_{3r_0}(\nu)(x) = \int_{B(x,3r_0)} \frac{d(x,y)}{\mu(B(x,d(x,y)))} d\nu(y).$$

Proof. Let $B_0 = B(x_0, r_0) \subset X$ be a ball. By (2) and by the doubling property of μ there exists $C > 0$ depending only on s and C_d such that

$$C_s r^s \le \mu(B(x,r)), \qquad \text{where} \qquad C_s = C \mu(B_0) r_0^{-s}, \tag{5}$$

for all balls $B(x,r) \subset X$ with $x \in 4B_0$ and $r < 8r_0$. We use C_s in this proof to clarify notations.

The claim is a consequence of the following inequality: There exist $a > 1$ and $b > 1$, depending only on s and the doubling constant of μ, such that for any $\lambda > 0$ and any $0 < \varepsilon < C_s^{1/s} {C_1}^{-1} {C_2}^{\frac{1-s}{s}}$,

$$\begin{aligned} \mu(\{\hat{I}_{3r_0}(\nu) > a\lambda\}) \le bC_s^{\frac{1}{1-s}} \varepsilon^{\frac{s}{s-1}} \mu(\{\hat{I}_{3r_0}(\nu) > \lambda\}) \\ + C\mu(\{x \in 4B_0 : M_1(\nu)(x) > \varepsilon\lambda\}), \end{aligned} \tag{6}$$

where $C \ge 1$ is depending only on the doubling constant of μ, $C_1 = C_1(C_d) \ge 1$ is from (8) and $C_2 = C_2(C_d, s) \ge 1$ is from (10). Indeed, multiplying both sides of (6) by λ^{p-1} and integrating with respect to λ, we obtain for any $R > 0$,

$$\begin{aligned} &\int_0^R \mu(\{\hat{I}_{3r_0}(\nu) > a\lambda\})\lambda^{p-1}\, d\lambda \\ &\qquad \le bC_s^{\frac{1}{1-s}} \varepsilon^{\frac{s}{s-1}} \int_0^R \mu(\{\hat{I}_{3r_0}(\nu) > \lambda\})\lambda^{p-1}\, d\lambda \\ &\qquad\qquad + C\int_0^R \mu(\{x \in 4B_0 : M_1(\nu)(x) > \varepsilon\lambda\})\lambda^{p-1}\, d\lambda. \end{aligned}$$

Thus by changing variables, we have

$$\begin{aligned} &a^{-p}\int_0^{aR} \mu(\{\hat{I}_{3r_0}(\nu) > \lambda\})\lambda^{p-1}\, d\lambda \\ &\qquad \le bC_s^{\frac{1}{1-s}} \varepsilon^{\frac{s}{s-1}} \int_0^R \mu(\{\hat{I}_{3r_0}(\nu) > \lambda\})\lambda^{p-1}\, d\lambda \\ &\qquad\qquad + C\varepsilon^{-p}\int_0^{\varepsilon R} \mu(\{x \in 4B_0 : M_1(\nu)(x) > \lambda\})\lambda^{p-1}\, d\lambda. \end{aligned}$$

All integrals above are finite, since $\hat{I}_{3r_0}(\nu) = 0$ in $X \setminus 4B_0$. Next we choose

$$\varepsilon = \min\left\{\frac{1}{2}, {C_1}^{-1} {C_2}^{\frac{1-s}{s}} \left(\frac{1}{2} a^{-p} b^{-1} C_s^{1/(s-1)}\right)^{\frac{s-1}{s}}\right\},$$

and it follows that

$$\begin{aligned} &a^{-p}\int_0^{aR} \mu(\{\hat{I}_{3r_0}(\nu) > \lambda\})\lambda^{p-1}\, d\lambda \\ &\qquad\qquad \le C\varepsilon^{-p}\int_0^{\varepsilon R} \mu(\{x \in 4B_0 : M_1(\nu) > \lambda\})\lambda^{p-1}\, d\lambda. \end{aligned}$$

Letting $R \to \infty$, we obtain

$$a^{-p}\int_{4B_0} \hat{I}_{3r_0}(\nu)^p d\mu \le C\varepsilon^{-p}\int_{4B_0} M_1(\nu)^p d\mu, \tag{7}$$

which proves the Lemma.

It remains to prove (6). First, we notice that for any $x \in 4B_0 \setminus \frac{5}{4}B_0$

$$
\begin{aligned}
\hat{I}_{3r_0}(\nu)(x) &\le \int\limits_{B(x,3r_0)\setminus B(x,\frac{1}{4}r_0)} \frac{d(x,y)}{\mu(B(x,d(x,y)))} d\nu(y) \\
&\le \frac{\nu(B_0)}{\mu(B(x,\frac{1}{4}r_0))} 3r_0 \le C_1 C_s^{-1/s} \frac{\nu(B(x,5r_0))}{\mu(B(x,5r_0))^{1-1/s}} \\
&\le C_1 C_s^{-1/s} M_1(\nu)(x),
\end{aligned} \tag{8}
$$

where in the third step we used the condition (5). The constant $C_1 \ge 1$ depends only on the doubling constant of μ. So if $x \in 4B_0 \setminus \frac{5}{4}B_0$ and $\hat{I}_{3r_0}(\nu)(x) > \lambda$, then $M_1(\nu)(x) > \varepsilon\lambda$, when $\varepsilon < C_s^{1/s} {C_1}^{-1}$. Thus, for any $0 < \varepsilon < C_s^{1/s} {C_1}^{-1}$

$$
\begin{aligned}
&\mu(\{x \in 4B_0 \setminus \tfrac{5}{4}B_0 : \hat{I}_{3r_0}(\nu)(x) > \lambda\}) \\
&\qquad\qquad \le \mu(\{x \in 4B_0 \setminus \tfrac{5}{4}B_0 : M_1(\nu)(x) > \varepsilon\lambda\}).
\end{aligned} \tag{9}
$$

