ONE-DIMENSIONAL FAMILIES OF PROJECTIONS

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Abstract. Let m and n be integers with $0 < m < n$. We consider the question of how much the Hausdorff dimension of a measure may decrease under typical orthogonal projections from $\mathbb{R}^n$ onto m-planes provided that the dimension of the parameter space is one. We verify the best possible lower bound for the dimension drop and illustrate the sharpness of our results by examples. The question stems naturally from the study of measures which are invariant under the geodesic flow.

1. Introduction

The behaviour of different dimensions under projection-type mappings has been intensively investigated for several decades. The study was initiated by Marstrand [Mar] in the 1950's. He proved a well-known preservation theorem for Hausdorff dimension, dim, according to which the Hausdorff dimension of a planar set is preserved under typical orthogonal projections. Later this pioneering result has been extended to different directions by numerous authors (for a detailed account of a variety of related results see [Mat4]): Kaufman [K] verified Marstrand's theorem in terms of potential theoretical methods and Mattila [Mat1] proved it in higher dimensions. For measures the following analogy of Marstrand's preservation principle was discovered by Kaufman [K], Mattila [Mat2], Hu and Taylor [HT], and Falconer and Mattila [FM]: Let m and n be integers with $0 < m < n$ and let μ be a Radon measure on $\mathbb{R}^n$ with compact support. Denoting by μ_V the image of μ under the orthogonal projection onto an m-plane V, we have for typical m-planes V

$$\dim \mu_V = \min\{\dim \mu, m\}. \tag{1.1}$$

The wide investigation of related topics culminated to the work of Peres and Schlag [PS]. Among other things, they proved (1.1) for Sobolev dimension and parametrized families of transversal mappings.

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All the above mentioned results hold for typical projections. This means that they do not provide information about any specified projection. Interestingly, as discovered by Ledrappier and Lindenstrauss [LL], similar potential theoretical methods work if one considers a measure on the Riemann surface which is invariant under the geodesic flow and one particular projection - the natural projection from the unit tangent bundle onto the Riemann surface. It turns out that Hausdorff dimension is preserved in this case [LL]. Quite surprisingly, in higher dimensional base manifolds the preservation principle is not necessarily valid. Indeed, by employing the methods of [PS], it was shown in [JJL] that Hausdorff dimension may drop in higher dimensions. This takes us to the natural question of how much it may drop.

In preservation principles of the type (1.1) it is not necessary to consider the whole Grassmann manifold $G(n,m)$ consisting of all m-dimensional linear subspaces of $\mathbb{R}^n$ as the parameter space. The essential assumption is that the parameter space may be identified with an open subset of $G(n,m)$. However, in the case of n-dimensional Riemann manifolds the local invariance of a measure leads to the study of a 1-dimensional parametrized family of projections from $2(n-1)$-dimensional space onto $(n-1)$-dimensional space [JJL]. Hence, the dimension of the parameter space is less than that of the Grassmann manifold $G(2(n-1), n-1)$.

In this note we address the question of how much the Hausdorff dimension of a measure may drop for typical orthogonal projections from $\mathbb{R}^n$ onto m-planes provided that the dimension k of the parameter space is less than that of the Grassmann manifold. In our setting one may conclude the following from the results of [PS]: Fubini's theorem implies that the Hausdorff dimension of a given measure is preserved for almost all projections in a typical k-dimensional family. However, in general, it is impossible to say whether a given family is typical for a given measure.

We restrict our consideration to the case $k = 1$ which is relevant for measures which are invariant under the geodesic flow (Theorem 3.2). Clearly, one could always parametrize exceptional projections with many parameters. To prevent this from happening, we need to make an assumption guaranteeing that the projection is changed when the parameter is being changed (Definition 2.3). As an auxiliary tool we need to investigate k-dimensional parametrized families of projections from $\mathbb{R}^n$ onto hyperplanes (Proposition 3.1). Using similar methods, we are able to deal with k-dimensional parametrized families of projections from $\mathbb{R}^n$ onto lines (Proposition 3.3). In all these three cases we give the best possible lower bounds for Hausdorff dimensions of typical projected measures. The optimality of the bounds is illustrated by examples. In the whole generality the question seems to be quite difficult.

The paper is organized as follows: In section 2 we introduce the notation and verify auxiliary results whilst section 3 is dedicated to the main results. In section

3 we also discuss the consequences of our results to measures which are invariant under the geodesic flow.

