SELF-IMPROVING PROPERTIES OF WEIGHTED HARDY INEQUALITIES

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ABSTRACT. Suppose that a domain $\Omega \subset \mathbb{R}^n$ admits the (p, β_0)-Hardy inequality, i.e. that $\int_\Omega |u|^p {d_\Omega}^{\beta_0 - p} \le C \int_\Omega |\nabla u|^p {d_\Omega}^{\beta_0}$ holds for all $u \in C_0^\infty(\Omega)$. Here $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. We show that then there exists $\varepsilon > 0$ such that Ω admits (q, β)-Hardy inequalities for all $p - \varepsilon < q < p + \varepsilon$ and all $\beta_0 - \varepsilon < \beta < \beta_0 + \varepsilon$.

1. Introduction

We consider in this note the weighted Hardy inequality

$$\int_\Omega \left(\frac{|u(x)|}{d_\Omega(x)} \right)^p d_\Omega(x)^\beta \, dx \le C_0 \int_\Omega |\nabla u(x)|^p \, d_\Omega(x)^\beta \, dx, \tag{1}$$

where $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. We say that a domain $\Omega \subset \mathbb{R}^n$ admits the (p, β)-Hardy inequality, if there exists a constant $C_0 = C_0(\Omega, p, \beta) > 0$ such that (1) holds for all $u \in C_0^\infty(\Omega)$. If $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, then, by the well-known result of Nečas [11], Ω admits the (p, β)-Hardy inequality for all $\beta < p - 1$. A more general sufficient condition for (p, β)-Hardy inequalities was given in [6]. Uniform p-fatness of the complement of Ω implies also Hardy inequalities for Ω, see [2], [9], and [14] for precise results and definitions; see also [3] and [4] for related results on pointwise inequalities. Notice also that if Ω admits the (p, β)-Hardy inequality, then, by approximation, (1) holds in fact for all Sobolev functions $u \in W_0^{1,p}(\Omega)$, and so in particular for all Lipschitz functions with compact support in Ω.

The main purpose of this note is to prove the following self-improving property of these Hardy inequalities.

Theorem 1. *Let $1 < p < \infty$ and $\beta_0 \in \mathbb{R}$, and suppose that $\Omega \subset \mathbb{R}^n$ admits the (p, β_0)-Hardy inequality with a constant $C_0 > 0$. Then there exists $\varepsilon = \varepsilon(C_0, p, \beta_0, n) > 0$ such that Ω admits the (q, β)-Hardy inequality whenever $p - \varepsilon < q < p + \varepsilon$ and $\beta_0 - \varepsilon < \beta < \beta_0 + \varepsilon$. Moreover, the constant in all these inequalities can be chosen to be $C = C(C_0, p, \beta_0, n) > 0$, independent of q and β.*

Theorem 1 is actually a generalization of earlier results, obtained in the unweighted case $\beta_0 = 0$. Koskela and Zhong [7] proved that the p-Hardy inequality (i.e. (1) with $\beta = 0$) implies q-Hardy inequalities for all $p - \varepsilon < q < p + \varepsilon$, with some small $\varepsilon > 0$. Also, Hajłasz [3] proved that the p-Hardy

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inequality implies always (p, β)-Hardy inequalities for $0 < \beta < \varepsilon$, again for some small positive ε.

In the case $q \geq p$, considered in Section 2, we prove in fact more general results than those of Theorem 1. In particular, we show that the (p, β_0)-Hardy inequality implies some weighted Hardy inequalities for all $q \geq p$, and moreover, in this case $\varepsilon = \varepsilon(C_0, p) > 0$ can be chosen to be independent of β_0. The proof of this part is quite straight-forward, as we only need to choose a suitable test-function for the (p, β_0)-inequality, and the result follows with a use of Hölder's inequality. In Section 3 we deal with the case $p - \varepsilon < q < p$, which turns out to be a bit more involved. The main idea of the proof is to apply so called "Lipschitz truncation" technique, using the level sets of both u/d_Ω and the maximal function of $|\nabla u|$. Theorem 1 is then proved at the end of Section 3 by combining the results of the cases $q \geq p$ and $p - \varepsilon < q < p$.

