Weak convergence of error processes in discretizations of stochastic integrals and Besov spaces

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Abstract

We consider the weak convergence of the rescaled error processes for Riemann discretizations of certain stochastic integrals and relate the integrability of their weak limit to the fractional smoothness of the stochastic integral.

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Introduction

Quantitative approximation problems for stochastic integrals arise naturally in connection with discrete time hedging in Stochastic Finance, like the variance optimal hedging where the approximation error is measured in L_2. Besides the L_2-norm there are many other criteria, like L_p-criteria or criteria related to the weak convergence, to measure the approximation error. Concepts of weak convergence are of particular interest in applications because they already provide the needed information in many cases and promise potentially better approximation rates than obtained under L_p-criteria. Results in this direction were obtained, for instance, in [8] and [9], for a more general overview see [10].

One starting point of our paper is a result of Gobet and Temam [8, Theorem 3] (see also [12]) concerning the weak convergence of the renormalized error processes appearing in Riemann discretizations of stochastic integrals: based on a general theorem of Rootzen [11] (see also [2] for related results) it is shown that the L_2-hedging error in the Black-Scholes model for the Binary option is $n^{-1/4}$ when equidistant time-nets are used in the discretization, whereas the rate of weak convergence is $1/\sqrt{n}$ in this case. On the other hand, it was shown in [5, 7] that by non-equidistant time-nets the L_2-approximation rate for the Binary option can be improved from $n^{-1/4}$ to $1/\sqrt{n}$.

The natural question was whether these new time nets also improve the weak convergence. The aim of this paper is to answer this question to the positive: in terms of Besov spaces we consider the fractional smoothness $\theta \in (0,1]$ of the stochastic integrals to be approximated. For example, for the Binary option we would get $1/2-\varepsilon$ for all $\varepsilon \in (0,1/2)$, see [7]. According to the fractional smoothness we choose special non-equidistant time-nets and show that the by $\sqrt{n}$ renormalized error processes converge weakly to a square integrable process. And the converse turns out to be true as well: if one has weak convergence towards a square integrable process, then the stochastic integral needs to have a certain fractional smoothness. In this way we develop further the ideas of Rootzen [11] and Gobet-Temam [8].

Hence, roughly speaking we obtain that the rate of weak convergence remains $1/\sqrt{n}$, but the integrability of the weak limit improves compared to the integrability obtained by equidistant time-nets.

1 Notation

Let $B=(B_t)_{t\in[0,1]}$ be a standard Brownian motion defined on a stochastic basis $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,1]})$, where $B_0\equiv 0$, all paths are assumed to be continuous, $(\mathcal{F}_t)_{t\in[0,1]}$ is the augmentation of the natural filtration of B, and $\mathcal{F}=\mathcal{F}_1$. Let X be either the Brownian motion or the geometric Brownian motion $S=(S_t)_{t\in[0,1]}$ with

$$S_t := e^{B_t-\frac{t}{2}}.$$

To treat both cases for X simultaneously we let $\sigma(x)\equiv 1$ if $X=B$ and $\sigma(x)=x$ if $X=S$, so that $dX_t=\sigma(X_t)dB_t$. Let g be a Borel function such that $\mathbb{E}g(X_1)^2<\infty$. Define the function G by setting

$$G(t,x) := \begin{cases} \mathbb{E}g(x+X_{1-t}) & : \quad X=B \\ \mathbb{E}g(xX_{1-t}) & : \quad X=S \end{cases}. \tag{1}$$

Then it follows that $G\in C^\infty([0,1)\times E)$, where $E=\mathbb{R}$ if $X=B$ and $E=(0,\infty)$ if $X=S$, and satisfies the partial differential equation

$$\frac{\partial G}{\partial t}(t,x)+\frac{\sigma(x)^2}{2}\frac{\partial^2 G}{\partial x^2}(t,x)=0$$

for $(t,x)\in[0,1)\times E$ with $G(1,x)=g(x)$ for $x\in E$. By Itô's formula,

$$g(X_1)=\mathbb{E}g(X_1)+\int_0^1\frac{\partial G}{\partial x}(u,X_u)dX_u \quad a.s.$$

Our interest is to approximate the stochastic integral $\int_0^1\frac{\partial G}{\partial x}(u,X_u)dX_u$ by a Riemann approximation. To this end, given a deterministic time net $\tau=(t_i)_{i=0}^n$ with $0=t_0<\cdots<t_n=1$, we define the error process

$$C_t(\tau):=\int_0^t\frac{\partial G}{\partial x}(u,X_u)\,dX_u-\sum_{i=0}^{n-1}\frac{\partial G}{\partial x}(t_i,X_{t_i})\left(X_{t_{i+1}\wedge t}-X_{t_i\wedge t}\right)$$

for $t \in [0,1]$ where we may assume that all paths are continuous. For $\beta \in (0,1]$ we introduce the special time-nets $\tau^{n,\beta} := \left(t_i^{n,\beta}\right)_{i=0}^n$ defined by

$$t_i^{n,\beta} := 1 - \left(1 - \frac{i}{n}\right)^{\frac{1}{\beta}}.$$

The smaller the β, the higher the concentration of the time-knots is near to one. In particular,

$$\frac{|t_{i+1}^{n,\beta} - u|}{(1-u)^{1-\beta}} \le \frac{|t_{i+1}^{n,\beta} - t_i^{n,\beta}|}{(1-t_i^{n,\beta})^{1-\beta}} \le \frac{1}{\beta n} \quad \text{for} \quad u \in [t_i^{n,\beta}, t_{i+1}^{n,\beta}) \tag{2}$$

and all $n = 1, 2, ...$ and $i = 0, ..., n-1$. The Besov spaces we use can be described by Hermite expansions as follows:

Definition 1.1. *Let $d\gamma(x) = (1/\sqrt{2\pi}) \exp(-x^2/2) dx$ be the standard Gaussian measure on $\mathbb{R}$ and let $(h_k)_{k=0}^\infty \subset L_2(\gamma)$ be the orthonormal basis consisting of Hermite polynomials. Given $\beta \in (0,1]$ and $f = \sum_{k=0}^\infty \alpha_k h_k$, we let $f \in B_{2,2}^\beta(\gamma)$ provided that*

$$\|f\|_{B_{2,2}^\beta(\gamma)} := \left(\sum_{k=0}^\infty (k+1)^\beta \alpha_k^2\right)^{\frac{1}{2}} < \infty.$$

The parameter β is the degree of fractional smoothness. In particular, we have that $B_{2,2}^1(\gamma)$ is the Malliavin Sobolev space $D_{1,2}(\gamma)$. To formulate our results, given $\beta \in (0,1]$ and $t \in [0,1)$, we define

$$\begin{aligned}
\nu_\beta(t) &:= \frac{1}{\beta}(1-t)^{1-\beta}, \\
A_\beta(t) &:= \frac{1}{2}\int_0^t \nu_\beta(u) \left[\left(\sigma^2 \frac{\partial^2 G}{\partial x^2}\right)(u, X_u)\right]^2 du, \\
Z_\beta(t) &:= W_{A_\beta(t)},
\end{aligned}$$

where G is obtained from the function g as in (1) and $W = (W_t)_{t \ge 0}$ is a standard Brownian motion starting at zero defined on some auxiliary probability space (M, μ), where we may and do assume that all paths are continuous. Finally, we extend the process A_β by

$$A_\beta(1) := \lim_{t \uparrow 1} A_\beta(t)$$

which might be an extended random variable. In the following $\Longrightarrow$ stands for the weak convergence, and $\Longrightarrow_{C[0,T]}$ for the weak convergence in $C[0,T]$ for some $T > 0$.