2. Definitions and preliminary results

In this section we introduce the notation we use throughout this paper. Let m and n be integers with $0 < m < n$ and let μ be a finite Radon measure on $\mathbb{R}^n$ with compact support. The Hausdorff dimension, dim, of μ is defined in terms of local dimensions as follows:

$$\dim\mu = \sup\{s \ge 0 \mid \liminf_{r\to 0} \frac{\log\mu(B(x,r))}{\log r} \ge s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n\}, \tag{2.1}$$

where $B(x,r)$ is the closed ball with centre at x and radius r. Equivalently,

$$\dim\mu = \inf\{\dim A \mid A \subset \mathbb{R}^n \text{ is a Borel set with } \mu(A) > 0\}. \tag{2.2}$$

It follows easily from (2.1) that

$$I_t(\mu) < \infty \implies \dim\mu \ge t, \tag{2.3}$$

where

$$I_t(\mu) = \iint |x-y|^{-t}\, d\mu(x)\, d\mu(y)$$

is the t-energy of μ.

Let k be an integer with $0 < k < m(n-m)$. Note that $m(n-m)$ is the dimension of the Grassmann manifold $G(n,m)$ of all m-dimensional linear subspaces of $\mathbb{R}^n$. Supposing that $\Lambda \subset \mathbb{R}^k$, we restrict our consideration to parametrized families $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m \mid \lambda \in \Lambda\}$ of orthogonal projections. The image of a measure μ under $P_\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is denoted by μ_λ, that is, $\mu_\lambda(A) = \mu(P_\lambda^{-1}(A))$ for all $A \subset \mathbb{R}^m$. Obviously,

$$\dim\mu - (n-m) \le \dim\mu_\lambda \le \min\{\dim\mu, m\} \tag{2.4}$$

for all $\lambda \in \Lambda$.

The following well known lemmas play a fundamental role in our approach. The proofs are included for the convenience of the reader. We use the notation $\mathcal{L}^k$ for the Lebesgue measure on $\mathbb{R}^k$.

Lemma 2.1. *Let $\Lambda \subset \mathbb{R}^k$ be bounded. Assume that there are positive constants s and C such that for all $x \ne y \in \mathbb{R}^n$ and for all $\delta > 0$*

$$\mathcal{L}^k(\{\lambda \in \Lambda \mid |P_\lambda(x-y)| \le \delta\}) \le C\delta^s |x-y|^{-s}.$$

Then for all $0 < t < s$ there is a constant c such that for all $x \ne y \in \mathbb{R}^n$

$$\int_\Lambda |P_\lambda(x-y)|^{-t}\, d\mathcal{L}^k(\lambda) \le c|x-y|^{-t}.$$

Proof. Using [Mat3, Corollary 1.15], we calculate as in [Mat3, Corollary 3.12]

$$\begin{aligned}\int_\Lambda |P_\lambda(x-y)|^{-t}\,d\mathcal{L}^k(\lambda) &= \int_0^\infty \mathcal{L}^k(\{\lambda\in\Lambda \mid |P_\lambda(x-y)|\le u^{-\frac{1}{t}}\})\,du\\ &\le \mathcal{L}^k(\Lambda)|x-y|^{-t}+C|x-y|^{-s}\int_{|x-y|^{-t}}^\infty u^{-\frac{s}{t}}\,du\\ &\le c|x-y|^{-t}\end{aligned}$$

where c depends on $\mathcal{L}^k(\Lambda)$, C, s, and t. □

Lemma 2.2. *Let $\Lambda\subset\mathbb{R}^k$ and let μ be a finite Radon measure on $\mathbb{R}^n$ with compact support and let l be a positive real number such that $\dim\mu\ge l$. Assume that for all $0<t<l$ there is a constant c such that for all $x\ne y\in\mathbb{R}^n$*

$$\int_\Lambda |P_\lambda(x-y)|^{-t}\,d\mathcal{L}^k(\lambda)\le c|x-y|^{-t}.$$

Then $\dim\mu_\lambda\ge l$ for $\mathcal{L}^k$-almost all $\lambda\in\Lambda$.

