Quasiregular mappings to generalized manifolds

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Abstract

We establish the basic analytic and geometric properties of quasiregular maps $f : \Omega \to X$, where $\Omega \subset \mathbb{R}^n$ is a domain and where X is a generalized n-manifold with a suitably controlled geometry. Generalizing the classical Väisälä and Poletsky inequalities, our main theorem shows that the path family method applies to these maps.

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1 Introduction

According to the so-called metric definition, quasiregular maps between Euclidean domains are branched coverings for which the distortion of infinitesimal balls is uniformly controlled, see Definition 4.1 below. Quasiregular maps can be equally defined by using the following analytic definition. A map $f : \Omega \to \mathbb{R}^n$ is K-quasiregular if it belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and if there exists $K \geqslant 1$ such that $|Df(x)|^n \leqslant KJ_f(x)$ almost everywhere. Martio, Rickman and Väisälä proved the equivalence of these two definitions in [20]. Their proof depends on Reshetnyak's fundamental theorem which shows that the analytic definition implies the branched covering property, see [25, I Theorem 4.1]. The theory of quasiregular maps is now well developed, see [16], [25]. It relies on varying methods, among which the geometric path family method is of utmost importance.

More recently, parts of this theory have been extended to cover a subclass of quasiregular maps, called mappings of bounded length distortion (BLD), between non-smooth spaces, see [14], [15]. The motivation for such extensions was to use BLD-maps to study problems in geometric topology. Also, in [12] the theory of quasiconformal maps; that is, the theory of quasiregular homeomorphisms, between general metric spaces with controlled geometry was developed. This theory also initiated a new way of looking at weakly differentiable maps between non-smooth spaces.

In this paper we develop a basic theory for quasiregular maps from Euclidean domains to metric spaces X. The target space is assumed to have locally controlled geometry (see Section 2), but, unlike in the case of homeomorphisms, a topological assumption is also required. We assume that X is an oriented generalized n-manifold as defined in Section 3. This class includes oriented topological manifolds as well as some interesting non-manifolds. The assumption guarantees the existence of a satisfactory degree theory, see Section 3.

Next, we discuss two interesting situations where our theory applies. First, in [29] Semmes constructed a three-dimensional space $X_1 \subset \mathbb{R}^4$ that has the properties assumed in Sections 2 and 3 below. More precisely, X_1 satisfies the following properties:

1. X_1 is Ahlfors 3-regular and LLC (see Section 2),

2. there exist quasiconformal homeomorphisms (and hence also non-injective quasiregular maps) from $\mathbb{R}^3$ to X_1, and

3. X_1 is not locally bi-Lipschitz equivalent to $\mathbb{R}^3$.

The third property stipulates that smooth or Lipschitz analysis cannot be applied to study maps with target X_1; thus, genuinely non-smooth methods are needed.

Next, we consider $X_2 = \{[(x,t)]\} = \Sigma H^3$; that is, the suspension of the Poincaré homology 3-sphere H^3. In this case X_2 can be equipped with a (polyhedral) metric satisfying the assumptions given in Section 2. The space X_2 is a generalized 4-manifold, yet not a manifold. Also, there are quasiregular maps from S^4, and consequently from $\mathbb{R}^4$ to X_2. Indeed, there is a natural surjective extension $\tilde{f} : S^4 \to X_2$, $\tilde{f}(x,t) = (f(x),t)$ of the covering map $f : S^3 \to H^3$ so that $\tilde{f}$ is quasiregular. Here the branch set of $\mathcal{B}_{\tilde{f}} \subset S^4$, the set of points where $\tilde{f}$ does not define a local homeomorphism, has precisely two points in it. This is in sharp contrast to the classical case; for Euclidean quasiregular maps the image of the branch set is either empty, or has positive $(n-2)$-measure, see [25, III Proposition 5.3].

The present work has three main purposes. First, we hope that our results will reveal which spaces X can receive quasiregular maps from $\mathbb{R}^n$. As a particular class of target spaces one can consider the "excellent package" constructions of Semmes [28]. For related questions see [3] and [14]. Second, our methods also provide a new way in studying some of the fundamental properties of Euclidean quasiregular maps. In particular, our treatment of the regularity properties of the mappings in question, as well as the properties of the branch set, does not use any differentiable structure on the target space. For instance, we give a new way of showing that the branch set of a quasiregular map has to be small. Our method also solves a problem of Bonk and Heinonen [4, Remark 3.5] concerning the size of the branch set of a Euclidean quasiregular map, see Theorems 9.7 and 9.8 below. Finally, the tools developed here can be applied to build a general theory for quasiregular maps $f : \Omega \to X$. The examples above show that such development may reveal new and interesting phenomena. In the last section we use our basic theory to give several equivalent characterizations of quasiregular maps $f : \mathbb{R}^n \to X$ with polynomial growth. Specifically, we show in Theorem 12.1 below that these maps behave like Euclidean maps in many ways when the global geometry of X is controlled, compare [11].

Our main result, Theorem 11.1, generalizes the classical Väisälä and Poletsky inequalities to our setting. Before we are able to prove Theorem 11.1 we have to establish several analytic and geometric properties of quasiregular maps that are of independent interest. While the basic philosophy used to prove Theorem 11.1 is similar to the one in the Euclidean case, we primarily have to find completely different methods. Here we benefit from recent work on analysis in metric measure spaces ([1], [12], [13], [18]), particularly from the method used in [1] to study regularity properties under mild assumptions, see Theorem 8.1.

2 Metric measure spaces

We will consider rectifiably connected metric measure spaces $X = (X, d, \mathcal{H}^n)$. Here and in what follows $\mathcal{H}^n$ stands for the Hausdorff n-measure. We denote the open ball with center x and radius r by $B(x, r)$, and write $S(x, r) = \{y \in X : d(x, y) = r\}$. The closure of a set $E \subset X$ is denoted by $\overline{E}$. Clearly, we have $\overline{B}(x, r) \subset B(x, r) \cup S(x, r)$, and the inclusion can be strict in general. We also use the notation $CB = B(x, Cr)$ when $B = B(x, r)$. We assume throughout this paper that X enjoys the following three properties:

1. X is proper; that is, every closed ball in X is compact,
2. X is locally Ahlfors n-regular, and
3. X is locally linearly locally connected (LLC).

A space X is said to be locally Ahlfors n-regular if there exists $\tau \geqslant 1$ such that for $x \in X$ and $0 < r < 1$ we have

$$\tau^{-1} r^n \leqslant \mathcal{H}^n(B(x, r)) \leqslant \tau r^n. \tag{2.1}$$

Moreover, X is locally LLC if there exists $\theta \geqslant 1$ such that for $x \in X$ and $0 < r < 1$ we have

(i) every two points a, $b \in B(x, r)$ can be joined in $B(x, \theta r)$, and

(ii) every two points a, $b \in X \setminus \overline{B}(x, r)$ can be joined in $X \setminus \overline{B}(x, \theta^{-1} r)$.

Here by joining a and b in B we mean that there exists a path $\gamma : [0, 1] \to B$ with $\gamma(0) = a$, $\gamma(1) = b$.

We denote the x-component of a ball $B(x, r)$ by $D(x, r)$. Then it follows from the assumptions 3. and 2. that for every ball $B(x, r)$ with $r < 1$,

$$B(x, \theta^{-1} r) \subset D(x, r) \subset B(x, r), \quad \text{and} \tag{2.2}$$

$$\tau^{-1} \theta^{-n} r^n \leqslant \mathcal{H}^n(D(x, r)) \leqslant \tau r^n. \tag{2.3}$$

Next we recall the definition of the modulus of a given path family. Let Γ be a family of paths in X. We define, for $1 \leqslant p < \infty$, the p-modulus $M_p\Gamma$ by

$$M_p\Gamma = \inf_{\rho \in T_\Gamma} \int_X \rho^p \, d\mathcal{H}^n,$$

where T_Γ is the set of all Borel functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho \, ds \geqslant 1 \quad \text{for every locally rectifiable } \gamma \in \Gamma.$$

We call such ρ a test function for Γ. Moreover, we denote M_n by M. We will use the fact that the modulus is subadditive: if (Γ_i) is a sequence of path families, then

$$M_p\Big(\bigcup_{i=1}^{\infty}\Gamma_i\Big) \leqslant \sum_{i=1}^{\infty} M_p\Gamma_i, \tag{2.4}$$

see [25, II Proposition 1.5].

If E and F are two disjoint sets in X, and $\Omega \subset X$ a domain, we denote by $\Delta(E,F;\Omega)$ the family of all paths joining E and F in $\overline{\Omega}$. The upper mass bound $\mathcal{H}^n(B(x,s)) \leqslant \tau s^n$ gives the following estimate (cf. [9, Lemma 7.18]). If $0 < r < R \leqslant 1$, then

$$M\Delta(\overline{B}(x,r), X\setminus B(x,R); B(x,R)) \leqslant C(\tau,n)\Big(\log\frac{R}{r}\Big)^{1-n}. \tag{2.5}$$

We will use the fact that balls in $\mathbb{R}^n$ (equipped with the standard metric and the Lebesgue n-measure) have the so-called Loewner property (cf. [9, Chapter 8]). Precisely, suppose that $E, F \subset B \subset \mathbb{R}^n$ are disjoint, compact and connected sets in a ball B. Denote

$$\psi(E,F) = \min\{\operatorname{diam} E, \operatorname{diam} F\}.$$

Then

$$M\Delta(E,F;B) \geqslant C(n)\log\Big(\frac{\operatorname{dist}(E,F)+\psi(E,F)}{\operatorname{dist}(E,F)}\Big), \tag{2.6}$$

where $C(n) > 0$ only depends on n.

Finally, we define Newtonian spaces which generalize Sobolev spaces to maps defined in metric measure spaces. For simplicity, we assume that X is as above, although the definition is useful also in more general settings (cf. [13]).

Suppose that $\Omega \subset X$, and that $u : \Omega \to \mathbb{R}$ is a measurable function. We call a Borel function $\rho : \Omega \to [0,\infty]$ an upper gradient of u if

$$\int_{\gamma_{x,y}} \rho\, ds \geqslant |u(x) - u(y)| \tag{2.7}$$

for every x, $y \in \Omega$ and every locally rectifiable path $\gamma_{x,y}$ joining x and y in Ω. Moreover, ρ is a p-weak upper gradient of u if (2.7) holds except for a path family Γ (not depending on x, y) of zero p-modulus.

We say that $u : \Omega \to \mathbb{R}$ belongs to the Newtonian space $N^{1,p}(\Omega)$ for some $1 \leqslant p < \infty$ if $u \in L^p(\Omega)$ and if there exists a p-weak, p-integrable upper gradient ρ of u. Moreover, $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(B)$ for every ball $B \subset\subset \Omega$. The notation $B \subset\subset \Omega$ means that $\overline{B} \subset \Omega$.

Now suppose $f : \Omega \to Y$, where $\Omega \subset X$ and $Y = (Y, d')$ is a metric space. We say that f belongs to the Newtonian space $N^{1,p}(\Omega, Y)$ if

$$f_{y_0} = d'(f(\cdot), y_0) \in L^p(\Omega) \quad \text{for every } y_0 \in Y,$$

and if there exists a Borel function $\rho : \Omega \to [0, \infty]$ in $L^p(\Omega)$ so that ρ is a p-weak upper gradient of f_{y_0} for every $y_0 \in Y$. Local Newtonian spaces are then defined the same way as above.

3 Discrete and open maps to generalized manifolds

We assume that X satisfies the assumptions 1.-3. given in Section 2. In order to be able to define quasiregular maps from Euclidean domains to X and develop their properties, we need to have degree calculus available. Such a calculus exists if we assume that X is an oriented topological manifold. In this paper we will use a weaker topological assumption, given below, which is satisfied by some interesting non-manifolds which fit into our framework. We follow [14, I.1- I.3] and [25, I.4 and II.3].

We denote by $H_c^*(X)$ the Alexander-Spanier cohomology groups of X with compact supports and coefficients in $\mathbb{Z}$. We then call X an oriented generalized n-manifold if it satisfies the following:

(a) the local cohomology groups of X are equivalent to $\mathbb{Z}$ in degree n and zero in degree $n-1$, and

(b) X is oriented, i.e. $H_c^n(X) \simeq \mathbb{Z}$ and an orientation is chosen.

It is worth recalling that our definition of a generalized manifold is not standard, see [14] for further comments. Now we assume that X is an oriented generalized n-manifold, and $f : \Omega \to X$ is a continuous map from a domain $\Omega \subset \mathbb{R}^n$. Then we can define the local degree $\mu(y, f, U)$ for any domain $U \subset\subset \Omega$ and $y \in X \setminus f(\partial U)$. In our notation $U \subset\subset \Omega$ means that the closure $\overline{U}$ of U is compact and satisfies $\overline{U} \subset \Omega$. Moreover, the degree satisfies the usual basic properties, see [14, I.2]. We call f sense-preserving if $\mu(y, f, U) > 0$ whenever $U \subset\subset \Omega$ and $y \in f(U) \setminus f(\partial U)$. Furthermore, f is discrete if $f^{-1}(y)$ is a discrete set in Ω for every $y \in X$, and open if $f(U) \subset X$ is open whenever $U \subset \Omega$ is open.

For the rest of this section we suppose that $f : \Omega \to X$ is a continuous, sense-preserving, discrete and open map. If $B(y, s) \subset X$ is a ball and $f(x) = y$ for some $x \in \Omega$, then we denote the x-component of $f^{-1}(D(y, s))$ by

$$U(x, s) = U(x, f, s).$$

We call a domain $U \subset\subset \Omega$ a normal domain for f if $f(\partial U) = \partial f(U)$. By openness of f, $\partial f(U) \subset f(\partial U)$ always holds. Moreover, a normal domain

U is called a normal neighborhood of $x \in \Omega$ if $U \cap f^{-1}(f(x)) = \{x\}$. If U is a normal domain, then we define $\mu(f, U) = \mu(y, f, U)$ for some $y \in f(U)$. This is well-defined because $\mu(y, f, U) = \mu(v, f, U)$ whenever y, $v \in f(U)$.

We next give some basic facts concerning normal domains and normal neighborhoods. The proofs are identical to the ones given in [25, I.4 and II Lemma 4.1].

Lemma 3.1. *Suppose that $V \subset X$ is a domain and $U \subset\subset \Omega$ a component of $f^{-1}(V)$. Then U is a normal domain and $f(U) = V$.*

Lemma 3.2. *Suppose that U is a normal domain. If $E \subset f(U)$ is a compact and connected set, then f maps every component of $f^{-1}(E) \cap U$ onto E. Moreover, if $F \subset f(U)$ is compact, then $f^{-1}(F) \cap U$ is compact.*

Lemma 3.3. *For each $x \in \Omega$ there exists $\sigma_x > 0$ so that, for every $0 < s < \sigma_x$, the following hold:*

(i) $U(x, s)$ *is a normal neighborhood of* x,

(ii) $\operatorname{diam} U(x, s) \to 0$ *as* $s \to 0$,

(iii) $U(x, s) = U(x, \sigma_x) \cap f^{-1}(D(f(x), s))$,

(iv) $\partial U(x, s) = U(x, \sigma_x) \cap f^{-1}(\partial D(f(x), s))$.

The local index $i(x, f)$ for $x \in \Omega$ can be defined as follows: if U is a normal neighborhood of x, then

$$i(x, f) = \mu(f, U); \tag{3.1}$$

$i(x, f)$ does not depend on the normal neighborhood U. The branch set $\mathcal{B}_f$ of f is the set of points $x \in \Omega$ for which $i(x, f) > 1$. Thus f defines a local homeomorphism at every $x \in \Omega \setminus \mathcal{B}_f$.