2 The result

The main result of the paper is

Theorem 2.1. *Let $\beta \in (0,1]$ and $g(X_1) \in L_2$. Then, for all $T \in [0,1)$,*

$$(\sqrt{n} C_t(\tau^{n,\beta}))_{t\in[0,T]} \Longrightarrow_{C[0,T]} (Z_\beta(t))_{t\in[0,T]} \quad as \quad n \to \infty.$$

Moreover, the following assertions are equivalent:

(i) *One has $g \in B^\beta_{2,2}(\gamma)$ for $X = B$ and $g(e^{\cdot - \frac{1}{2}}) \in B^\beta_{2,2}(\gamma)$ for $X = S$.*

(ii) *One has $\mathbb{E} A_\beta(1) < \infty$. Furthermore, letting $\widetilde{Z}_\beta(t) := W_{A_\beta(t)\chi_{\{A_\beta(1)<\infty\}}}$ for $t \in [0,1]$ it holds that*

$$(\sqrt{n} C_t(\tau^{n,\beta}))_{t\in[0,1]} \Longrightarrow_{C[0,1]} (\widetilde{Z}_\beta(t))_{t\in[0,1]}.$$

(iii) *Denoting, for $t \in [0,1)$, the law of the weak limit of $\sqrt{n} C_t(\tau^{n,\beta})$ as $n \to \infty$ by μ_t, one has*

$$\sup_{t\in[0,1)} \int_{\mathbb{R}} x^2 d\mu_t(x) < \infty.$$

The main implication in Theorem 2.1 is (i) $\Rightarrow$ (ii). Assertion (iii) is included to demonstrate to what extend (ii) can be weakened. Another version of (iii) would be:

(iii') *The weak limit of $\sqrt{n} C_1(\tau^{n,\beta})$ as $n \to \infty$ exists and its law has a finite second moment.*

However, we do not know whether (iii') is equivalent to (iii). As usual, having a weak convergence like in Theorem 2.1(ii) one obtains the weak convergence of functionals $\varphi((\sqrt{n} C_t(\tau^{n,\beta})_{t\in[0,1]})$ whenever $\varphi : C[0,1] \to \mathbb{R}$ is continuous. With respect to Value at Risk estimates the improvement of the integrability of the weak limit has the following consequence:

Corollary 2.2. *Let $\beta \in (0,1]$ and $g \in B^\beta_{2,2}(\gamma)$ for $X = B$ and $g(e^{\cdot - \frac{1}{2}}) \in B^\beta_{2,2}(\gamma)$ for $X = S$. Assume that there are no constants $a, b \in \mathbb{R}$ such that $g(X_1) = a + bX_1$ a.s. Then, for $\varepsilon \in (0,1)$,*

$$\limsup_n \mathbb{P}\left(\|\widetilde{Z}_\beta(1)\|_{L_2} (\varepsilon n)^{-\frac{1}{2}} + C_1(\tau^{n,\beta}) \le 0 \right) \le \varepsilon.$$

Proof. Our assumption about the non-linearity on g implies that $\|\widetilde{Z}_\beta(1)\|_{L_2} > 0$. In fact, $\|\widetilde{Z}_\beta(1)\|_{L_2} = 0$ would imply by a continuity argument that $\mathbb{E}\left|\left(\sigma^2(\partial^2 G/\partial x^2)\right)(u, X_u)\right|^2 = 0$ for $u \in [0,1)$ so that [7, Lemmas 3.9 and 3.10] and [3, Theorem 1.1] would imply that g is almost surely linear (there is also a shorter direct argument). Applying Theorem 2.1, we get that

$$\begin{aligned} & \limsup_n \mathbb{P}\left(\|\widetilde{Z}_\beta(1)\|_{L_2} (\varepsilon n)^{-\frac{1}{2}} + C_1(\tau^{n,\beta}) \le 0 \right) \\ = & \limsup_n \mathbb{P}\left(\sqrt{n} C_1(\tau^{n,\beta}) \le -\varepsilon^{-\frac{1}{2}} \|\widetilde{Z}_\beta(1)\|_{L_2} \right) \end{aligned}$$

$$
\begin{aligned}
&\le \mathbb{P}(\widetilde{Z}_\beta(1) \le -\varepsilon^{-\frac{1}{2}} \|\widetilde{Z}_\beta(1)\|_{L_2}) \\
&\le \varepsilon.
\end{aligned}
$$

□

Remark 2.3. (i) Corollary 2.2 says that, asymptotically, by an enlargement of the initial capital by $\|\widetilde{Z}_\beta(1)\|_{L_2} n^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}$ one gets an upper bound for the shortfall probability of ε. The factor $\|\widetilde{Z}_\beta(1)\|_{L_2}$ is a normalizing factor related to the size and complexity of g, the exponent for n comes from the rescaling factor $\sqrt{n}$, and the exponent for ε from the integrability of the weak limit of $\sqrt{n} C_1(\tau^{n,\beta})$.

(ii) In case that $g(X_1) = a + bX_1$ a.s. we get that $G(t,x) = a + bx$ so that $\widetilde{Z}_\beta(1) = 0$ a.s. and $C_1(\tau) = 0$ a.s. Consequently, Corollary 2.2 does not hold in this case.

3 Proof of Theorem 2.1

First let us recall a lemma which ensures some integrability properties needed later, maybe implicitly, in computations.

Lemma 3.1 ([3, Lemma 2.3]). *For a Borel function $g : \mathbb{R} \to \mathbb{R}$ such that $g(X_1) \in L_2$ and for $k, l \in \{0, 1, 2, ...\}$, $j \in \{1, 2\}$, and $b \in [0, 1)$ one has that*

$$
\mathbb{E} \sup_{0 \le s \le t \le b} |X_t|^k |X_s|^l \left(\frac{\partial^j G}{\partial x^j}(s, X_s) \right)^2 < \infty
$$

where G is given by (1) *and* $0^0 := 1$.

Moreover, we let

$$
H(t) := \left\| \left(\sigma^2 \frac{\partial^2 G}{\partial x^2} \right)(t, X_t) \right\|_{L_2} \quad \text{for} \quad t \in [0, 1)
$$

and obtain a continuous and non-decreasing function $H : [0, 1) \to [0, \infty)$ (see [7, Lemma 3.9]).

