Extrapolation of vector valued rearrangement operators

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Abstract

Given an injective map $\tau : \mathcal{D} \to \mathcal{D}$ between the dyadic intervals of the unit interval $[0,1)$, we study extrapolation properties of the induced rearrangement operator of the Haar system $\mathrm{Id}_X \otimes T_{p,\tau} : L^p_{X,0}([0,1)) \to L^p_X([0,1))$, where X is a Banach space and $L^p_{X,0}$ the subspace of mean zero random variables. If X is a UMD-space, then we prove that the property that $\mathrm{Id}_X \otimes T_{p,\tau}$ is an isomorphism for some $1 < p \neq 2 < \infty$ extrapolates across the entire scale of L^q_X-spaces with $1 < q < \infty$. In contrast, if only $\mathrm{Id}_X \otimes T_{p,\tau}$ is bounded and not its inverse, then we show that there can only exist one-sided extrapolation theorems.

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1 Introduction

In vector valued L^p-spaces we study rearrangement operators of the system

$$\{h_I/|I|^{1/p} : I \in \mathcal{D}\},$$

where $\mathcal{D}$ denotes the collection of all dyadic intervals included in $[0,1)$ and h_I is the L_∞-normalized Haar function with support I. These rearrangement

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operators are defined by an injective map $\tau : \mathcal{D} \to \mathcal{D}$ as extension of

$$\mathrm{Id}_X \otimes T_{p,\tau} : \sum_{I\in\mathcal{D}} a_I h_I/|I|^{1/p} \to \sum_{I\in\mathcal{D}} a_I h_{\tau(I)}/|\tau(I)|^{1/p},$$

where $(a_I)_{I\in\mathcal{D}} \subseteq X$ is finitely supported and X is a Banach space. This paper continues [11] and is related in spirit to [8]. In particular, we are motivated by extrapolation properties of vector valued martingale transforms, i.e. maps of type

$$\sum_{I\in\mathcal{D}} a_I h_I \to \sum_{I\in\mathcal{D}} c_I a_I h_I \tag{1}$$

where $(a_I)_{I\in\mathcal{D}} \subseteq X$ is finitely supported and $(c_I)_{I\in\mathcal{D}} \in \ell_\infty(\mathcal{D})$. Extrapolation theorems for these martingale transforms were widely studied in the literature and go back, for example, to Maurey [9] and Burkholder-Gundy [6] (see [5] for a general overview). In our setting these classical theorems state that if (1) is bounded on L^p_X for some $p \in (1,\infty)$, then it is bounded on L^q_X for all $q \in (1,\infty)$. The significance of those theorems can be already seen in the scalar valued setting: Since a martingale transform is trivially bounded on L^2, extrapolation yields its boundedness on each of the spaces L^q with $q \in (1,\infty)$. The aim of this paper is to analyze the extrapolation properties of the family $\mathrm{Id}_X \otimes T_{p,\tau}$.

In Section 3 we start by two examples. Example 3.1 shows that the continuity of a 'typical' permutation $\mathrm{Id}_X \otimes T_{p,\tau}$ already implies that X has to have the UMD-property. The second example provides a permutation such that the continuity of $\mathrm{Id}_X \otimes T_{p,\tau}$ with $p \in (1,2]$ implies the type p property of the Banach space X. As a consequence we deduce in Corollary 3.3 that one does not have an upwards extrapolation: For $X = \ell_p$ and $p \in (1,2)$ (so that X is, in particular, a UMD-space) there is a permutation τ such that $\mathrm{Id}_X \otimes T_{p,\tau}$ is continues, but $\mathrm{Id}_X \otimes T_{q,\tau}$ fails to be continuous for $q \in (p,2]$.

The natural question arises whether we still have a one-sided extrapolation meaning that the boundedness of $\mathrm{Id}_X \otimes T_{p,\tau}$ implies that one of $\mathrm{Id}_X \otimes T_{q,\tau}$ in the case $1 < q < p < 2$.

In Section 4 we answer this to the positive for permutations τ satisfying the assumption $|\tau(I)| = |I|$. The results are formulated in Theorem 4.2 and Corollary 4.3 and proved by transferring Maurey's classical argument [9] to the permutation case via Proposition 4.4. In Corollary 4.3 we extrapolate the boundedness of $\mathrm{Id}_X \otimes T_{p,\tau}$ for a UMD-space X and $p \in (1,2)$ *downwards to 1* to the boundedness of $\mathrm{Id}_X \otimes T_{q,\tau}$ for $q \in (1,p)$.

In Section 5 we do not assume anymore the condition $|\tau(I)| = |I|$. In Corollaries 5.6 and 5.7 we obtain a one-sided extrapolation as well. By duality Corollary 5.6 yields a two-sided extrapolation in Theorem 5.8: We show for a UMD-space X that if $\mathrm{Id}_X \otimes T_{p,\tau}$ is an isomorphism on some $L^p_{X,0}$ with $1 < p \neq 2 < \infty$, then the rearrangement $\mathrm{Id}_X \otimes T_{q,\tau}$ is an isomorphism on $L^q_{X,0}$ for each $q \in (1,\infty)$. Thus for a UMD-space valued rearrangement the property of being an isomorphism extrapolates across the entire scale of $L^q_{X,0}$ spaces, $q \in (1,\infty)$ – just as for martingale transforms or for scalar valued rearrangements $T_{q,\tau} : L^q_0 \to L^q_0$, see [11].

The extrapolation properties of *scalar* valued rearrangement operators are a direct consequence of Pisier's re-norming of H^1,

$$\|g\|_{H^1}^{1-\theta} \sim \sup\{\| \sum |g_I|^{1-\theta} |w_I|^\theta h_I \|_{L^p} : \|w\|_{L^2} = 1\},$$

where $p \in (1,2)$, $1/p = 1 - (\theta/2)$, $g = \sum g_I h_I$, and $w = \sum w_I h_I$. This well known fact is recorded for instance in [10] and was exploited further in [8]. As Pisier's re-norming of H^1 uses the lattice structure of L^p, our analysis of the *vector* valued case circumvents its use and relies instead on combinatorial and geometric properties of τ that hold when $T_{p,\tau}$ is an isomorphism [11].

2 Preliminaries

In the following we equip the unit interval $[0,1)$ with the Lebesgue measure λ. The set of dyadic intervals of length 2^{-k} is denoted by $\mathcal{D}_k$, the set of all dyadic intervals by $\mathcal{D}$, and $\mathcal{F}_k := \sigma(\mathcal{D}_k)$. Given $I \in \mathcal{D}$, we use $Q(I) := \{K \subseteq I : K \in \mathcal{D}\}$ and h_I denotes the L_∞-normalized Haar function supported on I. For a Banach space X we let $L^p_X = L^p_X([0,1))$ be the space of all Radon random variables $f : [0,1) \to X$ such that $\|f\|^p_{L^p_X} := \int_0^1 \|f(t)\|^p_X dt < \infty$ and $L^p_{X,0}$ be the sub-space of mean zero random variables, where $L^p = L^p_{\mathbb{K}}([0,1))$ and $L^p_0 = L^p_{\mathbb{K},0}([0,1))$ if nothing is said to the contrary with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. To avoid artificial special cases we assume that the Banach spaces are at least of dimension one.

Spaces of type and cotype. Let $1 \leq p \leq 2 \leq q < \infty$. A Banach space X is of *type* p (*cotype* q) provided that there is a constant $c > 0$ such that

for all $n = 1, 2, ...$ and $a_1, a_2, ..., a_n \in X$ one has that

$$\left\| \sum_{k=1}^{n} r_k a_k \right\|_{L_X^p} \leq c \left(\sum_{k=1}^{n} \|a_k\|_X^p \right)^{\frac{1}{p}} \left(\left(\sum_{k=1}^{n} \|a_k\|_X^q \right)^{\frac{1}{q}} \leq c \left\| \sum_{k=1}^{n} r_k a_k \right\|_{L_X^q} \right),$$

where $r_1, r_2, ...$ denote independent Bernoulli random variables. We let $\mathrm{Type_p(X)} := \inf c$ ($\mathrm{Cotype_q(X)} := \inf c$).