We would like to point out that Theorem 1 is sharp, in the following qualitative sense: Given $p \in (1, \infty)$, $\beta \in (p-n, p)$, and $\varepsilon > 0$ small enough, there exists a bounded domain $\Omega \subset \mathbb{R}^n$ which admits the (p, β)-Hardy inequality, but where the $(p, \beta \pm \varepsilon)$- and $(p \pm \varepsilon, \beta)$-Hardy inequalities fail. The construction of such domains is explained in Section 4. The above bounds for β are natural, since each proper subdomain $\Omega \subsetneq \mathbb{R}^n$ admits the (p, β)-Hardy inequality whenever $\beta < p - n$ (cf. [6]), and the (p, β)-Hardy inequality fails in a bounded domain for each $\beta \geq p$.

The constants in our results are not supposed to be optimal, in any sense. Nevertheless, we try to express the explicit formulas of these constants, given by our calculations, in order to emphasize their dependence of the given data. For the notation, we mention that C_0 denotes always the constant of the fixed (p, β_0)-Hardy inequality, but constants $C_1, C_2, \ldots$ will be used in each proof separately.

2. The case $q \geq p$

If a domain $\Omega \subset \mathbb{R}^n$ admits the (p, β)-Hardy inequality for some $1 < p < \infty$ and $\beta \in \mathbb{R}$, then Ω admits weighted Hardy inequalities for all $q \geq p$, but we have to add also to the weight exponent β the difference $q - p$. In addition, there is always a small $\delta_0 > 0$, depending on the given data, such that the new weight exponent may actually vary between $\beta + q - p - \delta_0$ and $\beta + q - p + \delta_0$. The next lemma gives a more precise formulation of these facts.

Lemma 2. *Let $1 < p < \infty$, $\beta \in \mathbb{R}$, and suppose that $\Omega \subset \mathbb{R}^n$ admits the (p, β)-Hardy inequality with a constant $C_0 > 0$. Then Ω admits the $(p + s, \beta + s + \delta)$-Hardy inequality for all $s \geq 0$ and all $|\delta| < \frac{p}{2} C_0^{-1/p}$, with the constant*

$$C(C_0, p, s, \delta) = \left(\frac{C_1 (1 + s/p)^p}{1 - C_1 (|\delta|/p)^p} \right)^{1+s/p}, \tag{2}$$

where $C_1 = 2^p C_0$.

The proof of Lemma 2 generalizes the ideas used in the proof of Theorem 2.3 in [7] and in a part of the proof of Theorem 1 in [3]; see also the proof of the main theorem in [14].

Proof. Let $u \in C_0^\infty(\Omega)$ and let $s \geq 0$ and $\delta \in \mathbb{R}$, with $|\delta|$ small. Define

$$v(x) = |u(x)|^{1+s/p} d_\Omega(x)^{\delta/p}.$$

Then it is easy to see that v is a Lipschitz function with compact support in Ω, and moreover

$$|\nabla v(x)| \leq \left(1 + \tfrac{s}{p}\right)|u(x)|^{\frac{s}{p}}|\nabla u(x)| d_\Omega(x)^{\frac{\delta}{p}} + \tfrac{|\delta|}{p}|u(x)|^{1+\frac{s}{p}} d_\Omega(x)^{\frac{\delta}{p}-1} \tag{3}$$

for almost every $x \in \Omega$, since $|\nabla d_\Omega| \leq 1$. If we use this estimate in the (p,β)-Hardy inequality for v, and denote $a = 1 + s/p$, $C_1 = 2^p C_0$, we obtain

$$\begin{aligned} \int |u(x)|^{p+s} d_\Omega(x)^{\delta+\beta-p}\, dx &\leq C_1 a^p \int |u(x)|^s |\nabla u|^p d_\Omega(x)^{\delta+\beta}\, dx \\ &\quad + C_1\left(\tfrac{|\delta|}{p}\right)^p \int |u(x)|^{p+s} d_\Omega(x)^{\delta-p+\beta}\, dx. \end{aligned} \tag{4}$$