3.1 Proof of (ii) $\Longrightarrow$ (iii)

Because of $\mu_t = \text{law}(Z_\beta(t))$ for $t \in [0, 1)$ we get that

$$
\int_{\mathbb{R}} x^2 d\mu_t(x) = \mathbb{E}\widetilde{Z}_\beta(t)^2 \le \mathbb{E}A_\beta(1) < \infty.
$$

3.2 Proof of (iii) $\Longrightarrow$ (i)

Our assumption implies that

$$
\frac{1}{2} \int_0^1 \nu_\beta(u) H(u)^2 du \quad = \quad \sup_{t \in [0,1)} \mathbb{E} \frac{1}{2} \int_0^t \nu_\beta(u) \left[\left(\sigma^2 \frac{\partial^2 G}{\partial x^2} \right)(u, X_u) \right]^2 du
$$

$$
\begin{aligned}
&= \sup_{t\in[0,1)} \mathbb{E}\widetilde{Z}_\beta(t)^2 \\
&= \sup_{t\in[0,1)} \int_{\mathbb{R}} x^2 d\mu_t(x)
\end{aligned}
$$

so that

$$
\int_0^1 (1-u)^{1-\beta} H(u)^2 du < \infty.
$$

Now assertion (i) follows from [7, Proof of Theorem 3.2].

3.3 Preparations for the proof of (i) $\Longrightarrow$ (ii)

We first decompose the error process. For $t \in [0,1]$ and a time net $\tau = (t_i)_{i=0}^n$, $0 = t_0 < \cdots < t_n = 1$, we obtain, $\mathbb{P}$-a.s., that

$$
\begin{aligned}
C_t(\tau) &= \left[\int_0^t \frac{\partial G}{\partial x}(u, X_u)\, dX_u - \sum_{i=0}^{n-1} \frac{\partial G}{\partial x}(t_i, X_{t_i}) \left(X_{t_{i+1}\wedge t} - X_{t_i \wedge t}\right)\right] \\
&= \sum_{i=0}^{n-1} \int_{t_i\wedge t}^{t_{i+1}\wedge t} \left[\frac{\partial G}{\partial x}(u, X_u) - \frac{\partial G}{\partial x}(t_i, X_{t_i}) \right. \\
&\qquad\qquad \left. - \frac{\partial^2 G}{\partial x^2}(t_i, X_{t_i})(X_u - X_{t_i})\right] dX_u \\
&\quad + \sum_{i=0}^{n-1} \int_{t_i\wedge t}^{t_{i+1}\wedge t} \left[\sigma(X_u) - \sigma(X_{t_i})\right] \frac{\partial^2 G}{\partial x^2}(t_i, X_{t_i})(X_u - X_{t_i})\, dB_u \\
&\quad + \sum_{i=0}^{n-1} \int_{t_i\wedge t}^{t_{i+1}\wedge t} \left(\sigma \frac{\partial^2 G}{\partial x^2}\right)(t_i, X_{t_i})(X_u - X_{t_i})\, dB_u \\
&=: I_t^1(\tau) + I_t^2(\tau) + I_t^3(\tau).
\end{aligned}
$$

The appropriate L_2-integrability of the integrands in the decomposition above is obtained by standard arguments (see, for example, Lemma 3.1 and its proof).

Estimation of $I^1(\tau)$ and $I^2(\tau)$. First we show that a Taylor expansion of order one of the integrand of the stochastic integral $\int_0^1 \frac{\partial G}{\partial x}(u, X_u)\, dX_u$ gives an L_2-approximation rate of $o(1/\sqrt{n})$ provided that appropriate time nets are taken.

Proposition 3.2. *Let $\beta \in (0,1]$, and $g \in B_{2,2}^\beta(\gamma)$ for $X = B$ and $g(e^{\cdot - \frac{1}{2}}) \in B_{2,2}^\beta(\gamma)$ for $X = S$. Then one has that*

$$
\lim_{n\to\infty} n\mathbb{E}|I_1^1(\tau^{n,\beta})|^2 = \lim_{n\to\infty} n\mathbb{E}\left|\sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} \left[\frac{\partial G}{\partial x}(u, X_u) - \frac{\partial G}{\partial x}\left(t_i^{n,\beta}, X_{t_i^{n,\beta}}\right) \right.\right.
$$
$$
\left.\left. - \frac{\partial^2 G}{\partial x^2}\left(t_i^{n,\beta}, X_{t_i^{n,\beta}}\right)\left(X_u - X_{t_i^{n,\beta}}\right)\right] dX_u\right|^2 = 0.
$$

Proof. (a) Let $0 \le a < b < 1$ and

$$
\begin{aligned}
\Phi(u,x) &= \Phi(u,x,\omega) \\
&:= \left[\frac{\partial G}{\partial x}(u,x) - \frac{\partial G}{\partial x}(a,X_a(\omega)) - \frac{\partial^2 G}{\partial x^2}(a,X_a(\omega))(x - X_a(\omega))\right]\sigma(x)
\end{aligned}
$$

for $u \in [a,b]$ and $x \in \mathbb{R}$ if $X = B$ and $x > 0$ if $X = S$, where we shall suppress ω in the following. By a computation we get that

$$
\frac{\partial \Phi}{\partial u}(u,x) + \frac{\sigma(x)^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(u,x) = -\frac{\partial^2 G}{\partial x^2}(a,X_a)\sigma'(x)\sigma(x)^2.
$$

From this we conclude that

$$
\begin{aligned}
&\frac{\partial \Phi^2}{\partial u}(u,x) + \frac{\sigma(x)^2}{2}\frac{\partial^2 \Phi^2}{\partial x^2}(u,x) \\
= \quad & 2\Phi(u,x)\left[\frac{\partial \Phi}{\partial u}(u,x) + \frac{\sigma(x)^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(u,x)\right] + \sigma(x)^2\left[\frac{\partial \Phi}{\partial x}(u,x)\right]^2 \\
= \quad & -2\Phi(u,x)\frac{\partial^2 G}{\partial x^2}(a,X_a)\sigma'(x)\sigma(x)^2 + \sigma(x)^2\left[\frac{\partial \Phi}{\partial x}(u,x)\right]^2 \\
= \quad & -2\Phi(u,x)\frac{\partial^2 G}{\partial x^2}(a,X_a)\sigma'(x)\sigma(x)^2 \\
& \qquad + \left[\Phi(u,x)\sigma'(x) + \left[\frac{\partial^2 G}{\partial x^2}(u,x) - \frac{\partial^2 G}{\partial x^2}(a,X_a)\right]\sigma(x)^2\right]^2.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\left|\frac{\partial \Phi^2}{\partial u}(u,x) + \frac{\sigma(x)^2}{2}\frac{\partial^2 \Phi^2}{\partial x^2}(u,x)\right| &\le 2\left|\Phi(u,x)\frac{\partial^2 G}{\partial x^2}(a,X_a)\sigma(x)^2\right| \\
&\quad + 2\Phi(u,x)^2 + 2\left[\frac{\partial^2 G}{\partial x^2}(u,x) - \frac{\partial^2 G}{\partial x^2}(a,X_a)\right]^2 \sigma(x)^4.
\end{aligned}
$$

Moreover, by Itô's formula,

$$
\mathbb{E}\Phi(b,X_b)^2 = \mathbb{E}\Phi(a,X_a)^2 + \mathbb{E}\int_a^b \left[\frac{\partial \Phi^2}{\partial u}(u,X_u) + \frac{\sigma(X_u)^2}{2}\frac{\partial^2 \Phi^2}{\partial x^2}(u,X_u)\right] du.
$$

