

Analytic capacity and quasiconformal mappings with $W^{1,2}$ Beltrami coefficient

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Abstract

We show that if ϕ is a quasiconformal mapping with compactly supported Beltrami coefficient in the Sobolev space $W^{1,2}$, then ϕ preserves sets with vanishing analytic capacity. It then follows that a compact set E is removable for bounded analytic functions if and only if it is removable for bounded quasiregular mappings with compactly supported Beltrami coefficient in $W^{1,2}$.

1 Introduction

A Beltrami coefficient is a measurable function μ such that $\|\mu\|_\infty < 1$. Given an open set $\Omega \subset \mathbb{C}$, we say that $f : \Omega \rightarrow \mathbb{C}$ is μ -quasiregular if it belongs to the Sobolev space $W_{loc}^{1,2}(\Omega)$ and satisfies the Beltrami equation

$$\bar{\partial}f(z) = \mu(z)\partial f(z), \quad a.e. z \in \Omega.$$

If moreover f is a homeomorphism, then we call it μ -quasiconformal. For any $K \geq 1$, we say that f is K -quasiregular (or K -quasiconformal if f is homeomorphism) for some Beltrami coefficient μ satisfying $\|\mu\|_\infty \leq \frac{K-1}{K+1}$.

Several works have focussed in the question of how these mappings distort measures and capacities. For instance, Ahlfors (see [Ah1]) proved that they always preserve sets of zero area. In a remarkable paper, Astala [As] obtained deep estimates for the area distortion under K -quasiconformal mappings. More precisely, if ϕ is any (conveniently normalized) K -quasiconformal mapping, then

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one has the estimate

$$|\phi(E)| \leq C|E|^{\frac{1}{K}}$$

where the constant C depends only on K . As a consequence, the author obtained also sharp results on integrability of K -quasiconformal mappings, which in turn led to the bounds on K -quasiconformal distortion of Hausdorff dimension. Namely, for any K -quasiconformal mapping ϕ and any compact set E ,

$$\frac{1}{K} \left(\frac{1}{\dim(E)} - \frac{1}{2} \right) \leq \frac{1}{\dim(\phi(E))} - \frac{1}{2} \leq K \left(\frac{1}{\dim(E)} - \frac{1}{2} \right). \quad (1)$$

Moreover, in [As] the author shows the sharpness of both inequalities.

It is well known that sometimes the regularity of the Beltrami coefficient μ is inherited by the mapping itself. For instance, when μ is a compactly supported C^∞ function, then every μ -quasiconformal mapping ϕ is also C^∞ . As a consequence, ϕ is locally bilipschitz, and then some set functions like Hausdorff measures, Riesz and Bessel capacities, are preserved.

Nevertheless, there are other situations which, even far from $\mu \in C^\infty$, give interesting improvements of equation (1). For instance, when μ belongs to the class VMO of functions of vanishing mean oscillation, then

$$\dim(\phi(E)) = \dim(E).$$

That is, the corresponding μ -quasiconformal mappings ϕ do not distort Hausdorff dimension (see for instance [Iw]). However, nothing is known on the ratio between the Hausdorff measures $\mathcal{H}^t(E)$ and $\mathcal{H}^t(\phi(E))$, for any $t \in [0, 2]$.

In this context, of special interest is the assumption that μ is a compactly supported function in the Sobolev class $W^{1,2}$ (notice that this implies $\mu \in VMO$). For such Beltrami coefficients, it is shown in [CFMOZ] that the corresponding μ -quasiconformal mappings ϕ preserve sets with zero length, that is

$$\mathcal{H}^1(E) = 0 \quad \iff \quad \mathcal{H}^1(\phi(E)) = 0, \quad (2)$$

The proof of this fact uses some BMO removability techniques, related to both the Cauchy-Riemann (i.e. $\bar{\partial}$) and the Beltrami ($\bar{\partial} - \mu\partial$) differential operators. The main tool is an extended version of Weyl's lemma. Recall that classical Weyl's Lemma asserts that distributional solutions to the Cauchy-Riemann equation are actually analytic functions. In the more general case of the Beltrami equation [CFMOZ, Theorem 1], an analogous result can be given provided that the Beltrami coefficient belongs to $W^{1,2}$.

Theorem. *Let μ be a compactly supported Beltrami coefficient in the Sobolev space $W^{1,2}(\mathbb{C})$. Let $f \in L^p_{loc}(\mathbb{C})$ for some $p > 2$, and suppose that*

$$\langle \bar{\partial}f - \mu \partial f, \varphi \rangle = 0$$

whenever $\varphi \in C^\infty$ is compactly supported. Then, f is μ -quasiregular.

In [CFMOZ], similar arguments to those in (2), replacing *BMO* by *VMO*, allowed the authors to prove that if $\mu \in W^{1,2}$ is any compactly supported Beltrami coefficient, and ϕ is μ -quasiconformal, then

$$\mathcal{H}^1(E) \text{ is } \sigma\text{-finite} \quad \iff \quad \mathcal{H}^1(\phi(E)) \text{ is } \sigma\text{-finite.} \quad (3)$$

Furthermore, these mappings ϕ are shown to map 1-rectifiable sets to 1-rectifiable sets (and purely 1-unrectifiable sets to purely 1-unrectifiable sets).

As we shall see in this paper, all these facts have interesting consequences when studying removability problems for bounded μ -quasiregular mappings, that is, the μ -quasiregular counterpart for the problem of Painlevé. Recall that a compact set E is said to be *removable (for bounded analytic functions)* if for any open set $\Omega \supset E$, every bounded function $f : \Omega \rightarrow \mathbb{C}$, analytic on $\Omega \setminus E$, admits an analytic extension to the whole of Ω . The *problem of Painlevé* consists of giving metric and geometric characterizations of these sets.