Second, we consider an easy case $\{\hat{I}_{3r_0}(\nu) > \lambda\} \supset B_0$. Then for any $x \in 4B_0$, we have by the weak-estimate of the Riesz potential, see [He, Theorem 3.22],

$$
\begin{aligned}
\mu(B_0) \le \mu(\{\hat{I}_{3r_0}(\nu) > \lambda\}) &\le C C_s^{\frac{1}{1-s}} \left(\frac{1}{a\lambda}\int_{B_0} d\nu\right)^{\frac{s}{s-1}} \\
&\le C_2 C_s^{\frac{1}{1-s}} \lambda^{\frac{s}{1-s}} M_1(\nu)(x)^{\frac{s}{s-1}} \mu(B_0),
\end{aligned} \tag{10}
$$

where we also used the doubling property of μ. Hence, for all $x \in 4B_0$

$$
M_1(\nu)(x) \ge C_s^{1/s} {C_2}^{\frac{1-s}{s}} \lambda,
$$

where $C_2 = C_2(C_d, s) > 1$ is from (10). Thus (6) is true with any $\varepsilon < C_s^{1/s} {C_2}^{\frac{1-s}{s}}$, since $\{\hat{I}_{3r_0}(\nu) > \lambda\} \subset 4B_0$.

Thus, we may assume that there exists $x \in B_0$ such that $\hat{I}_{3r_0}(\nu)(x) \le \lambda$. Let $\delta > 0$ be any small number. Let $A \subset 4B_0$ be an open set such that $\{\hat{I}_{3r_0}(\nu) > \lambda\} \subset A$ and $\mu(A) \le \mu(\{\hat{I}_{3r_0}(\nu) > \lambda\}) + \delta$. The set A has a Whitney covering with countable family of balls $\hat{\mathcal{W}} = \{B_i\}$, where the balls $\{\frac{1}{2}B : B \in \hat{\mathcal{W}}\}$ are pairwise disjoint, see [BBS2, Chapter 3] for the Whitney coverings in metric spaces. Now we only consider the balls which intersect the set $\frac{5}{4}B_0$ and we denote $\mathcal{W} = \{B \in \hat{\mathcal{W}} : B \cap \frac{5}{4}B_0 \ne \emptyset\}$. By the construction of the Whitney covering, for every $B_i \in \mathcal{W}$ we have $B_i \subset 2B_0$ and there exists $y_1 \in 4B_0$ such that $\hat{I}_{3r_0}(\nu)(y_1) \le \lambda$ and

$$
8r_i \le \operatorname{dist}(y_1, B_i) \le 16r_i. \tag{11}
$$

By a geometric argument, we know that: For any $B_i \in \mathcal{W}$ there exists $y_0 \in 2B_0$ such that $\hat{I}_{3r_0}(\nu)(y_0) \le \lambda$ and

$$
\operatorname{dist}(y_0, B_i) \le 54r_i. \tag{12}
$$

Indeed, when $y_1 \in 2B_0$ in (11) the claim is clear. So assume that $y_1 \in 4B_0 \setminus 2B_0$ in (11). Since $B_i \cap \frac{5}{4}B_0 \ne \emptyset$, we have $\operatorname{dist}(y_1, B_i) + \operatorname{diam} B_i \ge \frac{3}{4}r_0$. Now by (11) we get that $r_i \ge \frac{1}{24}r_0$. In addition, we know that there exists $y_0 \in B_0$ such that $\hat{I}_{3r_0}(\nu)(y_0) \le \lambda$ and therefore $\operatorname{dist}(y_0, B_i) \le \frac{9}{4}r_0 \le 54r_i$.

Let $B_1 = B(x_1, r_1) \in \mathcal{W}$ and let $a > 1$. Suppose B_1 intersects the set $\{M_1(\nu) \leq \varepsilon\lambda\}$. Let $B_2 = B(x_1, 56r_1)$ and denote $\nu_1 = \nu|_{B_2}$ and $\nu_2 = \nu - \nu_1$. By the weak-estimate of the Riesz potential, see [He, Theorem 3.22],

$$\mu(\{\hat{I}_{3r_0}(\nu_1) > a\lambda/2\}) \leq C C_s^{\frac{1}{1-s}} \left(\frac{1}{a\lambda} \int_X \, d\nu_1 \right)^{\frac{s}{s-1}}.$$

Let $x_3 \in B_1$ such that $M_1(\nu)(x_3) \leq \varepsilon\lambda$ and let $B_3 = B(x_3, 58r_1)$ so that $B_2 \subset B_3$. Then by the definition of the fractional maximal function M_1,

$$\int_X \, d\nu_1 = \int_{B_2} \, d\nu \leq \int_{B_3} \, d\nu \leq M_1(\nu)(x_3)\mu(B_3)^{\frac{s-1}{s}} \leq \varepsilon\lambda\mu(B_3)^{\frac{s-1}{s}}.$$

Thus by the doubling property,

$$\left(\frac{1}{a\lambda} \int_X \, d\nu_1 \right)^{\frac{s}{s-1}} \leq C \left(\frac{\varepsilon}{a} \right)^{\frac{s}{s-1}} \mu(\tfrac{1}{2}B_1).$$

So with $b = C/a^{s/(s-1)}$,

$$\mu(\{x \in B_1 : \hat{I}_{3r_0}(\nu_1)(x) > a\lambda/2\}) \leq b C_s^{\frac{1}{1-s}} \varepsilon^{\frac{s}{s-1}} \mu(\tfrac{1}{2}B_1). \tag{13}$$