One of the most important tools in the geometric theory of quasiregular maps is Väisälä's inequality for the conformal modulus of path families. In order to be able to effectively use this tool one needs the path lifting property as follows [25, II.3], [14, I.3.3]. Suppose that $f : \Omega \to X$ is a continuous, sense-preserving, discrete and open map as above, $\beta : [a, b) \to X$ a path, and $x \in f^{-1}(\beta(a))$. We call a path $\alpha : [a, c) \to \Omega$, $c \leqslant b$, a maximal f-lifting of β starting at x if $\alpha(a) = x$, $f \circ \alpha = \beta_{|[a,c)}$, and if the following holds: if $c < c' \leqslant b$, then there does not exist a path $\alpha' : [a, c') \to \Omega$ such that $\alpha = \alpha'_{|[a,c)}$ and $f \circ \alpha' = \beta_{|[a,c')}$.

Now let $x_1, \ldots x_k$ be k different points of $f^{-1}(\beta(a))$ so that

$$m = \sum_{j=1}^{k} i(x_j, f).$$

We say that the sequence $\alpha_1, \ldots, \alpha_m$ of paths is a maximal sequence of f-liftings of β starting at the points $x_1, \ldots, x_k$ if each α_L is a maximal f-lifting of β, so that

$$\begin{aligned} \operatorname{card}\{L : \alpha_L(a) = x_j\} &= i(x_j, f), \quad 1 \leqslant j \leqslant k, \text{ and} \\ \operatorname{card}\{L : \alpha_L(t) = x\} &\leqslant i(x, f) \quad \text{for each } x \in \Omega \text{ and } t. \end{aligned}$$

The existence of maximal sequences of f-liftings for Euclidean maps is proved in [25, II Theorem 3.2], and the proof generalizes to our setting.

Theorem 3.4. *Let $\beta : [a, b) \to X$ be a path, and let $x_1, \ldots, x_k$ be distinct points in $f^{-1}(\beta(a))$. Then β has a maximal sequence of f-liftings starting at $x_1, \ldots, x_k$.*

4 Quasiregular maps

In this section we give a definition of quasiregular mappings that take domains in Euclidean spaces into X, where $X = (X, d, \mathcal{H}^n)$ is an oriented generalized n-manifold satisfying the assumptions 1.-3. given in Section 2. The definition below corresponds to the so-called metric definition of Euclidean quasiregular mappings, see [25, II.6].

Suppose that $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a domain, and $f : \Omega \to X$ a continuous map. For $x \in \Omega$, define

$$H_f(x) = \limsup_{r \to 0} H(x, r) = \limsup_{r \to 0} \frac{L(x, r)}{l(x, r)},$$

where

$$L(x, r) = \max_{y \in \overline{B}(x, r)} d(f(y), f(x)),$$

and

$$l(x, r) = \min_{y \in S(x, r)} d(f(y), f(x)).$$

Notice that $L(x, r)$ does not need to equal $\max_{y \in S(x,r)} d(f(y), f(x))$ in general, even for homeomorphisms. Also, if f is not one-to-one, then $l(x, r)$ may equal zero.

Definition 4.1. We call a continuous map $f : \Omega \to X$ quasiregular if f is constant, or

- f is sense-preserving, discrete and open,
- there exists $H < \infty$ so that $H_f(x) \leqslant H$ for almost every $x \in \Omega$, and
- $H_f(x) < \infty$ for every $x \in \Omega$.

Below by data we mean n, H, τ and θ. In the theory of Euclidean quasiregular maps, the concept of monotonicity is often very useful. We call a map $f : \Omega \to X$ monotone if the following holds true with $T = 1$:

$$\operatorname{diam} f(B(x,r)) \leqslant T \operatorname{diam} f(S(x,r)) \tag{4.1}$$

for every $B(x,r) \subset\subset \Omega$. If there exists $1 \leqslant T < \infty$ such that (4.1) is satisfied, then the mapping f is said to be pseudomonotone. A continuous and open map f into $\mathbb{R}^n$ is monotone: indeed, the openness of f implies $\partial f(G) \subset f(\partial G)$ for every $G \subset\subset \Omega$. We next show that the LLC-assumption on X implies local pseudomonotonicity for maps with values in X.

Lemma 4.2. *Suppose that $f : \Omega \to X$ is continuous and open. Then for every $x \in \Omega$ there exists a radius $R = R(x) > 0$ so that $f_{|B(x,R)}$ is T-pseudomonotone, where $T \geqslant 1$ only depends on data.*

Proof. Fix $x \in \Omega$. Since f is open, it is non-constant. Thus, by the continuity of f, there exist a radius $0 < R < 1$ and a point $p \in \Omega$ so that $B(x,R) \subset\subset \Omega$, $5 \operatorname{diam} f(B(x,R)) < 1$ and

$$f(p) \notin B(f(x), 5 \operatorname{diam} f(B(x,R))). \tag{4.2}$$

Now fix $B = B(y,r) \subset B(x,R)$, and a point $w \in \partial f(B)$. Recall that the openness of f implies $\partial f(B) \subset f(\partial B)$, and thus

$$\operatorname{diam} f(\partial B) \geqslant \operatorname{diam} \partial f(B) =: A.$$

Then,

$$\operatorname{diam} f(B) = \sup_{u,v \in f(B)} d(u,v) \leqslant \sup_{u,v \in f(B)} (d(u,w) + d(v,w)) = 2 \sup_{v \in f(B)} d(v,w).$$

Hence the proof is complete if we can show that, given $v \in f(B)$, $d(v,w) \leqslant CA$, where $C \geqslant 1$ does not depend on y or r.

Fix a constant $1 \leqslant M \leqslant 2 \operatorname{diam} f(B)/A$ so that

$$v \notin B(w, MA). \tag{4.3}$$

Since $w \in \partial f(B) \subset f(\partial B)$, there exists a point $w' \in \partial B$ so that $f(w') = w$. Thus, by (4.2), there exists a path

$$\gamma : [0,1] \to (\Omega \setminus \overline{B}) \cup \{w'\}$$

so that $\gamma(0) = p$ and $\gamma(1) = w'$. We conclude that there exists a point $u \in f(|\gamma|)$ so that

$$u \in B(w, 3 \operatorname{diam} f(B)) \setminus (B(w, MA) \cup f(\overline{B})). \tag{4.4}$$

By (4.3), (4.4), and the local LLC-property, there exists a path

$$\alpha : [0,1] \to X \setminus B(w, \theta^{-1}MA)$$

so that $\alpha(0) = v \in f(B)$ and $\alpha(1) = u \notin \overline{f(B)}$. But then $M \leqslant \theta$, since otherwise $|\alpha| \cap \partial f(B) = \emptyset$, which contradicts the connectedness of α. We conclude that $v \in B(w, MA)$ for every $M \geqslant \theta$, i.e. $d(v,w) \leqslant 2\theta A$. The proof is complete. $\square$

5 Analytic properties of quasiregular maps

In this section we show that quasiregular maps $f : \Omega \to X$ belong to the Newtonian space $N^{1,n}_{\mathrm{loc}}(\Omega, X)$. It then follows from Lemma 4.2 and results in [13] and [18] that f maps sets of zero Lebesgue measure to sets of zero $\mathcal{H}^n$-measure, i.e. that f satisfies Condition *(N)*, and that the change of variables formula holds for f. We define the volume derivative J_f of f by

$$J_f(x) = \limsup_{r \to 0} \frac{\mathcal{H}^n\big(f(B(x,r))\big)}{\alpha_n r^n},$$

where α_n is the volume of the unit ball in $\mathbb{R}^n$. Similarly, we define

$$L_f(x) = \limsup_{r \to 0} \frac{L(x,r)}{r} = \limsup_{r \to 0} \frac{\max_{y \in \overline{B}(x,r)} d(f(y), f(x))}{r}.$$

Moreover, for $A \subset \Omega$, we denote

$$N(y, f, A) = \mathrm{card}(f^{-1}(y) \cap A).$$

Theorem 5.1. *Suppose that $f : \Omega \to X$ is quasiregular. Then $f \in N^{1,n}_{\mathrm{loc}}(\Omega, X)$.*

Proof. Fix $x_0 \in X$. Our mapping is continuous and thus we only have to show that f_{x_0} has a locally n-integrable n-weak upper gradient that does not depend on x_0. It suffices to consider f_{x_0} in a fixed domain $U \subset\subset \Omega$. We will first show that f_{x_0} has a p-integrable p-weak upper gradient for $1 < p < n$. For that we choose and fix $1 < p < n$ and a small $\epsilon > 0$.

Without loss of generality, we may assume that $H > 1$. We denote, for $j \in \mathbb{N}$,

$$A_j = \{x \in U : H^j < H_f(x) \leqslant H^{j+1}\},$$

and

$$A_0 = \{x \in U : 1 \leqslant H_f(x) \leqslant H\}.$$

Furthermore, for each j we fix a constant $\epsilon_j \in (0, \epsilon)$, to be chosen later. Notice that, by our definition of quasiregularity, $U = \cup_{j=0}^{\infty} A_j$, and $|A_j| = 0$ for all $j \in \mathbb{N}$. For each $j \in \mathbb{N} \cup \{0\}$, we choose $x \in A_j$ and a radius $0 < s_x < \epsilon_j$ such that

$$H(x, s_x) \leqslant 2H^{j+1}.$$

To simplify the notation, we write $s_{x_i} = s_i$. By the Besicovitch covering theorem [21, Theorem 2.7], we find a countable collection

$$\mathcal{B} = \{B(x_i, s_i)\} = \{B_i\}$$

of balls such that $5^{-1}B_i \cap 5^{-1}B_k = \emptyset$ whenever B_i, $B_k \in \mathcal{B}$ and $i \neq k$, and

$$1 = \chi_U(x) \leqslant \sum_{B \in \mathcal{B}} \chi_B(x) \leqslant C \quad \text{for every } x \in U. \tag{5.1}$$

Denote by $\mathcal{B}_j$ the subcollection of the balls $B(x_i, s_i) \in \mathcal{B}$ for which $x_i \in A_j$.

We define

$$\rho_\epsilon(x) = \sum_i \frac{L(x_i, s_i)}{s_i} \chi_{2B_i}(x).$$

Let Γ_ϵ denote all rectifiable paths $\gamma : [0,1] \to U$ such that $\operatorname{diam}|\gamma| > \epsilon$. Towards showing that f_{x_0} has a p-integrable p-weak upper gradient we first prove that, for all paths $\gamma \in \Gamma_\epsilon$, we have

$$|f_{x_0}(\gamma(1)) - f_{x_0}(\gamma(0))| \leqslant 2 \int_\gamma \rho_\epsilon \, ds \tag{5.2}$$

with

$$\int_U [\rho_\epsilon(x)]^p \, dx \leqslant M \tag{5.3}$$

where the constant M is independent of ϵ. For that, we fix $\gamma \in \Gamma_\epsilon$. By definition,

$$|f_{x_0}(\gamma(1)) - f_{x_0}(\gamma(0))| = |d(f(\gamma(1)), x_0) - d(f(\gamma(0)), x_0)|. \tag{5.4}$$

Notice that if $B_i \cap |\gamma| \neq \emptyset$, then $\mathcal{H}^1(|\gamma| \cap 2B_i) \geqslant s_i$. Hence (5.4) is bounded from above by

$$\begin{aligned} d(f(\gamma(1)), f(\gamma(0))) &\leqslant \sum_{B_i \cap |\gamma| \neq \emptyset} \operatorname{diam} f(B_i) \leqslant 2 \sum_{B_i \cap |\gamma| \neq \emptyset} s_i \frac{L(x_i, s_i)}{s_i} \\ &\leqslant 2 \sum_{B_i \cap |\gamma| \neq \emptyset} \int_{|\gamma|} \frac{L(x_i, s_i)}{s_i} \chi_{2B_i}(x) \, d\mathcal{H}^1(x) \leqslant 2 \int_\gamma \rho_\epsilon \, ds, \end{aligned}$$

as claimed at (5.2).

On the other hand, we have

$$\begin{aligned} \int_U [\rho_\epsilon(x)]^p \, dx &= \int_U \Big[\sum_{B_i \in \mathcal{B}} \frac{L(x_i, s_i)}{s_i} \chi_{2B_i}(x) \Big]^p dx \\ &\leqslant C \int_U \Big[\sum_{B_i \in \mathcal{B}} \frac{L(x_i, s_i)}{s_i} \chi_{1/5B_i}(x) \Big]^p dx. \end{aligned} \tag{5.5}$$

Here the inequality follows if one uses the $L^p - L^{\frac{p}{p-1}}$ duality and the boundedness of an appropriate restricted maximal function, see [2] (notice that if we replace p by 1, then the inequality (5.5) is obvious). Therefore, by the pairwise disjointness of the balls $\frac{1}{5}B_i$, we have

$$
\begin{aligned}
\int_U [\rho_\epsilon(x)]^p \, dx &\leqslant C \int_U \sum_{B_i \in \mathcal{B}} \left[\frac{L(x_i, s_i)}{s_i}\right]^p \chi_{1/5B_i}(x)\, dx \\
&\leqslant C \sum_{B_i \in \mathcal{B}} s_i^{n-p} [L(x_i, s_i)]^p \\
&= C \sum_{j=0}^{\infty} \sum_{B_i \in \mathcal{B}_j} \left[\frac{L(x_i, s_i)}{H^{2j}}\right]^p s_i^{n-p} H^{2pj} \qquad (5.6) \\
&\leqslant C \sum_{j=0}^{\infty} \sum_{B_i \in \mathcal{B}_j} [l(x_i, s_i)]^p s_i^{n-p} H^{2pj},
\end{aligned}
$$

where the last inequality follows from our choice of the sets A_j and the radii s_i. By Hölder's inequality, and Ahlfors regularity, the right hand term is smaller than

$$
\begin{aligned}
&C \sum_{j=0}^{\infty} H^{2pj} \Big(\sum_{B_i \in \mathcal{B}_j} l(x_i, s_i)^n \Big)^{\frac{p}{n}} \Big(\sum_{B_i \in \mathcal{B}_j} s_i^n \Big)^{\frac{n-p}{n}} \qquad (5.7) \\
\leqslant\ &C \sum_{j=0}^{\infty} H^{2pj} \Big(\sum_{B_i \in \mathcal{B}_j} \mathcal{H}^n(f(B_i)) \Big)^{\frac{p}{n}} \Big(\sum_{B_i \in \mathcal{B}_j} s_i^n \Big)^{\frac{n-p}{n}}.
\end{aligned}
$$

Since $|A_j| = 0$ for all $j \in \mathbb{N}$, we can choose the constants ϵ_j to be small enough, so that

$$
\Big(\sum_{B_i \in \mathcal{B}_j} s_i^n \Big)^{\frac{n-p}{n}} \leqslant H^{-2pj} 2^{-j}. \qquad (5.8)
$$

Also, by (5.1), and since $N(y, f, U) \leqslant N < \infty$ for all $y \in X$, we have

$$
\begin{aligned}
\Big(\sum_{B_i \in \mathcal{B}_j} \mathcal{H}^n(f(B_i)) \Big)^{\frac{p}{n}} &= \Big(\int_X \sum_{B_i \in \mathcal{B}_j} \chi_{f(B_i)}\, dx \Big)^{\frac{p}{n}} \\
&\leqslant (CN)^{\frac{p}{n}} \mathcal{H}^n(f(U))^{\frac{p}{n}}. \qquad (5.9)
\end{aligned}
$$

By combining (5.5), (5.6), (5.7), (5.8) and (5.9), we conclude the auxiliary claim (5.3). Having this, the weak compactness of L^p guarantees that there is $\rho \in L^p(U)$ and a sequence of ϵ_κ's where $\kappa = 1, 2, ...$ that decreases to zero such that ρ is an L^p-weak limit of ρ_{ϵ_κ}. Here we needed the fact that $p > 1$.