Now, if $C_1(|\delta|/p)^p < 1$ (i.e. $|\delta| < \frac{p}{2}C_0^{-1/p}$), we can move the last term in (4) to the left-hand side and then use Hölder's inequality on the right-hand side as follows:

$$\begin{aligned} \left(1 - C_1\frac{|\delta|^p}{p^p}\right) &\int \left(\frac{|u(x)|}{d_\Omega(x)}\right)^{p+s} d_\Omega(x)^{\beta+s+\delta}\, dx \\ &\leq C_1 a^p \int \left(|\nabla u|^p d_\Omega(x)^{\frac{1}{a}(\beta+s+\delta)}\right)\left(|u(x)|^s d_\Omega(x)^{\frac{a-1}{a}(\beta+\delta)-\frac{1}{a}s}\right) dx \\ &\leq C_1 a^p \left(\int |\nabla u|^{ap} d_\Omega(x)^{\beta+s+\delta}\, dx\right)^{1/a} \\ &\qquad \cdot \left(\int |u(x)|^{\frac{sa}{a-1}} d_\Omega(x)^{\beta+\delta-\frac{s}{a-1}}\, dx\right)^{\frac{a-1}{a}} \\ &\leq C_1 a^p \left(\int |\nabla u|^{p+s} d_\Omega(x)^{\beta+s+\delta}\, dx\right)^{1/a} \\ &\qquad \cdot \left(\int \left(\frac{|u(x)|}{d_\Omega(x)}\right)^{p+s} d_\Omega(x)^{\beta+s+\delta}\, dx\right)^{1-\frac{1}{a}}; \end{aligned} \tag{5}$$

notice that $ap = p + s = sa/(a-1)$ and $s/(a-1) = p$. From (5) we obtain that

$$\begin{aligned} \int &\left(\frac{|u(x)|}{d_\Omega(x)}\right)^{p+s} d_\Omega(x)^{\beta+s+\delta}\, dx \\ &\leq \left(\frac{C_1 a^p}{1 - C_1(|\delta|/p)^p}\right)^a \int |\nabla u|^{p+s} d_\Omega(x)^{\beta+s+\delta}\, dx, \end{aligned}$$

and this proves the lemma. □

The following theorem records two important consequences of Lemma 2.

Theorem 3. *Let $1 < p < \infty$ and $\beta_0 \in \mathbb{R}$. Suppose that $\Omega \subset \mathbb{R}^n$ admits the (p,β_0)-Hardy inequality with a constant $C_0 > 0$. Then*

(i) there exists $\delta = \delta(C_0,p) > 0$ such that, for each $q \geq p$, Ω admits the (q,β)-Hardy inequality whenever $\beta_1 - \delta < \beta < \beta_1 + \delta$, where $\beta_1 = \beta_0 + (q-p)$;

(ii) there exists $\varepsilon = \varepsilon(C_0, p) > 0$ such that Ω admits (q, β)-Hardy inequalities for all $p \le q < p + \varepsilon$ and $\beta_0 - \varepsilon < \beta < \beta_0 + \varepsilon$.

The constant in the (q, β)-Hardy inequalities of part (ii) depends only on C_0 and p; in part (i) the constant may depend also on q.

Proof. (i) Let $q \ge p$, define $s = q - p \ge 0$, and take $\delta = (p/2)(2C_0)^{-1/p}$. Then $1 + s/p = q/p$ and $1 - 2^p C_0(\delta/p)^p = 1/2$, and thus the constant in (2) for these q and δ is equal to $C_2 = \big(2^{p-1}C_0\big)^{q/p}(q/p)^q$. It now follows from Lemma 2 that Ω admits the (q, β)-Hardy inequality, with this constant C_2, for all β satisfying $\beta_0 + (q - p) - \delta < \beta < \beta_0 + (q - p) + \delta$.

(ii) Take $\varepsilon = \min\big\{1, (p/4)(2C_0)^{-1/p}\big\}$ and let $p \le q < p + \varepsilon$. The part (i) of the theorem (note that δ in the proof of part (i) is now greater or equal to 2ε) yields that Ω admits the (q, β)-Hardy inequality for all β such that $\beta_0 + (q - p) - 2\varepsilon < \beta < \beta_0 + (q - p) + 2\varepsilon$, and so especially for all $\beta_0 - \varepsilon < \beta < \beta_0 + \varepsilon$. Since $q < p + 1$, the constant in these inequalities can clearly be chosen to be independent of q. □

Remark. The proof of Theorem 3 shows that in the part (i) of the theorem we may take $\delta = \delta(p, C_0) = (p/2)(2C_0)^{-1/p}$, and the constant in these Hardy inequalities may be taken to be $C = \big(2^{p-1}C_0\big)^{q/p}(q/p)^q$. If $p = q$, then $C = 2^{p-1}C_0$. These explicit formulas will be later needed in the proof of the main theorem.