UMD-spaces. A Banach space X is called UMD-*space* provided that for some $p \in (1, \infty)$ (equivalently, for all $p \in (1, \infty)$) there is a constant $c_p > 0$ such that

$$\sup_{\theta_k \in [-1,1]} \left\| \sum_{k=1}^{n} \theta_k d_k \right\|_{L_X^p} \leq c_p \left\| \sum_{k=1}^{n} d_k \right\|_{L_X^p}$$

for all $n = 1, 2, ...$ and all martingale difference sequences $(d_k)_{k=1}^n \subseteq L_X^1(\mathcal{F}_n)$ with respect to $(\mathcal{F}_k)_{k=0}^n$, i.e. d_k is $\mathcal{F}_k$-measurable and $\mathbb{E}(d_k|\mathcal{F}_{k-1}) = 0$ for $k = 1, ..., n$. The infimum of all possible $c_p > 0$ is denoted by $\mathtt{UMD}_p(X)$.

Using [4, page 12] it follows that $\mathtt{UMD}_p(X) = \inf d_p$, where the infimum is taken over all $d_p > 0$ such that

$$\sup_{\theta_I \in [-1,1]} \left\| \sum_{I \in \mathcal{D}} \theta_I a_I h_I \right\|_{L_X^p} \leq d_p \left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_{L_X^p}$$

for all finitely supported $(a_I)_{I \in \mathcal{D}} \subseteq X$. An overview about UMD-spaces can be found in [5].

Hardy spaces. We recall the definition of Hardy spaces we shall use.

Definition 2.1. (i) A function $a \in L_{X,0}^1(\mathcal{F}_N)$, where $N \geq 1$, is called *atom* provided there exists a stopping time $\nu : \Omega \to \{+\infty, 0, ..., N\}$ such that

(a) $a_n := \mathbb{E}(a|\mathcal{F}_n) = 0$ on $\{n \leq \nu\}$ for $n = 0, ..., N$,

(b) $\|a\|_{L_X^\infty} \mathbb{P}(\nu < \infty) \leq 1$.

(ii) The space $H_X^{1,at}(\mathcal{F}_N)$ is given by the norm

$$\|f\|_{H_X^{1,at}} := \inf \sum_{k=1}^{\infty} |\mu_k|, \quad f \in L_{X,0}^1(\mathcal{F}_N),$$

where the infimum is taken over all sequences $(\mu_k)_{k=1}^{\infty} \subset [0,\infty)$ and atoms $(a^k)_{k=1}^{\infty}$ such that $f = \sum_{k=1}^{\infty} \mu_k a^k$ in $L_X^1(\mathcal{F}_N)$.

(iii) Given $p \in [1,\infty)$, the space $H_X^p(\mathcal{F}_N)$ is given by the norm

$$\|f\|_{H_X^p} := \left(\mathbb{E} \sup_{n=0,\dots,N} \|\mathbb{E}(f|\mathcal{F}_n)\|_X^p \right)^{\frac{1}{p}}, \quad f \in L_{X,0}^p(\mathcal{F}_N).$$

For an atom a we have that $a = 0$ on $\{\nu = \infty\}$, $\operatorname{supp}(a) \subseteq \{\nu < \infty\}$, and

$$\mathbb{E}\|a\|_X \le \|a\|_{L_X^\infty} \mathbb{P}(\nu < \infty) \le 1.$$

The following inequality is well-known (see [2] and [7], cf. [15]):

$$\|f\|_{H_X^1(\mathcal{F}_N)} \le \|f\|_{H_X^{1,at}(\mathcal{F}_N)} \le 18 \|f\|_{H_X^1(\mathcal{F}_N)}. \tag{2}$$

Rearrangement operators. Let $\tau : \mathcal{D} \to \mathcal{D}$ be an injective map. Given a Banach space X and $p \in [1,\infty)$, we define the rearrangement operator $\operatorname{Id}_X \otimes T_{p,\tau}$ on finite linear combinations of Haar functions as

$$\operatorname{Id}_X \otimes T_{p,\tau} : \sum a_I \frac{h_I}{|I|^{1/p}} \to \sum a_I \frac{h_{\tau(I)}}{|\tau(I)|^{1/p}}, \quad a_I \in X,$$

and let

$$\| \operatorname{Id}_X \otimes T_{p,\tau}\| := \sup \left\{ \left\| \sum_{I\in\mathcal{D}} a_I \frac{h_{\tau(I)}}{|\tau(I)|^{1/p}} \right\|_{L_X^p} : \left\| \sum_{I\in\mathcal{D}} a_I \frac{h_I}{|I|^{1/p}} \right\|_{L_X^p} \le 1 \right\}$$

where the supremum is taken over all finitely supported $(a_I)_{I\in\mathcal{D}} \subseteq X$. In the case $\| \operatorname{Id}_X \otimes T_{p,\tau}\| < \infty$ we say that $\operatorname{Id}_X \otimes T_{p,\tau}$ is bounded because it can be continuously extended to $L_{X,0}^p([0,1)) \to L_X^p([0,1))$. The dependence on p of the operator $T_{p,\tau}$ disappears when the injection $\tau : \mathcal{D} \to \mathcal{D}$ satisfies

$$|\tau(I)| = |I|, \quad I \in \mathcal{D},$$

so that we also use $T_\tau = T_{p,\tau}$.

Semenov's condition. For a non-empty collection $\mathcal{C}$ of dyadic intervals we let $\mathcal{C}^* := \bigcup_{I\in\mathcal{C}} I$. A rearrangement $\tau : \mathcal{D} \to \mathcal{D}$ with

$$|\tau(I)| = |I|$$

satisfies *Semenov's condition* if there is a $\kappa \in [1, \infty)$ such that

$$\sup_{\mathcal{C}\subseteq\mathcal{D}} \frac{|\tau(\mathcal{C})^*|}{|\mathcal{C}^*|} \leq \kappa < \infty. \tag{3}$$

Given $p \in (1, 2)$, Semenov's theorem [13, 14] asserts that under the restriction $|\tau(I)| = |I|$, condition (3) is equivalent to the boundedness of $T_\tau : L^p_0([0, 1)) \to L^p([0, 1))$.

Carleson's constant. For a non-empty collection $\mathcal{E} \subseteq \mathcal{D}$ the *Carleson constant* is given by

$$[\![\mathcal{E}]\!] := \sup_{I\in\mathcal{E}} \frac{1}{|I|} \sum_{J\subseteq I,\, J\in\mathcal{E}} |J|.$$

The Carleson constant is linked to rearrangement operators by the following theorem [11, Theorems 2 and 3]: For a bijection $\tau : \mathcal{D} \to \mathcal{D}$ the assertion that for some (all) $p \in (1, \infty)$ with $p \neq 2$ one has

$$\| \operatorname{Id}_{\mathbb{K}} \otimes T_{p,\tau} : L^p_{X,0} \to L^p_X\| \cdot \| \operatorname{Id}_{\mathbb{K}} \otimes T_{p,\tau^{-1}} : L^p_{X,0} \to L^p_X\| < \infty$$

is equivalent to the existence of an $A \geq 1$ such that

$$\frac{1}{A}[\![\mathcal{E}]\!] \leq [\![\tau(\mathcal{E})]\!] \leq A[\![\mathcal{E}]\!]$$

for all non-empty $\mathcal{E} \subseteq \mathcal{D}$.

3 Two examples

In this section we consider bijections $\tau : \mathcal{D} \to \mathcal{D}$ such that $|\tau(I)| = |I|$ for all $I \in \mathcal{D}$ and provide examples which show that $\mathtt{UMD}_p(X)$ and $\mathrm{Type}_p(X)$ may both be obstructions to the boundedness of

$$\operatorname{Id}_X \otimes T_\tau : L^p_{X,0} \to L^p_X.$$

From that it becomes clear that Semenov's boundedness criterion [13] does not have a direct correspondence in the vector valued case.

Example 3.1. Let $\tau_0 : \mathcal{D} \to \mathcal{D}$ be the injection that leaves invariant the intervals of the even numbered dyadic levels. On the odd numbered dyadic levels we define τ_0 to exchange the dyadic intervals contained in $[0, 1/2)$ with those contained in $[1/2, 1)$ by the shifts

$$\tau_0(I) = I + \frac{1}{2} \text{ if } I \subseteq [0, 1/2) \text{ and } \tau_0(I) = I - \frac{1}{2} \text{ if } I \subseteq [1/2, 1).$$

Then one has the following:

(i) The rearrangement $\tau_0 = \tau_0^{-1}$ satisfies Semenov's condition with $\kappa = 2$ so that T_{τ_0} is an isomorphism on L^p_0 for $p \in (1, \infty)$.