To simplify the notation, we write $\rho_{\epsilon_\kappa} = \rho_\kappa$. Then (5.2) gives

$$|f_{x_0}(\gamma(1)) - f_{x_0}(\gamma(0))| \leqslant 2 \int_\gamma \rho_\kappa \, ds \tag{5.10}$$

for each $\kappa \geqslant \ell$ when $\gamma \in \Gamma_{\epsilon_\ell}$. By Mazur's lemma, we find functions $\tilde{\rho}_\kappa$, each a convex combination of $\rho_\kappa, \rho_{\kappa+1}, \ldots$ such that the sequence $\{\tilde{\rho}_\kappa\}$ converges to ρ in $L^p(U)$. Now (5.10) also holds with ρ_κ replaced with $\tilde{\rho}_\kappa$ for every $\kappa \geqslant \ell$. By Fuglede's lemma (see [13, Lemma 3.4]), (5.10) holds for ρ for p-almost every $\gamma \in \cup_\ell \Gamma_{\epsilon_\ell}$. Since U was an arbitrary domain compactly contained in Ω, the above arguments give that f_{x_0} has a locally p-integrable p-weak upper gradient.

For the weak gradient ∇f_{x_0} we have, by absolute continuity, quasiregularity of f, and Ahlfors regularity,

$$\begin{aligned} |\nabla f_{x_0}(x)|^n &\leqslant L_f(x)^n = \limsup_{r \to 0} \frac{L(x,r)^n}{r^n} \leqslant H^n \limsup_{r \to 0} \frac{l(x,r)^n}{r^n} \\ &\leqslant C \limsup_{r \to 0} \frac{\mathcal{H}^n(f(B(x,r)))}{r^n} = C J_f(x) \end{aligned} \tag{5.11}$$

for almost every $x \in U$. Since $f_{|U}$ is N-to-1, the mapping $E \mapsto \mathcal{H}^n(f(E))$ is an N-additive set function from the Borel subsets of U to $\mathbb{R}$. Hence, by [20, Lemma 2.3],

$$\int_U |\nabla f_{x_0}(x)|^n \, dx \leqslant C \int_U J_f(x) \, dx \leqslant C N \mathcal{H}^n(fU).$$

We conclude that f_{x_0} belongs to the (classical) Sobolev space $W^{1,n}(U)$, and by Fuglede's lemma and (5.11), L_f is an n-weak upper gradient of f_{x_0}. Finally, we conclude that $f \in N^{1,n}_{\mathrm{loc}}(\Omega, X)$. □

The following theorem shows the usefulness of pseudomonotonicity. The theorem is proved in greater generality in [13, Theorem 7.2], using the ideas in [19].

Theorem 5.2. *Suppose that $f : \Omega \to X$ is a pseudomonotone map in $N^{1,n}_{\mathrm{loc}}(\Omega, X)$. Then f satisfies Condition (N).*

Notice that Condition *(N)* is a local property. Hence, combining Theorem 5.1, Theorem 5.2 and Lemma 4.2 yields

Corollary 5.3. *Suppose that $f : \Omega \to X$ is quasiregular. Then f satisfies Condition (N).*

Now we consider the change of variables formula. For Lipschitz maps the formula follows from a theorem of Kirchheim [18, Corollary 8].

Theorem 5.4. *Suppose that $A \subset \mathbb{R}^n$ is a measurable set and $f : A \to X$ a Lipschitz map. Then there exists a measurable function $\mathcal{J}(\cdot, f) : A \to [0, \infty]$ so that*

$$\int_A u(f(x))\mathcal{J}(x, f)\, dx = \int_X u(y)N(y, f, A)\, d\mathcal{H}^n(y) \tag{5.12}$$

for every measurable $u : X \to [0, \infty]$, whenever one of the integrals is finite.

In order to prove (5.12) for quasiregular maps, we need the following property of maps in Newtonian spaces, cf. [13, Proposition 4.6].

Theorem 5.5. *Suppose that $f : \Omega \to X$ belongs to $N^{1,p}_{\mathrm{loc}}(\Omega, X)$ for some $1 \leqslant p < \infty$. Then there exists a partition*

$$\Omega = \Big(\bigcup_{k=1}^{\infty} A_k\Big) \cup E, \quad E, A_1, A_2, \ldots \text{ pairwise disjoint,}$$

so that $f_{|A_k}$ is k-Lipschitz for every $k \in \mathbb{N}$, and $|E| = 0$. In particular, if f satisfies Condition (N), then $f(\Omega)$ is countably n-rectifiable.

Corollary 5.6. *Suppose that $f : \Omega \to X$ is quasiregular. Then* (5.12) *is valid, and $f(\Omega)$ is countably n-rectifiable.*

Proof. By Theorems 5.1, 5.4 and 5.5, (5.12) holds true for the restrictions $f_{|A_k}$ given in Theorem 5.5. On the other hand, by Corollary 5.3 and Theorem 5.5, $|E| = \mathcal{H}^n(f(E)) = 0$, and thus (5.12) follows by applying the formula for each $f_{|A_k}$ and summing both sides over k. The second claim follows from Corollary 5.3 and Theorem 5.5. □

Remark 5.7. By the definition of J_f and (5.12), $J_f(x) \leqslant \mathcal{J}(x, f)$ for almost every $x \in \Omega$. We will later show that f defines a local homeomorphism at almost every $x \in \Omega$, which implies that in fact $J_f(x) = \mathcal{J}(x, f)$ almost everywhere.

6 The K_O-inequality

Modulus inequalities for path families play a fundamental role in the theory of Euclidean quasiregular maps. In this section we show the validity of the so-called K_O-inequality, which controls the modulus of a path family by the modulus of its image under f, in our setting. The interested reader finds more about related questions in [6]. The proof of a reverse inequality, Väisälä's inequality, will be much more involved. In fact, one of our main goals in this paper is to develop enough theory so that we are able to generalize Väisälä's inequality to our setting.

First we prove a distortion inequality which corresponds to the inequality used to give the analytic definition of quasiregular maps ([25, I.1]).

Lemma 6.1. *Suppose that $f : \Omega \to X$ is a quasiregular map. Then*

$$L_f(x)^n \leqslant C J_f(x)$$

for almost every $x \in \Omega$, where $C > 0$ only depends on data.

Proof. We have

$$\begin{aligned} L_f(x)^n &= \limsup_{r\to 0} \frac{L(x,r)^n}{r^n} \\ &\leqslant \limsup_{r\to 0} \frac{\alpha_n L(x,r)^n}{\mathcal{H}^n(f(B(x,r)))} \cdot \limsup_{r\to 0} \frac{\mathcal{H}^n(f(B(x,r)))}{\alpha_n r^n} \end{aligned}$$

whenever the right hand side is well-defined. Here, the last term equals $J_f(x)$. On the other hand,

$$f(B(x,r)) \supset D(f(x), l(x,r)),$$

and so

$$\mathcal{H}^n(f(B(x,r))) \geqslant \mathcal{H}^n(D(f(x), l(x,r))).$$

Hence,

$$\begin{aligned} &\limsup_{r\to 0} \frac{\alpha_n L(x,r)^n}{\mathcal{H}^n(f(B(x,r)))} \\ \leqslant\ &\limsup_{r\to 0} \frac{L(x,r)^n}{l(x,r)^n} \cdot \limsup_{r\to 0} \frac{\alpha_n l(x,r)^n}{\mathcal{H}^n(D(f(x), l(x,r)))}. \end{aligned} \tag{6.1}$$

By the definition of quasiregularity, the first term on the right is bounded by H^n for almost every $x \in \Omega$. Also, by (2.3), the second term is bounded by $\tau\theta^n$. The proof is complete. □

Our next theorem generalizes the K_O-inequality to the current setting.

Theorem 6.2. *Suppose that $f : \Omega \to X$ is a non-constant quasiregular map, and Γ a path family in Ω. If ρ is a test function for $f(\Gamma)$, then*

$$M\Gamma \leqslant K \int_X N(y, f, \Omega)\rho(y)^n \, d\mathcal{H}^n(y), .$$

where $K > 0$ only depends on data. In particular, if $N(y, f, \Omega) \leqslant N < \infty$ for every $y \in X$, then

$$M\Gamma \leqslant NKMf(\Gamma).$$

Proof. Suppose that $\rho : X \to [0, \infty]$ is a test function for $f(\Gamma)$. Define $\rho' : \Omega \to [0, \infty]$,

$$\rho'(x) = (\rho \circ f)(x) L_f(x).$$

By Theorem 5.1 and the definition of a Newtonian space, f is absolutely continuous on every $\gamma \in \Gamma' = \Gamma \setminus \Gamma_0$, where $M\Gamma_0 = 0$. Thus,

$$\int_\gamma \rho' \, ds = \int_\gamma (\rho \circ f) L_f \, ds \geqslant \int_{f\circ\gamma} \rho \, ds \geqslant 1$$

for every $\gamma \in \Gamma'$. Since ρ' is a Borel function, we conclude that ρ' is a test function for Γ.

Now, we apply the change of variables formula. By Lemma 6.1 and Remark 5.7, $L_f(x)^n \leqslant C\mathcal{J}(x,f)$ for almost every $x \in \Omega$. Thus, by Corollary 5.6 and the subadditivity property of modulus (2.4),

$$\begin{aligned} M\Gamma &= M\Gamma' \leqslant \int_\Omega \rho'(x)^n \, dx = \int_\Omega \rho(f(x))^n L_f(x)^n \, dx \\ &\leqslant K \int_\Omega \rho(f(x))^n \mathcal{J}(x,f) \, dx = K \int_X N(y,f,\Omega)\rho(y)^n \, d\mathcal{H}^n(y). \end{aligned}$$

The proof is complete.

□

7 Local dilatation bounds

In this section we estimate the dilatation H_f and the inverse dilatation H_f^*, to be defined later. First we prove that H_f is locally bounded at every $x \in \Omega$. If $U \subset\subset \Omega$, we denote

$$N(f,U) = \max_{y \in X} N(y,f,U).$$

Theorem 7.1. *Suppose that $f : \Omega \to X$ is a non-constant quasiregular map. Then*

$$H_f(x) \leqslant H' \quad \textit{for every } x \in \Omega\,,$$

where H' only depends on data and on $i(x,f)$.

Proof. Fix $x \in \Omega$ and a radius $R < \sigma_x$, where σ_x is given in Lemma 3.3, such that

$$B(f(x), 10\theta^3 R) \subset f(\Omega).$$

By Lemmas 3.2 and 3.3 (i) and (iv), $\partial U(x,R)$ is compact, and $x \notin \partial U(x,R)$. Hence we may choose $r > 0$ to be small enough so that $B(x,2r) \subset U(x,R)$ and $B(x,2r) \cap \partial U(x,R) = \emptyset$.

By the definition of $l(x,r)$, and since $U(x,R)$ is a normal neighborhood of x, we can choose

$$0 < s < 2l(x,r) \tag{7.1}$$

so that

$$f(S(x,r)) \cap B(f(x),s) \neq \emptyset.$$

By (2.2),

$$B(f(x), s) \subset D(f(x), \theta s),$$

and so

$$f(S(x,r)) \cap D(f(x), \theta s) \neq \emptyset. \tag{7.2}$$

If necessary, we can further assume that r is so small that $\theta s < R$. By (7.2) and Lemma 3.3 (iii), $U(x, \theta s)$ intersects $S(x,r)$. Thus the same is true also for the x-component E of $\overline{U}(x, \theta s) \cap \overline{B}(x, 2r)$. We conclude that E is a compact and connected set in $\overline{B}(x, 2r)$ satisfying $x \in E$ and $S(x,r) \cap E \neq \emptyset$.

By Lemma 4.2, there exists a constant $M > 0$, only depending on data, so that the following holds: for every $x \in \Omega$ there exists $R_0 > 0$ so that for every $r < R_0$,

$$L(x,r) \leqslant \operatorname{diam} f(B(x,r)) \leqslant M \operatorname{diam} f(S(x,r)). \tag{7.3}$$

We choose r to be small enough so that (7.3) is satisfied. Then we are able to find a radius $t > 0$ so that

$$2Mt > L(x,r), \tag{7.4}$$

and

$$v \in f(S(x,r)) \cap (X \setminus \overline{B}(f(x), t)) \neq \emptyset. \tag{7.5}$$

By (7.1), (7.3) and (7.4), the theorem is proved provided we show that $t \leqslant Cs$, where $C \geqslant 1$ only depends on data and $i(x,f)$. We assume that $t > 2\theta^2 s$. Also, we assume that r is small enough so that $t < R$.

Choose a point $w \in \partial D(f(x), R)$. Then, by the LLC-condition, we can join v and w in (v is as in (7.5))

$$X \setminus B(f(x), \theta^{-1} t)$$

by a path γ. Choose $v' \in S(x,r)$ so that $f(v') = v$, and denote by $|\gamma'|$ the v'-component of $f^{-1}(|\gamma|)$. By Lemma 3.3 (iv), $|\gamma'|$ intersects $\partial U(x,R)$. Thus, the v'-component F of $|\gamma'| \cap \overline{B}(x, 2r)$ is a compact and connected set which intersects both $S(x,r)$ and $S(x, 2r)$.

Now we are ready to apply Theorem 6.2. We denote

$$\Gamma = \Delta(E, F; B(x, 3r)).$$

Since $\operatorname{diam} E, \operatorname{diam} F \geqslant r$ and both intersect $S(x,r)$, (2.6) yields

$$M\Gamma \geqslant C(n). \tag{7.6}$$

Also, since

$$f(E) \subset B(f(x), \theta s)$$

and

$$f(F) \cap B(f(x), \theta^{-1}t) = \emptyset,$$

(7.7)
$$Mf(\Gamma) \leqslant M\Gamma',$$

where

$$\Gamma' = \Delta\big(\overline{B}(f(x), \theta s), X \setminus B(f(x), \theta^{-1}t); B(f(x), \theta^{-1}t)\big).$$

By Ahlfors regularity and (2.5),

(7.8)
$$M\Gamma' \leqslant C\Big(\log \frac{t}{\theta^2 s}\Big)^{1-n}.$$

Combining Theorem 6.2 with (7.6), (7.7), (7.8) and (3.1) yields

$$C(n) \leqslant Ki(x, f)\Big(\log \frac{t}{\theta^2 s}\Big)^{1-n};$$

that is,

$$t \leqslant C(n, H, \tau, \theta, i(x, f))s.$$

The proof is complete. □

Next we prove a similar bound for the so-called inverse dilatation. Suppose that $f : \Omega \to X$ is a non-constant quasiregular map, and $U(x, r)$ is a normal neighborhood of a point $x \in \Omega$. We define

$$\begin{aligned} L^*(x, r) &= \max_{y \in \overline{U}(x,r)} |y - x|, \\ l^*(x, r) &= \min_{y \in X \setminus U(x,r)} |y - x|, \quad \text{and} \\ H^*_f(x) &= \limsup_{r \to 0} H^*(x, r) = \limsup_{r \to 0} \frac{L^*(x, r)}{l^*(x, r)}. \end{aligned}$$

We will state and use a modulus estimate which slightly generalizes (2.6). See [30, Theorem 10.12] for the proof.