3. The case $p - \varepsilon < q < p$ and the proof of Theorem 1

The (p, β)-Hardy inequality implies weighted Hardy inequalities also for some $q < p$, but here q can not be much smaller than p. In the next theorem we consider the case where the weight function remains unaltered; the possible variations for the weight exponent β are then included in the main theorem.

Theorem 4. *Let $1 < p < \infty$ and $\beta \in \mathbb{R}$. If Ω admits the (p, β)-Hardy inequality, then there exists $\varepsilon = \varepsilon(C_0, p, n) > 0$ such that Ω admits (q, β)-Hardy inequalities for all $p - \varepsilon < q \le p$, with a constant $C(C_0, p, \beta, n) > 0$ independent of q.*

The proof of Theorem 4 goes along the same lines as the proof of Theorem 2.2 in [7] (cf. also [1] and [10]), but some modifications are needed due to the additional weight in the inequality. For reader's convenience we present here the entire proof.

In the proof we need the restricted Hardy-Littlewood maximal function, defined as

$$M_R f(x) = \sup_{0 < r \le R} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)|\, dy$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The famous maximal function theorem of Hardy, Littlewood and Wiener (see e.g. [13]) states that $||M_R f||_p \le C_M(n, p)||f||_p$ for $1 < p < \infty$.

Proof of Theorem 4. Let $u \in C_0^\infty(\Omega)$. Let $\lambda > 0$ and denote

$$E_\lambda = \{x \in \Omega : |u(x)| \le \lambda d_\Omega(x)\},$$
$$G_\lambda = \{x \in \Omega : M_{d_\Omega(x)/2}|\nabla u(x)| \le \lambda\},$$

and $F_\lambda = E_\lambda \cap G_\lambda$. Let $x, y \in F_\lambda$ be such that $d_\Omega(x) \geq d_\Omega(y)$. If $d_\Omega(x) \geq 5|x-y|$, then $d_\Omega(y) \geq d_\Omega(x) - |x-y| \geq 4|x-y|$, and we obtain, using a well-known pointwise characterization of Sobolev functions, that

$$\begin{aligned}|u(x)-u(y)| &\leq C|x-y|\big(M_{2|x-y|}|\nabla u|(x) + M_{2|x-y|}|\nabla u|(y)\big)\\ &\leq C|x-y|\big(M_{d_\Omega(x)/2}|\nabla u|(x) + M_{d_\Omega(y)/2}|\nabla u|(y)\big)\\ &\leq 2C\lambda|x-y|,\end{aligned}$$

where $C = C(n) > 0$. On the other hand, if $d_\Omega(x) \leq 5|x-y|$, we have

$$\begin{aligned}|u(x)-u(y)| \leq |u(x)| + |u(y)| &\leq \lambda\big(d_\Omega(x) + d_\Omega(y)\big)\\ &\leq 10\lambda|x-y|.\end{aligned}$$

Thus $u|_{F_\lambda}$ is a $C_1\lambda$-Lipschitz function, where $C_1 = C_1(n) > 0$. We may now extend $u|_{F_\lambda}$ to a $C_1\lambda$-Lipschitz function $\tilde{u}$ in Ω, by the classical McShane extension

$$\tilde{u}(x) = \inf_{y\in F_\lambda}\big\{u(y) + C_1\lambda|x-y|\big\}.$$

Notice that also $\tilde{u}$ has a compact support in Ω. Indeed, if $\delta = d(\operatorname{spt}(u), \partial\Omega) > 0$ and $x \in \Omega$ is such that $d_\Omega(x) < \delta/2$, it follows that $x \in F_\lambda$ and $u(x) = 0$. Above $\operatorname{spt}(u)$ denotes the closure of the set $\{x \in \Omega : u > 0\}$.