Next we estimate $\hat{I}_{3r_0}(\nu_2)$ in B_1. If $B_0 \subset B_2$, then $\hat{I}_{3r_0}(\nu_2)(x) = 0$ for all $x \in B_1$ and (14) follows. Assume that $B_0 \setminus B_2 \neq \emptyset$. If $x_4 \in 2B_0$ is a point with $d(x_4, B_1) \leq 54r_1$, then because of the choice of B_2, for all $x \in B_1$ and for all $y \in X \setminus B_2$ we have $d(x_4, y) \leq d(x_4, x) + d(x, y) \leq 3d(x, y)$ and $d(x, y) \leq d(x, x_4) + d(x_4, y) \leq 56r_1 + d(x_4, y) \leq 57d(x_4, y)$. Thus, if in addition $\hat{I}_{3r_0}(\nu)(x_4) \leq \lambda$, then

$$\begin{aligned}
\hat{I}_{3r_0}(\nu_2)(x) &= \int_{B(x,3r_0)} \frac{d(x,y)}{\mu(B(x, d(x,y)))} \, d\nu_2(y) \\
&\leq \hat{C} \int_{B_0} \frac{d(x_4,y)}{\mu(B(x, 6d(x,y)))} \, d\nu_2(y) \\
&\leq \hat{C} \int_{B(x_4,3r_0)} \frac{d(x_4,y)}{\mu(B(x_4, d(x_4,y)))} \, d\nu_2(y) \\
&= \hat{C} \hat{I}_{3r_0}(\nu_2)(x_4) \leq \hat{C}\lambda,
\end{aligned}$$

where we used the doubling property of μ and the facts that $B(x_4, d(x_4, y)) \subset B(x, 6d(x, y))$, the support of ν_2 is in B_0 and $x \in 2B_0$. Thus, if a is chosen so that $a \geq 2\hat{C}$, then $\hat{I}_{3r_0}(\nu_2)(x) \leq a\lambda/2$. In other words, either

$$B_1 \subset \{x : M_1(\nu)(x) > \varepsilon\lambda\}$$

or

$$\{x \in B_1 : \hat{I}_{3r_0}(\nu)(x) > a\lambda\} \subset \{x \in B_1 : \hat{I}_{3r_0}(\nu_1)(x) > a\lambda/2\}. \tag{14}$$

In the second case, it follows from (13) that

$$\mu(\{x \in B_1 : \hat{I}_{3r_0}(\nu)(x) > a\lambda\}) \leq b C_s^{\frac{1}{1-s}} \varepsilon^{\frac{s}{s-1}} \mu(\tfrac{1}{2}B_1).$$

Now adding over all $B_1 \in \mathcal{W}$ and letting $\delta \to 0$, we get

$$\begin{aligned}
\mu(\{x \in \tfrac{5}{4}B_0 : \hat{I}_{3r_0}(\nu) > a\lambda\}) &\leq b C_s^{\frac{1}{1-s}} \varepsilon^{\frac{s}{s-1}} \mu(\{x \in 2B_0 : \hat{I}_{3r_0}(\nu) > \lambda\}) \\
&\quad + C\mu(\{x \in 2B_0 : M_1(\nu)(x) > \varepsilon\lambda\}),
\end{aligned}$$

where $C = C(C_d) > 1$. Now by this estimate and (9), we obtain (6). $\square$

4 Adams inequality for Riesz potential

Notice that [EKM, Thm. 6.2.1] has a result similar to the following theorem, but the potential they studied is different from ours. Still, if we make an assumption, as in Theorem 4.1, that $C_s r^s \le \mu(B(x,r))$ for all balls $B(x,r) \subset X$ of radius $r < \operatorname{diam} X$, and additional assumptions that $1/p - 1/q - 1/s \ge 0$ and $\mu(X) = \infty$, then Theorem 4.1 follows from [EKM, Thm. 6.2.1]. However the additional assumptions for the exponents and for the measure of the space is not needed in the following proof. Here the the proof is similar to that of [Zi, Thm. 4.7.2], but we need a different approach in many estimates, since our measure is not Ahlfors regular.

Theorem 4.1. *Let (X, d, μ) be a metric measure space, where μ is a doubling Radon measure, and $1 < p < s$. Let ν be a Radon measure in X. Suppose that there are positive constants M and C_s such that*

$$C_s r^s \le \mu(B(x,r)) \qquad \text{and} \qquad \frac{\nu(B(x,r))}{\mu(B(x,r))} \le M r^\alpha,$$

for all balls $B(x,r) \subset X$ of radius $r < \operatorname{diam} X$, where $\alpha = \frac{sq}{p} - s - q$ and $1 < p < q < \infty$. Then

$$\left(\int_X I_1(|f|)^q d\nu \right)^{1/q} \le C C_s^{\frac{1}{q} - \frac{1}{p}} M^{\frac{1}{q}} \left(\int_X |f|^p d\mu \right)^{1/p},$$

for all $f \in L^p(X, \mu)$, where $C = C(p, q, C_d, s) > 0$.

Proof. In this proof, we assume $\operatorname{diam} X = \infty$. When the diameter of X is finite, a few minor technical changes are needed in the following proof. Indeed, we only need to integrate over $[0, \operatorname{diam} X]$ instead of $[0, \infty[$ in (16).

For $t > 0$, let $A_t = \{x : I_1(|f|)(x) > t\}$ and $\nu_t = \nu|_{A_t}$. We may assume that $\nu(A_t) > 0$. First, we have by Fubini's theorem and the doubling property of μ that

$$\begin{aligned} t\nu(A_t) &\le \int_{A_t} I_1(|f|) d\nu = \int_X I_1(|f|) d\nu_t \\ &\le C_d \int_X \int_X \frac{|f(x)| d(x,y)}{\mu(B(y, 2d(x,y)))} \, d\mu(x) d\nu_t(y) \\ &\le C_d \int_X \int_X \frac{|f(x)| d(x,y)}{\mu(B(x, d(x,y)))} \, d\nu_t(y) d\mu(x) \\ &= C_d \int_X I_1(\nu_t)(x) |f(x)| \, d\mu(x). \end{aligned} \tag{15}$$

Next we estimate $I_1(\nu_t)$ in the following way. We set $r_j = 2^j$.