In fact, we apply Itô's formula on $[a,b]$ conditionally on $X_a = y$, obtain (conditionally) the equation for b replaced by τ_N defined as the minimum of b, $\inf\{s \in [a,b] : |(\partial(\Phi^2)/\partial x)(u,X_u)| \ge N\}$ and $\inf\{s \in [a,b] : |X_u - y| \ge N \text{ if } X = B, |X_u/y| \notin ((1/N),N) \text{ if } X = S\}$, and let $N \to \infty$ by the help of Lemma 3.1 (see also [6, proof of Theorem 6] for the conditioning argument). Now Gronwall's lemma gives

$$
\begin{aligned}
\mathbb{E}\Phi(b,X_b)^2 \le c_{(3)} &\left[\int_a^b \mathbb{E}\left|\Phi(u,X_u)\frac{\partial^2 G}{\partial x^2}(a,X_a)\sigma(X_u)^2\right| du \right. \\
&\left. + \int_a^b \mathbb{E}\left[\frac{\partial^2 G}{\partial x^2}(u,X_u) - \frac{\partial^2 G}{\partial x^2}(a,X_a)\right]^2 \sigma(X_u)^4 du\right] \qquad (3)
\end{aligned}
$$

for some absolute constant $c_{(3)} > 0$.

(b) Let $i \in \{0, ..., n-1\}$ and $u \in [t_i^{n,\beta}, t_{i+1}^{n,\beta})$, and set

$$\Phi_i^n(u,x) := \left[\frac{\partial G}{\partial x}(u,x) - \frac{\partial G}{\partial x}(t_i^{n,\beta}, X_{t_i^{n,\beta}}) - \frac{\partial^2 G}{\partial x^2}(t_i^{n,\beta}, X_{t_i^{n,\beta}})(x - X_{t_i^{n,\beta}}) \right] \sigma(x).$$

From step (a) we conclude that

$$\begin{aligned} n\mathbb{E} \Bigg| \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} \Bigg[\frac{\partial G}{\partial x}(u, X_u) &- \frac{\partial G}{\partial x}\left(t_i^{n,\beta}, X_{t_i^{n,\beta}}\right) \\ &- \frac{\partial^2 G}{\partial x^2}\left(t_i^{n,\beta}, X_{t_i^{n,\beta}}\right)\left(X_u - X_{t_i^{n,\beta}}\right) \Bigg] dX_u \Bigg|^2 \\ = \;& n \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} \mathbb{E}\Phi_i^n(u, X_u)^2 du \\ \le \;& c_{(3)} n \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} \Bigg[\int_{t_i^{n,\beta}}^{u} \mathbb{E} \left| \Phi_i^n(v, X_v) \frac{\partial^2 G}{\partial x^2}(t_i^{n,\beta}, X_{t_i^{n,\beta}}) \sigma(X_v)^2 \right| dv \\ &+ \int_{t_i^{n,\beta}}^{u} \mathbb{E} \left[\frac{\partial^2 G}{\partial x^2}(v, X_v) - \frac{\partial^2 G}{\partial x^2}(t_i^{n,\beta}, X_{t_i^{n,\beta}}) \right]^2 \sigma(X_v)^4 dv \Bigg] du \\ = \;& c_{(3)} n \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} \int_{t_i^{n,\beta}}^{u} A_i^n(v)^2 dv du \end{aligned}$$

with

$$\begin{aligned} A_i^n(v)^2 := \mathbb{E} \left| \Phi_i^n(v, X_v) \frac{\partial^2 G}{\partial x^2}(t_i^{n,\beta}, X_{t_i^{n,\beta}}) \sigma(X_v)^2 \right| + \\ \mathbb{E} \left[\frac{\partial^2 G}{\partial x^2}(v, X_v) - \frac{\partial^2 G}{\partial x^2}(t_i^{n,\beta}, X_{t_i^{n,\beta}}) \right]^2 \sigma(X_v)^4 \end{aligned}$$

for $v \in [t_i^{n,\beta}, t_{i+1}^{n,\beta})$. Using (2) we continue by (cf. also [7])

$$\begin{aligned} c_{(3)} n \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} \int_{t_i^{n,\beta}}^{u} A_i^n(v)^2 dv du &= c_{(3)} n \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} (t_{i+1}^{n,\beta} - u) A_i^n(u)^2 du \\ &\le \frac{c_{(3)}}{\beta} \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} (1-u)^{1-\beta} A_i^n(u)^2 du \\ &= \frac{c_{(3)}}{\beta} \int_0^1 (1-u)^{1-\beta} \psi_n(u) du \end{aligned}$$

with

$$\psi_n(u) := \sum_{i=0}^{n-1} \chi_{[t_i^{n,\beta}, t_{i+1}^{n,\beta})}(u) A_i^n(u)^2.$$

(c) Now we show that

$$\psi_n(u) \le c_{(4)} \left| H(u) \vee \|g(X_1)\|_{L_2} \right|^2 \tag{4}$$

for some absolute constant $c_{(4)} > 0$. Assume again $a = t_i^{n,\beta} \le u < t_{i+1}^{n,\beta}$. Since the process $\left(\sigma(X_u)^2 \frac{\partial^2 G}{\partial x^2}(u, X_u)\right)_{u \in [0,1)} \subseteq L_2(\Omega, \mathbb{P})$ is a martingale (the argument for $X = S$ is given in [4]; the case $X = B$ can be treated in the same way) we get that

$$\mathbb{E}\left[\frac{\partial^2 G}{\partial x^2}(u, X_u) - \frac{\partial^2 G}{\partial x^2}(a, X_a)\right]^2 \sigma(X_u)^4 \le c_{(5)} H(u)^2 \tag{5}$$

for some absolute constant $c_{(5)} > 0$. The first term of $A_i^n(u)^2$ can be bounded by

$$\begin{aligned}
& \mathbb{E}\left|\Phi_i^n(u, X_u)\sigma(X_u)^2 \frac{\partial^2 G}{\partial x^2}(a, X_a)\right| \\
= \; & \mathbb{E}\left|\left[\frac{\partial G}{\partial x}(u, X_u) - \frac{\partial G}{\partial x}(a, X_a) - \frac{\partial^2 G}{\partial x^2}(a, X_a)(X_u - X_a)\right]\sigma(X_u)\right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\sigma(X_u)^2 \frac{\partial^2 G}{\partial x^2}(a, X_a)\right| \\
\le \; & \mathbb{E}\left|\left[\sigma(X_u)\frac{\partial G}{\partial x}(u, X_u)\right]\left[\sigma(X_a)^2 \frac{\partial^2 G}{\partial x^2}(a, X_a)\right]\left[\frac{\sigma(X_u)}{\sigma(X_a)}\right]^2\right| \\
& + \mathbb{E}\left|\left[\sigma(X_a)\frac{\partial G}{\partial x}(a, X_a)\right]\left[\sigma(X_a)^2 \frac{\partial^2 G}{\partial x^2}(a, X_a)\right]\left[\frac{\sigma(X_u)}{\sigma(X_a)}\right]^3\right| \\
& + \mathbb{E}\sigma(X_a)^4 \left[\frac{\partial^2 G}{\partial x^2}(a, X_a)\right]^2 \frac{\sigma(X_u)^3 |X_u - X_a|}{\sigma(X_a)^4}.
\end{aligned}$$