Proof. We may assume that Λ is bounded. Let $\varepsilon>0$. Consider $0<d'<d<l\le\dim\mu$. By (2.1) there are a Borel set $A_{d,\varepsilon}\subset\mathbb{R}^n$ and a positive constant $C_{d,\varepsilon}$ such that $\mu(\mathbb{R}^n\setminus A_{d,\varepsilon})<\varepsilon$ and

$$\mu(B(x,r))\le C_{d,\varepsilon}r^d \text{ for all } r>0 \text{ and } x\in A_{d,\varepsilon}.$$

Define $\mu_{d,\varepsilon}=\mu|_{A_{d,\varepsilon}}$. Clearly,

$$\mu_{d,\varepsilon}(B(x,r))\le 2^dC_{d,\varepsilon}r^d \text{ for all } r>0 \text{ and } x\in\mathbb{R}^n,$$

which, in turn, implies by a straightforward calculation (see [Mat3, p. 109]) that $I_{d'}(\mu_{d,\varepsilon})<\infty$. Hence, by Fubini's theorem

$$\begin{aligned}\int_\Lambda I_{d'}((\mu_{d,\varepsilon})_\lambda)\,d\mathcal{L}^k(\lambda) &= \iiint_\Lambda |P_\lambda(x-y)|^{-d'}\,d\mathcal{L}^k(\lambda)\,d\mu_{d,\varepsilon}(x)\,d\mu_{d,\varepsilon}(y)\\ &\le cI_{d'}(\mu_{d,\varepsilon})<\infty.\end{aligned}$$

Therefore, by (2.3) $\dim(\mu_{d,\varepsilon})_\lambda\ge d'$ for $\mathcal{L}^k$-almost all $\lambda\in\Lambda$. Letting $\varepsilon\to 0$ and $d'\to l$ through countable sequences gives the claim since $\dim(\mu_{d,\varepsilon})_\lambda\to\dim\mu_\lambda$ by (2.2). □

We equip the Grassmann manifold $G(n,m)$ with a Riemann metric and continue by defining the type of projection families we are working with.

Definition 2.3. Let $\Lambda\subset\mathbb{R}^k$ be open and connected. A parametrized family $\{P_\lambda:\mathbb{R}^n\to\mathbb{R}^m \mid \lambda\in\Lambda\}$ of orthogonal projections is called full if there exist positive constants R_f, C_f, and c_f with the following properties:

(1) The mapping $\lambda\mapsto V_\lambda$ restricted to $B(\lambda_0,R_f)$ is an embedding with uniformly continuous derivative for all $\lambda_0\in\Lambda$.
(2) For all $\lambda=(\lambda_1,\dots,\lambda_k)\in\Lambda$ we have $|\frac{\partial V_\lambda}{\partial\lambda_i}|\le C_f$ for all $i=1,\dots,k$.

(3) Moreover, $\mathrm{vol}(\frac{\partial V_\lambda}{\partial \lambda_1}, \dots, \frac{\partial V_\lambda}{\partial \lambda_k}) \geq c_f$ for all $\lambda = (\lambda_1, \dots, \lambda_k) \in \Lambda$.

Here $V_\lambda = P_\lambda(\mathbb{R}^n) \in G(n,m)$ and $\mathrm{vol}(v_1, \dots, v_k)$ is the k-dimensional volume of the parallelepiped spanned by the vectors $v_1, \dots, v_k$.

Remark 2.4. (a) The third property of Definition 2.3 implies that there is a positive constant d_f such that $|\frac{\partial V_\lambda}{\partial \lambda_i}| \geq d_f$ for all $i = 1, \dots, k$.

(b) Consider $\lambda_0 \in \Lambda$ and $R > 0$ such that $\mathcal{V} = \cup_{\lambda \in B(\lambda_0, R)} V_\lambda$ is a smooth k-dimensional submanifold of $G(n,m)$. Let $\{e_1, \dots, e_m\}$ and $\{e_{m+1}, \dots, e_n\}$ be orthonormal bases of $V_{\lambda_0} \in G(n,m)$ and its orthogonal complement $V_{\lambda_0}^\perp \in G(n, n-m)$, respectively. One may choose local coordinates on $G(n,m)$ near V_{λ_0} in terms of rotations of the basis vectors $\{e_1, \dots, e_m\}$ in the following manner: For $1 \leq i \leq m$ and $m+1 \leq j \leq n$, let $-\frac{\pi}{4} < \alpha_{ij} < \frac{\pi}{4}$ be the components of $\alpha \in (-\frac{\pi}{4}, \frac{\pi}{4})^{m(n-m)}$. Rotating e_i by the angle α_{ij} towards e_j for all i and j gives local coordinates for the m-plane spanned by the rotated vectors. We fix a Riemann metric on $G(n,m)$ such that $\{\frac{\partial}{\partial \alpha_{ij}} \mid i = 1, \dots, m, j = m+1, \dots, n\}$ is an orthonormal basis of the tangent space $T_{V_{\lambda_0}} G(n,m)$. For simplicity, we refer to these basis vectors by $e_{ij} = \frac{\partial}{\partial \alpha_{ij}}$.