$$\begin{aligned} I_1(\nu_t)(x) &= \int_X \frac{d(x,y)}{\mu(B(x,d(x,y)))} d\nu_t(y) \\ &= \sum_{j=-\infty}^{+\infty} \int_{\{r_j < d(x,y) \le r_{j+1}\}} \frac{d(x,y)}{\mu(B(x,d(x,y)))} d\nu_t(y) \\ &\le \sum_{j=-\infty}^{+\infty} \frac{r_{j+1}}{\mu(B(x,r_j))} \nu_t(B(x,r_{j+1})) \\ &\le C_d \sum_{j=-\infty}^{+\infty} \frac{\nu_t(B(x,r_{j+1}))}{\mu(B(x,r_{j+1}))} r_{j+1} \le C_d^2 \int_0^\infty \frac{\nu_t(B(x,r))}{\mu(B(x,r))} dr, \end{aligned}$$

where in the last two steps we used the doubling property of μ. Now for some $R > 0$, to be fixed later, we have

$$\begin{aligned} t\nu(A_t) &\le C \int_0^R \int_X |f(x)| \frac{\nu_t(B(x,r))}{\mu(B(x,r))} d\mu(x) dr \\ &\quad + C \int_R^\infty \int_X |f(x)| \frac{\nu_t(B(x,r))}{\mu(B(x,r))} d\mu(x) dr = J_1 + J_2. \end{aligned} \tag{16}$$

We will estimate J_1 and J_2 in the following way. First, to estimate J_1, we have by the growth condition for the measures that, with $1/p + 1/p' = 1$,

$$\begin{aligned} \frac{\nu_t(B(x,r))}{\mu(B(x,r))} &= \left[\frac{\nu_t(B(x,r))}{\mu(B(x,r))}\right]^{1/p} \left[\frac{\nu_t(B(x,r))}{\mu(B(x,r))}\right]^{1/p'} \\ &\le (Mr^\alpha)^{1/p} \left[\frac{\nu_t(B(x,r))}{\mu(B(x,r))}\right]^{1/p'}. \end{aligned}$$

By the Hölder inequality and the above estimate,

$$\begin{aligned} J_1 &\le C \int_0^R ||f||_p \left(\int_X \left[\frac{\nu_t(B(x,r))}{\mu(B(x,r))}\right]^{p'} d\mu(x) \right)^{1/p'} dr \\ &\le C||f||_p M^{1/p} \int_0^R \left(\int_X \left[\frac{\nu_t(B(x,r))}{\mu(B(x,r))}\right] d\mu(x) \right)^{1/p'} r^{\alpha/p} dr. \end{aligned} \tag{17}$$

For $r > 0$, we define the set

$$E_r = (X \times X) \cap \{(x,y) : d(x,y) < r\}$$

and by Fubini's theorem, we obtain

$$\begin{aligned} \int_X \frac{\nu_t(B(x,r))}{\mu(B(x,r))} d\mu(x) &= \int_X \frac{1}{\mu(B(x,r))} \int_{B(x,r)} d\nu_t(y) d\mu(x) \\ &= \int_X \int_X \frac{\chi_{E_r}(x,y)}{\mu(B(x,r))} d\mu(x) d\nu_t(y) \le C_d \nu(A_t), \end{aligned} \tag{18}$$

where the last step follows from

$$\int_X \frac{\chi_{E_r}(x,y)}{\mu(B(x,r))} d\mu(x) \le C_d \tag{19}$$

for all $y \in X$, which in turn follows from the doubling property of μ. Indeed,

$$\begin{aligned}\int_X \frac{\chi_{E_r}(x,y)}{\mu(B(x,r))} d\mu(x) &= \int_{B(y,r)} \frac{1}{\mu(B(x,r))} d\mu(x) \\ &\le C_d \int_{B(y,r)} \frac{1}{\mu(B(x,2r))} d\mu(x) \\ &\le C_d \int_{B(y,r)} \frac{1}{\mu(B(y,r))} d\mu(x) = C_d,\end{aligned}$$

since $B(y,r) \subset B(x,2r)$. Combining (17) and (18), we arrive at

$$J_1 \le C||f||_p M^{1/p} \nu(A_t)^{1/p'} \int_0^R r^{\alpha/p} dr = C||f||_p M^{1/p} \nu(A_t)^{1/p'} R^{\alpha/p+1},$$

since $1 + \frac{\alpha}{p} = \frac{1}{p}(\frac{s}{p} - 1)(q - p) > 0$.

Next, we estimate J_2. By the assumption on the measure μ,

$$\begin{aligned}\frac{\nu_t(B(x,r))}{\mu(B(x,r))} &= \left[\frac{\nu_t(B(x,r))}{\mu(B(x,r))}\right]^{1/p} \left[\frac{\nu_t(B(x,r))}{\mu(B(x,r))}\right]^{1/p'} \\ &\le C_s^{-1/p} r^{-s/p} \nu(A_t)^{1/p} \left[\frac{\nu_t(B(x,r))}{\mu(B(x,r))}\right]^{1/p'},\end{aligned}$$

which gives us

$$\begin{aligned}J_2 &\le C \int_R^\infty ||f||_p \left(\int_X \left[\frac{\nu_t(B(x,r))}{\mu(B(x,r))}\right]^{p'} d\mu(x) \right)^{1/p'} dr \\ &\le CC_s^{-1/p} ||f||_p \; \nu(A_t)^{1/p} \int_R^\infty \left(\int_X \frac{\nu_t(B(x,r))}{\mu(B(x,r))} d\mu(x) \right)^{1/p'} r^{-s/p} dr.\end{aligned}$$

By (18),

$$J_2 \le CC_s^{-1/p} ||f||_p \nu(A_t) \int_R^\infty r^{-s/p} dr \le CC_s^{-1/p} ||f||_p \nu(A_t) R^{1-s/p},$$

since $1 - s/p < 0$.

Now we have

$$J_1 + J_2 \le C||f||_p \left(M^{1/p} \nu(A_t)^{1/p'} R^{\alpha/p+1} + C_s^{-1/p} \nu(A_t) R^{1-s/p} \right).$$

By choosing

$$R = \left(\frac{\nu(A_t)}{MC_s} \right)^{\frac{1}{\alpha+s}},$$

we arrive at

$$J_1 + J_2 \le CC_s^{1/q-1/p} ||f||_p M^{1/q} \nu(A_t)^{1-1/q}.$$

Now from (16) and the previous inequality, it follows

$$t\nu(A_t)^{1/q} \le CC_s^{1/q-1/p} M^{1/q} ||f||_p.$$

Thus the Riesz potential operator $I_1(\cdot)$ is of weak type (p,q), whenever

$$1 < p < q < \infty, \qquad p < s,$$

and the claim follows from the Marcinkiewicz Interpolation, Theorem 2.4. $\square$

When μ is a doubling measure on X and $B_0 = B(x_0, r_0) \subset X$, then

$$\tilde{C}_s r^s \leq \mu(B(x,r)), \qquad \text{where} \qquad \tilde{C}_s = \frac{c\mu(B_0)}{2^s r_0^s},$$

for all balls $B(x,r) \subset X$ with $x \in B_0$ and $r < 2r_0$. Here c is from (2).