Since $\left(\sigma(X_u)\frac{\partial G}{\partial x}(u, X_u)\right)_{u \in [0,1)}$ is an L_2-martingale (for a similar reason the process $\left(\sigma(X_u)^2 \frac{\partial^2 G}{\partial x^2}(u, X_u)\right)_{u \in [0,1)}$ shares this property) we finally get that

$$\begin{aligned}
& \mathbb{E}\left|\Phi_i^n(u, X_u)\sigma(X_u)^2 \frac{\partial^2 G}{\partial x^2}(t_i^{n,\beta}, X_{t_i^{n,\beta}})\right| \\
& \qquad\qquad \le c_{(6)}\left[H(u)\left\|\sigma(X_u)\frac{\partial G}{\partial x}(u, X_u)\right\|_{L_2} + H(u)^2\right]
\end{aligned} \tag{6}$$

for some absolute constant $c_{(6)} > 0$. Using $\mathbb{E}\left|(\sigma \frac{\partial G}{\partial x})(u, X_u)\right|^2 = \sum_{k=1}^{\infty} k\alpha_k^2 u^{k-1}$ for $g = \sum_{k=0}^{\infty} \alpha_k h_k$ if $X = B$ and $g(e^{\cdot - (1/2)}) = \sum_{k=0}^{\infty} \alpha_k h_k$ if $X = S$, where $(h_k)_{k=0}^{\infty}$ are the normalized Hermite polynomials, and [7, Lemma 3.9] we get that

$$\left\|\sigma(X_u)\frac{\partial G}{\partial x}(u, X_u)\right\|_{L_2} \le c_{(7)}\left[\|g(X_1)\|_{L_2} + H(u)\right], \tag{7}$$

where $c_{(7)} > 0$ is an absolute constant, so that

$$\begin{aligned}
\psi_n(u) & \le c_{(6)}\left[H(u)\left\|\sigma(X_u)\frac{\partial G}{\partial x}(u, X_u)\right\|_{L_2} + H(u)^2\right] + c_{(5)} H(u)^2 \\
& \le [c_{(5)} + c_{(6)}]H(u)^2 + c_{(6)}c_{(7)} H(u)\left[\|g(X_1)\|_{L_2} + H(u)\right]
\end{aligned}$$

and inequality (4) follows.

(d) Now we can conclude the proof. Because of [7, proof of Theorem 3.2] the condition $g \in B^{\beta}_{2,2}(\gamma)$ for $X = B$ and $g(e^{\cdot - \frac{1}{2}}) \in B^{\beta}_{2,2}(\gamma)$ for $X = S$, respectively, implies that

$$\int_0^1 (1-u)^{1-\beta} \left| H(u) \vee \|g(X_1)\|_{L_2} \right|^2 du < \infty,$$

it remains to show that

$$\lim_n \psi_n(u) = 0 \quad \text{for all} \quad u \in [0,1).$$

But this follows from

$$\lim_n \mathbb{E} \sum_{i=0}^{n-1} \left| \Phi_i^n(u, X_u) \frac{\partial^2 G}{\partial x^2}(t_i^{n,\beta}, X_{t_i^{n,\beta}}) \sigma(X_u)^2 \right| \chi_{[t_i^{n,\beta}, t_{i+1}^{n,\beta})}(u) = 0$$

and

$$\lim_n \mathbb{E} \sum_{i=0}^{n-1} \left[\frac{\partial^2 G}{\partial x^2}(u, X_u) - \frac{\partial^2 G}{\partial x^2}(t_i^{n,\beta}, X_{t_i^{n,\beta}}) \right]^2 \sigma(X_u)^4 \chi_{[t_i^{n,\beta}, t_{i+1}^{n,\beta})}(u) = 0$$

by dominated convergence and Lemma 3.1. □

Lemma 3.3. *For $\beta \in (0,1]$ and $g(X_1) \in L_2$ one has that*

$$\lim_{n\to\infty} n\mathbb{E}|I_1^2(\tau^{n,\beta})|^2 = \lim_{n\to\infty} n\mathbb{E} \Bigg| \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} \left[\sigma(X_u) - \sigma\left(X_{t_i^{n,\beta}}\right) \right] \frac{\partial^2 G}{\partial x^2}\left(t_i^{n,\beta}, X_{t_i^{n,\beta}}\right) \left(X_u - X_{t_i^{n,\beta}}\right) dB_u \Bigg|^2 = 0.$$

Proof. For $X = B$ the integrand vanishes so that we only have to check the case $X = S$. Here we get, for some absolute constant $c > 0$,

$$\begin{aligned}
& n\mathbb{E} \left| \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} \left[S_u - S_{t_i^{n,\beta}} \right]^2 \frac{\partial^2 G}{\partial x^2}\left(t_i^{n,\beta}, S_{t_i^{n,\beta}}\right) dB_u \right|^2 \\
= \; & n \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} \mathbb{E} \left| \left[S_u - S_{t_i^{n,\beta}} \right]^2 \frac{\partial^2 G}{\partial x^2}\left(t_i^{n,\beta}, S_{t_i^{n,\beta}}\right) \right|^2 du \\
= \; & n \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} \mathbb{E} \left| S^2_{t_i^{n,\beta}} \frac{\partial^2 G}{\partial x^2}\left(t_i^{n,\beta}, S_{t_i^{n,\beta}}\right) \right|^2 \mathbb{E} \left| \frac{S_u - S_{t_i^{n,\beta}}}{S_{t_i^{n,\beta}}} \right|^4 du \\
= \; & n \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} H(t_i^{n,\beta})^2 \mathbb{E}|S_{u - t_i^{n,\beta}} - 1|^4 du \\
\leq \; & cn \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} H(t_i^{n,\beta})^2 (u - t_i^{n,\beta})^2 du \\
= \; & cn \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} H(t_i^{n,\beta})^2 (t_{i+1}^{n,\beta} - u)^2 du
\end{aligned}$$

$$\leq \quad cn \sum_{i=0}^{n-1} \int_{t_i^{n,\beta}}^{t_{i+1}^{n,\beta}} H(u)^2 (t_{i+1}^{n,\beta} - u)^2 du$$

since H is continuous and non-decreasing. Because of

$$\begin{aligned}
\sum_{i=0}^{n-1} \chi_{[t_i^{n,\beta}, t_{i+1}^{n,\beta})}(u) H(u)^2 n (t_{i+1}^{n,\beta} - u)^2 &\leq \frac{1}{\beta} \sum_{i=0}^{n-1} \chi_{[t_i^{n,\beta}, t_{i+1}^{n,\beta})}(u) H(u)^2 (t_{i+1}^{n,\beta} - u) \\
&\leq \frac{1}{\beta}(1-u) H(u)^2
\end{aligned}$$

and

$$\sum_{i=0}^{n-1} \chi_{[t_i^{n,\beta}, t_{i+1}^{n,\beta})}(u) H(u)^2 n (t_{i+1}^{n,\beta} - u)^2 \leq \frac{1}{\beta^2 n} H(u)^2$$

for $u \in [0,1)$, where we have used (2), and because of

$$\int_0^1 (1-u) H(u)^2 du < \infty,$$

which is a consequence of [7, Lemma 3.9], we arrive at our assertion by dominated convergence. $\square$

Preparations for $I^3(\tau)$. The process $I^3(\tau)$ is responsible for the structure of the weak limit of the re-normalized error process. The next lemma is based on some principal ideas of [11].