Theorem 7.2. *Suppose that $0 < r < R$, and that $E, F \subset B(x, R) \subset \mathbb{R}^n$ are disjoint sets such that*

$$E \cap S(x, s) \neq \emptyset,\ F \cap S(x, s) \neq \emptyset$$

for every $s \in (r, R)$. Then

$$M\Delta(E, F; B(x, R)) \geqslant C(n) \log \frac{R}{r},$$

where $C(n) > 0$ only depends on n.

Now, we are ready to prove a bound for H^*_f.

Theorem 7.3. *Suppose that $f:\Omega\to X$ is a non-constant quasiregular map. Then*

$$H_f^*(x)\leqslant H^*$$

for every $x\in\Omega$, where $H^*\geqslant 1$ only depends on data and on $i(x,f)$.

Proof. Fix a radius $\delta_x>0$ such that $\delta_x<\sigma_x$, where σ_x is given in Lemma 3.3, and such that

$$H(x,s)\leqslant 2H' \tag{7.9}$$

for every $s>0$ for which $L(x,s)\leqslant \delta_x/(10\theta)$; this choice can be made by Theorem 7.1. Moreover, we can choose $r>0$ to be small enough such that

$$2L(x,L^*(x,r))<\delta_x, \tag{7.10}$$

and $2L^*(x,r)<R(x)$, where $R(x)>0$ is as in Lemma 4.2. Denote $L^*=L^*(x,r)$ and $l^*=l^*(x,r)$. By Lemma 4.2,

$$\operatorname{diam} f(S(x,t))\geqslant \frac{\operatorname{diam} f(B(x,t))}{M}\geqslant \frac{\operatorname{diam} f(B(x,l^*))}{M}\geqslant \frac{l(x,l^*)}{M}$$

for every $t\in(l^*,L^*)$, where $M>0$ only depends on data. Thus, for each such t we can choose points $a_t,b_t\in S(x,t)$ so that

$$d(f(a_t),f(b_t))=\operatorname{diam} f(S(x,t))\geqslant \frac{l(x,l^*)}{M}. \tag{7.11}$$

Denote

$$E=\{a_t : t\in(l^*,L^*)\},\quad F=\{b_t : t\in(l^*,L^*)\},$$

and

$$\Gamma=\Delta\big(E,F;B(x,L^*)\big).$$

Then

$$M\Gamma\geqslant C(n)\log\frac{L^*}{l^*} \tag{7.12}$$

by Theorem 7.2. Also,

$$f(|\gamma|)\subset B(f(x),L(x,L^*)),$$

and, by (7.11),

$$\mathcal{H}^1(f(|\gamma|))\geqslant \frac{l(x,l^*)}{M}$$

for every $\gamma\in\Gamma$. Thus $\rho:X\to[0,\infty]$,

$$\rho(z)=\chi_{B(f(x),L(x,L^*))}(z)\frac{M}{l(x,l^*)}$$

is a test function for $f(\Gamma)$. Hence

$$M f(\Gamma) \leqslant \frac{M^n \mathcal{H}^n(B(f(x), L(x, L^*)))}{l(x, l^*)^n} \leqslant \frac{M^n \tau L(x, L^*)^n}{l(x, l^*)^n} \tag{7.13}$$

by Ahlfors regularity. Combining (7.12), (7.13), Theorem 6.2 and (3.1) shows that the theorem is proved if we can show that

$$L(x, L^*) \leqslant C\, l(x, l^*),$$

where $C \geqslant 1$ only depends on data.

By (7.9),

$$L(x, L^*) \leqslant 2H' l(x, L^*), \quad L(x, l^*) \leqslant 2H' l(x, l^*),$$

and so it suffices to show that

$$l(x, L^*) \leqslant L(x, l^*). \tag{7.14}$$

By the definition of L^*, there exists a point

$$v \in S(x, L^*) \cap \partial U(x, r).$$

By Lemma 3.3 (iv), and since $\partial D(f(x), r) \subset S(f(x), r)$,

$$f(v) \in \partial D(f(x), r) \subset S(f(x), r).$$

Thus $l(x, L^*) \leqslant r$. Similarly, there exists a point

$$w \in S(x, l^*) \cap \partial U(x, r),$$

and so Lemma 3.3 (iv) implies

$$f(w) \in \partial D(f(x), r) \subset S(f(x), r).$$

Thus

$$L(x, l^*) \geqslant d(f(w), f(x)) = r.$$

The proof is complete. $\square$

8 Generalized local inverse map

Let $f : \Omega \to X$ be a non-constant quasiregular map, and suppose that U is a normal domain so that $f(U) = V$. We denote $m = \mu(f, U)$, and define an "inverse" mapping $g_U : V \to \mathbb{R}^n$ of f by setting

$$g_U(y) = \frac{1}{m} \sum_{x \in f^{-1}(y) \cap U} i(x, f) x. \tag{8.1}$$

Theorem 8.1. *For f and U as above, $g_U \in N^{1,n}(V, \mathbb{R}^n)$.*

The basic idea behind the proof of Theorem 8.1 is the same as in Theorem 5.1. However, the lack of the Besicovitch covering theorem on the target space X causes some difficulties. To overcome these difficulties, we recall a covering theorem by Balogh, Koskela and Rogovin, [1, Lemma 2.2].

Lemma 8.2. *Let $\mathcal{B}$ be a collection of balls $B(x, r_x)$ (open or closed) with $x \in V$ in a metric space X such that*

$$V \subset \cup_{B \in \mathcal{B}} B \subset\subset X.$$

Then there exists a finite or countable sequence $B_\nu = B(x_\nu, r_\nu) \in \mathcal{B}$ with the following properties:

1. *$V \subset \cup_\nu B_\nu$*
2. *if $\nu \neq \kappa$, $\nu, \kappa \in \mathbb{N}$, then either*
   - *$x_\nu \in X \setminus B_\kappa$ and $B_\kappa \setminus B_\nu \neq \emptyset$, or*
   - *$x_\kappa \in X \setminus B_\nu$ and $B_\nu \setminus B_\kappa \neq \emptyset$*
3. *$B(x_\nu, \frac{1}{3} r_\nu) \cap B(x_\kappa, \frac{1}{3} r_\kappa) = \emptyset$ when $\nu \neq \kappa$*

Proof of Theorem 8.1. First we notice that the mapping g_U is continuous. The proof of this fact is essentially the same as in the case $X = \mathbb{R}^n$, and thus is omitted here, see [25, Proof of Lemma II 5.3].

Fix $\epsilon > 0$. Combining Lemma 8.2 and Theorem 7.3 with the fact that the mapping f is discrete and open, we find a finite or countable family of balls, denoted by $\mathcal{B} = \{B(y_j, r_j)\} = \{B_j\}$, with the following properties: $r_j < \epsilon$ for every $j \in \mathbb{N}$, $V \subset \cup B_j$, and, if we denote

$$\{x_j^{i_j}\} = f^{-1}(y_j) \cap U, \quad i_j = 1, ..., k_j \leqslant m,$$

then for every $j \in \mathbb{N}$ and i_j, we have

1. $B_j \subset V$,
2. the $x_j^{i_j}$-components $U_j^{i_j}$ of $f^{-1}(B_j)$ are pairwise disjoint,
3. $H^*(x_j^{i_j}, s) \leqslant H^*$ for all $s \leqslant r_j$, and
4. the family $\mathcal{B}$ satisfies the properties 1., 2., and 3. of Lemma 8.2.

We start our proof with showing the following auxiliary estimate:

$$|g_U(z) - g_U(y_j)| \leqslant \max \left\{ L^*(x_j^{i_j}, r_j) \ : \ 1 \leqslant i_j \leqslant k_j \right\} \tag{8.2}$$

for every $j \in \mathbb{N}$ and all $z \in B_j$. To this end, fix $j \in \mathbb{N}$, let $z \in B_j$, and denote

$$\bigcup_{i_j=1}^{k_j} \bigcup_{\nu=1}^{k(z,i_j)} \{p_\nu^{i_j}\} = f^{-1}(z) \cap U,$$

where

$$k(z, i_j) = \operatorname{card}\{f^{-1}(z) \cap U_j^{i_j}\}.$$

Since

$$\sum_{\nu=1}^{k(z,i_j)} i(p_\nu^{i_j}, f) = i(x_j^{i_j}, f)$$

for each $i_j = 1, ..., k_j$, we have

$$\begin{aligned}
|g_U(z) - g_U(y_j)| &= \frac{1}{m}\left|\sum_{i_j=1}^{k_j} \sum_{\nu=1}^{k(z,i_j)} i(p_\nu^{i_j}, f)\, p_\nu^{i_j} - \sum_{i_j=1}^{k_j} i(x_j^{i_j}, f)\, x_j^{i_j}\right| \\
&= \frac{1}{m}\left|\sum_{i_j=1}^{k_j} \sum_{\nu=1}^{k(z,i_j)} i(p_\nu^{i_j}, f)\, \left[p_\nu^{i_j} - x_j^{i_j}\right]\right| \\
&\leqslant \frac{1}{m} \sum_{i_j=1}^{k_j} i(x_j^{i_j}, f) \max_{\nu, i_j} \left|p_\nu^{i_j} - x_j^{i_j}\right| \\
&\leqslant \max_{i_j} L^*(x_j^{i_j}, r_j), \qquad (8.3)
\end{aligned}$$

which implies (8.2).

We define

$$\rho_\epsilon(y) = 2 \sum_j \max_{1 \leqslant i_j \leqslant k_j} \frac{L^*(x_j^{i_j}, r_j)}{r_j} \chi_{2B_j}(y). \tag{8.4}$$

Let Γ_ϵ denote all rectifiable paths $\gamma : [0, 1] \to V$ with $\operatorname{diam} |\gamma| > \epsilon$. Towards showing that g_U has an n-integrable n-upper gradient we first prove that, for all paths $\gamma \in \Gamma_\epsilon$, we have

$$|g_U(\gamma(1)) - g_U(\gamma(0))| \leqslant \int_\gamma \rho_\epsilon \, ds \tag{8.5}$$

with

$$\int_V [\rho_\epsilon(y)]^n \, d\mathcal{H}^n(y) \leqslant C\, |U| \tag{8.6}$$

where the constant C does not depend on ϵ. For proving these, we fix $\gamma \in \Gamma_\epsilon$.

Notice that if $|\gamma| \cap B_j \neq \emptyset$, then $\mathcal{H}^1(|\gamma| \cap 2B_j) \geqslant r_j$. Combining this with (8.2), we have

$$\begin{aligned}\int_\gamma \rho_\epsilon \, ds &\geqslant 2 \sum_{|\gamma| \cap B_j \neq \emptyset} \max_{1 \leqslant i_j \leqslant k_j} L^*(x_j^{i_j}, r_j) \\ &\geqslant |g_U(\gamma(1)) - g_U(\gamma(0))| \end{aligned} \tag{8.7}$$

and (8.5) follows. For showing (8.6), we first compute

$$\begin{aligned}\int_V [\rho_\epsilon(y)]^n \, d\mathcal{H}^n(y) &= 2^n \int_V \Big[\sum_j \max_{1 \leqslant i \leqslant k_j} \frac{L^*(x_j^{i_j}, r_j)}{r_j} \chi_{2B_j}(y) \Big]^n d\mathcal{H}^n(y) \\ &\leqslant C \int_V \Big[\sum_j \max_{1 \leqslant i \leqslant k_j} \frac{L^*(x_j^{i_j}, r_j)}{r_j} \chi_{\frac{1}{3}B_j}(y) \Big]^n d\mathcal{H}^n(y). \end{aligned} \tag{8.8}$$

Here the last inequality follows if one uses the $L^n - L^{\frac{n}{n-1}}$ duality and the boundedness of an appropriate restricted maximal function, see [2] (notice that if we replace n by 1, then the inequality is obvious).

Now, using the fact that the balls $\frac{1}{3}B_j$ are pairwise disjoint, and Ahlfors regularity, we have

$$\int_V [\rho_\epsilon(y)]^n \, d\mathcal{H}^n(y) \leqslant C \sum_j \big[\max_{1 \leqslant i_j \leqslant k_j} L^*(x_j^{i_j}, r_j) \big]^n. \tag{8.9}$$

To simplify writing we denote

$$\max_{1 \leqslant i_j \leqslant k_j} L^*(x_j^{i_j}, r_j) = L^*(x_j^{i_j^\circ}, r_j) = L_j^*.$$

In order to show that the right hand side of (8.9) converges, we will argue the same way as in [1]. Precisely, we claim the following.

Claim $\Diamond$: Let $c = 10(H^*)^2$. Then the balls $B(x_j^{i_j^\circ}, L_j^*/c)$ are pairwise disjoint.

Proof of Claim $\Diamond$. By the symmetry of property 2. of Lemma 8.2, we may assume that $y_j \notin B_\nu$, and that there exists $z \in B_\nu \setminus B_j$. Therefore, we have

1. $x_j^{i_j^\circ} \notin U_\nu^{i_\nu^\circ}$
2. There exists $v \in U_\nu^{i_\nu^\circ} \setminus U_j^{i_j^\circ}$.

We write $x_j = x_j^{i_j^\circ}$, $x_\nu = x_\nu^{i_\nu^\circ}$, $U_j = U_j^{i_j^\circ}$ and $U_\nu = U_\nu^{i_\nu^\circ}$. The first part implies that

$$|x_j - x_\nu| > \frac{L^*(x_\nu, r_\nu)}{H^*}. \tag{8.10}$$

Therefore, if we suppose that

$$|x_j - x_\nu| > \frac{L^*(x_j, r_j)}{2H^*},$$

then the claim follows with $c = 5H^*$. Hence, we may now assume that

$$|x_j - x_\nu| \leqslant \frac{L^*(x_j, r_j)}{2H^*}. \tag{8.11}$$

The second property above implies

$$|v - x_j| > \frac{L^*(x_j, r_j)}{H^*}.$$

Combining this with our assumption (8.11) we have

$$L^*(x_\nu, r_\nu) \geqslant \frac{L^*(x_j, r_j)}{2H^*}.$$

This together with (8.10) implies

$$|x_j - x_\nu| \geqslant \frac{L^*(x_j, r_j)}{2(H^*)^2}.$$

Therefore, Claim $\Diamond$ follows from this and (8.10).

Finally, the second auxiliary inequality (8.6) follows. Indeed, combining (8.9) with Claim $\Diamond$, we have

$$\int_V [\rho_\epsilon(y)]^n \; d\mathcal{H}^n(y) \leqslant C \sum_j (L_j^*)^n \leqslant C|U|. \tag{8.12}$$

In order to remove the restriction diam $|\gamma| > \epsilon$ we argue as in Theorem 5.1. The weak compactness of L^n guarantees that there is $\rho \in L^n(V)$, and a sequence of ϵ_κ's, where $\kappa = 1, 2, ...$, that decreases to zero such that ρ is an L^n-weak limit of ρ_{ϵ_κ}. To simplify the notation, we write $\rho_{\epsilon_\kappa} = \rho_\kappa$. Then (8.5) gives

$$|g_U(\gamma(1)) - g_U(\gamma(0))| \leqslant \int_\gamma \rho_\kappa \, ds \tag{8.13}$$

for each $\kappa \geqslant \ell$ when $\gamma \in \Gamma_{\epsilon_\ell}$. By Mazur's lemma, we find functions $\tilde{\rho}_\kappa$, each a convex combination of $\rho_\kappa, \rho_{\kappa+1}, ...$, such that the sequence $\{\tilde{\rho}_\kappa\}$ converges to ρ in $L^n(V)$. Now (8.13) also holds with ρ_κ replaced with $\tilde{\rho}_\kappa$ for every $\kappa \geqslant \ell$. By Fuglede's lemma (see [13], Lemma 3.4), (8.13) holds for ρ for n-almost every $\gamma \in \cup_\ell \Gamma_{\epsilon_\ell}$. Thus g_U has an n-integrable n-weak upper gradient and, therefore, due to the continuity of g_U this finishes the proof of Theorem 8.1. $\square$

We have also shown the following estimate for the integral of the function ρ.