Now $|\nabla\tilde{u}| \leq |\nabla u|\chi_{F_\lambda} + C_1\lambda\chi_{\Omega\setminus F_\lambda}$, and since the (p,β)-Hardy inequality holds for $\tilde{u}$, we obtain

$$\begin{aligned}&\int_{F_\lambda}|\tilde{u}(x)|^p\,d_\Omega(x)^{\beta-p}\,dx\\ &\qquad\leq C_0\int_{F_\lambda}|\nabla u(x)|^p\,d_\Omega(x)^\beta\,dx + C_0C_1^p\int_{\Omega\setminus F_\lambda}\lambda^p\,d_\Omega(x)^\beta\,dx.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{E_\lambda}|u(x)|^p\,d_\Omega(x)^{\beta-p}\,dx \;&\leq\; C_0\int_{F_\lambda}|\nabla u(x)|^p\,d_\Omega(x)^\beta\,dx\\ &\quad + C_0C_1^p\int_{\Omega\setminus F_\lambda}\lambda^p\,d_\Omega(x)^\beta\,dx + \int_{E_\lambda\setminus F_\lambda}|u(x)|^p\,d_\Omega(x)^{\beta-p}\,dx\\ &\leq C_0\int_{G_\lambda}|\nabla u(x)|^p\,d_\Omega(x)^\beta\,dx\\ &\quad + C_0C_1^p\int_{\Omega\setminus E_\lambda}\lambda^p\,d_\Omega(x)^\beta\,dx + 2C_0C_1^p\int_{\Omega\setminus G_\lambda}\lambda^p\,d_\Omega(x)^\beta\,dx,\end{aligned}\tag{6}$$

where we have used the definition of E_λ and the fact that $\Omega\setminus F_\lambda = \Omega\setminus E_\lambda \cup E_\lambda\setminus G_\lambda$. The next step is to multiply (6) by $\lambda^{-\varepsilon-1}$, where $0 < \varepsilon < p-1$, and then integrate with respect to λ over $(0,\infty)$. With the change of the order of the integration on the left-hand side, this leads us to

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_\Omega \left(\frac{|u(x)|}{d_\Omega(x)}\right)^{p-\varepsilon} d_\Omega(x)^\beta\,dx \\
&\qquad \le C_0 \int_0^\infty \lambda^{-\varepsilon-1}\int_{G_\lambda} |\nabla u(x)|^p\, d_\Omega(x)^\beta\,dx\,d\lambda \\
&\qquad\qquad + C_0C_1^p \int_0^\infty \lambda^{p-\varepsilon-1}\int_{\Omega\setminus E_\lambda} d_\Omega(x)^\beta\,dx\,d\lambda \\
&\qquad\qquad + 2C_0C_1^p \int_0^\infty \lambda^{p-\varepsilon-1}\int_{\Omega\setminus G_\lambda} d_\Omega(x)^\beta\,dx\,d\lambda.
\end{aligned}
\tag{7}
$$

The first term on the right-hand side of (7) can be estimated, by the definition of G_λ and the trivial inequality $|\nabla u| \le M_R|\nabla u|$, to be less than

$$\frac{C_0}{\varepsilon}\int_\Omega |\nabla u(x)|^{p-\varepsilon}\, d_\Omega(x)^\beta\,dx,$$

and the middle integral, by the definition of E_λ, is less than

$$\frac{1}{p-\varepsilon}\int_\Omega \left(\frac{|u(x)|}{d_\Omega(x)}\right)^{p-\varepsilon} d_\Omega(x)^\beta\,dx.$$

Now, if $0 < r \le d_\Omega(x)/2$ and $y \in B(x,r)$, we have that $d_\Omega(x)/2 \le d_\Omega(y) \le 2d_\Omega(x)$, and thus $d_\Omega(x)^\beta \le 2^{|\beta|}d_\Omega(y)^\beta$ for all such y, and all $\beta \in \mathbb{R}$. Using this fact, the definition of G_λ, and the maximal function theorem (recall that $p-\varepsilon > 1$), we can estimate the last integral in (7) as follows:

$$
\begin{aligned}
\int_0^\infty \lambda^{p-\varepsilon-1}\int_{\Omega\setminus G_\lambda} d_\Omega(x)^\beta\,dx\,d\lambda &= \int_\Omega d_\Omega(x)^\beta \int_0^{M_{d_\Omega(x)/2}|\nabla u|(x)} \lambda^{p-\varepsilon-1}\,d\lambda\,dx \\
&\le \frac{1}{p-\varepsilon}\int_\Omega d_\Omega(x)^\beta \big(M_{d_\Omega(x)/2}|\nabla u|(x)\big)^{p-\varepsilon}\,dx \\
&\le \frac{2^{|\beta|}}{p-\varepsilon}\int_\Omega \Big(M_{d_\Omega(x)/2}\big(|\nabla u|{d_\Omega}^{\beta/(p-\varepsilon)}\big)(x)\Big)^{p-\varepsilon}\,dx \\
&\le C_M\frac{2^{|\beta|}}{p-\varepsilon}\int_\Omega |\nabla u(x)|^{p-\varepsilon}\, d_\Omega(x)^\beta\,dx,
\end{aligned}
$$