3. Main results

In this section we state and prove our main results. We start with the case of k-dimensional families of projections onto hyperplanes, which is an essential tool in the proof of Theorem 3.2.

Proposition 3.1. *Let $\Lambda \subset \mathbb{R}^k$ be an open and connected set and let μ be a finite Radon measure on $\mathbb{R}^n$ with compact support. Assume that $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^{n-1} \mid \lambda \in \Lambda\}$ is a full family of orthogonal projections. Then for $\mathcal{L}^k$-almost all $\lambda \in \Lambda$*

$$\dim \mu_\lambda \geq \begin{cases} \dim \mu, & \text{if } \dim \mu < k \\ k, & \text{if } k \leq \dim \mu < k+1 \\ \dim \mu - 1, & \text{if } \dim \mu \geq k+1. \end{cases}$$

Proof. By (2.4) it is sufficient to consider the case $\dim \mu < k$. Writing Λ as a countable union of open balls, we may assume that Λ is an open ball. We make this assumption to avoid some technical problems caused by the boundary of Λ when using the inverse function theorem.

Our claim is that there is a constant C depending only on C_f and c_f such that for all $x \neq y \in \mathbb{R}^n$ and for all $\delta > 0$

$$\mathcal{L}^k(\{\lambda \mid |P_\lambda(x-y)| \leq \delta\}) \leq C \delta^k |x-y|^{-k}. \tag{3.1}$$

To verify (3.1) we may assume that $|x-y| = 1$. Furthermore, it is enough to prove (3.1) for $0 < \delta < \delta_0$ where δ_0 is a constant depending only on C_f and c_f. Note that hyperplanes can be parametrized by their orthogonal complements

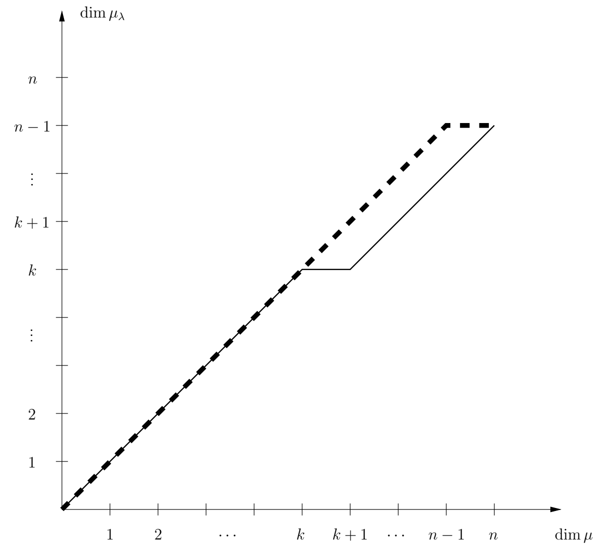

FIGURE 1. The case of a full k-dimensional parametrized family of projections onto hyperplanes: the lower bound given by Proposition 3.1 and the upper one given by (2.4) are illustrated by —— and - - -, respectively.

and $G(n, n-1)$ may be locally identified with an open subset of the unit sphere S^{n-1}. By Definition 2.3,

$$\operatorname{vol}(\frac{\partial V_\lambda^\perp}{\partial \lambda_1}, \dots, \frac{\partial V_\lambda^\perp}{\partial \lambda_k}) \geq c_f$$

for all $\lambda = (\lambda_1, \dots, \lambda_k) \in \Lambda$.