When studying removable sets, it is natural to talk about analytic capacity. Recall that given a compact set E , the *analytic capacity* of E is defined as

$$\gamma(E) = \sup \{ |f'(\infty)|; f \in H^\infty(\mathbb{C} \setminus E), \|f\|_\infty \leq 1 \}.$$

Here, by $H^\infty(\Omega)$ we mean the space of bounded analytic functions on the open set Ω , and $f'(\infty) = \lim_{z \rightarrow \infty} z(f(\infty) - f(z))$. For a set $A \subset \mathbb{C}$ which may be non compact, one defines

$$\gamma(A) = \sup_{E \subset A \text{ compact}} \gamma(E).$$

Ahlfors [Ah2] proved that E is removable for bounded analytic functions if and only if $\gamma(E) = 0$. Furthermore, it is not difficult to show that $\gamma(E) \leq C \mathcal{H}^1(E)$, while $\dim(E) > 1$ implies $\gamma(E) > 0$. It took long time to have a precise geometric characterization of the zero sets for γ . In [Da1], G. David proved that if E has finite length then

$$\gamma(E) = 0 \quad \iff \quad E \text{ is purely 1-unrectifiable.}$$

Later, in [To2], X. Tolsa characterized sets with vanishing analytic capacity in terms of Menger curvature (see Theorem 5 below for more details).

In this paper, as well as in [CFMOZ], our main objects of study are the removable singularities for bounded solutions to a fixed Beltrami equation. Namely, we say that a compact set E is *removable for bounded μ -quasiregular mappings*, or simply *μ -removable*, if for any open set Ω , any bounded function $f : \Omega \rightarrow \mathbb{C}$, μ -quasiregular on $\Omega \setminus E$, admits a μ -quasiregular extension to the whole of Ω . By means of Stoilow's factorization Theorem, one easily shows that E is μ -removable if and only if $\gamma(\phi(E)) = 0$ for any μ -quasiconformal mapping ϕ . In connection with this question, the following result is proved in [CFMOZ].

Theorem. *Let $\mu \in W^{1,2}(\mathbb{C})$ be a compactly supported Beltrami coefficient, and let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be a μ -quasiconformal mapping. Then,*

$$\gamma(E) = 0 \quad \iff \quad \gamma(\phi(E)) = 0 \quad (4)$$

for any compact set E with σ -finite $\mathcal{H}^1(E)$.

Pekka Koskela suggested us that the σ -finiteness assumption might be removed in the preceding result. In this paper we do the job.

Theorem 1. *Let $\mu \in W^{1,2}(\mathbb{C})$ be a compactly supported Beltrami coefficient, and let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be a μ -quasiconformal mapping. Then,*

$$\gamma(E) = 0 \quad \iff \quad \gamma(\phi(E)) = 0$$

for any compact set E .

It follows from Theorem 1 that if $\mu \in W^{1,2}$ is compactly supported, then being removable and being μ -removable are equivalent notions.

Corollary 2. *Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient. Then, a compact set E is removable for bounded μ -quasiregular mappings if and only if $\gamma(E) = 0$.*

Theorem 1 implies that, given a compactly supported Beltrami coefficient $\mu \in W^{1,2}(\mathbb{C})$, the corresponding μ -quasiconformal mappings preserve the removable sets for bounded analytic functions. This fact is closely related to a question of J. Verdera [Ve1] on the preservation of removable sets under some planar homeomorphisms. More precisely, the author wondered how analytic capacity is distorted under bilipschitz mappings. Recall that a mapping $\phi : \mathbb{C} \rightarrow \mathbb{C}$ is said to be L -bilipschitz if

$$\frac{1}{L}|z - w| \leq |\phi(z) - \phi(w)| \leq L|z - w|$$

for any pair of points $z, w \in \mathbb{C}$. This question was solved in [To2]:

Theorem. *Let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be an L -bilipschitz mapping. Then,*

$$\gamma(\phi(E)) \simeq \gamma(E) \tag{5}$$

with constants that depend only on L .

Furthermore, it is shown in [To2] that any planar homeomorphism $\phi : \mathbb{C} \rightarrow \mathbb{C}$ satisfying (5) must be a bilipschitz mapping. It is well known that L -bilipschitz mappings are μ -quasiconformal for some Beltrami coefficient μ with $\|\mu\|_\infty$ depending only on L , but in general this does not imply any $W^{1,2}$ regularity for μ . In fact, there is not a precise description of the class of compactly supported Beltrami coefficients μ that produce bilipschitz μ -quasiconformal mappings. It was remarked in [CFMOZ, Example 4] that there are non bilipschitz μ -quasiconformal mappings with compactly supported $\mu \in W^{1,2}$. At the same time, the example $\mu(z) = \frac{1}{2} \chi_{\mathbb{D}}(z)$ gives a bilipschitz μ -quasiconformal mapping with $\mu \notin W^{1,2}$. Thus, there is no relation between bilipschitz μ -quasiconformal mappings and Beltrami coefficients $\mu \in W^{1,2}$.

For the proof of the Theorem 1, our main tool is the following improved version of the preceding theorem on the bilipschitz invariance of analytic capacity.

Theorem 3. *Given $E, F \subset \mathbb{C}$, let $\phi : E \rightarrow F$ be a bilipschitz homeomorphism. That is, there exists $L > 0$ such that*

$$\frac{1}{L} |z - w| \leq |\phi(z) - \phi(w)| \leq L |z - w| \tag{6}$$

whenever $z, w \in E$. Then there exists some constant C depending only on L such that

$$\frac{1}{C} \gamma(F) \leq \gamma(E) \leq C \gamma(F).$$

Notice that in this result we assume the mapping ϕ to be bilipschitz only on E , not in the whole complex plane. From this theorem we deduce

Corollary 4. *Assume that $\phi : \mathbb{C} \rightarrow \mathbb{C}$ is a planar homeomorphism, locally bilipschitz in a measurable set $\Omega \subset \mathbb{C}$. That is, there are constants $C > 0$ and $\delta > 0$ such that*

$$\frac{1}{C} |z - w| \leq |\phi(z) - \phi(w)| \leq C |z - w|$$

whenever $z, w \in \Omega$ and $|z - w| < \delta$. Then

$$\gamma(E \cap \Omega) = 0 \quad \iff \quad \gamma(\phi(E \cap \Omega)) = 0$$

for any compact set $E \subset \mathbb{C}$.