where $C_M = C_M(p-\varepsilon,n)$. If we now assume that ε is so small that $C_0C_1^p\frac{\varepsilon}{p-\varepsilon} \le 1/2$, it follows from (7) and the previous estimates that

$$
\begin{aligned}
&\int_\Omega \left(\frac{|u(x)|}{d_\Omega(x)}\right)^{p-\varepsilon} d_\Omega(x)^\beta\,dx \\
&\qquad \le 2\left(C_0 + 2C_0C_1^pC_M\frac{\varepsilon\,2^{|\beta|}}{p-\varepsilon}\right)\int_\Omega |\nabla u(x)|^{p-\varepsilon}\, d_\Omega(x)^\beta\,dx \\
&\qquad \le \big(2C_0 + C_M2^{|\beta|}\big)\int_\Omega |\nabla u(x)|^{p-\varepsilon}\, d_\Omega(x)^\beta\,dx.
\end{aligned}
\tag{8}
$$

If we in addition require that $\varepsilon \le (p-1)/2$, we may in fact assume that the constant C_M in (8) depends only on p and n. Thus Ω admits the $(p-\varepsilon,\beta)$-Hardy inequality for all $0 < \varepsilon \le \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(C_0,p,n) > 0$, with the constant $C(C_0,p,\beta,n) = 2C_0 + C_M2^{|\beta|} > 0$. This proves the theorem. □

Using Theorems 3 and 4, we are now able to prove our main theorem.

Proof of Theorem 1. Since Ω admits the (p,β_0)-Hardy inequality with the constant $C_0>0$, we obtain from the part (ii) of Theorem 3 a small constant $\varepsilon_1=\varepsilon_1(C_0,p)$ such that Ω admits (q,β)-Hardy inequalities whenever $p\le q<p+\varepsilon_1$ and $\beta_0-\varepsilon_1<\beta<\beta_0+\varepsilon_1$. Moreover, the constant in all these inequalities can be taken to be $C_1=C_1(C_0,p)>0$.

From Theorem 4 we obtain, on the other hand, a small $\varepsilon_2=\varepsilon_2(C_0,p,n)$ such that Ω admits the (q,β_0)-Hardy inequality with a constant $C_2=C_2(C_0,p,\beta_0,n)$ whenever $p-\varepsilon_2\le q<p$. If we now take $\delta(q)=\delta(q,C_2)=(q/2)(2C_2)^{-1/q}$ for each $p-\varepsilon_2\le q<p$, we conclude, by Theorem 3(i) (cf. also the remark after Theorem 3), that Ω admits the (q,β)-Hardy inequality for all $\beta_0-\delta(q)<\beta<\beta_0+\delta(q)$, with the constant $C_3(q)=2^{q-1}C_2\le 2^{p-1}C_2$. But since $\delta(q)$ is now an increasing function of q, we obtain, e.g. by choosing $\varepsilon_3=\min\{\varepsilon_2,\delta(p-\varepsilon_2)\}$, that the (q,β_0)-Hardy inequality holds with the constant $2^{p-1}C_2$ whenever $p-\varepsilon_3<q<p$ and $\beta_0-\varepsilon_3<\beta<\beta_0+\varepsilon_3$; notice that in particular $\varepsilon_3=\varepsilon_3(C_0,p,\beta_0,n)$. We finish the proof by taking $\varepsilon=\min\{\varepsilon_1,\varepsilon_3\}$ and $C=\max\{C_1,2^{p-1}C_2\}$. $\square$

4. Qualitative sharpness of Theorem 1

In this section we construct examples which show that, given $1<p<\infty$, $p-n<\beta<p$, and any sufficiently small $\varepsilon>0$, there exists a domain $\Omega\subset\mathbb{R}^n$ which admits the (p,β)-Hardy inequality, but fails to admit the $(p,\beta\pm\varepsilon)$- and $(p\pm\varepsilon,\beta)$-Hardy inequalities. For simplicity, we give the construction only for $n=2$, but the essential ideas for higher dimensional examples are exactly the same as in the planar case. We would like to mention that a more general (and more detailed) treatment of domains having properties similar to those in the following construction is given in [8].