(ii) For $p \in (1, \infty)$ one has

$$\frac{1}{3}\mathtt{UMD}_p(X) \leq \| \operatorname{Id}_X \otimes T_{\tau_0} : L^p_{X,0} \to L^p_X \| \leq 2\mathtt{UMD}_p(X) \tag{4}$$

so that the boundedness of $\operatorname{Id}_X \otimes T_{\tau_0}$ on $L^p_{X,0}$, $p \in (1, \infty)$, holds precisely when X satisfies the UMD-property.

Proof. Assertion (i) is obvious so that let us turn to (ii) and let $N \geq 2$ be even and recall that $\mathcal{D}_k$ is the set of dyadic intervals of length 2^{-k}. For $k \geq 1$ define

$$\mathcal{D}_k^- := \{I \in \mathcal{D}_k : I \subseteq [0, 1/2)\}.$$

The testing functions by which we link the boundedness of $\operatorname{Id}_X \otimes T_{\tau_0}$ to the UMD-property of X are

$$f = \sum_{k=1}^{N} \sum_{I \in \mathcal{D}_k^-} a_I h_I \quad \text{and} \quad g = \sum_{k=1}^{N/2} \sum_{I \in \mathcal{D}_{2k}^-} a_I h_I,$$

where $a_I \in X$. Note that g is obtained from f by deleting every second dyadic level from the Haar expansion of f starting with level 1. Consequently,

$$\begin{aligned} \left\| \sum_{k=1}^{N} (-1)^k \sum_{I \in \mathcal{D}_k^-} a_I h_I \right\|_{L^p_X} &= \|f - 2g\|_{L^p_X} \\ &\leq \|f\|_{L^p_X} + 2\|g\|_{L^p_X} \\ &\leq \|f\|_{L^p_X} + 2\|(\operatorname{Id}_X \otimes T_{\tau_0})f\|_{L^p_X} \end{aligned}$$

$$\leq \big(1+2\| \operatorname{Id}_X \otimes T_{\tau_0} : L^p_{X,0} \to L^p_X\|\big)\, \|f\|_{L^p_X}$$
$$\leq 3\| \operatorname{Id}_X \otimes T_{\tau_0} : L^p_{X,0} \to L^p_X\| \left\| \sum_{k=1}^{N} \sum_{I\in\mathcal{D}_k^-} a_I h_I \right\|_{L^p_X}.$$

In our definition of $\mathtt{UMD}_p(X)$ it is sufficient to consider ± 1 transforms (this is a well-known extreme point argument). Furthermore, by an appropriate augmentation of the filtration we can even restrict ourselves to alternating sequences of signs ± 1. Hence we obtain the left hand side of (4) (in fact, we can think to work on $[0,1/2)$ as probability space after re-normalization).

For the right hand side of (4) we fix some $N \geq 1$ and observe that the action of the above rearrangement is an isometry when restricted to $\sum_{k \text{ odd}, 0\leq k\leq N} \sum_{k\in\mathcal{D}_k} a_I h_I$ and an isometry when restricted to $\sum_{k \text{ even}, 0\leq k\leq N} \sum_{k\in\mathcal{D}_k} a_I h_I$. Using the $\mathtt{UMD}$-property of X, we merge this information to obtain the boundedness of the rearrangement operator on the entire space $L^p_{X,0}$.

■

Example 3.2. There exists a rearrangement $\tau_0 : \mathcal{D} \to \mathcal{D}$ with $|\tau_0(I)| = |I|$ satisfying the Semenov condition (3), such that for all $p \in (1,2]$ and all Banach spaces X one has that

$$\operatorname{Type}_p(X) \leq \| \operatorname{Id}_X \otimes T_{\tau_0} : L^p_{X,0} \to L^p_X\|.$$

PROOF. (a) Fix $n \geq 1$ and assume disjoint dyadic intervals $I_0, ..., I_n$ of the same length, one after each other starting with I_0. Let

$$\mathcal{A}_k := \big\{ I \in \mathcal{D} : I \subseteq I_k, |I| = 2^{-k}|I_k| \big\}$$

for $k = 1, ..., n$. We define a permutation $\tau_n : \mathcal{D} \to \mathcal{D}$ such that

(i) $\mathcal{A}_k$ is shifted from I_k to I_0 for each $k = 1, ..., n$,

(ii) all subintervals of I_0 of length $2^{-k}|I_0|$, $k = 1, ..., n$, are shifted to I_1,

(iii) all subintervals of I_1 of length $2^{-k}|I_1|$, $k = 2, ..., n$, are shifted to I_2, ...,

(iv) all subintervals of I_{n-1} of length $2^{-n}|I_{n-1}|$ are shifted to I_n.

On all other intervals τ_n acts as an identity. One can check that τ_n satisfies Semenov's condition with $\kappa = 3$. Moreover, for $a_1, ..., a_n \in X$,

$$\begin{aligned}\int_0^1 \left\| \sum_{k=1}^n \sum_{I\in\mathcal{A}_k} a_k h_I(t) \right\|_X^p dt &= |I_0| \sum_{k=1}^n \|a_k\|_X^p, \\ \int_0^1 \left\| \sum_{k=1}^n \sum_{I\in\mathcal{A}_k} a_k h_{\tau_n(I)}(t) \right\|_X^p dt &= |I_0| \left\| \sum_{k=1}^n r_k a_k \right\|_{L_X^p}^p,\end{aligned}$$

so that

$$\left\| \sum_{k=1}^n r_k a_k \right\|_{L_X^p} \leq \| \operatorname{Id}_X \otimes T_{\tau_n} : L_{X,0}^p \to L_X^p \| \left(\sum_{k=1}^n \|a_k\|^p \right)^{\frac{1}{p}}$$

where $r_1, ..., r_n$ are independent Bernoulli random variables.

(b) Now we 'glue together' the permutations $\tau_1, \tau_2, ...$: to this end we find pairwise disjoint dyadic intervals $I_0^1, I_1^1 \subseteq [0, 1/2)$, $I_0^2, I_1^2, I_2^2 \subseteq [1/2, 3/4)$, $I_0^3, I_1^3, I_2^3, I_3^3 \subseteq [3/4, 7/8), \ldots$, where $I_0^n, ..., I_n^n$ is a collection as in part (a). Defining the permutation τ_0 on $I_0^n, ..., I_n^n$ as in (a) for all $n = 1, 2, ...$ and elsewhere as identity, we arrive at our desired permutation τ_0.

■

Corollary 3.3. *For the permutation τ_0 from Example 3.2, $p \in (1, 2)$, and $X := \ell_p$ one has*

$$\| \operatorname{Id}_X \otimes T_{\tau_0} : L_{X,0}^p \to L_X^p \| < \infty$$

but

$$\| \operatorname{Id}_X \otimes T_{\tau_0} : L_{X,0}^q \to L_X^q \| = \infty \quad \textit{for all} \quad q \in (p, 2].$$

Proof. The first relation follows from Fubini's theorem and the Semenov condition. On the other side, $X = \ell_p$ is not of type q as long as $q \in (p, 2]$ so that T_{τ_0} fails to be bounded in $L_{X,0}^q$.

■

4 Maurey's extrapolation method and the Semenov condition

By Corollary 3.3 we have seen that an extrapolation from p to q fails in general if $q \in (p, 2]$. Here one should note that the boundedness of $\operatorname{Id}_X \otimes T_\tau$:

$L^p_{X,0} \to L^p_X$ implies the boundedness of $T_\tau : L^p_0 \to L^p$, hence the Semenov condition. The aim of this section is to show that, by Maurey's extrapolation method [9], one has an extrapolation from p to q in the case that $q \in (1, p)$.