As explained above, the μ -quasiconformal mappings we deal with are not bilipschitz in the whole plane, in general. However, they are locally bilipschitz on the level sets of its Jacobian determinant, modulo some *small* set of bad points. This follows from the quasimetricity and from the fact that these μ -quasiconformal mappings belong to some second order Sobolev spaces. Moreover, it turns out that they are strongly differentiable everywhere except on a set of Hausdorff dimension 0.

The paper is structured as follows. In Section 2 we prove Theorem 3, and in Section 3 its Corollary 4. In Section 4, we use this result to prove Theorem 1.

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2 Proof of Theorem 3

2.1 Analytic capacity and curvature

We need to recall the notion of curvature of a measure. Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x, y, z) = \frac{1}{R(x, y, z)},$$

where $R(x, y, z)$ is the radius of the circumference passing through x, y, z (with $R(x, y, z) = \infty$, $c(x, y, z) = 0$ if x, y, z lie on the same line). If two among these points coincide, we set $c(x, y, z) = 0$. For a positive finite Borel measure σ on \mathbb{C} , we define the *curvature of σ* as

$$c^2(\sigma) = \int \int \int c(x, y, z)^2 d\sigma(x) d\sigma(y) d\sigma(z). \quad (7)$$

We recall the characterization of γ in terms of curvature from [To2]:

Theorem 5. *For any compact $E \subset \mathbb{C}$ we have*

$$\gamma(E) \simeq \sup \sigma(E),$$

where the supremum is taken over all Borel measures σ supported on E such that $\sigma(B(x, r)) \leq r$ for all $x \in E$, $r > 0$ and $c^2(\sigma) \leq \sigma(E)$.

We will prove the following result.

Theorem 6. *Let σ be a Borel measure supported on a compact set $E \subset \mathbb{C}$, such that $\sigma(B(x, r)) \leq r$ for all $x \in E$, $r > 0$ and $c^2(\sigma) < \infty$. Let $\phi : E \rightarrow \phi(E)$ be*

a bilipschitz mapping. There exists a positive constant C depending only on the bilipschitz constant of ϕ such that

$$c^2(\phi_{\#}\sigma) \leq C(\sigma(E) + c^2(\sigma)),$$

where $\phi_{\#}\sigma$ stands for the image measure of σ by ϕ .

It is easy straightforward check that Theorem 3 is a direct consequence of Theorems 5 and 6. We remark that Theorem 6 was proved in [To2, Theorem 1.3] under the stronger assumption that ϕ is bilipschitz on the whole complex plane. The next Subsections 2.2, 2.3 and 2.4 deal with the proof of this result.

2.2 Additional notation and terminology

By a square we mean a square with sides parallel to the axes. Moreover, we assume the squares to be half closed - half open. The side length of a square Q is denoted by $\ell(Q)$. Given $a > 0$, aQ denotes the square concentric with Q with side length $a\ell(Q)$. The average (linear) density of a Borel measure σ on Q is

$$\theta_{\sigma}(Q) := \frac{\sigma(Q)}{\ell(Q)}. \quad (8)$$

We say that σ has linear growth if there exists some constant C such that

$$\sigma(B(x, r)) \leq Cr \quad \text{for all } x \in \mathbb{C}, r > 0.$$

A square $Q \subset \mathbb{C}$ is called 4-dyadic if it is of the form $[j2^{-n}, (j+4)2^{-n}) \times [k2^{-n}, (k+4)2^{-n})$, with $j, k, n \in \mathbb{Z}$. So a 4-dyadic square with side length $4 \cdot 2^{-n}$ is made up of 16 dyadic squares with side length 2^{-n} . Given $a, b > 1$, we say that Q is (a, b) -doubling if $\sigma(aQ) \leq b\sigma(Q)$. If we don't want to specify the constant b , we say that Q is a -doubling. Given two squares $Q \subset R$, we set

$$\delta_{\sigma}(Q, R) := \int_{R_Q \setminus 2Q} \frac{1}{|y - x_Q|} d\sigma(y),$$

where x_Q stands for the center of Q , and R_Q is the smallest square concentric with Q that contains R . Given a bilipschitz mapping $\phi : \mathbb{C} \rightarrow \mathbb{C}$ and a square Q , in [To2] one says that that $\phi(Q)$ is a ϕ -square, and then one defines its side length as $\ell(\phi(Q)) := \ell(Q)$. Now we only know that ϕ bilipschitz from E onto $\phi(E)$, and thus $\phi(Q)$ is not defined in general. So we have to change the notion of ϕ -square. A first attempt would consist in saying that P is a ϕ -square if $P = \phi(Q \cap E)$ for some square Q . This definition has a serious drawback: we cannot set $\ell(P) := \ell(Q)$ because it may happen that $P = \phi(Q \cap E) = \phi(R \cap E)$ for two different squares Q, R . However there is an easy solution: a ϕ -square

is not a subset of \mathbb{C} , but a pair of the form $P = (Q, \phi(Q \cap E))$, for some square $Q \subset \mathbb{C}$. We denote $\ell(P) := \ell(Q)$. On the other hand, abusing language sometimes we will identify the ϕ -square $P = (Q, \phi(Q \cap E))$ with the set $\phi(Q \cap E)$, and so we will use notations such as $\text{diam}(P)$ (notice that $\text{diam}(P) \lesssim \ell(P)$), or $\phi_{\#}\sigma(P)$. If Q_0 is a dyadic (or 4-dyadic) square, we say that $(Q_0, \phi(Q_0 \cap E))$ is a dyadic (or 4-dyadic) ϕ -square. If $Q = (Q_0, \phi(Q_0 \cap E))$ is a ϕ -square, we denote $\lambda Q = (\lambda Q_0, \phi(\lambda Q_0 \cap E))$, for $\lambda > 0$. To simplify notation, we set $\tau := \phi_{\#}\sigma$ and $F := \phi(E)$. A ϕ -square Q is said to be λ -doubling if $\tau(\lambda Q) \leq C\tau(Q)$ for some fixed $C \geq 1$. We also set