Let $1<p<\infty$ and $p-n<\beta<p$, and let

$$0<\varepsilon<\min\{p-1,\, p-\beta,\, 2-p+\beta\},$$

so that $1<p-\varepsilon$ and $p-2<\beta\pm\varepsilon<p$. Let us first consider the case $p-1<\beta+\varepsilon<p$. Denote $\mu=\beta+\varepsilon-p+2$ and $\lambda=\mu-2\varepsilon$ (i.e. $\lambda=\beta-\varepsilon-p+2$). Let K_μ be the usual μ-dimensional von Koch -snowflake curve with $\operatorname{diam}(K_\mu)=1$ (see e.g. the construction in [5, Section 2]). Replace the long (short) edges of the square $(0,5)\times(-2,2)$ by 5 (4) copies of K_μ, and furthermore, remove the set $K_\lambda+(2,0)$, where K_λ is the λ-dimensional snowflake curve if $\lambda>1$, and the standard λ-dimensional Cantor set (on the interval [0,1] of the real line, but embedded in $\mathbb{R}^2$) if $0<\lambda\le 1$. Let Ω denote this new domain; also, denote $\Omega_\lambda=\Omega\cap[1,4]\times[-1,1]$ and $\Omega_\mu=\Omega\setminus\Omega_\lambda$.

Let then $u\in C_0^\infty(\Omega)$. Since $\beta<p-2+\mu$, the results of [6] imply that there exist $C>0$ and $1<q<p$, independent of u, such that the pointwise (p,β)-Hardy inequality

$$|u(x)|\le C d_\Omega(x)^{1-\frac{\beta}{p}} M_{2d_\Omega(x),q}\big(|\nabla u| {d_\Omega}^{\beta/p}\big)(x)$$

holds for all $x\in\Omega_\mu$. Hence, by the maximal function theorem, it is easy to see that

$$\int_{\Omega_\mu}\left(\frac{|u(x)|}{d_\Omega(x)}\right)^p d_\Omega(x)^\beta\,dx\le C\int_\Omega|\nabla u(x)|^p\,d_\Omega(x)^\beta\,dx. \tag{9}$$

Let us then consider the part Ω_λ. Let $\mathcal{W}$ be a Whitney decomposition of Ω (cf. [13]), and denote

$$\mathcal{F} = \{Q \in \mathcal{W} : Q \cap \Omega_\lambda \neq \emptyset \neq Q \cap \Omega_\mu\};$$

notice in this definition that the cubes $Q \in \mathcal{W}$ are all closed. Also denote $\mathcal{W}_\lambda = \{Q \in \mathcal{W} : Q \subset \Omega_\lambda\}$. Then, for every cube $Q \in \mathcal{W}_\lambda$, there exists a chain of cubes $\widetilde{Q} \in \mathcal{W}_\lambda$, denoted $P(Q)$, joining Q to some $Q_0 \in \mathcal{F}$ (depending on Q), with the following properties: (i) The shadows $S(\widetilde{Q}_1)$ and $S(\widetilde{Q}_2)$ (here $S(\widetilde{Q}) = \{Q \in \mathcal{W}_\lambda : \widetilde{Q} \in P(Q)\}$) of cubes $\widetilde{Q}_1, \widetilde{Q}_2 \in \mathcal{W}_\lambda \cup \mathcal{F}$ are either disjoint, or we have that $S(\widetilde{Q}_1) \subset S(\widetilde{Q}_2)$ or $S(\widetilde{Q}_2) \subset S(\widetilde{Q}_1)$; (ii) For each $\widetilde{Q} \in \mathcal{W}_\lambda \cup \mathcal{F}$

$$\#\{Q \in S(\widetilde{Q}) : \operatorname{diam}(Q) \approx 2^{-j}\} \leq C 2^{\lambda j} \operatorname{diam}(\widetilde{Q})^\lambda. \tag{10}$$

Using (10) and the fact that $\beta - p + 2 - \lambda > 0$, we obtain

$$\sum_{Q \in S(\widetilde{Q})} \operatorname{diam}(Q)^{\beta - p + 2} \leq C \operatorname{diam}(\widetilde{Q})^{\beta - p + 2} \tag{11}$$

for all $\widetilde{Q} \in \mathcal{W}_\lambda \cup \mathcal{F}$.