Now we have the following local version of Adams inequality for Riesz potentials.

Corollary 4.2. *Let (X, d, μ) be a metric measure space, where μ is a doubling Radon measure, and $1 < p < s$. Assume that ν is a Radon measure such that*

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \leq M r^{\frac{sq}{p} - s - q}$$

for all balls $B(x,r) \subset X$ of radius $r < \operatorname{diam} X$, where M is a positive constant and $1 < p < q < \infty$. If $f \in L^p(B_0, \mu)$ for some ball $B_0 = B(x_0, r_0) \subset X$, we have

$$\left(\int_{B_0} I_{1,B_0}(|f|)^q d\nu \right)^{1/q} \leq C\mu(B_0)^{1/q - 1/p} \; r_0^{\frac{s}{p} - \frac{s}{q}} M^{1/q} \left(\int_{B_0} |f|^p d\mu \right)^{1/p},$$

where $C = C(p, q, C_d, s) > 0$ is a constant.

5 Proofs of Theorem 1.3 and Theorem 1.4

First we prove the Adams type inequality in a case $p = 1$.

Proof of Theorem 1.3. Let $u \in \operatorname{Lip}_0(X)$. First, we consider the case $q = 1$. We may assume that $u \geq 0$. For $t > 0$, define $E_t = \{x : u(x) > t\}$. The set E_t is open and bounded, since u is continuous and has compact support. In addition, the set E_t is of finite perimeter for a.e. $t \in [0, \infty[$. Lemma 3.1 imply that for all such t there exists a covering of E_t by a sequence of balls $B_i := B(x_i, r_i)$ such that

$$\sum_{i=1}^{\infty} \frac{\mu(B_i)}{r_i} \leq CP(E_t, X).$$

Hence by the assumption on the measure ν in the theorem, we have

$$\nu(E_t) \leq \sum_i \nu(B_i) \leq M \sum_i \frac{\mu(B_i)}{r_i} \leq CMP(E_t, X).$$

Now applying the co-area formula (Theorem 2.3), we get

$$\int_X u \, d\nu = \int_0^\infty \nu(E_t) dt \leq CM \int_0^\infty P(E_t, X) dt \leq CM \int_X \operatorname{Lip} u \, d\mu.$$

Next, we prove the case $q > 1$. Let $f \in L^{q'}(X, \nu)$, $f \geq 0$ and $B(x,r) \subset X$. By Hölder inequality we get

$$\begin{aligned} \int_{B(x,r)} f \, d\nu &\leq \left(\int_{B(x,r)} f^{q'} \, d\nu \right)^{1/q'} \nu(B(x,r))^{1/q} \\ &\leq M^{1/q} ||f||_{q';\nu} \frac{\mu(B(x,r))}{r}. \end{aligned}$$

So the measure $f d\nu$ satisfies the assumptions of Theorem 1.3 with $q = 1$, which was proved above. Hence if $u \in \mathrm{Lip}_0(X)$,

$$\int_X u f d\nu \leq C M^{1/q} ||f||_{q';\nu} \int_X \mathrm{Lip}\, u \; d\mu,$$

for all $f \in L^{q'}(X, \nu)$, $f \geq 0$. Because $u \geq 0$ and hence

$$||u||_{q;\nu} = \sup \left\{ \int_X u f d\nu : ||f||_{q';\nu} \leq 1, f \geq 0 \right\},$$

we have

$$\left(\int_X |u|^q d\nu \right)^{1/q} \leq C M^{1/q} \int_X \mathrm{Lip}\, u \; d\mu.$$

□

Next we prove a version of Theorem 1.4 for all Lipschitz functions, not necessarily with compact support. From now on, we assume that $p > 1$.

Theorem 5.1. *Suppose that the assumptions in Theorem 1.4 hold for the space X and for measures μ and ν. Let $u \in \mathrm{Lip}(X)$. For all balls $B = B(x, r) \subset X$*

$$\left(\int_B |u - u_B|^q d\nu \right)^{1/q} \leq C \mu(B)^{1/q - 1/p} \; r^{\frac{t-1}{t} + \frac{s}{p} - \frac{s}{q}} M^{1/q} \left(\int_{2\tau B} (\mathrm{Lip}\, u)^p d\mu \right)^{1/p},$$

where $C = C(p, q, s, t, C_d, C_p, \tau) > 0$ is a constant.

Proof. Fix a ball $B = B(x, r) \subset X$. By Theorem 3.2, we have for each $y \in B$

$$|u(y) - u_{B(z_y, r/8)}|^t \leq C r^{t-1} I_{1, B(x, 2r)}((\mathrm{Lip}\, u)^t)(y), \tag{20}$$

for some ball $B(z_y, r/8) \subset B(x, 2r)$.