$$\theta_{\tau}(Q) := \frac{\tau(Q)}{\ell(Q)}$$

and if R is another ϕ -square which contains Q , we put

$$\delta_{\tau}(Q, R) := \int_{R_Q \setminus 2Q} \frac{1}{|y - x_Q|} d\tau(y),$$

where x_Q stands for some fixed (arbitrary) point of Q and R_Q is the smallest ϕ -square concentric with Q that contains R . That is to say, if $Q = (Q_0, \phi(Q_0 \cap E))$, $R = (R_0, \phi(R_0 \cap E))$, and S is the smallest square concentric with Q_0 that contains R_0 , we set $R_Q = (S, \phi(S \cap E))$. An Ahlfors regular curve is a curve Γ such that $\mathcal{H}^1(\Gamma \cap B(x, r)) \leq Cr$ for all $x \in \Gamma$, $r > 0$, and some fixed $C > 0$. We say that Γ is a chord arc curve if it is a bilipschitz image of an interval in \mathbb{R} . If the bilipschitz constant of the map is L , we say that Γ is an L -chord arc curve.

2.3 The corona decomposition

Theorem 6 will be proved by means of a corona type decomposition for σ similar to the one in [To2]. In Lemma 7 below, where we prove the existence of this decomposition, we will introduce a family $\text{Top}(E)$ of 4-dyadic squares (the top squares) satisfying some precise properties. Given any square $Q \in \text{Top}(E)$, we denote by $\text{Stop}(Q)$ the subfamily of the squares $P \in \text{Top}(E)$ satisfying

- (a) $P \cap 3Q \neq \emptyset$,
- (b) $\ell(P) \leq \frac{1}{8}\ell(Q)$,
- (c) P is maximal, in the sense that there doesn't exist another square $P' \in \text{Top}(E)$ satisfying (a) and (b) which contains P .

We also denote by $Z(\sigma)$ the set of points $x \in \mathbb{C}$ such that there does not exist a sequence of (70, 5000)-doubling squares $\{Q_n\}_n$ centered at x with $\ell(Q_n) \rightarrow 0$ as $n \rightarrow \infty$, so that moreover $\ell(Q_n) = 2^{-k_n}$ for some $k_n \in \mathbb{Z}$. We have $\sigma(Z(\sigma)) = 0$

(see [To2, Remark 2.1]). The set of good points for Q is defined as

$$G(Q) := 3Q \cap \text{supp}(\sigma) \setminus \left[Z(\sigma) \cup \bigcup_{P \in \text{Stop}(Q)} P \right].$$

Lemma 7 (The corona decomposition). *Let σ be a Borel measure supported on $E \subset \mathbb{C}$ such that $\sigma(B(x, r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$ and $c^2(\sigma) < \infty$. There exists a family $\text{Top}(E)$ of 4-dyadic $(16, 5000)$ -doubling squares (called top squares) which satisfy the packing condition*

$$\sum_{Q \in \text{Top}(E)} \theta_\sigma(Q)^2 \sigma(Q) \leq C(\sigma(E) + c^2(\sigma)), \quad (9)$$

and such that for each square $Q \in \text{Top}(E)$ there is a family of C_1 -chord arc curves Γ_Q^i , $i = 1, \dots, N_0$, such that if we set $\Gamma_Q = \bigcup_{i=1}^{N_0} \Gamma_Q^i$, we have

- (a) $G(Q) \subset \Gamma_Q \cap E$.
- (b) For each $P \in \text{Stop}(Q)$ there exists some square \tilde{P} containing P such that $\delta_\sigma(P, \tilde{P}) \leq C_2 \theta_\sigma(Q)$ and $\tilde{P} \cap \Gamma_Q \cap E \neq \emptyset$.
- (c) If P is a square with $\ell(P) \leq \ell(Q)$ such that either $P \cap G(Q) \neq \emptyset$ or there is another square $P' \in \text{Stop}(Q)$ such that $P \cap P' \neq \emptyset$ and $\ell(P') \leq \ell(P)$, then $\sigma(P) \leq C_3 \theta_\sigma(Q) \ell(P)$.

Moreover, $\text{Top}(E)$ contains some 4-dyadic square R_0 such that $E \subset R_0$. The constants C_1, C_2, C_3, N_0 are absolute.

Notice that the chord arc constant of the curves Γ_Q^i in the lemma is uniformly bounded above by C_1 .

Proof. The proof of this lemma is very similar to the one of Main Lemma 3.1 in [To2], and so we will only describe in detail the required modifications for the proof. Notice that, in Lemma 7, Γ_Q is made up of a finite union of chord arc curves, while in [To2, Main Lemma 3.1] Γ_Q is an Ahlfors regular curve. On the other hand, in the statement (b) above we ask $\tilde{P} \cap \Gamma_Q \cap E \neq \emptyset$, while in (b) of Main Lemma 3.1 in [To2] one asks only $\tilde{P} \cap \Gamma_Q \neq \emptyset$. These are the only differences between both lemmas.

First we will explain the arguments to show that Γ_Q is made up of a finite number of chord arc curves. First we need some notation. Given a set $K \subset \mathbb{C}$ and a square Q , let V_Q be an infinite strip (or line in the degenerate case) of smallest possible width which contains $K \cap 3Q$, and let $w(V_Q)$ denote the width of V_Q . Denote

$$\beta_K(Q) = \frac{w(V_Q)}{\ell(Q)}.$$

Recall the following version of Jones' traveling salesman theorem [Jo]:

Theorem 8 (P. Jones). *A set $K \subset \mathbb{C}$ is contained in an Ahlfors regular curve if and only if there exists some constant C_4 such that for every dyadic square Q*

$$\sum_{P \in \mathcal{D}, P \subset Q} \beta_K(P)^2 \ell(P) \leq C_4 \ell(Q).$$

The regularity constant of the curve depends on C_4 .