Lemma 3.4. *Let $T \in (0,1]$ and $\mu_n(\omega) = \mu_n^+(\omega) - \mu_n^-(\omega)$, where $\mu_n^+(\omega)$ and $\mu_n^-(\omega)$ are finite Borel measures on $[0,T]$ for $\omega \in \Omega$. Assume that*

- *(i) $\mu_n \pm([0,t])$ are measurable for all $t \in [0,T]$,*
- *(ii) $\sup_n \mathbb{E}|(\mu_n^+ + \mu_n^-)([0,T])|^p < \infty$ for some $p \in (0,\infty)$, and that*
- *(iii) there is a finite Borel measure μ on $[0,T]$ such that, in probability,*

$$\lim_n \sup_{t \in [0,T]} |\mu_n([0,t]) - \mu([0,t])| = 0.$$

Then, given a continuous process $(a_s)_{s\in[0,T]}$ of $\mathcal{F}$-measurable random variables, one has that

$$\lim_n \sup_{t \in [0,T]} \left| \int_0^t a_s d\mu_n(s) - \int_0^t a_s d\mu(s) \right| = 0 \quad \text{in probability.} \tag{8}$$

Proof. (a) For $N = 1, 2, ...$ let

$$\begin{aligned}
a_t^N &:= a_0 \chi_{\left[0, \frac{T}{2^N}\right]}(t) + \sum_{l=1}^{2^N - 1} a_{\frac{l}{2^N}T} \chi_{\left(\frac{l}{2^N}T, \frac{l+1}{2^N}T\right]}(t) \\
&= a_{T\frac{2^N-1}{2^N}} \chi_{[0,T]}(t) + \left(a_{T\frac{2^N-2}{2^N}} - a_{T\frac{2^N-1}{2^N}} \right) \chi_{\left[0, T\frac{2^N-1}{2^N}\right]}(t) + \cdots \\
&\quad + \left(a_0 - a_{\frac{T}{2^N}} \right) \chi_{\left[0, T\frac{1}{2^N}\right]}(t).
\end{aligned}$$

To check (8) for $a^N = (a_t^N)_{t\in[0,T]}$ it is enough to verify (8) for a^N replaced by $b = \left(\varphi\chi_{[0,r]}(t)\right)_{t\in[0,T]}$ with $r \in [0,T]$ and an $\mathcal{F}$-measurable random variable φ. Since φ is a constant factor, an easy argument shows that it is sufficient to check the case $\varphi \equiv 1$. But then we can use assumption (iii) and obtain (8) for a^N.

(b) To replace a^N by a we observe that

$$\begin{aligned}\sup_{t\in[0,T]}\left|\int_0^t a_s d\mu_n(s) - \int_0^t a_s d\mu(s)\right| &\le \sup_{t\in[0,T]} |a_t - a_t^N|(\mu_n^+ + \mu_n^- + \mu)([0,T]) \\ &\quad + \sup_{t\in[0,T]}\left|\int_0^t a_s^N d\mu_n(s) - \int_0^t a_s^N d\mu(s)\right|.\end{aligned}$$

Because of (ii) and $\sup_{t\in[0,T]}|a_t(\omega) - a_t^N(\omega)| \to 0$ as $N\to\infty$ for all $\omega\in\Omega$ step (a) implies the assertion. □

The aim of the lemma before is to prove the following counterpart of [11, Lemma 1.5].

Lemma 3.5. *Let $k\in\{1,2\}$, $T\in(0,1]$, and let $a = (a_t)_{t\in[0,T]}$ be a path-wise continuous process of $\mathcal{F}$-measurable random variables. Define*

$$\psi_s^{n,k}(a) \quad := \quad n^{\frac{k}{2}} \sum_{i=0}^{n-1} a_{t_i^{n,\beta}} \left(\frac{X_s - X_{t_i^{n,\beta}}}{\sigma\left(X_{t_i^{n,\beta}}\right)}\right)^k \mathcal{X}_{[t_i^{n,\beta}, t_{i+1}^{n,\beta})}(s)$$

for $s\in[0,T]$. Then

$$\lim_n \left[\sup_{t\in[0,T]}\left|\int_0^t \psi_s^{n,1}(a)ds\right| + \sup_{t\in[0,T]}\left|\int_0^t \psi_s^{n,2}(a)ds - \frac{1}{2}\int_0^t \nu_\beta(s)a_s ds\right|\right] = 0$$

in probability where $\nu(s) = (1/\beta)(1-s)^{1-\beta}$.

Proof. Define the random measures

$$\mu_n^k := n^{\frac{k}{2}} \sum_{i=0}^{n-1} \delta_{\{s_i^{n,\beta}\}} \int_{s_i^{n,\beta}}^{s_{i+1}^{n,\beta}} \left(\frac{X_s - X_{s_i^{n,\beta}}}{\sigma\left(X_{s_i^{n,\beta}}\right)}\right)^k ds$$

for $k\in\{1,2\}$ and $s_i^{n,\beta} := t_i^{n,\beta}\wedge T$ and let $(\mu_n^k)^\pm$ be the positive and negative parts (ω-wise), respectively. By a standard computation one checks that

$$\sup_n \mathbb{E} n^{\frac{k}{2}} \sum_{i=0}^{n-1} \int_{s_i^{n,\beta}}^{s_{i+1}^{n,\beta}} \left|\frac{X_s - X_{s_i^{n,\beta}}}{\sigma\left(X_{s_i^{n,\beta}}\right)}\right|^k ds < \infty$$

so that $\sup_n \mathbb{E}((\mu_n^k)^+ + (\mu_n^k)^-)([0,T]) < \infty$. Moreover, using (2),

$$\sup_{t\in[0,T]}\left|\int_0^t \psi_s^{n,k}(a)ds - \int_0^t a_s d\mu_n^k(s)\right|^2 \le \sup_{\substack{0\le i\le n-1 \\ t\in[s_i^{n,\beta}, s_{i+1}^{n,\beta}]}} \left|\int_t^{s_{i+1}^{n,\beta}} \psi_s^{n,k}(a)ds\right|^2$$

$$\leq \sup_{0\leq i\leq n-1}\left(s^{n,\beta}_{i+1}-s^{n,\beta}_{i}\right)\int_0^T\left(\psi^{n,k}_s(a)\right)^2 ds \leq \frac{(a^*)^2}{\beta n}\int_0^T\left(\psi^{n,k}_s(1)\right)^2 ds,$$

where $a^* := \sup_{t\in[0,T]}|a_t|$ and $\sup_{m\geq 1}\mathbb{E}\int_0^T\left(\psi^{m,k}_s(1)\right)^2 ds<\infty$, so that

$$\lim_n \sup_{t\in[0,T]}\left|\int_0^t \psi^{n,k}_s(a)ds-\int_0^t a_s d\mu^k_n(s)\right|=0 \tag{9}$$

in probability. In view of (9) and Lemma 3.4 we only need to verify

$$\lim_n \sup_{t\in[0,T]}|\mu^1_n([0,t])|=0 \quad \text{and} \quad \lim_n \sup_{t\in[0,T]}\left|\mu^2_n([0,t])-\frac{1}{2}\int_0^t \nu_\beta(s)ds\right|=0. \tag{10}$$