Definition 4.1. Let $\tau : \mathcal{D} \to \mathcal{D}$ be a permutation with $|\tau(I)| = |I|$. An operator A which maps $f \in L^1_{X,0}(\mathcal{F}_n)$ into a non-negative random variable $A(f) : [0,1) \to [0,\infty)$ and which is homogeneous (i.e. $A(\mu f) = |\mu| A(f)$, λ-a.s., for all $\mu \in \mathbb{K}$), where $n \geq 1$, is τ-monotone with constant $c > 0$ provided that one has, λ-a.s., that

$$A\left(\sum_{k=1}^n \gamma_k d_k\right) \leq c \sup_{k=1,\dots,n} |P_{k-1,\tau}(\gamma_k)| A\left(\sum_{k=1}^n d_k\right) \tag{5}$$

for all

$$d_k(t) = \sum_{I \in \mathcal{D}_{k-1}} a_I h_I(t), \quad a_I \in X,$$

and non-decreasing $(\gamma_k)_{k=1}^n$ with

$$\gamma_k(t) = \sum_{I \in \mathcal{D}_{k-1}} \gamma_k(I) I_I(t), \quad \gamma_k(I) \geq 0,$$

where $P_{k-1,\tau}(\gamma_k) := \sum_{I \in \mathcal{D}_{k-1}} \gamma_k(I) I_{\tau(I)}(t)$.

Note that $P_{k,\tau}(\gamma)$ is correctly defined for all $\gamma : [0,1) \to \mathbb{R}$ that are constant on the dyadic intervals of length 2^{-k}.

Theorem 4.2. *For a permutation $\tau : \mathcal{D} \to \mathcal{D}$ with $|\tau(I)| = |I|$ the following assertions are equivalent:*

(i) *The permutation τ satisfies the Semenov condition* (3).

(ii) *For all $1 < q < p < \infty$, Banach spaces X, $n = 1, 2, ...$, and τ-monotone operators A, defined on $L^1_{X,0}(\mathcal{F}_n)$, with constant $c > 0$ one has that*

$$\|A : L^q_{X,0}(\mathcal{F}_n) \to L^q([0,1))\| \leq d \|A : L^p_{X,0}(\mathcal{F}_n) \to L^p([0,1))\|$$

where $d = d(p,q,c) > 0$ and

$$\|A\|_r = \|A : L^r_{X,0}(\mathcal{F}_n) \to L^r([0,1))\| := \sup\left\{ \|A(f)\|_{L^r} : \|f\|_{L^r_{X,0}} \leq 1 \right\}.$$

Before we give the proof of Theorem 4.2 we apply it to our original extrapolation problem.

Corollary 4.3. *Let X be a* UMD*-space and let $\tau : \mathcal{D} \to \mathcal{D}$ be a permutation such that*

$$|\tau(I)| = |I|.$$

If, for some $p \in (1,2)$, one has that

$$\mathrm{Id}_X \otimes T_\tau : L^p_{X,0} \to L^p_X$$

is bounded, then

$$\mathrm{Id}_X \otimes T_\tau : L^q_{X,0} \to L^q_X$$

is bounded for all $q \in (1,p)$.

Proof. Because our assumption implies that $\mathrm{Id}_{\mathbb{K}} \otimes T_\tau : L^p_0 \to L^p$ is bounded it has to satisfy the Semenov condition. We fix $n \geq 1$ and apply the previous theorem to the operator A defined, for $d_k = \sum_{I \in \mathcal{D}_{k-1}} a_I h_I$ with $a_I \in X$, as

$$A\left(\sum_{k=1}^n d_k\right) := \int_\Omega \left\| (\mathrm{Id}_X \otimes T_\tau)\left(\sum_{k=1}^n r_k(\omega) d_k\right)\right\|_X d\mathbb{P}(\omega)$$

where $r_1, ..., r_n$ are independent Bernoulli random variables. It is easy to see that A satisfies (5) with $c = 1$. Moreover by the UMD-property we have

$$\left\| A\left(\sum_{k=1}^n d_k\right)\right\|_{L^p} \sim \left\| (\mathrm{Id}_X \otimes T_\tau)\left(\sum_{k=1}^n d_k\right)\right\|_{L^p_X},$$

where the multiplicative constants do not depend on n. Hence Theorem 4.2 yields the assertion.

■

The maximal inequality of the following Proposition 4.4 provides the link between rearrangements satisfying Semenov's condition and Maurey's extrapolation technique in [9].

Proposition 4.4. *Assume that Semenov's condition (3) is satisfied for a permutation τ with $|\tau(I)| = |I|$ and that $0 \leq Z_0 \leq Z_1 \leq \cdots \leq Z_n$ is a sequence of functions $Z_k : [0,1) \to [0,\infty)$, where Z_k is constant on all dyadic intervals of length $1/2^k$. Then one has that*

$$\int_0^1 \sup_{k=0,\ldots,n} (P_{k,\tau}(Z_k))(t)dt \leq \kappa \int_0^1 Z_n(t)dt.$$

PROOF. Let $\Delta_0 := Z_0$ and $\Delta_k := Z_k - Z_{k-1}$ for $k = 1, ..., n$, and let us write

$$\Delta_k = \sum_{I \in \mathcal{D}_k} a_I 1_I$$

with $a_I \geq 0$. Fix $k \in \{0, ..., n\}$ and observe that, point wise,

$$P_{k,\tau} 1_I \leq 1_{\tau(Q(I))^*} \quad \text{with} \quad Q(I) = \{K \subseteq I : K \in \mathcal{D}\}$$

for $I \in \mathcal{D}_{k'}$ with $k' = 0, ..., k$ (note that 1_I is constant on the dyadic intervals of length 2^{-k} so that we may apply $P_{k,\tau}$). This implies that

$$P_{k,\tau}\left(\sum_{k'=0}^{k} \sum_{I \in \mathcal{D}_{k'}} a_I 1_I\right) \leq \sum_{k'=0}^{k} \sum_{I \in \mathcal{D}_{k'}} a_I 1_{\tau(Q(I))^*}.$$

Because the expression on the right-hand side is monotone in k we conclude that

$$\sup_{k=0,\ldots,n} P_{k,\tau}\left(\sum_{k'=0}^{k} \sum_{I \in \mathcal{D}_{k'}} a_I 1_I\right) \leq \sum_{k'=0}^{n} \sum_{I \in \mathcal{D}_{k'}} a_I 1_{\tau(Q(I))^*}.$$

Integration gives

$$\int_0^1 \left[\sup_{k=0,\ldots,n} P_{k,\tau}\left(\sum_{k'=0}^{k} \sum_{I \in \mathcal{D}_{k'}} a_I 1_I\right)(t)\right] dt \leq \sum_{k'=0}^{n} \sum_{I \in \mathcal{D}_{k'}} a_I |\tau(Q(I))^*|.$$

Our hypothesis gives $|\tau(Q(I))^*| \leq \kappa |I|$ so that

$$\sum_{k'=0}^{n} \sum_{I \in \mathcal{D}_{k'}} a_I |\tau(Q(I))^*| \leq \kappa \sum_{k'=0}^{n} \sum_{I \in \mathcal{D}_{k'}} a_I |I| = \kappa \int_0^1 \left[\sum_{k=0}^{n} \Delta_k(t)\right] dt$$

and we are done because

$$\int_0^1 \left[\sup_{k=0,\dots,n} P_{k,\tau} \left(\sum_{k'=0}^{k} \sum_{I \in \mathcal{D}_{k'}} a_I 1_I \right)(t) \right] dt = \int_0^1 \sup_{k=0,\dots,n} (P_{k,\tau} Z_k)(t) dt$$

and

$$\int_0^1 \left[\sum_{k=0}^{n} \Delta_k(t) \right] dt = \int_0^1 Z_n(t) dt.$$

■

Proof of Theorem 4.2. (i) $\Longrightarrow$ (ii) We let $\frac{1}{q} = \frac{1}{r} + \frac{1}{p}$ and

$$d_k := \sum_{I \in \mathcal{D}_{k-1}} \alpha_I h_I \quad \text{so that} \quad T_\tau d_k = \sum_{I \in \mathcal{D}_{k-1}} \alpha_I h_{\tau(I)}.$$