Moreover, in [Jo] the author claims that there exists some constant $\eta > 0$ small enough such that if $\beta_K(Q) \leq \eta$ for every square $Q \in \mathcal{D}$, then K is contained in quasicircle. So we have

Theorem 9. *There exists some absolute constant $\eta > 0$ such that if, for every dyadic square Q , $K \subset \mathbb{C}$ satisfies*

$$\sum_{P \in \mathcal{D}, P \subset Q} \beta_K(P)^2 \ell(P) \leq \eta \ell(Q),$$

then K is contained in a chord arc curve.

Although the preceding result is not proved in [Jo], it follows from easy modifications of the author's arguments (we suggest the reader to look also at [GM, Chapter 10, Theorem 2.3]).

Given $R \in \text{Top}(E)$, in [To2] the existence of the curve Γ_R follows from an application of Theorem 8. To this end, in [To2, Lemma 4.5] one constructs a set K which contains $G(R)$ and which, in a sense, approximates $\text{supp}(\sigma)$ on the squares $P \in \text{Stop}(R)$. Then one proves in [To2, Subsection 4.4] that for any square Q with $\ell(Q) \leq C_5 \ell(R)$ (where $C_5 < 1$ is some small positive constant),

$$\sum_{P \in \mathcal{D}, P \subset Q} \beta_K(P)^2 \ell(P) \leq C \theta_\sigma(R)^{-3} \iiint_{(x,y,z) \in (3Q)^3 \cap R^\ell} c(x,y,z)^2 d\sigma(x) d\sigma(y) d\sigma(z). \quad (10)$$

See p. 1266 of [To2] for the precise definition of R^ℓ . For the reader's convenience, let us say that the triple integral on the right hand side is some truncated version of the curvature $c^2(\sigma|_{3Q})$. Moreover, from the construction of the stopping squares, it turns out that

$$C \theta_\sigma(R)^{-3} \iiint_{(x,y,z) \in (3Q)^3 \cap R^\ell} c(x,y,z)^2 d\sigma(x) d\sigma(y) d\sigma(z) \leq C_6 \ell(Q). \quad (11)$$

From (10), (11), and Theorem 8 one infers the existence of the regular curve Γ_Q .

However, a careful examination of the proof of Main Lemma 3.1 in [To2] shows that the constant C_6 in (11) can be taken so that $C_6 \leq \eta$ (in fact, as small as

we want). To this end, one has to take the parameter ε_0 small enough in the construction of the high curvature squares (see p. 1252 of [To2]). We leave the details for the reader. Then Theorem 9 can be applied for the squares Q with $Q \cap 3R \neq \emptyset$ such that with $\ell(Q) \leq C_5 \ell(R)$. As a consequence, one deduces that K is contained in a finite number N_0 of chord arc curves (we only have to cover $3R$ by a finite number of squares with side length $C_5 \ell(R)$ and then we apply Theorem 9).

On the other hand, it is easy to check that the curve Γ_Q constructed in the proof of Main Lemma 3.1 satisfies $\tilde{P} \cap \Gamma_Q \cap E \neq \emptyset$, as required above in (b). Of course, the same happens with the “new” curves Γ_Q described above since the method of construction has not changed. This is due to the fact that the set K obtained in [To2, Lemma 4.5] is contained in $\text{supp}(\mu)$. This is not stated in [To2, Lemma 4.5], but it is easily seen. \square

Remark. *It is easy to check that one can always assume $\tilde{P} \subset 16Q$ in the statement (b) of Lemma 7.*

2.4 The curvature of $\phi_{\#}\sigma$

In Lemma 7 we have shown how to construct a corona type decomposition for a measure σ with linear growth and finite curvature. We will see below that ϕ sends this corona type decomposition into another corona decomposition in terms of ϕ -squares, and we will prove that that the existence of such a decomposition implies that the curvature is finite. These will be the basic ingredients for the proof of Theorem 6.

First we introduce some notation. Given a family $\text{Top}(F)$ of 4-dyadic ϕ -squares and a fixed $Q \in \text{Top}(F)$, we denote by $\text{Stop}(Q)$ the subfamily of ϕ -squares which satisfy the properties (a), (b), (c) stated at the beginning of Subsection 2.3 (with squares replaced by ϕ -squares). The set $G(Q)$ is also defined as in Subsection 2.3, with ϕ -squares instead of squares.

Lemma 10. *Let τ be a Borel measure supported on a compact set $F \subset \mathbb{C}$. Suppose that $\tau(B(x, r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$. Let $\text{Top}(F)$ be a family of 4-dyadic 16-doubling ϕ -squares which contains some 4-dyadic ϕ -square R_0 such that $F = R_0$, and such that for each $Q \in \text{Top}(F)$ there exists a C_7 -AD regular curve Γ_Q satisfying:*

- (a) τ -almost every point in $G(Q)$ belongs to Γ_Q .

(b) For each $P \in \text{Stop}(Q)$ there exists some ϕ -square \tilde{P} containing P such that $\delta_\tau(P, \tilde{P}) \leq C\theta_\tau(Q)$ and $\tilde{P} \cap \Gamma_Q \neq \emptyset$.

(c) If P is a ϕ -square with $\ell(P) \leq \ell(Q)$ such that either $P \cap G(Q) \neq \emptyset$ or there is another ϕ -square $P' \in \text{Stop}(Q)$ such that $P \cap P' \neq \emptyset$ and $\ell(P') \leq \ell(P)$, then $\tau(P) \leq C\theta_\tau(Q)\ell(P)$.

Then,

$$c^2(\tau) \leq C \sum_{Q \in \text{Top}(F)} \theta_\tau(Q)^2 \tau(Q).$$

The proof of this lemma is almost the same as the one of [To2, Main Lemma 8.1], and so we omit the details.

In order to show that ϕ transforms a corona type decomposition like the one in Lemma 7 into another like the one of the preceding lemma we need the following result of MacManus [MM]:

Theorem 11. *Any M -bilipschitz map of a subset of a line or a circle into the plane has an extension to a $C(M)$ -bilipschitz map from the plane onto itself.*

We are ready to prove Theorem 6 now:

Proof of Theorem 6. We consider the measure σ and its corona type decomposition given by Lemma 7. It is straightforward to check that $\tau := \phi_\# \sigma$ has linear growth. We take the family $\text{Top}(F) = \phi(\text{Top}(E))$, and for $Q \in \text{Top}(F)$ with $Q = (Q_0, \phi(Q_0 \cap E))$, we define $\text{Stop}(Q) = \phi(\text{Stop}(Q_0))$.