Let $E_1:=0$, $E_2:=1/2$, and $b^{n,k}_s:=\psi^{n,k}_s(1)$. To show (10) we upper bound

$$\begin{aligned}
&\sup_{0\leq i\leq n}\left|\int_0^{s^{n,\beta}_i} b^{n,k}_s ds - E_k\int_0^{s^{n,\beta}_i}\nu_\beta(s)ds\right|\\
&\qquad + \sup_{1\leq i\leq n}\sup_{t\in[s^{n,\beta}_{i-1},s^{n,\beta}_i]}\left|E_k\int_t^{s^{n,\beta}_i}\nu_\beta(s)ds\right|\\
&\leq \sup_{0\leq i\leq n}\left|\int_0^{s^{n,\beta}_i} b^{n,k}_s ds - \mathbb{E}\int_0^{s^{n,\beta}_i} b^{n,k}_s ds\right| &(11)\\
&\qquad + \sup_{0\leq i\leq n}\left|\mathbb{E}\int_0^{s^{n,\beta}_i} b^{n,k}_s ds - E_k\int_0^{s^{n,\beta}_i}\nu_\beta(s)ds\right| &(12)\\
&\qquad +1/(2\beta^2 n)
\end{aligned}$$

where we used (2) again.

Term (11): By Doob's maximal inequality for martingales we have that

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq i\leq n}\left|\int_0^{s^{n,\beta}_i} b^{n,k}_s ds - \mathbb{E}\int_0^{s^{n,\beta}_i} b^{n,k}_s ds\right|^2\\
\leq\;& 4n^k\sum_{i=0}^{n-1}\left(s^{n,\beta}_{i+1}-s^{n,\beta}_i\right)\int_{s^{n,\beta}_i}^{s^{n,\beta}_{i+1}}\mathbb{E}\left(\frac{X_s-X_{s^{n,\beta}_i}}{\sigma\left(X_{s^{n,\beta}_i}\right)}\right)^{2k} ds\\
\leq\;& c_k n^k\sum_{i=0}^{n-1}\left(s^{n,\beta}_{i+1}-s^{n,\beta}_i\right)\int_{s^{n,\beta}_i}^{s^{n,\beta}_{i+1}}(s-s^{n,\beta}_i)^k ds \to_{n\to\infty} 0.
\end{aligned}$$

Term (12): Because for $k=1$ the term is zero we assume that $k=2$ and get

$$\begin{aligned}
&\sup_{0\leq i\leq n}\left|\mathbb{E}\int_0^{s^{n,\beta}_i} b^{n,2}_s ds - \frac{1}{2}\int_0^{s^{n,\beta}_i}\nu_\beta(s)ds\right|\\
=\;& \sup_{1\leq i\leq n}\left|\sum_{j=0}^{i-1}\int_{s^{n,\beta}_j}^{s^{n,\beta}_{j+1}} n\mathbb{E}\left(\frac{X_s-X_{s^{n,\beta}_j}}{\sigma\left(X_{s^{n,\beta}_j}\right)}\right)^2 ds - \frac{1}{2}\int_0^{s^{n,\beta}_i}\nu_\beta(s)ds\right|
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{1\le i\le n}\Bigg|\sum_{j=0}^{i-1}\int_{s_j^{n,\beta}}^{s_{j+1}^{n,\beta}} n\left(s-s_j^{n,\beta}+m(s-s_j^{n,\beta})\ \left(s-s_j^{n,\beta}\right)^2\right)ds \\
&\qquad\qquad -\frac{1}{2}\int_0^{s_i^{n,\beta}} \nu_\beta(s)ds\Bigg| \\
&\le \sum_{j=0}^{n-1}\int_{s_j^{n,\beta}}^{s_{j+1}^{n,\beta}} nm(s-s_j^{n,\beta})\left(s-s_j^{n,\beta}\right)^2 ds \\
&\quad +\frac{1}{2}\sum_{j=0}^{n-1}\int_{s_j^{n,\beta}}^{s_{j+1}^{n,\beta}}\left|n\left(s_{j+1}^{n,\beta}-s_j^{n,\beta}\right)-\nu_\beta(s)\right|ds \\
&\le \frac{e}{3\beta^3 n}+\frac{1}{2}\sum_{j=0}^{n-1}\int_{s_j^{n,\beta}}^{s_{j+1}^{n,\beta}}\left|n\left(s_{j+1}^{n,\beta}-s_j^{n,\beta}\right)-\nu_\beta(s)\right|ds \\
&\le \frac{e}{3\beta^3 n}+\frac{1}{2}\int_{[0,t_{j_{0,n}}^{\beta,n})}\frac{1}{\beta}\sup_{u\in[0,1-\frac{1}{n}]}\left|\left(u+\frac{1}{n}\right)^{\frac{1}{\beta}-1}-u^{\frac{1}{\beta}-1}\right|ds \\
&\qquad +\frac{1}{2}\int_{[t_{j_{0,n}}^{\beta,n},T]}\left|n\left(s_{j+1}^{n,\beta}-s_j^{n,\beta}\right)-\nu_\beta(s)\right|ds
\end{aligned}
$$

where $m:[0,1]\to[0,e]$ is a continuous function and $j_{0,n}$ is the largest $j\in\{0,1,....,n\}$ such that $s_j^{n,\beta}=t_j^{n,\beta}$. Finally, by (2) we can bound the last term by

$$\frac{1}{\beta}|T-t_{j_{0,n}}^{n,\beta}|\le\frac{1}{\beta^2 n}$$

so that the term (12) converges to zero as $n\to\infty$ and the proof is complete. □

Lemma 3.6. *For $T\in(0,1)$ one has that*

$$
\begin{aligned}
&\left\|\sup_{t\in[T,1]}\left|\sqrt{n}C_t(\tau^{n,\beta})-\sqrt{n}C_T(\tau^{n,\beta})\right|\right\|_{L_2} \\
&\qquad\qquad \le \frac{c}{\sqrt{\beta}}\left(\int_{(T-\frac{1}{\beta n})^+}^1(1-s)^{1-\beta}H(s)^2ds\right)^{\frac{1}{2}}
\end{aligned}
$$

where $H(s)^2=\mathbb{E}\left(\sigma^2(X_s)\frac{\partial^2 G}{\partial x^2}(s,X_s)\right)^2$ and $c>0$ is an absolute constant.