Define $X_0 := 0$, $X_k := d_1 + \cdots + d_k$ for $k = 1, ..., n$, $X_k^* := \sup_{l=0,\dots,k} \|X_l\|_X$ for $k = 0, ..., n$, ${}^*X_k := X_{k-1}^* + \sup_{l=1,\dots,k} \|d_l\|_X$ for $k = 1, ..., n$,

$$\gamma_k := ({}^*X_k + \delta)^\alpha$$

for some $\delta > 0$,

$$\alpha := 1 - \frac{q}{p},$$

and

$$\beta_k := P_{k-1,\tau} \gamma_k.$$

By definition we have that

$$\frac{T_\tau(d_k)}{\beta_k} = T_\tau \left(\frac{d_k}{\gamma_k} \right).$$

From the monotonicity assumption on the operator A it follows that

$$\left\| A \left(\sum_{k=1}^{n} d_k \right) \right\|_{L^q} \le c \|\beta_n^*\|_{L^r} \left\| A \left(\sum_{k=1}^{n} \frac{d_k}{\gamma_k} \right) \right\|_{L^p} \le c \|A\|_p \|\beta_n^*\|_{L^r} \left\| \sum_{k=1}^{n} \frac{d_k}{\gamma_k} \right\|_{L_X^p}.$$

From [9, Lemma A] we know that

$$\left\| \sum_{k=1}^{n} \frac{d_k}{\gamma_k} \right\|_{L_X^p} \le \frac{p}{q} \left(\mathbb{E}({}^*X_n + \delta)^q \right)^{\frac{1}{p}}$$

$$\leq \frac{p}{q} 3^{\frac{q}{p}} \left(\mathbb{E}(X_n^* + \delta)^q\right)^{\frac{1}{p}}.$$

Finally, applying Proposition 4.4 we get

$$\|\beta_n^*\|_{L^r}^r = \int_0^1 \sup_{k=1,\dots,n} |(P_{k-1,\tau}(\gamma_k))(t)|^r dt = \int_0^1 \sup_{k=1,\dots,n} (P_{k-1,\tau}(|\gamma_k|^r))(t) dt$$
$$\leq \kappa \int_0^1 |\gamma_n(t)|^r dt = \kappa \int_0^1 |{}^*X_n(t) + \delta|^{\alpha r} dt \leq 3^{\alpha r} \kappa \int_0^1 |X_n^*(t) + \delta|^{\alpha r} dt.$$

Combining all estimates, we get

$$\left\| A\left(\sum_{k=1}^n d_k\right)\right\|_{L^q} \leq c\|A\|_p 3^\alpha \kappa^{\frac{1}{r}} \left(\mathbb{E}|X_n^* + \delta|^{\alpha r}\right)^{\frac{1}{r}} \frac{p}{q} 3^{\frac{q}{p}} \left(\mathbb{E}|X_n^* + \delta|^q\right)^{\frac{1}{p}}.$$

By $\delta \downarrow 0$ and Doob's maximal inequality this implies

$$\left\| A\left(\sum_{k=1}^n d_k\right)\right\|_{L^q} \leq c\|A\|_p \frac{3p}{q-1} \kappa^{\frac{1}{r}} \|d_1 + \cdots + d_n\|_{L_X^q}.$$

(ii) $\Longrightarrow$ (i) We fix $X = \mathbb{K}$, $n \in \{1, 2, ...\}$, and a permutation τ with $|\tau(I)| = |I|$. Let $A\left(\sum_{k=1}^n d_k\right) := \left(\sum_{k=1}^n (T_\tau d_k)^2\right)^{\frac{1}{2}}$ which is τ-monotone with constant $c = 1$. Clearly, $\|Af\|_{L^2} = \|f\|_{L^2}$. If we have an extrapolation to some $q \in (1, 2)$, then by the square function inequality the usual permutation operator is bounded in L^q with a constant not depending on n, so that by Semenov's theorem [13] condition (3) has to be satisfied.

■

5 Extrapolation and the Carleson condition

In this section we consider rearrangement operators induced by bijections $\tau : \mathcal{D} \to \mathcal{D}$ that preserves the Carleson packing condition, that is there is an $A \geq 1$ such that

$$\frac{1}{A}[\![\mathcal{E}]\!] \leq [\![\tau(\mathcal{E})]\!] \leq A[\![\mathcal{E}]\!]$$

for all non-empty $\mathcal{E} \subseteq \mathcal{D}$. In particular, we do not rely anymore on the a-priori hypothesis that $|\tau(I)| = |I|$. The corresponding extrapolation results are formulated in Corollary 5.6, Corollary 5.7, and Theorem 5.8, where we

obtain in Corollary 5.7 an alternative proof of Corollary 4.3 that works without X being a UMD-space. To shorten the notation we let $\mathcal{D}_0^N := \bigcup_{k=0}^N \mathcal{D}_k$ for $N \geq 0$. Because we use complex interpolation we shall assume that all Banach spaces are complex.

We start with a technical condition which ensures a one-sided extrapolation. The condition will be justified by Examples 5.2 and 5.3 below.

Definition 5.1. Let X be a Banach space, $\tau : \mathcal{D}_0^N \to \mathcal{D}_0^L$ be an injection, $\gamma_I > 0$ for $I \in \mathcal{D}_0^N$, $p \in (1,\infty)$, and $\kappa > 0$. We say that condition $C(X,p,\kappa)$ is satisfied, provided that for all $J_0 \in \mathcal{D}_0^N$ there is a decomposition

$$\big\{I \in \mathcal{D}_0^N : I \subseteq J_0\big\} = \bigcup_i \mathcal{K}_i,$$

$\mathcal{K}_i \neq \emptyset$, such that the following is satisfied:

(C1) $\sum_i |\mathcal{K}_i^*| \leq \kappa |J_0|$.

(C2) For $1 = \frac{1}{p} + \frac{1}{q}$ and

$$\beta_i := \sup\left\{ \left\| \sum_{I\in\mathcal{K}_i} \gamma_I^{\frac{1}{q}} a_I h_I \right\|_{L_X^p}^q : \left\| \sum_{I\in\mathcal{K}_i} a_I h_I \right\|_{L_X^p} = 1 \right\}$$

one has that $\sum_i \beta_i |\tau(\mathcal{K}_i)^*| \leq \kappa |J_0|$.

(C3) There exists $p_* \in [p,\infty)$ such that

$$\left(\sum_i \left\| \sum_{I\in\mathcal{K}_i} a_I h_I \right\|_{L_X^{p_*}}^{p_*} \right)^{\frac{1}{p_*}} \leq \kappa \left\| \sum_{J_0 \supseteq I \in \mathcal{D}_0^N} a_I h_I \right\|_{L_X^{p_*}}.$$

Example 5.2. We assume that $\tau : \mathcal{D} \to \mathcal{D}$ with $|\tau(I)| = |I|$ satisfies the Semenov condition (3) with constant $\kappa \in [1,\infty)$, restrict τ to $\tau_N : \mathcal{D}_0^N \to \mathcal{D}_0^N$, and take $\gamma_I = 1$ for all $I \in \mathcal{D}_0^N$. Let X be arbitrary, $p \in (1,\infty)$, and $J_0 \in \mathcal{D}_0^N$. Because of

$$\left| \bigcup_{J_0 \supseteq I \in \mathcal{D}_0^N} \tau_N(I) \right| \leq \kappa |J_0|$$

we can take

$$\mathcal{K}_1 := \left\{ I \in \mathcal{D}_0^N : I \subseteq J_0 \right\}$$

and conditions (C1), (C2), and (C3) (for any p^*) are satisfied with constant κ uniformly in N.

Example 5.3. Let $\tau : \mathcal{D} \to \mathcal{D}$ be a bijection and assume that there is an $A \geq 1$ such that

$$\frac{1}{A}[\![\mathcal{E}]\!] \leq [\![\tau(\mathcal{E})]\!] \leq A[\![\mathcal{E}]\!]$$

for all non-empty $\mathcal{E} \subseteq \mathcal{D}$. Let X be a UMD-space and $\gamma_I := |I|/|\tau(I)|$. As shown in [11, Theorem 1], the permutation $\sigma = \tau^{-1}$ satisfies the following property P: There exists an $M > 0$ such that for all dyadic intervals $J_0 \in \mathcal{D}$ there exists a decomposition as disjoint union

$$\{I \in \mathcal{D} : I \subseteq J_0\} = \sigma(\mathcal{D}) \cap J_0 = \bigcup_i \sigma(\mathcal{L}_i) \cup \bigcup_i \mathcal{E}_i$$

such that

(1) $[\![\bigcup_i \mathcal{E}_i]\!] \leq M$,

(2) $\sup_{K \in \mathcal{L}_i} \frac{|\sigma(K)|}{|K|} \leq M \frac{|\sigma(\mathcal{L}_i)^*| + |\mathcal{E}_i^*|}{|\mathcal{L}_i^*|}$ for $\mathcal{L}_i \neq \emptyset$,

(3) $\sum_i |\sigma(\mathcal{L}_i)^*| \leq M|J_0|$.