By the Minkowski inequality,

$$\begin{aligned} \left(\int_B |u(y) - u_B|^q d\nu(y) \right)^{1/q} &\leq \left(\int_B |u(y) - u_{B(z_y, r/8)}|^q d\nu(y) \right)^{1/q} \\ &+ \left(\int_B |u_{B(z_y, r/8)} - u_B|^q d\nu(y) \right)^{1/q}. \end{aligned} \tag{21}$$

To estimate last two terms in (21), we observe that

$$\begin{aligned} |u_{B(z_y, r/8)} - u_B| &\leq |u_{B(z_y, r/8)} - u_{2B}| + |u_B - u_{2B}| \\ &\leq \fint_{B(z_y, r/8)} |u - u_{2B}| \; d\mu + \fint_B |u - u_{2B}| \; d\mu \\ &\leq C \fint_{2B} |u - u_{2B}| \; d\mu, \end{aligned}$$

where in the last step, we used the doubling property of μ. Thus by the Poincaré inequality,

$$\begin{aligned} \left(\int_B |u_{B(z_y, r/8)} - u_B|^q d\nu(y) \right)^{1/q} &\leq C \nu(B)^{1/q} \fint_{2B} |u - u_{2B}| d\mu \\ &\leq C \nu(B)^{1/q} r \left(\fint_{2\tau B} (\mathrm{Lip}\, u)^p d\mu \right)^{1/p} \\ &\leq C M^{1/q} r^{\frac{s}{p} - \frac{s}{q} - \frac{1}{t}} \mu(B)^{\frac{1}{q} - \frac{1}{p}} r \left(\int_{2\tau B} (\mathrm{Lip}\, u)^p d\mu \right)^{1/p}. \end{aligned}$$

To estimate other term in (21) we apply the pointwise estimate (20) and Corollary 4.2, where in this case $\tilde{q} = q/t$ and $\tilde{p} = p/t$

$$\begin{aligned}\left(\int_B |u(y) - u_{B(z_y,r/8)}|^q d\nu(y)\right)^{1/q} &= \left(\int_B (|u(y) - u_{B(z_y,r/8)}|^t)^{q/t} d\nu(y)\right)^{1/q} \\ &\le Cr^{\frac{t-1}{t}} \left(\int_{2B} (I_{1,2B}((\operatorname{Lip} u)^t))^{q/t} d\nu\right)^{1/q} \\ &\le C\mu(B)^{\frac{1}{q}-\frac{1}{p}}\, r^{\frac{t-1}{t}+\frac{s}{p}-\frac{s}{q}} M^{1/q} \left(\int_{2B} ((\operatorname{Lip} u)^t)^{p/t} d\mu\right)^{1/p}.\end{aligned}$$

The claim follows from (21) and the two estimates above. □

Proof of Theorem 1.4. From Remark 3.3: For each $y \in B_0$,

$$|u(y)|^t \le Cr_0^{t-1} I_{1,B_0}((\operatorname{Lip} u)^t)(y).$$

By this estimate and by Corollary 4.2,

$$\begin{aligned}\left(\int_{B_0} |u(y)|^q d\nu(y)\right)^{1/q} &\le Cr_0^{\frac{t-1}{t}} \left(\int_{B_0} (I_{1,B_0}((\operatorname{Lip} u)^t))^{q/t} d\nu\right)^{1/q} \\ &\le C\mu(B_0)^{\frac{1}{q}-\frac{1}{p}}\, r_0^{\frac{t-1}{t}+\frac{s}{p}-\frac{s}{q}} M^{1/q} \left(\int_{B_0} ((\operatorname{Lip} u)^t)^{p/t} d\mu\right)^{1/p}.\end{aligned}$$

□

6 Borderline cases

Here we prove some results in the borderline case $p = s$. First, a version of Adams type inequality for Riesz potential is considered. Here we are only able to prove a weak-type local inequality for the Riesz potential; it would be interesting to know if a strong-type inequality can be achieved.

Theorem 6.1. *Let (X, d, μ) be a metric measure space, where μ is a doubling Radon measure. Let $B_0 = B(x_0, r_0) \subset X$ and suppose that ν is a Radon measure in B_0 with*

$$\nu(B(x,r)) \le M \left(\log \frac{r_0}{r}\right)^{\frac{1-s}{s}q}, \tag{22}$$

for all balls $B(x,r) \subset X$ such that $x \in 2B_0$ and $r < r_0/2$. Here $1 < s < q < \infty$ and M is a positive constant. Then

$$t\nu(\{x \in B_0 : I_{1,B_0}(|f|)(x) > t\})^{1/q} \le Cr_0\mu(B_0)^{-1/s} M^{\frac{1}{q}} \left(\int_{B_0} |f|^s d\mu\right)^{1/s},$$

for all $t > 0$ and all $f \in L^s(B_0, \mu)$, where $C = C(s, q, C_d)$ is a positive constant.

Proof. We use the same techniques here as in the proof of Theorem 4.1, but we need a different approach to obtain some estimates. For $t > 0$, let $A_t = \{x \in$

$B_0 : I_{1,B_0}(|f|)(x) > t\}$ and $\nu_t = \nu|_{A_t}$. We may assume that $\nu(A_t) > 0$. By the estimate (15) and by the Hölder inequality

$$
\begin{aligned}
t\nu(A_t) &\leq C_d \int_{B_0} I_{1,B_0}(\nu_t)(x)|f(x)|\, d\mu(x) \\
&\leq C_d ||f||_s \left(\int_{B_0} I_{1,B_0}(\nu_t)^{s'} d\mu \right)^{1/s'} \\
&\leq C_d ||f||_s \left(\int_{4B_0} \hat{I}_{3r_0}(\nu_t)^{s'} d\mu \right)^{1/s'},
\end{aligned} \tag{23}
$$

where $\hat{I}_{3r_0}$ is the local Riesz potential as in Lemma 3.4. We shall apply Lemma 3.4 to estimate the norm of $\hat{I}_{3r_0}$. First, for each $x \in 4B_0$ and $0 < r < 5r_0$,

$$
\begin{aligned}
\frac{\nu_t(B(x,r))}{\mu(B(x,r))^{1-1/s}} &\leq C \left(\int_r^{2r} \left[\frac{\nu_t(B(x,\xi))}{\mu(B(x,\xi))^{1-1/s}} \right]^{s'} \frac{d\xi}{\xi} \right)^{1/s'} \\
&\leq C \left(\int_0^{10r_0} \left[\frac{\nu_t(B(x,\xi))}{\mu(B(x,\xi))^{1-1/s}} \right]^{s'} \frac{d\xi}{\xi} \right)^{1/s'}.
\end{aligned}
$$