Consider $\lambda^1 \in \Lambda$ such that $|P_{\lambda^1}(x-y)| < \delta_0$. Making δ_0 sufficiently small, a quantitative version of the inverse function theorem (see [PS, Lemma 7.6]) guarantees that there is a neighbourhood U_1 of λ^1 with the following properties (for further details see [JJN, Lemma 2.2]):

(1) The set U_1 contains a ball B_1 the radius of which depends only on C_f and c_f.
(2) The restriction of $\lambda \mapsto P_\lambda(x-y)$ to U_1 is a diffeomorphism onto its image.
(3) There are $\hat{\lambda}^1 \in \Lambda$ and a constant M depending only on C_f and c_f such that

$$\{\lambda \in U_1 \mid P_\lambda(x-y) \leq \delta\} \subset B(\hat{\lambda}^1, M\delta)$$

for all $0 < \delta < \delta_0$.
(4) For all $\lambda \in \partial U_1$ we have $|P_\lambda(x-y)| \geq 4\delta_0$, and for all $\lambda \in B_1$ we have $|P_\lambda(x-y)| < 2\delta_0$.

Note that if λ^1 is close to the boundary of Λ we may have to extend the family outside of Λ in order to find $\tilde{\lambda}^1$ and B_1. This is the reason why we assume that Λ is a ball.

Having selected open sets $U_1, \ldots, U_k$ such that the above properties (1)–(4) are valid, we proceed inductively by taking $\lambda^{k+1} \in \Lambda \setminus \cup_{i=1}^k U_i$ with $|P_{\lambda^{k+1}}(x-y)| < \delta_0$. Choose a neighbourhood U_{k+1} of λ^{k+1} having properties (1)–(4). Since the balls $B_1, B_2, \ldots, B_{k+1}$ selected in (1) are disjoint by (4), the process terminates after a finite number of steps, say k_0. Using the fact that $|P_\lambda(x-y)| \geq 4\delta_0$ when $\lambda \in \Lambda \setminus \cup_{i=1}^{k_0} U_i$, we get from (3) that for all $0 < \delta < \delta_0$

$$\{\lambda \in \Lambda \mid P_\lambda(x-y) \leq \delta\} \subset \bigcup_{i=1}^{k_0} \{\lambda \in U_i \mid P_\lambda(x-y) \leq \delta\} \subset \bigcup_{i=1}^{k_0} B(\hat{\lambda}^i, M\delta).$$

From this one easily deduces (3.1). Note that by (1), the constant k_0 depends only on C_f and c_f

Since $\dim \mu < k$, Lemma 2.2 combined with Lemma 2.1 gives with the choice $l = \dim \mu$ that $\dim \mu_\lambda \geq \dim \mu$ for $\mathcal{L}^k$-almost all $\lambda \in \Lambda$. □

Now we are ready to consider the case of 1-dimensional families of projections onto m-planes. The proof is based on Proposition 3.1.

Theorem 3.2. *Let $\Lambda \subset \mathbb{R}$ be an open interval and let μ be a finite Radon measure on $\mathbb{R}^n$ with compact support. Assume that $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m \mid \lambda \in \Lambda\}$ is a full family of orthogonal projections. Then for $\mathcal{L}$-almost all $\lambda \in \Lambda$*

$$\dim \mu_\lambda \geq \begin{cases} \max\{0, \dim \mu - (n-m-1)\}, & \text{if } \dim \mu < n-m \\ 1, & \text{if } n-m \leq \dim \mu < n-m+1 \\ \dim \mu - (n-m), & \text{if } \dim \mu \geq n-m+1. \end{cases}$$

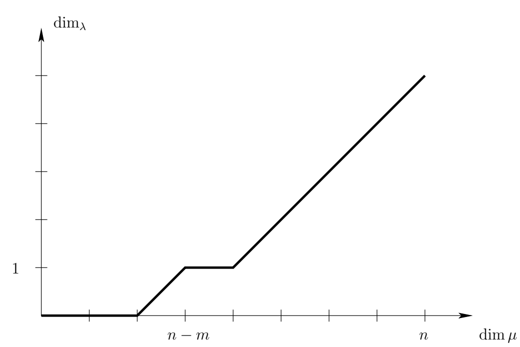

FIGURE 2. The case of a full 1-dimensional parametrized family of projections onto m-planes.

Proof. By (2.4) it is enough to consider the case $\dim \mu < n - m$. Fix $\lambda^0 \in \Lambda$. Let $\frac{\partial V_\lambda}{\partial \lambda}|_{\lambda=\lambda^0} = \sum_{ij} \lambda_{ij} e_{ij}$ where $\lambda_{ij} \in \mathbb{R}$ and $\{e_{ij}\}$ is the orthonormal basis of the tangent space $T_{V_{\lambda_0}} G(n,m)$ given in Remark 2.4. After renaming the coordinates we may assume that $|\lambda_{1n}| = \max_{ij} |\lambda_{ij}|$. By Definition 2.3, there is a constant c depending only on c_f, n, and m such that $|\lambda_{1n}| \geq c$.