Remark 8.3. The inverse mapping g_U has an n-weak upper gradient ρ which satisfies the following estimate

$$\int_V [\rho(y)]^n \, d\mathcal{H}^n(y) \leqslant C|U| \,. \tag{8.14}$$

9 The size of $f(\mathcal{B}_f)$

In this section we show that if $f : \Omega \to X$ is a quasiregular map, then $\mathcal{H}^n(f(\mathcal{B}_f)) = 0$. It is true that also $|\mathcal{B}_f| = 0$ for a non-constant map, but to prove this we need the results given in the following sections. Our method of proof is new even in the case of Euclidean quasiregular maps. In fact, the method gives a stronger result, and yields an answer to a problem of Bonk and Heinonen [4, Remark 3.5] on the size of the branch set of a Euclidean quasiregular map, see Theorem 9.8 below.

First we observe that the proof of Theorem 7.3 gives a stronger result than stated.

Lemma 9.1. *Suppose that $f : \Omega \to X$ is a non-constant quasiregular map. Then for every $x \in \Omega$ and $\eta \geqslant 1$ there exist a radius $\delta_{x,\eta} > 0$ and a constant $\kappa > 0$, only depending on data, $i(x, f)$ and η, so that*

$$L^*(x, \eta r) \leqslant \kappa l^*(x, r)$$

for every $r < \delta_{x,\eta}$.

Proof. The proof goes exactly like the proof of Theorem 7.3, with the following exception: instead of $L^* = L^*(x, r)$ consider $\tilde{L}^* = L^*(x, \eta r)$. Then, instead of (7.14),

$$l(x, \tilde{L}^*) \leqslant \eta L(x, l^*)$$

holds. We leave the details to the reader. □

We will prove a porosity estimate for $f(\mathcal{B}_f)$. This estimate will then imply that $\mathcal{H}^n(f(\mathcal{B}_f)) = 0$. We call a set $E \subset X$ λ-porous, $0 < \lambda < 1$, if

$$\liminf_{r \to 0} r^{-1} \sup\{t > 0 : \text{ there exists } B(y, t) \subset B(x, r) \setminus E\} \geqslant \lambda$$

for every $x \in E$. Porosities imply size estimates as follows. See [5, Lemma 3.12], or [17] for the proof.

Lemma 9.2. *Suppose that X is an Ahlfors n-regular metric space, and $E \subset X$. If E is λ-porous for some $\lambda \in (0, 1)$, then*

$$\dim_{\mathcal{H}} E \leqslant n - \epsilon,$$

where $\epsilon > 0$ only depends on n, λ and the Ahlfors regularity constant of X, quantitatively.

In order to prove porosity estimates we use a method similar to a one used in [24]. The following topological result turns out to be very convenient. See [22, Theorem 2] for the proof (the statement there is a bit different, but the proof applies).

Lemma 9.3. *Suppose that $f : \Omega \to X$ is a continuous, sense-preserving, discrete and open map. Assume that $x \in \mathcal{B}_f$, and that $U(x, r)$ is a normal neighborhood of x. Then there exists a point $y \in \partial D(f(x), r)$ so that*

$$\operatorname{diam} f^{-1}(y) \geqslant l^*(x, r).$$

Now we are ready to prove the main result of this section. For a given $f : \Omega \to X$, and $m \geqslant 2$, we denote

$$\mathcal{B}_m = \{x \in \mathcal{B}_f : i(x, f) = m\}.$$

Theorem 9.4. *Suppose that $f : \Omega \to X$ is a quasiregular map and $m \geqslant 2$. Then for every $x_0 \in \mathcal{B}_m$ there exists a radius $R_0 > 0$ such that the set*

$$f(\mathcal{B}_m \cap U(x_0, R_0))$$

is λ-porous, where $\lambda \in (0, 1)$ and only depends on data and m, quantitatively.

Proof. We choose the radius $R_0 > 0$ to be small enough so that $U(x_0, R_0)$ is a normal neighborhood of x_0. We fix $x \in \mathcal{B}_m \cap U(x_0, R_0)$ and a radius $r > 0$ so that $4\theta r < \min\{\sigma_x, \delta_{x,2\theta}\}$, where σ_x and $\delta_{x,2\theta}$ are as in Lemmas 3.3 and 9.1, respectively. Moreover, we assume that

$$B(f(x), 4\theta r) \subset D(f(x_0), R_0).$$

Our goal is to show that there exists a constant $a > 0$, only depending on data and m, so that

$$B(y, ar) \subset B(f(x), 2\theta r) \setminus f(\mathcal{B}_m \cap U(x_0, R_0)) \tag{9.1}$$

for some $y \in B(f(x), 2\theta r)$.

By Lemmas 3.3 (iii) and 9.3, there exists a point $y \in \partial D(f(x), r)$ so that

$$\operatorname{diam}(f^{-1}(y) \cap U(x, 4\theta r)) \geqslant l^*(x, r). \tag{9.2}$$

Fix $s \in (0, r/2)$. We will show that if s is small enough, then

$$f^{-1}(D(y, s)) \cap U(x, 4\theta r)$$

consists of at least two different components. Suppose that there exists

$$z \in f^{-1}(y) \cap U(x, 4\theta r)$$

so that

$$U(z,s) = f^{-1}(D(y,s)) \cap U(x, 4\theta r).$$

Then, by (9.2),

$$\operatorname{diam} U(z,s) \geqslant l^*(x,r). \tag{9.3}$$

We denote

$$\Gamma = \Delta(U(z,s), \partial U(x, 2\theta r); U(x, 4\theta r)).$$

Then every $\gamma \in f(\Gamma)$ joins $D(y,s)$ and $\partial D(f(x), 2\theta r)$ in $D(f(x), 4\theta r)$. By triangle inequality and (2.2),

$$B(y, r/2) \subset B(f(x), 2r) \subset D(f(x), 2\theta r).$$

Thus

$$Mf(\Gamma) \leqslant M\Gamma^*,$$

where

$$\Gamma^* = \Delta(B(y,s), X \setminus B(y, r/2); D(x_0, R_0)).$$

Then (2.5) yields

$$Mf(\Gamma) \leqslant C(n,\tau)\Big(\log \frac{r}{2s}\Big)^{1-n}. \tag{9.4}$$

On the other hand,

$$\operatorname{diam} \partial U(x, 2\theta r) \geqslant l^*(x,r),$$

and

$$\operatorname{dist}(\partial U(x, 2\theta r), U(z,s)) \leqslant L^*(x, 2\theta r).$$

Hence, (2.6), Lemma 9.1 and (9.3) yield

$$M\Gamma \geqslant C(n) \log \frac{L^*(x, 2\theta r) + l^*(x,r)}{L^*(x, 2\theta r)} \geqslant C, \tag{9.5}$$

where $C > 0$ only depends on data and m. By (9.4), (9.5) and Theorem 6.2 we conclude that $s \geqslant ar$, where $a > 0$ only depends on data and m. We have proved that the set

$$f^{-1}(D(y, ar)) \cap U(x, 4\theta r) \tag{9.6}$$

consists of at least two different components.

Since $U(x_0, R_0)$ is a normal neighborhood of x_0, and $i(x_0, f) = m$, $\mu(w, f, U(x_0, R_0)) = m$ for every $w \in D(f(x_0), R_0)$. Now suppose that (9.1) does not hold when a is chosen as in (9.6). Then there exists a point $v_1 \in U_1 \cap \mathcal{B}_m$, where U_1 is a component of $f^{-1}(D(y, ar))$ in $U(x_0, R_0)$.

By (9.6), there exists another component U_2 of $f^{-1}(D(y, ar))$ in $U(x_0, R_0)$. Moreover, by Lemma 3.1 there exists a point

$$v_2 \in f^{-1}(f(v_1)) \cap U_2.$$

But now

$$m = \mu(f(v_1), f, U(x_0, R_0)) \geqslant i(v_1, f) + i(v_2, f) \geqslant m + 1.$$

This is a contradiction. The proof is complete. $\square$

By combining Lemma 9.2 and Theorem 9.4, we have

Corollary 9.5. *Suppose that $f : \Omega \to X$ is a quasiregular map, and $m \geqslant 2$. Then*

$$\dim_{\mathcal{H}} f(\mathcal{B}_m) < n - \epsilon,$$

where $\epsilon > 0$ only depends on data and m, quantitatively.

Corollary 9.6. *Suppose that $f : \Omega \to X$ is a quasiregular map. Then $\mathcal{H}^n(f(\mathcal{B}_f)) = 0$.*

Proof. For every $m \geqslant 2$ we can cover $\mathcal{B}_m$ by countably many sets $U(x_j^m, R_j^m)$ as in Theorem 9.4. By Corollary 9.5, $\mathcal{H}^n(f(\mathcal{B}_m \cap U(x_j^m, R_j^m))) = 0$ for every j. Thus also $\mathcal{H}^n(f(\mathcal{B}_f)) = 0$. $\square$

In [4], Bonk and Heinonen solved a long-standing open problem by proving the following theorem.

Theorem 9.7 ([4, Theorem 1.3])**.** *Suppose that $f : \Omega \to \mathbb{R}^n$ is a non-constant K-quasiregular map. Then*

$$\dim_{\mathcal{H}} \mathcal{B}_f \leqslant n - \epsilon(n, K),$$

where $\epsilon(n, K) > 0$ only depends on n and K.

Their method was to show that there exist $m \geqslant 2$ and $\lambda \in (0, 1)$, only depending on n and K, so that the set

$$\mathcal{B}_f \cap \{x \in \Omega : \, i(x, f) \geqslant m\} \tag{9.7}$$

is λ-porous, quantitatively. On the other hand, an earlier theorem by Sarvas [27] says that for every $m \geqslant 2$ there exists $\lambda_m \in (0, 1)$, only depending on n, K and m, so that the set

$$\mathcal{B}_f \cap \{x \in \Omega : \, i(x, f) \leqslant m\} \tag{9.8}$$

is λ_m-porous. Combining (9.7), (9.8) and Lemma 9.2 then yields Theorem 9.7. However, the known proofs of (9.8), and subsequently Theorem 9.7, are purely qualitative. Hence Bonk and Heinonen asked [4, Remark 3.5] for a direct quantitative proof. Our next theorem solves this problem.

Theorem 9.8. *Suppose that $f : \Omega \to \mathbb{R}^n$ is a non-constant quasiregular map. Then $\mathcal{B}_m$ is λ_m-porous for every $m \geqslant 2$, where $\lambda_m \in (0,1)$ only depends on n, K and m, quantitatively.*

Proof. Recall that in the Euclidean n-space

$$M\Delta(S(x,r),S(x,R);B(x,R)) = \omega_{n-1}\Big(\log\frac{R}{r}\Big)^{1-n} \tag{9.9}$$

whenever $0 < r < R$, compare (2.5). Here $\omega_{n-1} = \mathcal{H}^{n-1}(S(0,1))$. We fix a point $x \in \mathcal{B}_m$ and a radius $R_x > 0$ so that $R_x < \sigma_x$, where σ_x is as in Lemma 3.3. Moreover, we require the following:

$$H^*(x,s) \leqslant H^* \tag{9.10}$$

for every $s \leqslant R_x$, where H^* only depends on n and K, and for every $s \leqslant R_x$ there exists a point y_s so that

$$f(U(x,R_x)\cap\mathcal{B}_m)\cap B(y_s,\alpha s) = \emptyset \quad \text{and} \quad B(y_s,\alpha s) \subset B(f(x),s), \tag{9.11}$$

where $\alpha \in (0,1)$ only depends on n, K and m. These requirements can be made by [25, III Lemma 4.1] and Theorem 9.4, respectively.

Now consider $R < R_x$. Our goal is to show that there exists a ball

$$B(u,\beta L^*(x,R)) \subset B(x,L^*(x,R)) \setminus \mathcal{B}_m, \tag{9.12}$$

where $\beta \in (0,1)$ only depends on n, K and m. Let $\delta > 0$ be small enough so that $L^*(x,\delta R) < l^*(x,R)$. We denote

$$\Gamma = \Delta(U(x,\delta R),\partial U(x,R);U(x,R)).$$

Then every $\gamma \in f(\Gamma)$ joins $S(f(x),\delta R)$ and $S(f(x),R)$ in $B(f(x),R)$. Thus, by (9.9),

$$Mf(\Gamma) = \omega_{n-1}\Big(\log\frac{1}{\delta}\Big)^{1-n}. \tag{9.13}$$

On the other hand,

$$\begin{aligned} M\Gamma &\geqslant M\Delta\big(S\big(x,l^*(x,\delta R)\big),S\big(x,L^*(x,R)\big);B\big(x,L^*(x,R)\big)\big) \\ &= \omega_{n-1}\left(\log\frac{L^*(x,R)}{l^*(x,\delta R)}\right)^{1-n} \geqslant \omega_{n-1}\left(\log\frac{(H^*)^2 l^*(x,R)}{L^*(x,\delta R)}\right)^{1-n}, \end{aligned} \tag{9.14}$$

where the last inequality follows by applying (9.10) to both $L^*(x,R)$ and $l^*(x,\delta R)$. Combining (9.13), (9.14) and the K_O-inequality $M\Gamma \leqslant K_O m M f\Gamma$ gives

$$L^*(x,\delta R) \leqslant (H^*)^2\delta^{(K_O m)^{1/(n-1)}} l^*(x,R).$$

Hence, if we choose

$$\delta = \min\left\{ \left(2(H^*)^2\right)^{-(K_O m)^{\frac{-1}{n-1}}}, \frac{1}{2} \right\},$$

then

$$L^*(x, \delta R) \leqslant l^*(x, R)/2. \tag{9.15}$$

By (9.11) there exists a normal domain $U \subset U(x, \delta R)$ so that $\mathcal{B}_m \cap U = \emptyset$ and $f(U) = B(y_{\delta R}, \alpha\delta R)$. We denote $y_{\delta R} = y$. We choose a point $u \in U$ so that $f(u) = y$, and denote by r the largest radius so that $B(u, r) \subset U$. Then (9.12) follows if we can show that $r \geqslant \beta L^*(x, R)$, where $\beta > 0$ only depends on n, K and m.