Thus

$$
M_1(\nu_t)(x) \leq C \left(\int_0^{10r_0} \left[\frac{\nu_t(B(x,r))}{\mu(B(x,r))^{1-1/s}} \right]^{s'} \frac{dr}{r} \right)^{1/s'},
$$

where M_1 is the fractional maximal operator as in Lemma 3.4. Now by Lemma 3.4,

$$
\int_{4B_0} \hat{I}_{3r_0}(\nu_t)^{s'} d\mu \leq C r_0^{s'} \mu(B_0)^{1-s'} \int_{4B_0} \int_0^{10r_0} \frac{\nu_t(B(x,r))^{s'}}{\mu(B(x,r))} \frac{dr}{r}\, d\mu(x). \tag{24}
$$

We now divide integration with respect to r into two parts. Let R_1 be any number with $0 < R_1 \leq r_0/2$, to be fixed later. We estimate integrals on the right hand side of (24) in the following way. First, to estimate the integral with respect to r from 0 to R_1, we use the growth condition of ν, Fubini's theorem and (18), to get

$$
\begin{aligned}
&\int_{4B_0} \int_0^{R_1} \frac{\nu_t(B(x,r))^{s'}}{\mu(B(x,r))} \frac{dr}{r}\, d\mu(x) \\
&\quad \leq \int_0^{R_1} M^{s'-1} \left(\log \frac{r_0}{r} \right)^{-q/s} r^{-1} \int_{4B_0} \frac{\nu_t(B(x,r))}{\mu(B(x,r))}\, d\mu(x)\, dr \\
&\quad \leq C M^{s'-1} \nu(A_t) \int_0^{R_1} r^{-1} \left(\log \frac{r_0}{r} \right)^{-q/s} dr \\
&\quad \leq C M^{s'-1} \nu(A_t) \left(\log \frac{r_0}{R_1} \right)^{1-q/s}.
\end{aligned} \tag{25}
$$

Second, we get the estimate for the remaining part of the integral in (24) by

(18), as follows

$$
\begin{aligned}
&\int_{4B_0}\int_{R_1}^{10r_0}\frac{\nu_t(B(x,r))^{s'}}{\mu(B(x,r))}\frac{dr}{r}\;d\mu(x)\\
&\qquad\le\int_{R_1}^{10r_0}\nu(A_t)^{s'-1}r^{-1}\int_{4B_0}\frac{\nu_t(B(x,r))}{\mu(B(x,r))}\;d\mu(x)\;dr\\
&\qquad\le C\nu(A_t)^{s'}\int_{R_1}^{10r_0}r^{-1}\;dr\\
&\qquad\le C\nu(A_t)^{s'}\log\frac{r_0}{R_1}.
\end{aligned}\tag{26}
$$

Now from (24), we get by (25) and (26) that

$$
\begin{aligned}
\int_{4B_0}\hat{I}_{3r_0}(\nu_t)^{s'}d\mu\le Cr_0^{s'}\mu(B_0)^{1-s'}\Bigg[&M^{s'-1}\nu(A_t)\left(\log\frac{r_0}{R_1}\right)^{1-q/s}\\
&+\nu(A_t)^{s'}\log\frac{r_0}{R_1}\Bigg].
\end{aligned}
$$

Now, we choose $0<R_1\le r_0/2$ such that

$$
\log\frac{r_0}{R_1}=\left(\frac{M}{\nu(A_t)}\right)^{s'/q}.
$$

This is always possible, if $M\ge(\log 2)^{q/s'}\nu(B_0)$, which we may assume. Indeed, (22) shows that $M\ge c\nu(B_0)$ for some constant $c>0$, independent of ν and B_0. Thus, multiplying M by a constant, we may assume that $M\ge(\log 2)^{q/s'}\nu(B_0)$.

Therefore we arrive at

$$
\int_{4B_0}\hat{I}_{3r_0}(\nu_t)^{s'}d\mu\le Cr_0^{s'}\mu(B_0)^{1-s'}M^{s'/q}\nu(A_t)^{s'(1-1/q)}.
$$

Now by the estimate above and (23), we obtain that

$$
t\nu(A_t)^{1/q}\le Cr_0\mu(B_0)^{-1/s}M^{1/q}||f||_s.
$$

This completes the proof of the Theorem. □

Next we obtain the following Adams inequality for Lipschitz functions when $p=s$.

Theorem 6.2. *Let (X,d,μ) be a complete metric measure space such that it supports weak $(1,1)$-Poincaré inequality and μ is a doubling Radon measure. Let $B_0=B(x_0,r_0)\subset X$ such that $r_0<\operatorname{diam}X/10$ and suppose that ν is a Radon measure in B_0 with*

$$
\nu(B(x,r))\le M\left(\log\frac{r_0}{r}\right)^{\frac{1-s}{s}q},
$$

for all balls $B(x,r)\subset X$ such that $x\in 2B_0$ and $r<r_0/2$. Here $1<s<q<\infty$ and M is a positive constant. Then

$$
\left(\int_{B_0}|u|^q d\nu\right)^{1/q}\le Cr_0\mu(B_0)^{-1/s}M^{1/q}\left(\int_{B_0}(\operatorname{Lip}u)^s d\mu\right)^{1/s},
$$

for all $u\in\operatorname{Lip}_0(B_0)$, where $C=C(q,s,C_d,C_p,\tau)>0$ is a constant.