Define a $(n-m)$-dimensional family $\{\widetilde{V}_{\tilde{\lambda}} \mid \tilde{\lambda} \in \widetilde{\Lambda} \subset \mathbb{R}^{n-m}\}$ of hyperplanes in the following way: By the uniform continuity of $\lambda \mapsto \frac{\partial V_\lambda}{\partial \lambda}$ we find $0 < \delta < \frac{\pi}{8}$ such that

$$|(\frac{\partial V_\lambda}{\partial \lambda} \mid \frac{\partial}{\partial \alpha_{1n}}|_{V_\lambda})| > \frac{c}{2} \text{ for all } \lambda \in (\lambda^0 - \delta, \lambda^0 + \delta)$$

where $\frac{\partial}{\partial \alpha_{1n}}|_{V_\lambda} \in T_{V_\lambda} G(n,m)$. Let $\widetilde{\Lambda} = (\lambda^0 - \delta, \lambda^0 + \delta) \times (-\delta, \delta)^{n-m-1}$. Given $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n-m}) \in \widetilde{\Lambda}$, let $\widetilde{V}_{\tilde{\lambda}}$ be the $(n-1)$-plane spanned by $V_{\tilde{\lambda}_1}$ and $e_{m+1} \cos \tilde{\lambda}_2 + e_n \sin \tilde{\lambda}_2, \ldots, e_{n-1} \cos \tilde{\lambda}_{n-m} + e_n \sin \tilde{\lambda}_{n-m}$. Here $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$ used in the construction of the basis $\{e_{ij}\}$.

For $\tilde{\lambda}^0 = (\lambda^0, 0 \ldots, 0)$ we have

$$\frac{\partial \widetilde{V}_{\tilde{\lambda}}}{\partial \tilde{\lambda}_i}|_{\tilde{\lambda}=\tilde{\lambda}^0} = e_{m-1+i,n} \text{ for all } i = 2, \ldots, n-m.$$

Clearly, $\lambda \mapsto \frac{\partial \widetilde{V}_{\tilde{\lambda}}}{\partial \tilde{\lambda}_i}$ is uniformly continuous for all $i = 1, \ldots, n-m$. Hence, by the choice of δ, $\widetilde{V}_{\tilde{\lambda}}$ determines a full $(n-m)$-dimensional family of projections $\widetilde{P}_{\tilde{\lambda}} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ with $\widetilde{P}_{\tilde{\lambda}}(\mathbb{R}^n) = \widetilde{V}_{\tilde{\lambda}}$. Denoting by $\widetilde{P}_{\tilde{\lambda}}\mu$ the image of μ under $\widetilde{P}_{\tilde{\lambda}}$ and applying Proposition 3.1 gives

$$\dim \widetilde{P}_{\tilde{\lambda}}\mu = \dim \mu \text{ for } \mathcal{L}^{n-m}\text{-almost all } \tilde{\lambda} \in \widetilde{\Lambda}. \tag{3.2}$$

By Fubini's theorem, for $\mathcal{L}$-almost all $\tilde{\lambda}_1 \in (\lambda^0 - \delta, \lambda^0 + \delta)$ there is $(\tilde{\lambda}_2, \ldots, \tilde{\lambda}_{n-m}) \in (-\delta, \delta)^{n-m-1}$ such that (3.2) holds for $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n-m})$. Combining (2.4) with the fact that $V_{\tilde{\lambda}_1} \subset \widetilde{V}_{\tilde{\lambda}}$ implies the claim since

$$\begin{aligned} \dim \mu_{\tilde{\lambda}_1} &= \dim P_{\tilde{\lambda}_1}\mu = \dim(P_{\tilde{\lambda}_1} \widetilde{P}_{\tilde{\lambda}})\mu \\ &\geq \dim \widetilde{P}_{\tilde{\lambda}}\mu - (n-m-1) = \dim \mu - (n-m-1) \end{aligned}$$

for $\mathcal{L}$-almost all $\tilde{\lambda}_1 \in (\lambda^0 - \delta, \lambda^0 + \delta)$. □

Finally, we consider k-dimensional families of projections onto lines.