To construct a regular curve Γ_Q such as the one required in Lemma 10, we consider the union of chord arc curves $\Gamma_{Q_0} = \bigcup_{i=1}^{N_0} \Gamma_{Q_0}^i$ of Lemma 7. Notice that we cannot set $\Gamma_Q = \phi(\Gamma_{Q_0})$ because ϕ is not defined on the whole set Γ_{Q_0} . By MacManus' theorem we can solve this problem easily. For each $i = 1, \dots, N_0$, let $\rho_i : \mathbb{R} \supset I \rightarrow \Gamma_{Q_0}^i$ be a bilipschitz parametrization of the chord arc curve $\Gamma_{Q_0}^i$. Consider the subset $\Gamma_{Q_0}^i \cap E$ and the bilipschitz map $\phi \circ \rho_i$, defined on $\rho_i^{-1}(\Gamma_{Q_0}^i \cap E) \subset \mathbb{R}$. By Theorem 11, $\phi \circ \rho_i$ has a bilipschitz extension f_i onto the whole complex plane. We consider the chord arc curve $\Gamma_i := f_i(\mathbb{R})$, and we set $\Gamma_Q := \bigcup_{i=1}^{N_0} \Gamma_i$ (and we add a finite number of segments if necessary to ensure that Γ_Q is a regular curve).

We have to show that the assumptions of Lemma 10 hold for the family $\text{Top}(F)$, their corresponding stopping squares, and the curves Γ_Q , $Q \in \text{Top}(F)$. Indeed, (a) and (c) are the translation of the corresponding statements (a) and (c) of Lemma 7. On the other hand, (b) is a consequence of the fact that if

$P_0 \in \text{Stop}(Q_0)$ for some $Q_0 \in \text{Top}(E)$, and $\tilde{P}_0 \subset 16Q_0$ (recall Remark 2.3), and moreover we have

$$\delta_\sigma(P_0, \tilde{P}_0) \leq C_2 \theta_\sigma(Q_0),$$

then $P = (P_0, \phi(P_0 \cap E))$ and $\tilde{P} = (\tilde{P}_0, \phi(\tilde{P}_0 \cap E))$ satisfy

$$\delta_\tau(P, \tilde{P}) \lesssim \theta_\tau(Q). \quad (12)$$

To prove this estimate, recall that

$$\delta_\tau(P, \tilde{P}) := \int_{\tilde{P}_P \setminus 2P} \frac{1}{|y - z_P|} d\tau(y),$$

where z_P is some fixed point of P and \tilde{P}_P is the smallest ϕ -square concentric with P that contains \tilde{P} . Then, if we set $z_{P_0} = \phi^{-1}(z_P)$, we have

$$\delta_\tau(P, \tilde{P}) = \int_{(\tilde{P}_0)_{P_0} \setminus 2P_0} \frac{1}{|\phi(y) - \phi(z_{P_0})|} d\sigma(y) \simeq \int_{(\tilde{P}_0)_{P_0} \setminus 2P_0} \frac{1}{|y - z_{P_0}|} d\sigma(y).$$

Since $z_{P_0} \in P_0$, from the property (c) of Lemma 7, it follows easily that

$$\begin{aligned} \int_{(\tilde{P}_0)_{P_0} \setminus 2P_0} \frac{1}{|y - z_{P_0}|} d\sigma(y) &\leq \int_{(\tilde{P}_0)_{P_0} \setminus 2P_0} \frac{1}{|y - x_{P_0}|} d\sigma(y) + C\theta_\sigma(Q_0) \\ &= \delta(P_0, \tilde{P}_0) + C\theta_\sigma(Q_0) \leq C\theta_\sigma(Q_0) \simeq \theta_\tau(Q). \end{aligned}$$

(recall that x_{P_0} is the center of P_0 , which may not coincide with z_{P_0}), and so (12) holds.

Thus from Lemmas 10 and 7 we infer that

$$c^2(\tau) \lesssim C \sum_{Q \in \text{Top}(F)} \theta_\tau(Q)^2 \tau(Q) \simeq \sum_{Q_0 \in \text{Top}(E)} \theta_\sigma(Q_0)^2 \sigma(Q_0) \lesssim \sigma(E) + c^2(\sigma).$$

□

3 Proof of Corollary 4

This is an immediate consequence of Theorem 3. Let $F \subset E \cap \Omega$ be compact. Obviously, $\gamma(F) = 0$. Cover F by a finite number of closed balls B_i , $1 \leq i \leq N$, of diameter $\delta/2$. Since $\phi : F \cap B_i \rightarrow \phi(F \cap B_i)$ is bilipschitz, we have

$$\gamma(\phi(F \cap B_i)) \simeq \gamma(F \cap B_i) = 0$$

for all i . This implies that $\gamma(\phi(F)) = 0$ (this is a consequence of the semiadditivity of γ , but it can be proven by much simpler arguments). Since this holds for any compact set $\phi(F) \subset \phi(E \cap \Omega)$, we have $\gamma(\phi(E \cap \Omega)) = 0$. □

4 Proof of Theorem 1

In all this section, μ is a compactly supported Beltrami coefficient belonging to the Sobolev space $W^{1,2}(\mathbb{C})$, and $\phi : \mathbb{C} \rightarrow \mathbb{C}$ is a μ -quasiconformal mapping. There is no restriction if we normalize ϕ in such a way that $\phi(z) - z = O(1/|z|)$ as $|z| \rightarrow \infty$. We can assume also that $\text{supp}(\mu) \subset \mathbb{D}$.

We start by giving some auxiliary results. The first one gives some information on the distortion of Riesz capacities under μ -quasiconformal mappings. Recall that if E is any compact set on the plane, the $(1, p)$ -Riesz capacity of E is defined as

$$C_{1,p}(E) = \inf \{ \|D\psi\|_p \}$$

where the infimum is taken over all compactly supported $\psi \in C^\infty(\mathbb{C})$ with $\psi \geq \chi_E$. One obtains the same quantity if we simply assume $\psi \in W^{1,p}(\mathbb{C})$. For more details about Riesz capacities, see [AH], for example.