Now there exists a point $v \in \partial U \cap S(u, r)$, so $f(v) \in S(y, \alpha\delta R)$. We denote by I the segment joining u and v. Then

$$\operatorname{diam} f(I) \geqslant \alpha\delta R,$$

which together with (2.6) implies

$$M\Gamma_1 = M\Delta(f(I), S(f(x), R); B(f(x), R)) \geqslant C(n, K, m). \tag{9.16}$$

We denote by Γ' the family of all lifts γ' of $\gamma \in \Gamma_1$ starting at I. Then every $\gamma' \in \Gamma'$ joins I and $\partial U(x, R)$ by Lemma 3.3 (iv). Also,

$$I \subset B(u, r) \subset B(u, l^*(x, R)/2) \subset B(x, l^*(x, R)) \subset U(x, R),$$

where the third inclusion follows by (9.15). Hence

$$\begin{aligned} M\Gamma' &\leqslant M\Delta(S(u, r), S(u, l^*(x, R)/2); B(u, l^*(x, R)/2)) \\ &= \omega_{n-1}\Big(\log \frac{l^*(x, r)}{2r}\Big)^{1-n}. \end{aligned} \tag{9.17}$$

Combining (9.16), (9.17) and Poletsky's inequality $M\Gamma_1 \leqslant K_I M\Gamma'$ (see [25, II (8.2)] and Theorem 11.1 below) yields

$$C(n, K, m) \leqslant K_I \omega_{n-1}\Big(\log \frac{l^*(x, r)}{2r}\Big)^{1-n}. \tag{9.18}$$

Applying (9.18) and (9.10) gives (9.12).

In order to complete the proof we need to show that for every $t < L^*(x, R_x)$ there exists $0 < R < R_x$ so that $L^*(x, R) = t$. Suppose that this is not the case. Then there exists $R < R_x$ and a sequence (R_i) converging to R so that $(L^*(x, R_i))$ does not converge to $L^*(x, R)$. We may assume that (R_i) either decreases or increases to R. In the first case we have a contradiction because

$$\overline{U}(x, R) = \bigcap_{i=1}^{\infty} \overline{U}(x, R_i)$$

is a compact and connected set. In the latter case we choose a point

$$p \in \partial U(x,R) \cap S(x, L^*(x,R)),$$

and a radius $\epsilon < \sigma_p$. Then $U(p,\epsilon) \cap U(x,R_i) \neq \emptyset$ for large enough i by Lemma 3.3 (iii). Hence we have a contradiction when $\epsilon \to 0$. The proof is complete. $\square$

A quantitative bound for $\epsilon(n,K)$ in Theorem 9.7 now immediately follows from (9.7), Theorem 9.8 and Lemma 9.2. By using the techniques in [24] one can also give a quantitative proof for (9.8); we omit the proof since it is more technical than the proof of Theorem 9.8.

10 Poletsky's lemma

Theorem 5.1 tells us that a quasiregular mapping $f : \Omega \to X$ lies in the Newtonian space $N^{1,n}_{\mathrm{loc}}(\Omega, X)$ and, therefore, it is absolutely continuous outside a path family of zero n-modulus. In this section we prove a substitute for this fact in the "inverse" direction, called Poletsky's lemma. This lemma is a consequence of Theorem 8.1. To state it we need some terminology. We refer to [30] for the definitions concerning paths and path integrals, such as path length and Condition *(N)*, used below.

Let $\beta : I_0 \to X$ be a closed rectifiable path, and let $\alpha : I \to \Omega$ be a path such that $f \circ \alpha \subset \beta$. This means that $f \circ \alpha$ is the restriction of β to some subinterval of I_0. If the length function $s_\beta : I_0 \to [0, l(\beta)]$ is constant on some interval $J \subset I$, β is also constant on J, and the discreteness of f implies that also α is constant on J. It follows that there is a unique mapping $\alpha^* : s_\beta(I) \to \Omega$ such that $\alpha = \alpha^* \circ (s_\beta|_I)$. We say that α^* is the f-representation of α with respect to β and f is absolutely precontinuous on α if α^* is absolutely continuous.

Theorem 10.1. *Suppose that Γ is a family of paths γ in Ω such that $f \circ \gamma$ is locally rectifiable and there is a closed subpath α of γ on which $f : \Omega \to X$ is not absolutely precontinuous. Then $Mf(\Gamma) = 0$.*

The rest of this section is almost parallel to the proof in the Euclidean case, see [25, pages 46-48]. Before going to the proof of Theorem 10.1 we need to introduce some notation. First, we fix a domain $G \subset\subset \Omega$ and set $\mathcal{B}_k = \{x \in G : i(x,f) = k\}$, $k \geqslant 2$. We choose pairwise disjoint open cubes Q_j, $j \in \mathbb{N}$, such that $2Q_j \subset G \setminus \mathcal{B}_f$, $f\big|_{2Q_j}$ is one-to-one, $G \setminus \mathcal{B}_f \subset \bigcup_{j=1}^{\infty} \overline{Q}_j$. Then we have the homeomorphic inverse mappings $h_j : f(2Q_j) \to 2Q_j$. By Theorem 8.1 we know that $h_j \in N^{1,n}(f(2Q_j), \mathbb{R}^n)$. We choose an n-weak upper gradient ρ_j of h_j, set $\rho_j(y) = 0$ for $y \in X \setminus f(2Q_j)$, and define

$$\rho(y) = \sup\Big\{\rho_j \chi_{f(2Q_j)}(y) : j \in \mathbb{N}\Big\}.$$

By Remark 8.3 the functions ρ_j can be chosen so that

$$\int_{f(2Q_j)} \big[\rho_j(y)\big]^n \, d\mathcal{H}^n(y) \leqslant C|Q_j|\,, \qquad \text{for all } j = 1, 2, ... \tag{10.1}$$

Similarly, for each $x \in \mathcal{B}_k$ we choose a normal neighborhood $U \subset G$ of x. We cover $\mathcal{B}_k$ by such normal neighborhoods U_{ki}, $i \in \mathbb{N}$, and let g_{ki} denote the "inverse" map given at (8.1). By Theorem 8.1, we have $g_{ki} \in N^{1,n}(f(U_{ki}), \mathbb{R}^n)$. Finally, we fix a set $F \subset X$ of zero $\mathcal{H}^n$-measure which contains all the points where at least one ρ_j is not finite and which also contains the set $f(\mathcal{B}_f)$ (Corollary 9.6).

Proof of Theorem 10.1. We follow the notation given above. Let Γ be a family of closed paths $\gamma : I \to G$ such that $f \circ \gamma$ is rectifiable for every $\gamma \in \Gamma$ and the following three properties are satisfied:

1. $\int_{f\circ\gamma} \chi_F \, ds = 0$ for every $\gamma \in \Gamma$.
2. If $\alpha : I' \to G$ is a closed subpath of some $\gamma \in \Gamma$ and if $|\alpha| \subset 2Q_j$, then
$$|h_j\big(f(\alpha(t_1))\big) - h_j\big(f(\alpha(t_2))\big)| \leqslant \int_{f\circ\alpha} \rho \, ds < \infty$$
for all $t_1, t_2 \in I'$.
3. There is $\zeta_{ki} \in L^n(f(U_{ki}))$ such that if $\alpha : I' \to G$ is a closed subpath of some $\gamma \in \Gamma$ and if $|\alpha| \subset U_{ki}$, then
$$|g_{ki}\big(f(\alpha(t_1))\big) - g_{ki}\big(f(\alpha(t_2))\big)| \leqslant \int_{f\circ\alpha} \zeta_{ki} \, ds < \infty$$
for all $t_1, t_2 \in I'$.

Our first claim is that these choices are legitimate in terms of Theorem 10.1. Precisely, we claim the following.
Claim 1. *Let $\Gamma_\circ$ be a family of closed paths γ in G such that at least one of the above conditions 1.-3. is not satisfied. Then $Mf(\Gamma_\circ) = 0$.*

Proof of Claim 1. Let Γ_q, $q = 1, ..., 3$, be the family of paths $\gamma \in \Gamma_0$ for which the Condition q. is not valid. The first subclaim, $Mf(\Gamma_1) = 0$, follows because one can choose a test-function to be infinity in F and zero otherwise. The Subclaims 2. and 3. are direct consequences of Theorem 8.1 and the definition of the Newtonian space $N^{1,n}$. For proving $Mf(\Gamma_2) = 0$ one needs also notice that

$$\begin{aligned} \int_{f(G)} \rho(y)^n \, d\mathcal{H}^n(y) &= \sum_{j=1}^{\infty} \int_{f(2Q_j)} \rho_j(y)^n \, d\mathcal{H}^n(y) \\ &\leqslant C \sum_{j=1}^{\infty} |Q_j| \leqslant C|G|. \end{aligned} \tag{10.2}$$

Here we used (10.1). This completes the proof of Claim 1.
Claim 2. If $\gamma : I \to G$ is a closed path such that $f \circ \gamma \notin f(\Gamma_0)$, then the f-representation γ^* of γ satisfies Condition *(N).*

Proof of Claim 2. We denote $I' = s_\beta I$. Let $E \subset I'$ be a set with $\mathcal{H}^1(E) = 0$. We cover the set $I' \setminus (\gamma^*)^{-1}(\mathcal{B}_f)$ by a family $\{I_\mu\,;\ \mu = 1, 2, ...\}$ of closed intervals with disjoint interiors in $I' \setminus (\gamma^*)^{-1}(\mathcal{B}_f) =: A_H$ such that $\gamma^* I_\mu$ is contained in some $2Q_{j_\mu}$, $\mu = 1, 2, ...$ Clearly, $\gamma^*(t) = h_{j_\mu}(f \circ \gamma^*)(t)$ for all $t \in I_\mu$. Combining this with the Condition 2., we have $\mathcal{H}^1\big(\gamma^*(E \cap A_H)\big) = 0$. To complete the proof of Claim 2. next we turn our attention to the branch set. This time $\gamma^*(t) = g_{ki}(f \circ \gamma^*)(t)$ for all $t \in (\gamma^*)^{-1}\big(\mathcal{B}_k\ \cap U_{ki}\big) =: A_{ki}$ and applying 3., we have $\mathcal{H}^1(\gamma^*(E \cap A_{ki})) = 0$. Since

$$(\gamma^*)^{-1}\mathcal{B}_f = \cup_{k \geqslant 2} \cup_i (\gamma^*)^{-1}(\mathcal{B}_k \cap U_{ki}),$$

we have $\mathcal{H}^1(\gamma^*(E \cap (\gamma^*)^{-1}\mathcal{B}_f)) = 0$. Therefore,

$$\mathcal{H}^1(\gamma^* E) \leqslant \mathcal{H}^1(\gamma^*(E \setminus (\gamma^*)^{-1}\mathcal{B}_f)) + \mathcal{H}^1(\gamma^*(E \cap (\gamma^*)^{-1}\mathcal{B}_f)) = 0.$$

Claim 3. The path γ^* is differentiable a.e. in I' and $\int_{I'} |(\gamma^*)'(t)|\, dt < \infty$.

Proof of Claim 3. By 1., $\mathcal{H}^1((\gamma^*)^{-1}(\mathcal{B}_f)) = 0$. Therefore, it is enough to consider γ^* in $I' \setminus (\gamma^*)^{-1}(\mathcal{B}_f)$. Following the notation above, we cover this set by a family $\{I_\mu\,;\ \mu = 1, 2, ...\}$ of closed intervals with disjoint interiors in $I' \setminus (\gamma^*)^{-1}(\mathcal{B}_f)$ such that $\gamma^* I_\mu$ is contained in some $2Q_{j_\mu}$, $\mu = 1, 2, ...$ For all $t \in I_\mu$, we have $\gamma^*(t) = h_{j_\mu}(f \circ \gamma^*)(t)$. Therefore, the Condition 2. gives for $t_1, t_2 \in I_\mu$ that

$$|\gamma^*(t_1) - \gamma^*(t_2)| \leqslant \int_{f \circ \alpha} \rho\, ds < \infty. \tag{10.3}$$

Here $\alpha = \gamma|_{[t_1, t_2]}$. Changing variables on the right hand side of (10.3), we obtain

$$|\gamma^*(t_1) - \gamma^*(t_2)| \leqslant \int_{t_1}^{t_2} \rho\big(\gamma^*(t)\big)\, dt < \infty. \tag{10.4}$$

This estimate together with the Rademacher-Stepanov theorem gives that γ^* is differentiable a.e in I_μ, and $|(\gamma^*)'(t)| \leqslant \rho(t)$ for a.e. $t \in I_\mu$.

Now, first by Bary's theorem [26, p. 285] we find that γ^* is absolutely continuous in G. Second, exhausting the domain Ω by an increasing sequence of domains D_i which are compactly contained in Ω we see that the claim of Theorem 10.1 holds. This completes the proof. $\square$

As in the classical case, [23], also in our setting it follows from Poletsky's lemma that the branch set of a nonconstant quasiregular mapping has measure zero.

Corollary 10.2. *If $f : \Omega \to X$ is a nonconstant quasiregular mapping, then*

- $J_f > 0$ *a.e.*
- $|\mathcal{B}_f| = 0$.
- *for any measurable set $E \subset \mathbb{R}^n$, $|E| = 0$ if and only if $\mathcal{H}^n(fE) = 0$.*

A large part of the proof is taken with minor modifications that are needed in our setting from [23].

Proof. First we will show that $J_f > 0$ almost everywhere. On the contrary we suppose that there is a set A with positive measure, contained in a closed cube $Q \subset \Omega$, such that $J_f = 0$ on this set A. Write $Q = I \times Q_\circ$, where $Q_\circ \subset \mathbb{R}^{n-1}$ and $I \subset \mathbb{R}$. Let Γ be the family of paths $\gamma_z(t) = (t, z)$, $z \in Q_\circ$, such that

$$\int_{\gamma_z} \chi_A \, ds > 0\,. \tag{10.5}$$

Then Γ has positive n-modulus. This simply follows from the assumption $|A| > 0$. In view of Theorem 6.2 we see that also the family $f(\Gamma)$ has a positive n-modulus. Combining this with Theorem 10.1 we find that $Mf(\Gamma') > 0$, where Γ' is the family of paths in Γ on which f is absolutely precontinuous. Then

$$\mathcal{H}^1((\gamma^*)^{-1}A) > 0 \tag{10.6}$$

for every $\gamma \in \Gamma'$. On the other hand, $\mathcal{H}^n(f(A)) = 0$, which, when combined with (10.6), yields $Mf(\Gamma') = 0$, a contradiction. Therefore, $J_f > 0$ almost everywhere, as claimed.

In order to verify the last statement in this corollary we need only show that $|E| = 0$ provided $\mathcal{H}^n(fE) = 0$, see Corollary 5.3. We may assume that $N(f, E) \leqslant N < \infty$. Then, if we denote $E_i = \{x \in E : J_f(x) \geqslant 1/i\}$ for $i \in \mathbb{N}$, and $E_0 = \{x \in E : J_f(x) = 0\}$, we have

$$|E_i|/i \leqslant \int_{E_i} J_f(x)\, dx \leqslant N\mathcal{H}^n(f(E_i)) = 0$$

for each $i \in \mathbb{N}$. Here we applied Corollary 5.6. Therefore, $|E| \leqslant \sum_{i=0}^{\infty} |E_i| = 0$, as claimed. Now especially choosing $E = \mathcal{B}_f$ and employing Corollary 9.6, it follows that the measure of the branch set is zero. $\square$

11 The Poletsky and Väisälä Inequalities

In this section we establish the classical Poletsky and Väisälä inequalities using Poletsky's lemma, Theorem 10.1. Recall that by data we mean H, n, θ and τ.