Proposition 3.3. *Let $\Lambda \subset \mathbb{R}^k$ be an open and connected set and let μ be a finite Radon measure on $\mathbb{R}^n$ with compact support. Assume that $\{P_\lambda : \mathbb{R}^n \to \mathbb{R} \mid \lambda \in \Lambda\}$ is a full family of orthogonal projections. Then for $\mathcal{L}^k$-almost all $\lambda \in \Lambda$*

$$\dim \mu_\lambda \geq \begin{cases} \max\{0, \dim \mu - (n-k-1)\}, & \textit{if } \dim \mu < n-k \\ 1, & \textit{if } \dim \mu \geq n-k. \end{cases}$$

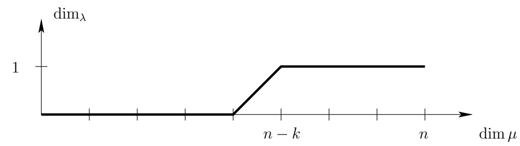

FIGURE 3. The case of a full k-dimensional parametrized family of projections onto lines.

Proof. Clearly it is enough to consider the case $n-k-1<\dim\mu<n-k$. Since $G(n,1)$ may be locally identified with S^{n-1} the tangent space $T_{V_\lambda}G(n,1)$ may be embedded naturally in $\mathbb{R}^n$. Fix $\lambda^0\in\Lambda$. The assumption that the family $\{P_\lambda:\mathbb{R}^n\to\mathbb{R}\mid\lambda\in\Lambda\}$ is full implies that the vectors $\frac{\partial V_\lambda}{\partial\lambda_1}|_{\lambda=\lambda^0},\dots,\frac{\partial V_\lambda}{\partial\lambda_k}|_{\lambda=\lambda^0}$ span a k-dimensional plane K. Choose an orthonormal basis $(e_1,\dots,e_n)$ of $\mathbb{R}^n$ such that e_1 spans V_{λ^0} and the vectors $e_{n-k+1},\dots,e_n$ span K. Fix $0<\delta<\frac{\pi}{8}$ such that the mapping $\lambda\mapsto V_\lambda$ restricted to $B(\lambda^0,\delta)$ is a diffeomorphism onto its image and the angle between $\frac{\partial V_\lambda}{\partial\lambda_i}$ and K is less than $\frac{\pi}{8}$ for all $i=1,\dots,k$ and $\lambda\in B(\lambda^0,\delta)$.

We define a $k(n-k)$-dimensional family $\{\widetilde{V}_{\tilde\lambda}\mid\tilde\lambda\in\widetilde\Lambda\subset\mathbb{R}^{k(n-k)}\}$ of $(n-k)$-planes as follows: Let $\widetilde\Lambda=B(\lambda^0,\delta)\times(-\delta,\delta)^{k(n-k-1)}$. Given $\tilde\lambda\in\widetilde\Lambda$, denote the first k components of $\tilde\lambda$ by λ and the remaining $k(n-k-1)$ components by $\tilde\lambda_{ij}$ for $i=2,\dots,n-k$ and $j=n-k+1,\dots,n$, and define $\widetilde{V}_{\tilde\lambda}$ to be the $(n-k)$-plane spanned by V_λ and $e_2\cos\tilde\lambda_{2,n-k+1}+e_{n-k+1}\sin\tilde\lambda_{2,n-k+1},\dots,e_{n-k}\cos\tilde\lambda_{n-k,n}+e_n\sin\tilde\lambda_{n-k,n}$.

Now $\{\widetilde{V}_{\tilde\lambda}\mid\tilde\lambda\in\widetilde\Lambda\}$ is an open subset of $G(n,n-k)$ and (1.1) implies that

$$\dim\widetilde{P}_{\tilde\lambda}\mu=\dim\mu\text{ for }\mathcal{L}^{k(n-k)}\text{-almost all }\tilde\lambda\in\widetilde\Lambda.$$

The claim follows by Fubini's theorem and (2.4) as in the proof of Theorem 3.2. □

Remark 3.4. (a) It is easy to see that all the above lower bounds may be achieved. To verify this, let $\{e_1,\dots,e_n\}$ be the standard basis of $\mathbb{R}^n$.