Lemma 12. *Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient, and let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be a μ -quasiconformal mapping. Then,*

$$C_{1,q}(\phi(E))^{\frac{1}{q}} \lesssim C_{1,p}(E)^{\frac{1}{p}}$$

for any compact set $E \subset \mathbb{D}$, and any $1 < q < p < 2$.

Proof. It is not hard to see that if ψ is a $W^{1,p}$ function, continuous and compactly supported inside of \mathbb{D} , and $\psi \geq 1$ on E , then $\psi \circ \phi^{-1}$ is a continuous, compactly supported function of class $W^{1,q}$, whenever $q < p$. Indeed,

$$\left(\int |D(\psi \circ \phi^{-1})|^q \right)^{\frac{1}{q}} \leq \left(\int |D\psi|^p \right)^{\frac{1}{p}} \left(\int_{\mathbb{D}} J(\cdot, \phi)^{\frac{1-\frac{q}{p}}{1-\frac{q}{2p}}} \right)^{\frac{1}{q} - \frac{1}{p}}$$

and here we must remark that the last integral converges since $q < p < 2$ and $\mu \in W^{1,2}$. Therefore,

$$C_{1,q}(\phi(E))^{\frac{1}{q}} \leq \left(\int |D\psi|^p \right)^{\frac{1}{p}} \left(\int_{\mathbb{D}} J(\cdot, \phi)^{\frac{1-\frac{q}{p}}{1-\frac{q}{2p}}} \right)^{\frac{1}{q} - \frac{1}{p}}$$

Thus,

$$C_{1,q}(\phi(E))^{\frac{1}{q}} \leq C_{1,p}(E)^{\frac{1}{p}} \left(\int_{\mathbb{D}} J(\cdot, \phi)^{\frac{1-\frac{q}{p}}{1-\frac{q}{2p}}} \right)^{\frac{1}{q} - \frac{1}{p}}$$

and the result follows. \square

The following result establishes that the means of the Jacobian determinant of a quasiconformal mapping behave precisely as an incremental quotient.

Lemma 13. *Let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be a K -quasiconformal mapping. Let $z, w \in \mathbb{C}$ be such that $z \neq w$, and $D = D(z, |z - w|)$. Then,*

$$\frac{|\phi(z) - \phi(w)|}{|z - w|} \simeq \left(\frac{1}{|D|} \int_D J(z, \phi) dA(z) \right)^{\frac{1}{2}}$$

with constants that depend only on K .

Proof. Let $r = |w - z|$. Then,

$$\begin{aligned} |\phi(w) - \phi(z)| &\leq \max_{|\zeta - z| = r} |\phi(\zeta) - \phi(z)| \leq C_K \min_{|\zeta - z| = r} |\phi(\zeta) - \phi(z)| \\ &\leq C_K \left(\frac{|\phi(D(z, r))|}{\pi} \right)^{\frac{1}{2}} \\ &= C_K |z - w| \left(\frac{1}{|D(z, r)|} \int_{D(z, r)} J(\zeta, \phi) dA(\zeta) \right)^{\frac{1}{2}} \end{aligned}$$

The converse inequality can be obtained analogously. □

Proof of Theorem 1. Since $\mu \in W^{1,2}(\mathbb{C})$ is compactly supported, we know (see for instance [CFMOZ, Proposition 3]) that $\phi \in W_{loc}^{2,p}$ for every $p < 2$. Thus, writing the Jacobian determinant as $J(\cdot, \phi) = |\partial\phi|^2 - |\bar{\partial}\phi|^2$, we get $J(\cdot, \phi) \in W_{loc}^{1,p}$ for every $p < 2$, and in particular $J(\cdot, \phi)$ admits a $C_{1,p}$ -quasicontinuous representative, for each $p < 2$. Moreover, modulo a set of small $C_{1,p}$ -capacity, every point is a Lebesgue point for $J(\cdot, \phi)$ (for more details see [AH, Chapter 6]). Then, given $\varepsilon > 0$ we can find an open set $U \subset \mathbb{D}$ such that:

- $C_{1,p}(U) < \varepsilon$.
- $J(\cdot, \phi)$ is continuous on $\bar{\mathbb{D}} \setminus U$.
- $\lim_{r \rightarrow 0} \frac{1}{|D(z, r)|} \int_{D(z, r)} J(w, \phi) dA(w) = J(z, \phi)$ uniformly on $z \in \bar{\mathbb{D}} \setminus U$.

As a consequence of the third point, together with Lemma 13, we can find $\delta = \delta(\varepsilon) > 0$ such that whenever $z, w \in \bar{\mathbb{D}} \setminus U$ satisfy $|z - w| < \delta$, we have

$$J(z, \phi)^{\frac{1}{2}} \simeq \frac{|\phi(w) - \phi(z)|}{|w - z|}$$

with constants that depend on K and ε . At this point, we could feel tempted to say that ϕ is locally bilipschitz on $\bar{\mathbb{D}} \setminus U$, but this may fail because $J(\cdot, \phi)$ may vanish at some points on the boundary of $\bar{\mathbb{D}} \setminus U$. To solve this, for each integer $n \in \mathbb{N}$ we set

$$F_n = \left\{ z \in \bar{\mathbb{D}} \setminus U : \frac{1}{n} < J(z, \phi) < n \right\}$$