If we now denote $v = |u|^p$, an application of a standard chaining argument (cf. [12, Lemma 8]) gives the estimate

$$|v_Q - v_{Q_0}| \leq C \sum_{\widetilde{Q} \in P(Q)} \operatorname{diam}(\widetilde{Q}) \fint_{\widetilde{Q}} |\nabla v| \tag{12}$$

for $Q \in S(Q_0)$, where $Q_0 \in \mathcal{F}$. Since $|v(x)| \leq |v(x) - v_Q| + |v_Q - v_{Q_0}| + |v_{Q_0}|$ for $x \in Q \in S(Q_0)$, we obtain, using Poncaré's inequality, estimate (12), inequality (11), and finally, the pointwise (p,β)-Hardy inequality for the cubes $Q_0 \in \mathcal{F}$, that

$$\begin{aligned} &\int_{\Omega_\lambda} \left(\frac{|u(x)|}{d_\Omega(x)} \right)^p d_\Omega(x)^\beta \, dx \\ &\quad \leq C \left[\int_{\Omega_\lambda} |u(x)|^{p-1} |\nabla u| d_\Omega(x)^{\beta - p + 1} \, dx + \int_\Omega |\nabla u(x)|^p \, d_\Omega(x)^\beta \, dx \right]. \end{aligned} \tag{13}$$

The (p,β)-Hardy inequality for u then follows from (9) and (13) with a simple use of Hölder's inequality and an elementary observation. See [8] for more details concerning these calculations.

In order to see that $(\tilde{p},\tilde{\beta})$-Hardy inequalities fail in Ω when $\tilde{p} - \tilde{\beta} = 2 - \lambda$, it suffices to consider functions $u_j \in C_0^\infty(\Omega_\lambda)$ such that $u_j(x) = 1$ for $x \in \big([1.5, 3.5] \times [-0.5, 0.5]\big) \setminus \mathcal{N}_{2^{-j}}(K_\lambda)$, $|\nabla u_j| \lesssim 2^j$ in $\mathcal{N}_{2^{-j}}(K_\lambda)$, and $|\nabla u_j| \leq C$ elsewhere in $\operatorname{spt}(|\nabla u_j|)$. Here $\mathcal{N}_\delta(A) = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, A) < \delta\}$. Then it is easy to show that

$$\int_\Omega |u_j|^{\tilde{p}} {d_\Omega}^{\tilde{\beta} - \tilde{p}} \xrightarrow{j \to \infty} \infty,$$

but

$$\int_\Omega |\nabla u_j|^{\tilde{p}} {d_\Omega}^{\tilde{\beta}} \leq C$$

for all $j \in \mathbb{N}$. Similarly, when $\tilde{p} - \tilde{\beta} = 2 - \mu$, $(\tilde{p},\tilde{\beta})$-Hardy inequalities fail for functions $v_j \in C_0^\infty(\Omega)$ with the properties that $v_j(x) = 1$ for $x \in$

$\Omega \setminus \mathcal{N}_{2^{-j}}(\partial\Omega)$, and $|\nabla v_j| \lesssim 2^j$ in $\mathcal{N}_{2^{-j}}(\partial\Omega)$. We conclude in particular that Ω fails to admit the $(p, \beta \pm \varepsilon)$- and $(p \pm \varepsilon, \beta)$-Hardy inequalities.

Finally, let us briefly consider the case $p - 1 \geq \beta + \varepsilon$. Denote again $\mu = \beta + \varepsilon - p + 2$ and $\lambda = \mu - 2\varepsilon$, and let C_μ and C_λ be the standard μ- and λ-dimensional Cantor sets on $[0, 1]$, embedded to $\mathbb{R}^2$. Then e.g. the domain

$$\Omega = \big((0, 5) \times (-2, 2)\big) \setminus \big((C_\mu + (1, 0)) \cup (C_\lambda + (3, 0))\big)$$

admits the (p, β)-Hardy inequality (even the pointwise inequality holds for x not too close to the λ-dimensional Cantor set), but the $(p, \beta \pm \varepsilon)$- and $(p \pm \varepsilon, \beta)$-Hardy inequalities fail to hold in Ω; the calculations are similar to those in the previous case.

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