Theorem 11.1. *Suppose that $f : \Omega \to X$ is a nonconstant quasiregular mapping. Let Γ be a path family in Ω, Γ' a path family in X, and m a positive integer such that the following is true. For every path $\beta : I \to X$ in Γ' there are paths $\alpha_1, ..., \alpha_m$ in Γ such that $f \circ \alpha_j \subset \beta$ for all j and such that for every $x \in \Omega$ and $t \in I$ the equality $\alpha_j(t) = x$ holds for at most $i(x, f)$ indices j. Then*

$$M(\Gamma') \leqslant \frac{C}{m} M(\Gamma), \tag{11.1}$$

where C only depends on data.

Before going to the proof we give two important corollaries of Theorem 11.1. The first one, the Poletsky inequality, we obtain simply taking $\Gamma' = f(\Gamma)$.

Corollary 11.2. *Let $f : \Omega \to X$ be a nonconstant quasiregular mapping and Γ a family of paths in Ω. Then*

$$M f(\Gamma) \leqslant C\, M(\Gamma), \tag{11.2}$$

where C only depends on data.

The second corollary follows from Theorem 11.1 and the path lifting result, Theorem 3.4.

Corollary 11.3. *Suppose that $f : \Omega \to X$ is a nonconstant quasiregular mapping. Let D be a normal domain for f, Γ' a family of paths in $f(D)$ and Γ the family of paths α in D such that $f \circ \alpha \in \Gamma'$. Then*

$$M(\Gamma') \leqslant \frac{C}{N(f, D)} M(\Gamma), \tag{11.3}$$

where the constant C depends on data.

Proof of Theorem 11.1. To simplify writing we denote the set $f\big(\mathcal{B}_f \cup \{x \in \Omega : J_f(x) = 0\}\big)$ by B. Corollary 9.6 tells us that the n-Hausdorff measure of the set B is zero. Combining this with Theorem 10.1 we may assume that for every $\beta \in \Gamma'$ we have

- β is locally rectifiable,
- if α is a path in Ω with $f \circ \alpha \subset \beta$, then f is locally absolutely precontinuous on α,
- $\int_\beta \chi_B \, ds = 0$.

Suppose that $\rho : \Omega \to [0,\infty]$ is a test function for Γ; that is, $\rho \in T_\Gamma$. We define $\rho' : X \to [0,\infty]$ by

$$\rho'(y) = \frac{2}{m}\chi_{f(\Omega)}(y) \sup_{C(y)} \sum_{x \in C(y)} \sigma(x) \tag{11.4}$$

where $C(y)$ runs over all subsets of $f^{-1}(y)$ such that $\operatorname{card} C(y) \leqslant m$ and

$$\sigma(x) = \begin{cases} \rho(x)\, L_f^*(x) & \text{if } x \in \Omega \setminus f^{-1}(B) \\ 0 & \text{if } x \in f^{-1}(B)\,. \end{cases} \tag{11.5}$$

Here and in what follows we use the notation

$$L_f^*(x) = \limsup_{r \to 0} \frac{L^*(x,r)}{r}\,. \tag{11.6}$$

It follows that $L_f^*(x)$ is finite almost everywhere in the set $\Omega \setminus f^{-1}(B)$. Indeed, by using Theorem 7.1 one can show that

$$L_f^*(x) \leqslant (H')^n \tau \frac{1}{J_f(x)} \qquad \text{a.e. in } \Omega \setminus f^{-1}(B), \tag{11.7}$$

where the constant H' is as in Theorem 7.1. Notice that we applied Theorem 7.1 with $i(x,f) = 1$.

We need to show that ρ' is a legitimate test function for Γ'; that is, $\rho' \in T_{\Gamma'}$. The same arguments as in [25, p. 49] show that ρ' is a Borel function. Suppose that $\beta : I_\circ \to X$ is a closed path in Γ'. There exist paths $\alpha_1, ..., \alpha_m$ in Γ such that $f \circ \alpha_j \subset \beta$ and $\operatorname{card}\{j : \alpha_j(t) = x\} \leqslant i(x,f)$ for all $t \in I_\circ$ and $x \in \Omega$. Let $\alpha_j^* : I_j \to \Omega$ be the f-representation of α_j with respect to β. Thus $\alpha_j(t) = \alpha_j^* \circ s_\beta(t)$ and $f \circ \alpha_j^* \subset \beta^\circ$, where $s_\beta : I_\circ \to [0, l(\beta)]$ is the length function and β° the normal representation of β; that is, $\beta^\circ : [0, l(\beta)] \to X$ and $\beta = \beta^\circ \circ s_\beta$. We have

$$1 \leqslant \frac{1}{m}\sum_{j=1}^m \int_{\alpha_j} \rho\, ds = \frac{1}{m}\sum_{j=1}^m \int_{I_j} \rho\big(\alpha_j^*(t)\big)\, \big|(\alpha_j^*)'(t)\big|\, dt\,. \tag{11.8}$$

By the definition of L_f^* it follows that $\Big|(\alpha_j^*)'(t)\Big| \leqslant 2L_f^*\big(\alpha_j^*(t)\big)$ for almost every $t \in I_j$. Combining this with (11.8) we find that

$$1 \leqslant \frac{2}{m}\sum_{j=1}^m \int_{I_j} \rho\big(\alpha_j^*(t)\big) L_f^*\big(\alpha_j^*(t)\big)\, dt = \frac{2}{m}\sum_{j=1}^m \int_{I_j} \sigma\big(\alpha_j^*(t)\big)\, dt\,.$$

Since $I_j \subset [0, l(\beta)]$, we have

$$1 \leqslant \frac{2}{m}\sum_{j=1}^m \int_0^{l(\beta)} \sigma\big(\alpha_j^*(t)\big)\chi_{I_j}(t)\, dt\,. \tag{11.9}$$

The condition $\int_\beta \chi_{f(\mathcal{B}_f)}\, ds = 0$ gives that for almost every $t \in [0, l(\beta)]$ the points $\alpha_j^*(t)$, $j \in \{i\,:\, t \in I_i\}$, are distinct points in $f^{-1}\big(\beta^\circ(t)\big)$. Therefore

$$\rho'\big(\beta^\circ(t)\big) \geqslant \frac{2}{m}\sum_{j=1}^{m} \sigma\big(\alpha_j^*(t)\big)\chi_{I_j}(t),$$

and so

$$1 \leqslant \int_0^{l(\beta)} \rho'\big(\beta^\circ(t)\big)\, dt = \int_\beta \rho'\, ds\,.$$

We have proved that ρ' is a legitimate test function for Γ'.

Let (Ω_i) be an exhaustion of Ω, and set $\rho_i = \rho\chi_{\overline{\Omega}_i}$, $\sigma_i = \sigma\chi_{\overline{\Omega}_i}$, and $\rho_i' = \rho'\chi_{f(\overline{\Omega}_i)}$. Suppose $y_0 \in f(\overline{\Omega}_i) \setminus B$. Then there is a connected neighborhood V of y_0 and k inverse mappings $g_\mu : V \to D_\mu$ with

$$\overline{\Omega}_i \cap f^{-1}(V) = \bigcup\{\overline{\Omega}_i \cap D_\mu :\ 1 \le \mu \le k\}.$$

For each $y \in V$, we define a set $L_y \subset J := \{1, \ldots, k\}$ as follows. If $k \le m$, then $L_y = J$. If $k > m$, then $\operatorname{card} L_y = m$, and for each $\mu \in L_y$, $\nu \in J \setminus L_y$, either $\sigma_i(g_\mu(y)) > \sigma_i(g_\nu(y))$ or $\sigma_i(g_\mu(y)) = \sigma_i(g_\nu(y))$ and $\mu > \nu$. Then

$$\rho_i'(y) = \frac{2}{m}\sum_{\mu\in L_y} \sigma_i(g_\mu(y))$$

for $y \in V$. Furthermore, for $L \subset J$, the sets $V_L = \{y \in V :\ L_y = L\}$ are pairwise disjoint Borel sets. By Hölder's inequality for series,

$$[\rho_i'(y)]^n \le \frac{2^n}{m}\sum_{\mu\in L_y} \sigma_i(g_\mu(y))^n\,.$$

Now

$$\int_{V_L} [\rho_i'(y)]^n\, d\mathcal{H}^n(y) \leqslant \frac{2^n}{m}\sum_{\mu\in L}\int_{V_L} (\sigma_i \circ g_\mu)^n(y)\, d\mathcal{H}^n(y)\,.$$

The change of variables formula gives

$$\int_{V_L} [\rho_i'(y)]^n\, d\mathcal{H}^n(y) \leqslant \frac{2^n}{m}\sum_{\mu\in L}\int_{g_\mu(V_L)} \sigma_i^n(x) J_f(x)\, dx. \tag{11.10}$$

Here we applied Corollary 5.6 together with Remark 5.7. The inequality (11.7) gives

$$\int_{V_L} [\rho_i'(y)]^n\, d\mathcal{H}^n(y) \leqslant \frac{C}{m}\sum_{\mu\in L}\int_{g_\mu(V_L)} \rho^n(x)\, dx. \tag{11.11}$$

where the constant C depends as claimed in Theorem 11.1. As in [25, pp. 51-52] we conclude that

$$\int_X [\rho_i'(y)]^n\, d\mathcal{H}^n(y) \leqslant \frac{C}{m}\int_{\mathbb{R}^n} \rho_i^n(x)\, dx. \tag{11.12}$$

Letting $i \to \infty$, we obtain Theorem 11.1.

12 Applications

In this section we characterize quasiregular maps $f : \mathbb{R}^n \to X$ with polynomial growth, assuming that the geometry of X is suitably controlled. Namely, we show in Theorem 12.1 below that the characterization given in [11] in the Euclidean case can be generalized to our setting. This is done by using the theory established in the previous sections, which allows us to apply suitable techniques from the Euclidean theory.

We now recall some terminology needed in this section. We assume that X is as in the previous sections. First, following [12], we define the n-Loewner property of X, which was already mentioned before (2.6). For disjoint, compact and connected sets E, $F \subset X$, denote

$$\zeta(E,F) = \frac{\operatorname{dist}(E,F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}.$$

Then X is called an n-Loewner space if there exists a decreasing homeomorphism $\phi : (0,\infty) \to (0,\infty)$ so that

$$M\Delta(E,F;X) \geqslant \phi(\zeta(E,F))$$

for every non-degenerate $E, F \subset X$. If X is (globally) Ahlfors n-regular; that is, if (2.1) holds for every ball $B(x,r) \subset X$ with constants only depending on X, and n-Loewner, then

$$M\Delta(E,F;X) \geqslant C\big(\log\zeta(E,F)\big)^{1-n} \tag{12.1}$$

when $\zeta(E,F)$ is large enough, where $C > 0$ only depends on data. Also, X is then (globally) LLC. For the proofs of these facts, see [12].

Now consider a locally integrable function (weight) $\omega : \mathbb{R}^n \to [0,\infty]$. We say that ω is doubling if there exists a constant $C > 0$ so that

$$\int_{B(a,2r)} \omega(x)\,dx \leqslant C \int_{B(a,r)} \omega(x)\,dx$$

for every $a \in \mathbb{R}^n$ and $r > 0$. If ω is doubling, then there exist C, $L > 0$ so that

$$\int_{B(0,R)} \omega(x)\,dx \leqslant CR^L$$

when R is large enough. Moreover, ω is an A_∞-weight if there exist $C > 0$ and $\epsilon > 0$ so that

$$\Big(\frac{1}{|B(a,r)|}\int_{B(a,r)} \omega(x)^{1+\epsilon}\,dx\Big)^{1/(1+\epsilon)} \leqslant \frac{C}{|B(a,r)|}\int_{B(a,r)} \omega(x)\,dx$$

for every $a \in \mathbb{R}^n$ and $r > 0$. Every A_∞-weight is doubling, cf. [10, Chapter 15] and the references therein. Finally, an A_∞-weight ω is called a strong

A_∞-weight if the following holds: if we define $\delta_\omega : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$,

$$\delta_\omega(x,y) = \Big(\int_{\overline{B}(x,y)} \omega(z)\, dz \Big)^{1/n},$$

where $\overline{B}(x,y)$ is the smallest closed ball containing x and y, then there exists a metric d_ω on $\mathbb{R}^n$, and a constant $C > 0$, so that

$$C^{-1}\delta_\omega(x,y) \leqslant d_\omega(x,y) \leqslant C\delta_\omega(x,y)$$

for every $x, y \in \mathbb{R}^n$.

Theorem 12.1. *Suppose that $f : \mathbb{R}^n \to X$ is a non-constant quasiregular map, where X is Ahlfors n-regular and n-Loewner. Then the following conditions are equivalent:*

(a) J_f is doubling,

(b) $N(y, f, \mathbb{R}^n) \leqslant N < \infty$ for every $y \in X$,

(c) J_f is an A_∞-weight,

(d) J_f is a strong A_∞-weight,

(e) for any $a \in X$, $d(f(x), a) \to \infty$ as $|x| \to \infty$.

We divide the proof of Theorem 12.1 into four propositions stated below. These propositions, combined with the fact that A_∞-weights are doubling, prove Theorem 12.1. We need the following auxiliary results. We assume that X satisfies the assumptions in Theorem 12.1. Notice in particular that X is assumed to be globally Ahlfors n-regular, hence unbounded.

Lemma 12.2. *Suppose that $f : \mathbb{R}^n \to X$ is a non-constant quasiregular map. Then $f(\mathbb{R}^n)$ is unbounded. Moreover, there exists a constant $\theta \geqslant 1$ so that for every $y \in X \setminus \{f(0)\}$ there exists a path*

$$\gamma : [0,\infty) \to X \setminus B(f(0), \theta^{-1} d(f(0), y)),$$

starting at y, so that $|\gamma|$ is unbounded.

Proof. Fix $y \in X$ as in the second claim, and a sequence (p_i) of points in X, so that

$$d(f(0), y) \leqslant d(f(0), p_1)$$

and so that $d(f(0), p_i)$ increases to infinity. If we denote $p_0 = y$, then, for each $i \in \mathbb{N}$, p_{i-1} and p_i can be joined by a path $\gamma_i : [0,1] \to X \setminus B(f(0), \theta^{-1} d(f(0), y))$ by the LLC-property of X. The second claim follows.

Now suppose that $f(\mathbb{R}^n) \subset B(a, R)$ for some $a \in X$ and $R > 0$. Denote by Γ the family of all paths joining $f(B(0,1))$ and $X \setminus B(a, 2R)$ in X, and by Γ' the family of all corresponding maximal f-liftings starting at $B(0,1)$. Then $|\gamma'|$ is unbounded for every $\gamma' \in \Gamma'$, and so $M\Gamma' = 0$. On the other hand, every $f \circ \gamma'$ is a subpath of some $\gamma \in \Gamma$, and so

$$Mf(\Gamma') \geqslant M\Gamma.$$

By the Loewner property of X, and the second claim, $M\Gamma > 0$, which is a contradiction by Theorem 11.1. We conclude that $f(\mathbb{R}^n)$ is unbounded.

□

Lemma 12.3. *Suppose that $f : \mathbb{R}^n \to X$ is a quasiregular map satisfying (a). Then there exist $C, k > 0$ so that*

$$L(0,R) \leqslant CR^k$$

for every $R > 0$.