Now we check the counterparts of (C1), (C2), and (C3) for the 'infinite' permutation τ.

Condition (C3): As X is a UMD-space (and therefore super-reflexive) there is a $p_0 \in [2, \infty)$ such that for all $p_* \in [p_0, \infty)$ the space X has cotype p_*. This cotype and the UMD-property imply (C3) (the constant may depend on p_*).

Condition (C1): We write

$$\bigcup_i \mathcal{E}_i = \left\{ \widetilde{I}_1, \widetilde{I}_2, ... \right\} \quad \text{and} \quad \widetilde{\mathcal{L}}_j := \{\tau(\widetilde{I}_j)\}$$

so that

$$\{I \in \mathcal{D} : I \subseteq J_0\} = \bigcup_i \sigma(\mathcal{L}_i) \cup \bigcup_j \sigma(\widetilde{\mathcal{L}}_j) =: \bigcup_i \mathcal{K}_i \cup \bigcup_j \widetilde{\mathcal{K}}_j.$$

Now

$$\sum_i |\mathcal{K}_i^*| + \sum_j |\widetilde{\mathcal{K}}_j^*| = \sum_i |\sigma(\mathcal{L}_i)^*| + \sum_j |\widetilde{I}_j| \le M|J_0| + [\![\bigcup_i \mathcal{E}_i]\!]|J_0| \le 2M|J_0|.$$

Condition (C2): let $p \in (1,\infty)$ be arbitrary and recall that

$$\beta_i = \sup\left\{ \left\| \sum_{I\in\mathcal{K}_i} \gamma_I^{\frac{1}{q}} a_I h_I \right\|_{L_p^X}^q : \left\| \sum_{I\in\mathcal{K}_i} a_I h_I \right\|_{L_p^X} = 1 \right\},$$

where we assume that the sums over I are finitely supported, and let

$$\widetilde{\beta}_j := \sup\left\{ \left\| \sum_{I\in\widetilde{\mathcal{K}}_j} \gamma_I^{\frac{1}{q}} a_I h_I \right\|_{L_p^X}^q : \left\| \sum_{I\in\widetilde{\mathcal{K}}_j} a_I h_I \right\|_{L_p^X} = 1 \right\} = \gamma_{\widetilde{I}_j}.$$

Because $\gamma_I = |I|/|\tau(I)|$, the $\mathtt{UMD}$-property of X gives

$$\beta_i \le \mathtt{UMD}_p(X)^q \sup_{I\in\mathcal{K}_i} \frac{|I|}{|\tau(I)|}.$$

Since

$$\sup_{I\in\mathcal{K}_i} \frac{|I|}{|\tau(I)|} \le M \frac{|\mathcal{K}_i^*| + |\mathcal{E}_i^*|}{|\tau(\mathcal{K}_i)^*|}$$

for $\mathcal{L}_i \neq \emptyset$ we get

$$\begin{aligned}
\sum_i \beta_i |\tau(\mathcal{K}_i)^*| &\le \mathtt{UMD}_p(X)^q \sum_i \sup_{I\in\mathcal{K}_i} \frac{|I|}{|\tau(I)|} |\tau(\mathcal{K}_i)^*| \\
&\le \mathtt{UMD}_p(X)^q \sum_i M \frac{|\mathcal{K}_i^*| + |\mathcal{E}_i^*|}{|\tau(\mathcal{K}_i)^*|} |\tau(\mathcal{K}_i)^*| \\
&= M\mathtt{UMD}_p(X)^q \sum_i [|\mathcal{K}_i^*| + |\mathcal{E}_i^*|] \\
&\le 2M^2 \mathtt{UMD}_p(X)^q |J_0|.
\end{aligned}$$

In the same way,

$$\sum_j \widetilde{\beta}_j |\tau(\widetilde{\mathcal{K}}_j)^*| = \sum_j |\widetilde{I}_j| \le M|J_0|.$$

Finally, if we restrict τ to $\tau_N : \mathcal{D}_0^N \to \mathcal{D}_0^{L_N}$ with L_N chosen such that $\tau(\mathcal{D}_0^N) \subseteq \mathcal{D}_0^{L_N}$, then (C1), (C2), and (C3) are satisfied with the same constant uniformly in N.

In the following we use the notation

$$L^r_X(\mathcal{D}^N_0) := L^r_{X,0}(\mathcal{F}_{N+1}),\ \ H^{1,at}_X(\mathcal{D}^N_0) := H^{1,at}_X(\mathcal{F}_{N+1}),$$

and $H^1_X(\mathcal{D}^N_0) := H^1_X(\mathcal{F}_{N+1})$ for $N = 0, 1, ...$ to avoid a permanent shift in N because we are working with the sets $\mathcal{D}^N_0$ rather than with the σ-algebras $\mathcal{F}_N$. Now fix Banach spaces X and Y and a bounded linear operator $S : X \to Y$, and define the family of operators $A_p : L^p_X\left(\mathcal{D}^N_0\right) \to L^p_Y\left(\mathcal{D}^L_0\right)$ by

$$A_p\left(\sum_{I\in\mathcal{D}^N_0} a_I h_I\right) := \sum_{I\in\mathcal{D}^N_0} S a_I \gamma_I^{\frac{1}{p}} h_{\tau(I)},$$

where $\gamma_I > 0$. We aim at extrapolation theorems for this family of operators and extrapolate - under the condition $C(X, p, \kappa)$ - from L^p downwards to H^1 in a first step:

Theorem 5.4. *If $p \in (1, \infty)$ and if assumption $C(X, p, \kappa)$ holds, then*

$$\|A_1 : H^1_X(\mathcal{D}^N_0) \to H^1_Y(\mathcal{D}^L_0)\| \le \frac{18p}{p-1}\kappa^{1+\frac{1}{q_*}}\|A_p : L^p_X(\mathcal{D}^N_0) \to L^p_Y(\mathcal{D}^L_0)\|$$

where $1 = (1/p_*) + (1/q_*)$ and p_* is taken from the definition of $C(X, p, \kappa)$.

Proof. Let $1 = \frac{1}{p} + \frac{1}{q}$ and let $a \in H^{1,at}_X(\mathcal{D}^N_0)$ be an atom with associated stopping time ν (like in Definition 2.1) and assume first that $\{\nu < \infty\} = J_0 \in \mathcal{D}^N_0$. For J_0 we choose the sets $\mathcal{K}_i$ like in Definition 5.1. Moreover, we use

$$D_q a := \sum_{I\in\mathcal{D}^N_0} \gamma_I^{\frac{1}{q}} a_I h_I \quad \text{and} \quad a_i := \sum_{I\in\mathcal{K}_i} a_I h_I$$

for $a = \sum_{I\in\mathcal{D}^N_0} a_I h_I$ and

$$\beta_i := \sup\left\{\left\|\sum_{I\in\mathcal{K}_i} \gamma_I^{\frac{1}{q}} a_I h_I\right\|^q_{L^p_X} : \left\|\sum_{I\in\mathcal{K}_i} a_I h_I\right\|_{L^p_X} = 1\right\}.$$