In the case of k-dimensional families of projections onto hyperplanes, denote by W the $(k+1)$-plane spanned by $e_1,\dots,e_k,e_n$. Rotate the vectors $e_1,\dots,e_k$ towards e_n independently and consider the k-dimensional family of projections onto hyperplanes V_λ spanned by $e_{k+1},\dots,e_{n-1}$ and rotations of $e_1,\dots,e_k$. Assuming that μ is a measure on W, we have

$$\dim\mu_\lambda=\min\{\dim\mu,k\}\text{ for }\mathcal{L}^k\text{-almost all }\lambda.$$

On the other hand, taking $\mu=\nu_1\times\nu_2$ where ν_1 is the restriction of $\mathcal{L}^{k+1}$ to the unit ball of W and ν_2 is a measure on $W^\perp$ with $\dim\nu_2=t$, we get

$$\dim\mu_\lambda=k+t=\dim\mu-1\text{ for }\mathcal{L}^k\text{-almost all }\lambda.$$

This gives the sharpness of Proposition 3.1.

Next we consider the case of 1-dimensional family of projections onto m-planes V_λ spanned by $e_2, \dots, e_m$ and rotations of e_1 towards e_{m+1}. Denote by W_1 the 2-plane spanned by e_1 and e_{m+1}, and by W_2 the $(m+1)$-plane spanned by $e_1, \dots, e_{m+1}$. Let $0 < t \le 2$. Defining $\mu = \nu_1 \times \nu_2$ where ν_1 is a measure on W_1 with $\dim \nu_1 = t$ and ν_2 is the restriction of $\mathcal{L}^{n-m-1}$ to the unit ball of $W_2^\perp$, we obtain

$$\dim \mu_\lambda = \dim \nu_t = \min\{t, 1\} \text{ for } \mathcal{L}\text{-almost all } \lambda.$$

Hence, the two first lower bounds of Theorem 3.2 may be achieved. The fact that the remaining lower bound in Theorem 3.2 is sharp can be verified similarly.

Finally, the sharpness of Proposition 3.3 follows by considering the k-dimensional family of projections onto lines spanned by rotating e_1 towards $e_2, \dots, e_{k+1}$ and by defining the measure μ as in the case of m-planes with m replaced by k.

(b) Representing Λ as a countable union of compact sets one may replace the uniform bounds in Definition 2.3 by local ones.

(c) As indicated in (a), the lower bounds given in Propositions 3.1, 3.2 and 3.3 are the best possible ones in the sense that for each $0 < d < n$ there is a measure μ with $\dim \mu = d$ and a family of projections such that the corresponding lower bounds are achieved. However, this does not mean that for any family of projections and for any $0 < d < n$ one could construct a measure achieving the lower bounds. Indeed, let $n = 4$, $m = 2$, and let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$. Consider the family of 2-planes V_λ spanned by $e_1 \cos\lambda + e_3 \sin\lambda$ and $e_2 \cos\lambda + e_4 \sin\lambda$ for $\lambda \in (-\frac{\pi}{8}, \frac{\pi}{8})$. It is easy to see that

$$\mathcal{L}(\{\lambda \mid |P_\lambda(z)| < \delta\}) \le \delta |z|^{-1}$$

for any $z \in \mathbb{R}^4$. This implies that $\dim \mu_\lambda = \dim \mu$ for $\mathcal{L}$-almost all λ provided that $\dim \mu \le 1$ whilst the lower bound given by Theorem 3.2 is zero.

(d) The study of projections of measures which are invariant under the geodesic flow on an n-dimensional Riemann manifold leads to a study of 1-dimensional parametrized families of mappings from a $2(n-1)$-dimensional manifold onto an $(n-1)$-dimensional manifold (see [JJL]).

In the case of an n-dimensional torus T the setting is as follows: Let Π be the natural projection from the unit tangent bundle of T onto T and let μ be an invariant measure under the geodesic flow. Then μ is locally of the form $\nu \times \mathcal{L}$ and the image $\Pi_*\mu$ of μ under Π is locally of the form $\nu_t \times \mathcal{L}$, where ν_t is the image of ν under a projection from $\mathbb{R}^{2(n-1)}$ onto $\mathbb{R}^{n-1}$. According to Theorem 3.2 and [JJL, Lemma 2.2],

$$\dim \Pi_*\mu \ge \dim \mu - (n-2)$$

provided that $\dim \nu \le n-1$, that is, $\dim \mu \le n$. On the other hand, if $\dim \nu \ge n$ then

$$\dim \Pi_*\mu \ge \dim \mu - (n-1).$$

In fact, since the setting in this case is as in (c) we have $\dim \Pi_*\mu = \dim \mu$ if $\dim \nu \leq 1$.

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