Proof. Fix a large number $R > 0$, and a point $a \in \overline{B}(0,R)$ so that

$$d(f(a), f(0)) = L(0,R).$$

Moreover, choose $\gamma : [0,\infty) \to X \setminus B(f(0), \theta^{-1}L(0,R))$, starting at $f(a)$, as in Lemma 12.2, and a maximal f-lifting γ' of γ starting at a. Then $|\gamma'|$ is unbounded, and so (12.1) yields

$$M\Gamma = M\Delta(\overline{B}(0,1), |\gamma'| \cap \overline{B}(0,2R); B(0,2R)) \geqslant C \log^{1-n} R. \tag{12.2}$$

Since every path $\eta \in f(\Gamma)$ intersects $f(\overline{B}(0,1))$ and $X \setminus B(f(0), \theta^{-1}L(0,R))$, the function $\rho : X \to [0,\infty]$,

$$\rho(y) = 2\theta L(0,R)^{-1} \chi_{f(B(0,2R))}$$

is a test function for $f(\Gamma)$ when R is large enough. Also, by (a),

$$\begin{aligned} \int_X N(y,f,B(0,2R))\rho^n(y)\, d\mathcal{H}^n(y) &= 2^n\theta^n L(0,R)^{-n} \int_{B(0,2R)} J_f(x)\, dx \\ &\leqslant CL(0,R)^{-n}R^L \end{aligned} \tag{12.3}$$

for some C, $L > 0$. The claim follows by (12.2), (12.3) and Theorem 6.2. □

Lemma 12.4. *Suppose that $f : \mathbb{R}^n \to X$ is a non-constant quasiregular map, and assume that (b) holds. Then there exists a constant $\kappa > 0$ so that*

$$L(a,2r)^n \leqslant \kappa \mathcal{H}^n(f(B(a,r)))$$

for every $a \in \mathbb{R}^n$ and $r > 0$.

Proof. We denote $U = U(a,s)$, where

$$s = \max\{l > 0 : U(a,l) \subset B(a,r)\}.$$

Moreover, we choose a point $q \in \overline{B}(a,2r)$ so that

$$d(f(q), f(a)) = L(a,2r).$$

Then, by Lemma 12.2 we find a path

$$\gamma : [0,\infty) \to X \setminus B(f(a), \theta^{-1}L(a,2r)),$$

starting at $f(q)$, and a lift γ' of γ, starting at q, so that $|\gamma'|$ is unbounded. Hence, by the n-Loewner property of $\mathbb{R}^n$,

$$M\Gamma = M\Delta(\overline{U}, |\gamma'| \cap \overline{B}(a,4r); \mathbb{R}^n) \geqslant C, \tag{12.4}$$

where $C > 0$ only depends on n. On the other hand, each $\eta \in f(\Gamma)$ intersects $B(f(a),s)$ and $X \setminus B(f(a),\theta^{-1}L(a,2r))$ (provided that $s < \theta^{-1}L(a,2r)$, which we can assume), and so

$$Mf(\Gamma) \leqslant C\Big(\log\frac{L(a,2r)}{\theta s}\Big)^{1-n} \tag{12.5}$$

by (2.5), where $C > 0$ does not depend on a or r. Combining (12.4), (12.5), (b) and Theorem 6.2 yields

$$L(a,2r) \leqslant Cs,$$

where $C > 0$ does not depend on a or r. Since $f(U) = D(a,s) \subset f(B(a,r))$,

$$s^n \leqslant \theta\tau\mathcal{H}^n(D(a,s)) \leqslant \theta\tau\mathcal{H}^n(f(B(a,r)))$$

by Ahlfors regularity. The proof is complete. $\square$

Lemma 12.5. *Suppose that $f : \mathbb{R}^n \to X$ is a non-constant quasiregular map. Then there exist $C \geqslant 1$ and $\alpha > 0$, only depending on data, so that*

$$L(a,\delta r) \leqslant C\delta^{\alpha}L(a,r)$$

for every $a \in \mathbb{R}^n$, $r > 0$ and $\delta \in (0,1/2)$.

Proof. We choose a point $y \in S(f(a),\theta L(a,r))$. Then, by Lemma 12.2 we find a path

$$\gamma : [0,\infty) \to X \setminus B(f(a), L(a,r)),$$

starting at y, so that $|\gamma|$ is unbounded. Then, by (12.1),

$$M\Gamma = M\Delta(f(\overline{B}(a,\delta r)), |\gamma|; X) \geqslant C_0\Big(\log\frac{\theta L(a,r)}{L(a,\delta r)}\Big)^{1-n}, \tag{12.6}$$

where $C_0 > 0$ does not depend on a or r. We denote by Γ' the family of all maximal f-liftings of paths in Γ, starting at $B(a, \delta r)$. Then, each $\eta \in \Gamma'$ intersects $\mathbb{R}^n \setminus B(a, r)$, and thus

$$M\Gamma' \leqslant \omega_{n-1} \Big(\log \frac{1}{\delta} \Big)^{1-n}. \tag{12.7}$$

The claim follows from (12.6), (12.7) and Theorem 11.1. The proof is complete. $\square$

Proposition 12.6. *Conditions* (a) *and* (b) *are equivalent.*

Proof. The first part of the proof adapts a method due to Väisälä [31] to our setting. We first assume (a), and suppose that (b) does not hold true. We fix a large $m \in \mathbb{N}$, to be determined later. Then we find a point $y \in X$ and a radius $M > 0$ so that y has m preimage points $x_1, \ldots, x_m$ inside $B(0, M) \subset \mathbb{R}^n$. By Lemma 3.3 we can choose $\delta > 0$ so that $U(x_i, \delta) \subset B(0, M)$ is a normal neighborhood for each $i = 1, \ldots, m$, and so that the sets $U(x_i, \delta)$ are pairwise disjoint.

By Lemma 12.2 we can choose a point $f(q) \in f(\mathbb{R}^n)$ so that $d(y, f(q))$ is as large as desired, and a path

$$\gamma : [0, \infty) \to X \setminus B(y, \theta^{-1} d(y, f(q))), \tag{12.8}$$

starting at $f(q)$, so that $|\gamma|$ is unbounded. Then by (12.1) and (2.2),

$$M\Gamma = M\Delta(\overline{D}(y, \delta), |\gamma|; X) \geqslant C_1 \Big(\log \frac{d(y, f(q))}{\delta} \Big)^{1-n}, \tag{12.9}$$

where $C > 0$ does not depend on q.

By (12.8) and Lemma 12.3, there exists $\alpha > 0$ so that

$$d(0, f^{-1}(|\gamma|)) \geqslant d(y, f(q))^{\alpha} \tag{12.10}$$

when the right hand term is large enough. For each $\eta \in \Gamma$ there are (at least) m maximal f-liftings η_i starting at the points $x_i \in B(0, M)$. Moreover, by (12.10) each of them intersects $\mathbb{R}^n \setminus B(0, d(y, f(q))^{\alpha})$. We denote the family of all such lifts by Γ'. Then, by Theorem 11.1 and (2.5),

$$M\Gamma \leqslant \frac{C}{m} M\Gamma' \leqslant \frac{C_2}{m} \Big(\log \frac{d(y, f(q))^{\alpha}}{M} \Big)^{1-n}. \tag{12.11}$$

Combining (12.9) and (12.11) yields

$$d(y, f(q))^{\alpha} \leqslant \frac{M}{\delta^{\beta}} d(y, f(q))^{\beta}, \quad \text{where } \beta = \Big(\frac{C_2}{C_1 m} \Big)^{1/(n-1)}.$$

Hence, if we fix m to be large enough so that $\beta < \alpha$, we have a contradiction when $d(y, f(q)) \to \infty$. We conclude that (a) implies (b).

Now we assume (b), and fix $x \in \mathbb{R}^n$ and $r > 0$. By Lemma 12.4, Ahlfors regularity and the change of variables formula,

$$\begin{aligned}\int_{B(x,2r)} J_f(y)\,dy &\leqslant N\mathcal{H}^n(f(B(x,2r))) \leqslant CL(x,2r)^n \\ &\leqslant C\mathcal{H}^n(f(B(x,r))) \leqslant C\int_{B(x,r)} J_f(y)\,dy.\end{aligned}$$

Hence (a) holds true. The proof is complete. □

Proposition 12.7. *Conditions* (a) *and* (b) *imply Condition* (c).

Proof. By Gehring's lemma [8], it suffices to show that there exists $C > 0$ so that

$$\frac{1}{|B(a,r)|}\int_{B(a,r)} J_f(x)\,dx \leqslant C\Big(\frac{1}{|B(a,r)|}\int_{B(a,r)} J_f(x)^{1/n}\,dx\Big)^n \tag{12.12}$$

for every $a \in \mathbb{R}^n$ and $r > 0$. We denote $B = B(a,r)$, and

$$f_B = \frac{1}{|B|}\int_B f_a(x)\,dx.$$

We claim that there exists $\mu > 0$, not depending on a or r, so that

$$\frac{1}{|B|}\int_B |f_a(x) - f_B|\,dx \geqslant \mu L(a,r/2). \tag{12.13}$$

We first consider (12.13) under the assumption $f_B \geqslant L(a,r/2)/2$. Then, by Lemma 12.5, $f_a(x) \leqslant L(a,r/2)/4$ for every $x \in B(a,\delta r)$. Thus

$$\frac{1}{|B|}\int_B |f_a(x) - f_B|\,dx \geqslant \frac{|B(a,\delta r)|}{4|B|}L(a,r/2) \geqslant \mu L(a,r/2)$$

where $\mu > 0$ does not depend on a and r.

Next we assume that $f_B < L(a,r/2)/2$, and fix $\epsilon > 0$, to be chosen later. Moreover, we choose a point $b \in B(a,r/2)$ so that $d(f(b),f(a)) = L(a,r/2)$. By Lemma 12.5,

$$d(f(x),f(b)) \leqslant \epsilon L(b,r/2)$$

for every $x \in B(b,\delta r)$, where $\delta > 0$ depends on ϵ. Therefore,

$$\begin{aligned}|f_a(x) - f_B| &\geqslant d(f(b),f(a)) - d(f(x),f(b)) - f_B \\ &\geqslant L(a,r/2)/2 - \epsilon L(b,r/2)\end{aligned} \tag{12.14}$$

whenever $x \in B(b,\delta r)$. On the other hand, by Lemma 12.4 and Ahlfors regularity,

$$L(b,r/2)^n \leqslant L(a,r)^n \leqslant \kappa\mathcal{H}^n(f(B(a,r/2))) \leqslant \kappa L(a,r/2)^n. \tag{12.15}$$

Then, if we choose $\epsilon = \kappa^{-1/n}/4$, (12.14) and (12.15) yield

$$\frac{1}{|B|}\int_B |f_a(x) - f_B| \geqslant \frac{|B(b, Cr)|}{4|B|} L(a, r/2) \geqslant \mu L(a, r/2),$$

where $\mu > 0$ does not depend on a or r. Hence (12.13) holds true.

In order to prove (12.12) we first use Lemma 12.4 and (b) to obtain

$$\frac{1}{|B|}\int_B J_f(x)\,dx \leqslant \frac{CL(a,r)^n}{|B|} \leqslant \frac{C\kappa}{|B|}\mathcal{H}^n(f(B(a,r/2))) \leqslant \frac{C\kappa}{|B|}L(a,r/2)^n. \tag{12.16}$$

On the other hand, the Poincaré inequality and (5.11) yield

$$\begin{aligned}\Big(\frac{1}{|B|}\int_B |f_a(x) - f_B|\,dx\Big)^n &\leqslant Cr^n\Big(\frac{1}{|B|}\int_B |\nabla f_a|\Big)^n \\ &\leqslant Cr^n\Big(\frac{1}{|B|}\int_B J_f^{1/n}\Big)^n.\end{aligned} \tag{12.17}$$

Combining (12.13), (12.16) and (12.17) gives (12.12). The proof is complete. □

Proposition 12.8. *Conditions* (b) *and* (c) *imply Condition* (d).

Proof. The proof of [7, Proposition 1.8] gives the claim if we can verify the following properties:

(i) there exists $p > n$ so that f is absolutely continuous on p-almost every path in $\mathbb{R}^n$,

(ii) $L_f^n(x) \leqslant CJ_f(x)$ for almost every $x \in \mathbb{R}^n$,

(iii) $\int_{f^{-1}(B(y,r))} J_f(x)\,dx \leqslant Cr^n$ for every $y \in X$ and $r > 0$.

The statement in [7] concerns maps with Euclidean targets, but the proof extends to our setting. Since we assume Condition (c), $J_f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for some $p > n$. Thus, by (5.11) and Fuglede's lemma, $f \in N^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, X)$, and (i) follows. Property (ii) is Lemma 6.1, and (iii) follows from (b), the Ahlfors regularity of X, the change of variables formula (5.12) and Remark 5.7. The proof is complete. □

Proposition 12.9. *Conditions* (b) *and* (e) *are equivalent.*

Proof. We first assume (e), and fix a point $y \in f(\mathbb{R}^n)$. Since f is discrete, there exists a ball $B = B(0, r) \subset \mathbb{R}^n$ so that

$$N(y, f, \mathbb{R}^n) = N(y, f, B) \leqslant \sum_{x \in f^{-1}(y)} i(x, f) = M < \infty. \tag{12.18}$$

Now suppose that there exists a point $v \in X$ with

$$\sum_{x \in f^{-1}(v)} i(x, f) > M,$$

and choose a compact path γ starting at v and ending at y. Then there are at least $M+1$ lifts γ^j of γ starting at $f^{-1}(v)$, and each of them either ends at some $x \in f^{-1}(y)$, or leaves every compact subset of $\mathbb{R}^n$. The latter cannot happen for any j by (e). Also, the former can happen for at most M γ^j:s by (12.18), which is a contradiction. Thus (b) follows from (e).

Now we assume (b), and suppose that (e) does not hold. Then there exists a sequence (a_i) of points in $\mathbb{R}^n$, so that $|a_i|$ increases to infinity but

$$\limsup_i d(f(a_i), f(0)) = R < \infty. \tag{12.19}$$

We may assume that $d(f(a_1), f(0)) = R/2$. Then

$$\operatorname{diam} f(B(0, |a_i|)) \geqslant R/2 \tag{12.20}$$

for every $i \in \mathbb{N}$. Since X is globally LLC, (12.20) and the proof of Lemma 4.2 imply that

$$\operatorname{diam} f(S(0, |a_i|)) \geqslant CR, \tag{12.21}$$

where $C > 0$ does not depend on i.

Next we fix $\delta > 0$ so that $U(x_j, \delta)$ is a normal neighborhood of x_j for every $x_j \in f^{-1}(f(0))$. Then

$$\bigcup_{x_j \in f^{-1}(f(0))} U(x_j, \delta) \subset B(0, t) \tag{12.22}$$

for some $t > 0$. When $|a_i| > t$, we denote by Γ_i the family of all paths joining $D(f(0), \delta)$ and $f(S(0, |a_i|))$ in X. Then by (12.19), (12.21), and the n-Loewner property of X, there exists $\epsilon > 0$ so that $M\Gamma_i \geqslant \epsilon$ for every i. We denote by Γ_i' the family of all lifts γ' of $\gamma \in \Gamma_i$ starting at $S(0, |a_i|)$. By Theorem 11.1, a contradiction to (12.19) follows if $M\Gamma_i' \to 0$ as $i \to \infty$.

By (12.22) every $\gamma' \in \Gamma_i'$ either intersects $B(0, t)$ or leaves every compact set in $\mathbb{R}^n$. The n-modulus of the family of all paths for which the latter happens is zero. All the other paths start at $S(0, |a_i|)$ and intersect $S(0, t)$, so

$$M\Gamma_i' \leqslant C(n)\Big(\log \frac{|a_i|}{t}\Big)^{1-n} \to 0 \quad \text{as } i \to \infty$$

by (2.4) and (2.5). We have the desired contradiction and thus (e) follows from (b). The proof is complete.

□

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