Denote by Z the set of points $z \in \overline{\mathbb{D}} \setminus U$ such that either ϕ is not differentiable at z , or ϕ^{-1} is not differentiable at $\phi(z)$. Then, it is straightforward to show that

$$\overline{\mathbb{D}} \setminus U = Z \cup \bigcup_{n=1}^{\infty} F_n$$

Since $J(\cdot, \phi)$ is continuous on $\overline{\mathbb{D}} \setminus U$, F_n is open in the topology of $\overline{\mathbb{D}} \setminus U$, so that $(\overline{\mathbb{D}} \setminus U) \setminus F_n$ is closed in the same topology. Therefore, since $\overline{\mathbb{D}} \setminus U$ is closed as a subset of \mathbb{C} , also $(\overline{\mathbb{D}} \setminus U) \setminus F_n$ is closed. Thus, $(\overline{\mathbb{D}} \setminus U) \setminus F_n$ is a decreasing sequence of compact sets and then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} C_{1,p}((\overline{\mathbb{D}} \setminus U) \setminus F_n) &= C_{1,p} \left(\bigcap_{n=1}^{\infty} (\overline{\mathbb{D}} \setminus U) \setminus F_n \right) \\ &= C_{1,p} \left((\overline{\mathbb{D}} \setminus U) \setminus \bigcup_{n=1}^{\infty} F_n \right) \\ &\leq C_{1,p}(Z) \end{aligned}$$

But both ϕ and ϕ^{-1} are differentiable $C_{1,p}$ -almost everywhere, for any $p < 2$, so that $\dim(Z) = 0$ and therefore the above limit also vanishes, that is,

$$\lim_{n \rightarrow \infty} C_{1,p}((\overline{\mathbb{D}} \setminus U) \setminus F_n) = 0$$

Now assume that $E \subset \mathbb{D}$ is a compact set with $\gamma(E) = 0$. We can split E into several small pieces. First of all, we consider $E \cap U$. By Lemma 12, and since $1 < q < 2$, one easily shows that

$$\gamma(\phi(E \cap U)) \lesssim C_{1,q}(\phi(E \cap U))^{\frac{1}{q}} \lesssim C_{1,p}(E \cap U)^{\frac{1}{p}} \leq C_{1,p}(U)^{\frac{1}{p}} < \varepsilon^{\frac{1}{p}}$$

with constants that may depend on ϕ , but not on ε . Secondly, since ϕ does not distort Hausdorff dimension,

$$\dim(\phi(E \cap Z)) = \dim(E \cap Z) = 0$$

For the remaining part, it follows from Lemma 13 that ϕ is locally bilipschitz on every set F_n . In fact, we can split F_n into a finite union of subsets $F_{n,k}$ with diameter small enough so that ϕ is bilipschitz on $F_{n,k}$ for each k . Thus, applying Corollary 4 we get

$$\gamma(\phi(E \cap F_n)) = 0$$

because $\gamma(E \cap F_n) \leq \gamma(E) = 0$. Summarizing, we get for any $n \in \mathbb{N}$

$$\begin{aligned} \gamma(\phi(E \cap ((\overline{\mathbb{D}} \setminus U) \setminus F_n))) &\leq C_{1,q}(\phi(E \cap ((\overline{\mathbb{D}} \setminus U) \setminus F_n)))^{\frac{1}{q}} \\ &\lesssim C_{1,p}(E \cap ((\overline{\mathbb{D}} \setminus U) \setminus F_n))^{\frac{1}{p}} \\ &\leq C_{1,p}((\overline{\mathbb{D}} \setminus U) \setminus F_n)^{\frac{1}{p}} \end{aligned}$$

Therefore, by the semiadditivity of analytic capacity [To2],

$$\begin{aligned} \gamma(\phi(E)) &\lesssim \gamma(\phi(E \cap U)) + \gamma(\phi(E \cap (\overline{\mathbb{D}} \setminus U))) \\ &\lesssim \gamma(\phi(E \cap U)) + \gamma(\phi(E \cap Z)) + \gamma(\phi(E \cap F_n)) + \gamma(\phi(E \cap ((\overline{\mathbb{D}} \setminus U) \setminus F_n))) \\ &\lesssim \varepsilon^{\frac{1}{p}} + 0 + 0 + C_{1,p}(E \cap ((\overline{\mathbb{D}} \setminus U) \setminus F_n))^{\frac{1}{p}} \end{aligned}$$

for each $n \in \mathbb{N}$. In the right hand side, the last term converges to 0 as $n \rightarrow \infty$. Thus, $\gamma(\phi(E))$ can be made as small as we wish. \square

References

- [AH] D. R. Adams, L. I. Hedberg, *Function spaces and potential theory*, Grundle Math. Wiss. 314, Springer-Verlag, Berlin (1996).
- [Ah1] L. V. Ahlfors, *Lectures on quasiconformal mappings*, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA (1987).
- [Ah2] L. V. Ahlfors, *Bounded analytic functions*, Duke Math. J. 14 (1947), 1-11.
- [As] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. 173 (1994), 37–60.
- [CFMOZ] A. Clop, D. Faraco, J. Mateu, J. Orobitg, X. Zhong, *Beltrami equations with coefficient in the Sobolev space $W^{1,p}$* , submitted.
- [Da1] G. David, *Unrectifiable 1-sets have vanishing analytic capacity*, Revista Mat. Iberoamericana 14(2) (1998), 369–479.
- [GM] J. B. Garnett, D. E. Marshall, *Harmonic measure*, New Mathematical Monographs, 2. Cambridge University Press, Cambridge (2005).
- [Iw] T. Iwaniec, *L^p -theory of quasiregular mappings*, in *Quasiconformal space mappings*, 39–64, Lecture Notes in Math., 1508, Springer, Berlin (1992).
- [Jo] P.W. Jones, *Rectifiable sets and the travelling salesman problem*, Invent. Math. 102 (1990), 1–15.
- [MM] P. MacManus, *Bi-Lipschitz extensions in the plane*, J. Anal. Math. 66 (1995), 85–115.
- [To2] X. Tolsa, *Painlevé’s problem and the semiadditivity of analytic capacity*, Acta Math. 190:1 (2003), 105–149.

- [To2] X. Tolsa, *Bilipschitz maps, analytic capacity, and the Cauchy integral*, Ann. of Math. 162 (2005), 1243–1304.
- [Ve1] J. Verdera, *Removability, capacity and approximation*. In “Complex Potential Theory”, (Montreal, PQ, 1993), NATO Adv. Sci. Int. Ser. C Math. Phys. Sci. 439, Kluwer Academic Publ., Dordrecht, 1994, pp. 419–473.

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