Proof. By Remark 3.3 and Theorem 6.1

$$
\begin{aligned}
\nu(\{x \in B_0 : &|u(x)| > t\})t^q \\
&\le \hat{C}^q \nu(\{x \in B_0 : I_{1,B_0}(\operatorname{Lip} u)(x) > t/\hat{C}\})(t/\hat{C})^q \\
&\le CA \left(\int_{B_0} (\operatorname{Lip} u)^s d\mu \right)^{q/s},
\end{aligned}
\tag{27}
$$

where $A = r_0^q \mu(B_0)^{-q/s} M$ and $\hat{C}$ is a constant from Remark 3.3. We may assume that $u \ge 0$, since positive and negative parts of u can be estimated separately in the following way. We use a truncation argument to prove a strong-type inequality from the weak-type inequality above. By the truncation property, see [HaK, Chapter 2], for every $0 < t_1 < t_2 < \infty$ the weak Poincaré inequality holds for the pair $u_{t_1}^{t_2}$, $(\operatorname{Lip} u)\chi_{\{t_1 < u \le t_2\}}$, where $u_{t_1}^{t_2} = \min\{\max\{0, u - t_1\}, t_2 - t_1\}$ and χ_E is the characteristic function of the set E. Thus (27) also holds for this pair. Now

$$
\begin{aligned}
\int_{B_0} u^q \, d\nu &\le \sum_{k=-\infty}^{\infty} 2^{kq} \nu(\{u \ge 2^{k-1}\}) \\
&= \sum_{k=-\infty}^{\infty} 2^{kq} \nu(\{u_{2^{k-2}}^{2^{k-1}} \ge 2^{k-2}\}) \\
&\le CA \sum_{k=-\infty}^{\infty} \left(\int_{B_0} (\operatorname{Lip} u)^s \chi_{\{2^{k-2} \le u < 2^{k-1}\}} d\mu \right)^{q/s} \\
&\le CA \left(\sum_{k=-\infty}^{\infty} \int_{B_0} (\operatorname{Lip} u)^s \chi_{\{2^{k-2} \le u < 2^{k-1}\}} d\mu \right)^{q/s} \\
&\le CA \left(\int_{B_0} (\operatorname{Lip} u)^s d\mu \right)^{q/s},
\end{aligned}
$$

where in the second to the last step we used the inequality $q/s \ge 1$. □

References

[Ad] Adams, David R. *A trace inequality for generalized potentials.* Studia Math. 48 (1973), 99–105.

[AH] Adams, D., Hedberg, L. *Function spaces and potential theory.* Grundlehren der Mathematischen Wissenschaften, 314, Springer-Verlag, Berlin, 1996.

[Am] Ambrosio, L. *Fine properties of sets of finite perimeter in doubling metric measure spaces.* Set-Valued Anal. 10 (2002), no. 2-3, 111–128.

[BBS] Björn, A., Björn, J., Shanmugalingam, N. *The Dirichlet problem for p-harmonic functions on metric spaces.* J. Reine Angew. Math. 556 (2003), 173–203.

[BBS2] Björn, A., Björn, J., Shanmugalingam, N. *Sobolev extensions of Hölder continuous and characteristic functions on metric spaces,* to appear in Canadian Journal of Mathematics.

[BMS] Björn, J., MacManus, P., Shanmugalingam, N. *Fat sets and pointwise boundary estimates for p-harmonic functions in metric spaces.* J. Anal. Math. 85 (2001), 339–369.

[Ch] Cheeger, J. *Differentiability of Lipschitz functions on metric measure spaces.* Geom. Funct. Anal., 9 (1999), no. 3, 428–517.

[EKM] Edmunds, D.E., Kokilashvili, V., Meskhi, A. *Bounded and compact integral operators.* Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[Ha] Hajłasz, P. *Sobolev spaces on an arbitrary metric space.* Potential Anal. 5 (1996), no. 4, 403–415.

[HaK] Hajłasz, P., Koskela, P. *Sobolev met Poincaré.* Mem. Amer. Math. Soc., 145 (2000), no. 688.

[He] Heinonen, J. *Lectures on analysis on metric spaces.* Springer-Verlag, (2001).

[HeK] Heinonen, J., Koskela, P., *Quasiconformal maps in metric spaces with controlled geometry.* Acta Math. 181 (1998), no. 1, 1–61.

[KZ] Keith, S., Zhong X. *The Poincare inequality is an open ended condition.* To appear in Ann. of Math.

[KKM] Kilpeläinen, T., Kinnunen, J., Martio, O. *Sobolev spaces with zero boundary values on metric spaces.* Potential Anal. 12 (2000), no. 3, 233–247.

[KK] Kinnunen, J., Korte, R. *Characterizations of Sobolev inequalities on metric spaces.* Preprint, 2007.

[KKST] Kinnunen, J., Korte, R., Shanmugalingam, N., Tuominen, H. *Lebesgue points and capacities via boxing inequalities in metric spaces.* Indiana Univ. Math. J., to appear.

[KM1] Kinnunen, J., Martio, O., *Nonlinear potential theory on metric spaces.* Illinois J. Math. 46 (2002), no. 3, 857–883.

[KM2] Kinnunen, J., Martio, O., *Potential theory of quasiminimizers.* Ann. Acad. Sci. Fenn. Math. 28 (2003), no. 2, 459–490.

[KS] Kinnunen, J., Shanmugalingam, N. *Regularity of quasi-minimizers on metric spaces.* Manuscr. Math. 105 (2001), 401–423.

[KoM] Koskela, P., MacManus, P. *Quasiconformal mappings and Sobolev spaces.* Studia Math. 131 (1998), no. 1, 1–17.

[Ma] Maz'ja, V.G. *Sobolev spaces.* Springer-Verlag, Berlin, 1985.

[Mi] Miranda Jr., M. *Functions of bounded variation on "good" metric spaces.* J. Math. Pures Appl. (9) 82 (2003), no. 8, 975–1004.

[MW] Muckenhoupt, B., Wheeden, R. *Weighted norm inequalities for fractional integrals.* Trans. Amer. Math. Soc. 192 (1974), 261–274.

[Sh] Shanmugalingam, N. *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces.* Rev. Mat. Iberoamericana 16 (2000), no. 2, 243–279.

[Sh2] Shanmugalingam, N. *Harmonic functions on metric spaces.* Illinois J. Math., 45 (2001), 1021–1050.

[St] Stein, E.M. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1993.

[Tu] Turesson, B.O. *Nonlinear potential theory and weighted Sobolev spaces.* Lecture Notes in Mathematics, 1736. Springer-Verlag, Berlin, 2000.

[Zi] Ziemer, W.P. *Weakly differentiable functions.* Volume 120 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1989.

Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), Fin-40014 University of Jyväskylä, Finland.
E-mail address: **tjmakala@jyu.fi**