We get that

$$
\begin{aligned}
\|A_1 a\|_{H^1_Y} &\leq \sum_i \|A_1 a_i\|_{H^1_Y} \\
&= \sum_i \|A_p D_q a_i\|_{H^1_Y} \\
&\leq \sum_i |\tau(\mathcal{K}_i)^*|^{\frac{1}{q}} \|A_p D_q a_i\|_{H^p_Y} \\
&\leq \frac{p}{p-1} \sum_i |\tau(\mathcal{K}_i)^*|^{\frac{1}{q}} \|A_p D_q a_i\|_{L^p_Y} \\
&\leq \frac{p}{p-1} \|A_p\| \sum_i |\tau(\mathcal{K}_i)^*|^{\frac{1}{q}} \|D_q a_i\|_{L^p_X} \\
&\leq \frac{p}{p-1} \|A_p\| \sum_i [|\tau(\mathcal{K}_i)^*|\beta_i]^{\frac{1}{q}} \|a_i\|_{L^p_X} \\
&\leq \frac{p}{p-1} \|A_p\| \sum_i [|\tau(\mathcal{K}_i)^*|\beta_i]^{\frac{1}{q}} |\mathcal{K}_i^*|^{\frac{1}{p}-\frac{1}{p_*}} \|a_i\|_{L^{p_*}_X} \\
&\leq \frac{p}{p-1} \|A_p\| \left(\sum_i \left| [|\tau(\mathcal{K}_i)^*|\beta_i]^{\frac{1}{q}} |\mathcal{K}_i^*|^{\frac{1}{p}-\frac{1}{p_*}} \right|^{q_*} \right)^{\frac{1}{q_*}} \\
&\qquad\qquad \left(\sum_i \|a_i\|^{p_*}_{L^{p_*}_X} \right)^{\frac{1}{p_*}}
\end{aligned}
$$

with $1 = \frac{1}{q_*} + \frac{1}{p_*}$. Letting $r := \frac{q}{q_*}$ and $1 = \frac{1}{r} + \frac{1}{s}$ we obtain that

$$
\sum_i \left| [|\tau(\mathcal{K}_i)^*|\beta_i]^{\frac{1}{q}} |\mathcal{K}_i^*|^{\frac{1}{p}-\frac{1}{p_*}} \right|^{q_*} \leq \left(\sum_i [|\tau(\mathcal{K}_i)^*|\beta_i] \right)^{\frac{1}{r}} \left(\sum_i |\mathcal{K}_i^*| \right)^{\frac{1}{s}} \leq \kappa |J_0|
$$

(with the obvious modification for $q = q_*$) and

$$
\begin{aligned}
\|A_1 a\|_{H^1_Y} &\leq \frac{p}{p-1} \kappa^{\frac{1}{q_*}} \|A_p\| |J_0|^{\frac{1}{q_*}} \left(\sum_i \|a_i\|^{p_*}_{L^{p_*}_X} \right)^{\frac{1}{p_*}} \\
&\leq \frac{p}{p-1} \kappa^{1+\frac{1}{q_*}} \|A_p\| |J_0|^{\frac{1}{q_*}} \|a\|_{L^{p_*}_X} \\
&\leq \frac{p}{p-1} \kappa^{1+\frac{1}{q_*}} \|A_p\| |J_0| \|a\|_{L^\infty_X} \\
&\leq \frac{p}{p-1} \kappa^{1+\frac{1}{q_*}} \|A_p\|.
\end{aligned}
$$

It is not difficult to check that any atom $a \in H^{1,at}_X(\mathcal{D}^N_0)$ can be written as finite convex combination of atoms considered in this proof so far. Using this and (2) we end up with

$$\|A_1 a\|_{H^1_Y} \leq \frac{p}{p-1}\kappa^{1+\frac{1}{q_*}}\|A_p\|\|a\|_{H^{1,at}_X} \leq \frac{18p}{p-1}\kappa^{1+\frac{1}{q_*}}\|A_p\|\|a\|_{H^1_X}$$

for all $a \in H^1_X(\mathcal{D}^N_0)$.

■

Now we interpolate between H^1 and L^p:

Lemma 5.5. *Let $1 < q < p < \infty$ and $\frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta}{p}$. If Y is a* UMD*-space, then one has*

$$\begin{aligned}\|A_q : L^q_X(\mathcal{D}^N_0) \to L^q_Y(\mathcal{D}^L_0)\| \\ \leq c\|A_1 : H^1_X(\mathcal{D}^N_0) \to H^1_Y(\mathcal{D}^L_0)\|^{1-\theta}\|A_p : L^p_X(\mathcal{D}^N_0) \to L^p_Y(\mathcal{D}^L_0)\|^{\theta}\end{aligned}$$

where $c > 0$ depends at most on Y, p, and q. In the case $\gamma_I \equiv 1$ the UMD*-property of Y is not needed and $c > 0$ does not depend on Y.*

Proof. Because we work with probability spaces consisting of a finite number of atoms only, we can replace (for simplicity) X and Y by finite dimensional subspaces $E \subseteq X$ and $F \subseteq Y$ such that $S(E) \subseteq F$, where we will see that the constant c can be chosen uniformly for all subspaces E and F. The family $(A_q)_{q\in[1,p]}$ is embedded into an analytic family of operators. Let V denote the vertical strip $V = \{x + it : x \in (0,1), t \in \mathbb{R}\}$ and let

$$J_z(a) := \sum_{I\in\mathcal{D}^N_0} Sa_I \gamma_I^{1-z(1-\frac{1}{p})} h_{\tau(I)}.$$

As $\frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta}{p}$ we have

$$J_\theta = A_q.$$

Since

$$\Re\left(1 - it\left(1-\frac{1}{p}\right)\right) = 1 \qquad \text{and} \qquad \Re\left(1-(1+it)\left(1-\frac{1}{p}\right)\right) = \frac{1}{p},$$

we have

$$\|J_{1+it}(f)\|_{L^p_F(\mathcal{D}^L_0)} \leq 2\mathtt{UMD}_p(Y)\|A_p(f)\|_{L^p_F(\mathcal{D}^L_0)} \tag{6}$$

and

$$\|J_{it}(f)\|_{H^1_F(\mathcal{D}^L_0)} \le c\|A_1(f)\|_{H^1_F(\mathcal{D}^L_0)} \tag{7}$$

for some $c > 0$ depending on Y only. The latter estimate (Y is a UMD-space) is folklore and can be derived in various ways. For example, one can follow [9, Remarque 2]. Following the proof that the complex interpolation method with parameter θ yields an exact interpolation functor of exponent θ, for example presented in [1, Theorem 4.1.2], we get that

$$\begin{aligned}
&\|J_\theta(f)\|_{(H^1_F(\mathcal{D}^L_0),L^p_F(\mathcal{D}^L_0))_\theta}\\
\le\ &\sup_{t\in\mathbb{R}}\|J_{it} : H^1_E(\mathcal{D}^N_0) \to H^1_F(\mathcal{D}^L_0)\|^{1-\theta}\sup_{t\in\mathbb{R}}\|J_{1+it} : L^p_E(\mathcal{D}^N_0) \to L^p_F(\mathcal{D}^L_0)\|^\theta\\
&\qquad\qquad \|f\|_{(H^1_E(\mathcal{D}^N_0),L^p_E(\mathcal{D}^N_0))_\theta}\\
\le\ &c^{1-\theta}(2\mathtt{UMD}_p(Y))^\theta\|A_1 : H^1_E(\mathcal{D}^N_0) \to H^1_F(\mathcal{D}^L_0)\|^{1-\theta}\\
&\qquad\qquad \|A_p : L^p_E(\mathcal{D}^N_0) \to L^p_F(\mathcal{D}^L_0)\|^\theta\|f\|_{(H^1_E(\mathcal{D}^N_0),L^p_E(\mathcal{D}^N_0))_\theta}
\end{aligned}$$

where $(Z_0, Z_1)_\theta$ denotes the interpolation space obtained by the complex method as in [1, p. 88]. Using

$$(H^1_E(\mathcal{D}^N_0), L^p_E(\mathcal{D}^N_0)_\theta = L^q_E(\mathcal{D}^N_0) \quad \text{and} \quad (H^1_F(\mathcal{D}^L_0), L^p_F(\mathcal{D}^L_0)_\theta = L^q_F(\mathcal{D}^L_0) \tag{8}$$

with multiplicative constants not depending on (N, L, X, Y) we arrive at our assertion. In the case $\gamma_I = 1$ we have $J_{it} = A_1$ and $J_{1+it} = A_p$ so that the UMD-property in (6) and (7) is not needed. The equivalences (8) are folklore, see [3, p. 334]. One can deduce them via the real interpolation method by exploiting $(H^1_Z(\mathcal{D}^M_0), L^r_Z(\mathcal{D}^M_0))_{\eta,s} = L^s_Z(\mathcal{D}^M_0)$ for $\eta \in (0,1)$, $r, s \in (1,\infty)$ with $(1/s) = 1 - \eta + (\eta/r)$, $Z \in \{E, F\}$, and $M \ge 0$, where the multiplicative constants in the norm estimates depend on (η, r, s) only (see [16] and the references therein), and the connection between the real and complex interpolation method presented in the second statement of [1, Theorem 4.7.2], where we use that the proof for the first inclusion works as well with $\theta_0 = 0$, $p_0 = 1$, and $(\overline{A})_{\theta_0,p_0}$ replaced by A_0.

■

Corollary 5.6. *Let $\tau : \mathcal{D} \to \mathcal{D}$ be a bijection such that there is an $A \geq 1$ with*

$$\frac{1}{A}[\![\mathcal{E}]\!] \leq [\![\tau(\mathcal{E})]\!] \leq A[\![\mathcal{E}]\!] \tag{9}$$

for all non-empty $\mathcal{E} \subseteq \mathcal{D}$. Furthermore, let X be a UMD*-space, $\gamma_I := |I|/|\tau(I)|$, and $1 < q < p < \infty$. Then the boundedness of*

$$\mathrm{Id}_X \otimes T_{p,\tau} : L^p_{X,0} \to L^p_X$$

implies the boundedness of

$$\mathrm{Id}_X \otimes T_{q,\tau} : L^q_{X,0} \to L^q_X.$$

In case of $|\tau(I)| = |I|$ the UMD*-property is not needed.*

Proof. (a) For all $N \geq 0$ we choose $L_N \geq 0$ such that

$$\tau(\mathcal{D}_0^N) \subseteq \mathcal{D}_0^{L_N}.$$

Then we can consider the restrictions $\tau_N : \mathcal{D}_0^N \to \mathcal{D}_0^{L_N}$ for $N \geq 0$. According to Example 5.3 the property $C(X, p, \kappa)$ for some $\kappa > 0$ is satisfied uniformly in N. Applying Lemma 5.5 and Theorem 5.4 gives that

$$\begin{aligned}
& \|T_{q,\tau_N} : L^q_X(\mathcal{D}_0^N) \to L^q_X(\mathcal{D}_0^{L_N})\| \\
\leq\ & c_{(5.5)} \|T_{1,\tau_N} : H^1_X(\mathcal{D}_0^N) \to H^1_X(\mathcal{D}_0^{L_N})\|^{1-\theta} \\
& \qquad \|T_{p,\tau_N} : L^p_X(\mathcal{D}_0^N) \to L^p_X(\mathcal{D}_0^{L_N})\|^{\theta} \\
\leq\ & c_{(5.5)} \left(\frac{18p}{p-1}\kappa^{1+\frac{1}{q_*}}\right)^{1-\theta} \|T_{p,\tau_N} : L^p_X(\mathcal{D}_0^N) \to L^p_X(\mathcal{D}_0^{L_N})\| \\
=:\ & c\|T_{p,\tau_N} : L^p_X(\mathcal{D}_0^N) \to L^p_X(\mathcal{D}_0^{L_N})\| \\
\leq\ & c\|T_{p,\tau} : L^p_{X,0} \to L^p_X\|.
\end{aligned}$$

(b) Now we consider a strictly increasing sequence of integers $B_N \geq 1$ such that

$$\tau(\mathcal{D}_0^{B_N}) \supseteq \mathcal{D}_0^N.$$

For $a = \sum_{I \in \mathcal{D}} a_I h_I$, where $(a_I)_{I \in \mathcal{D}} \subseteq X$ is finitely supported, we get

$$\|T_{q,\tau} a\|_{L^q_X} \ = \ \sup_N \|E(T_{q,\tau} a | \mathcal{F}_N)\|_{L^q_X}$$

$$
\begin{aligned}
&= \sup_N \|E(T_{q,\tau_{B_N}} a_{B_N}|\mathcal{F}_N)\|_{L^q_X} \\
&\leq \sup_N \|T_{q,\tau_{B_N}} a_{B_N}\|_{L^q_X} \\
&\leq \sup_N \|T_{q,\tau_{B_N}} : L^q_X(\mathcal{D}_0^{B_N}) \to L^q_X(\mathcal{D}_0^{L_{B_N}})\| \|a_{B_N}\|_{L^q_X(\mathcal{D}_0^{B_N})} \\
&\leq c\|T_{p,\tau} : L^p_{X,0} \to L^p_X\| \|a\|_{L^q_{X,0}}
\end{aligned}
$$

where $\tau_{B_N} : \mathcal{D}_0^{B_N} \to \mathcal{D}_0^{L_{B_N}}$ is the restriction of τ considered in (a) and a_{B_N} the restriction of a to $\mathcal{D}_0^{B_N}$.

■

Modifying slightly the first step in the proof of Corollary 5.6 we can remove the assumption that X is a UMD-space in Corollary 4.3:

Corollary 5.7. *Let X be a Banach space and let $\tau : \mathcal{D} \to \mathcal{D}$ be a permutation such that $|\tau(I)| = |I|$. Then, for $1 < q < p < 2$, the boundedness of*

$$\mathrm{Id}_X \otimes T_\tau : L^p_{X,0} \to L^p_X$$

implies the boundedness of $\mathrm{Id}_X \otimes T_\tau : L^q_{X,0} \to L^q_X$.

Proof. Our assumption implies $\gamma_I = 1$ and that τ satisfies Semenov's condition with some $\kappa \in [1, \infty)$. By Example 5.2 the restrictions $\tau_N : \mathcal{D}_0^N \to \mathcal{D}_0^N$ satisfy condition $c(X, p, \kappa)$ for all $p \in (1, \infty)$. Now we can follow the proof of Corollary 5.6 with $L_N = B_N = N$ and $\gamma_I = 1$ so that the UMD-property in Lemma 5.5 is not needed.

■

We close with an extrapolation theorem for rearrangement operators that are isomorphisms on $L^p_{X,0}$. For real valued rearrangements, i.e. when $X = \mathbb{R}$, the following theorem is well known. It can be obtained by different methods, the most direct route [10] going via Pisier's re-norming in L^p.

Theorem 5.8. *Let $\tau : \mathcal{D} \to \mathcal{D}$ be a bijection and $\gamma_I := |I|/|\tau(I)|$. Assume that X is a UMD-space. If there exists a $p \in (1, \infty)$ with $p \neq 2$ such that*

$$\|\mathrm{Id}_X \otimes T_{p,\tau} : L^p_{X,0} \to L^p_X\| \cdot \|\mathrm{Id}_X \otimes T_{p,\tau^{-1}} : L^p_{X,0} \to L^p_X\| < \infty, \tag{10}$$

then for each $q \in (1, \infty)$ one has that

$$\|\mathrm{Id}_X \otimes T_{q,\tau} : L^q_{X,0} \to L^q_X\| \cdot \|\mathrm{Id}_X \otimes T_{q,\tau^{-1}} : L^q_{X,0} \to L^q_X\| < \infty. \tag{11}$$

PROOF. (a) First we observe that our assumption implies that (10) holds for $X = \mathbb{C}$ and $X = \mathbb{R}$. If $p \in (2, \infty)$, then [11, Theorems 2 and 3] imply condition (9). In case of $p \in (1, 2)$ duality implies (10) for $X = \mathbb{R}$ and p replaced by the conjugate index $p' \in (2, \infty)$. Hence we have (9) as well.

(b) From Corollary 5.6 and (a) we immediately get (11) for $q \in (1, p)$.

(c) Let $q \in (p, \infty)$. It is easy to see that for a bijection $\sigma : \mathcal{D} \to \mathcal{D}$ and $r \in (1, \infty)$ the boundedness of

$$\| \operatorname{Id}_X \otimes T_{r,\sigma} : L^r_{X,0} \to L^r_X \| \quad \text{and} \quad \| \operatorname{Id}_{X'} \otimes T_{r',\sigma^{-1}} : L^{r'}_{X',0} \to L^{r'}_{X'} \|$$

are equivalent to each other where $1 = (1/r) + (1/r')$ (note, that X is in particular reflexive because of the UMD-property). Using this observation our assumption (10) holds for p' and X' and the conclusion for $q' \in (1, p')$ and X'. By duality we come back to q and X.

■

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