Proof. Let $T_n^\beta:=\sup\{t_i^{n,\beta}:t_i^{n,\beta}\le T, i=0,...,n-1\}$. Then, by Doob's maximal inequality and [6, Proof of Theorem 6],

$$
\begin{aligned}
&\left\|\sup_{t\in[T,1]}\left|C_t(\tau^{n,\beta})-C_T(\tau^{n,\beta})\right|\right\|_{L_2} \\
&\qquad \le 4\left\|C_1(\tau^{n,\beta})-C_{T_n^\beta}(\tau^{n,\beta})\right\|_{L_2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq c \left(\sum_{i=0}^{n-1} \int_{t_i^{n,\beta} \vee T_n^\beta}^{t_{i+1}^{n,\beta} \vee T_n^\beta} (t_{i+1}^{n,\beta} - s) H(s)^2 ds \right)^{\frac{1}{2}} \\
&\leq c \sup_{\substack{i=0,\dots,n-1 \\ s \in [t_i^{n,\beta}, t_{i+1}^{n,\beta})}} \left| \frac{t_{i+1}^{n,\beta} - s}{(1-s)^{1-\beta}} \right|^{\frac{1}{2}} \left(\int_{T_n^\beta}^1 (1-s)^{1-\beta} H(s)^2 ds \right)^{\frac{1}{2}} \\
&\leq \frac{c}{\sqrt{\beta n}} \left(\int_{\left(T - \frac{1}{\beta n}\right)^+}^1 (1-s)^{1-\beta} H(s)^2 ds \right)^{\frac{1}{2}}
\end{aligned}
$$

where $c > 0$ is an absolute constant and we have used (2). □

Theorem 3.7 ([11, Theorem 1.2])**.** *Let $T \in [0,1]$. Suppose that $\psi^n = (\psi_t^n)_{t \in [0,T]}$, $n = 1, 2, \ldots$, are progressively measurable processes and that*

$$\lim_n \sup_{t \in [0,T]} \left| \int_0^t [\psi_s^n]^2 \, ds - A_t \right| = 0 \quad \textit{in probability}$$

for some continuous process $A = (A_t)_{t \in [0,T]}$ and that

$$\lim_n \sup_{t \in [0,T]} \left| \int_0^t \psi_s^n ds \right| = 0 \quad \textit{in probability.}$$

Then

$$\left(\int_0^t \psi_s^n dB_s \right)_{t \in [0,T]} \Longrightarrow_{C[0,T]} (W_{A_t})_{t \in [0,T]} \quad \textit{for} \quad n \to \infty,$$

where the Brownian motion W is independent from $\mathcal{F}$.

Proof of Theorem 2.1: Combining Theorem 3.7 and Lemma 3.5 for $a_t := \left(\sigma^2 \frac{\partial^2 G}{\partial x^2}\right)(t, X_t)$ in case $k = 1$, $a_t := \left[\left(\sigma^2 \frac{\partial^2 G}{\partial x^2}\right)(t, X_t)\right]^2$ in case $k = 2$, and $A_t := A_\beta(t)$ yields to

$$(\sqrt{n} I_t^3(\tau^{n,\beta}))_{t \in [0,T]} \Longrightarrow_{C[0,T]} (W_{A_\beta(t)})_{t \in [0,T]} = (Z_\beta(t))_{t \in [0,T]}$$

for all $T \in [0,1)$. Because of Proposition 3.2, Lemma 3.3, and Doob's maximal inequality (note that $(I_t^1(\tau^{n,\beta}))_{t \in [0,T]}$ and $(I_t^2(\tau^{n,\beta}))_{t \in [0,T]}$ are L_2-martingales (cf. Lemma 3.1) so that $\sqrt{n} \sup_{t \in [0,T]} |I_t^k(\tau^{n,\beta})| \to_{L_2} 0$ as $n \to \infty$ for $k = 1, 2$) we can deduce that

$$(\sqrt{n} C_t(\tau^{n,\beta}))_{t \in [0,T]} \Longrightarrow_{C[0,T]} (Z_\beta(t))_{t \in [0,T]} \quad \text{as} \quad n \to \infty. \tag{13}$$

Proof of (i) $\Rightarrow$ (ii): Firstly, we observe that (i) implies

$$\int_0^1 (1-s)^{1-\beta} H(s)^2 ds < \infty$$

according to [7, proof of Theorem 3.2] so that $\mathbb{E} A_\beta(1) < \infty$. Given a continuous and bounded $\varphi : C[0,1] \to \mathbb{R}$ we have to prove that

$$\lim_n \mathbb{E} \varphi(Y^n) = \mathbb{E} \varphi(\widetilde{Z}_\beta)$$

where $Y_t^n := \sqrt{n} C_t(\tau^{n,\beta})$. We can restrict ourselves to uniformly continuous and bounded φ (cf. [1]). Let $T \in (0,1)$, $Y^{T,n} := (Y_{t\wedge T}^n)_{t\in[0,1]}$, and $\widetilde{Z}_\beta^T := (\widetilde{Z}_\beta(t\wedge T))_{t\in[0,1]}$. Then

$$\begin{aligned}\left|\mathbb{E}\varphi(Y^n) - \mathbb{E}\varphi(\widetilde{Z}_\beta)\right| \;&\le\; \left|\mathbb{E}\varphi(Y^n) - \mathbb{E}\varphi(Y^{T,n})\right| + \left|\mathbb{E}\varphi(Y^{T,n}) - \mathbb{E}\varphi(\widetilde{Z}_\beta^T)\right| \\ &\quad + \left|\mathbb{E}\varphi(\widetilde{Z}_\beta^T) - \mathbb{E}\varphi(\widetilde{Z}_\beta)\right|.\end{aligned}$$

We fix $\varepsilon > 0$ and find a $\delta > 0$ such that $|\varphi(f) - \varphi(g)| < \varepsilon$ for $\|f-g\|_{C[0,1]} < \delta$. Then

$$\begin{aligned}&\left|\mathbb{E}\varphi(Y^n) - \mathbb{E}\varphi(Y^{T,n})\right| \\ \le\;& \int_{\{\|Y^n - Y^{T,n}\|_{C[0,1]}\ge\delta\}} |\varphi(Y^n) - \varphi(Y^{T,n})| d\mathbb{P} + \\ &\qquad\qquad + \int_{\{\|Y^n - Y^{T,n}\|_{C[0,1]}<\delta\}} |\varphi(Y^n) - \varphi(Y^{T,n})| d\mathbb{P} \\ \le\;& 2\|\varphi\|_\infty \mathbb{P}\left(\|Y^n - Y^{T,n}\|_{C[0,1]} \ge \delta\right) + \varepsilon \\ \le\;& 2\|\varphi\|_\infty \frac{c_{(3.6)}^2}{\delta^2\beta} \int_{\left(T-\frac{1}{\beta n}\right)^+}^1 (1-s)^{1-\beta} H(s)^2 ds + \varepsilon\end{aligned}$$

where we have used Lemma 3.6. Let $T_0 \in (0,1)$ be such that

$$2\|\varphi\|_\infty \frac{c_{(3.6)}^2}{\delta^2\beta} \int_{T_0}^1 (1-s)^{1-\beta} H(s)^2 ds \le \varepsilon,$$

and

$$|\mathbb{E}\varphi(\widetilde{Z}_\beta^T) - \mathbb{E}\varphi(\widetilde{Z}_\beta)| \le \varepsilon$$

for $T \in [T_0, 1)$ (note that $\|\widetilde{Z}_\beta^T(\omega) - \widetilde{Z}_\beta(\omega)\|_{C[0,1]} \to 0$ as $T \uparrow 1$ for all $\omega \in \Omega$). Fix $n_0 \ge 1$ such that $1/(\beta n_0) \le (1-T_0)/2$. Hence, for $T \in [(T_0+1)/2, 1)$ and $n \ge n_0$ one has $T - \frac{1}{\beta n} \ge T_0$ and

$$\left|\mathbb{E}\varphi(Y^n) - \mathbb{E}\varphi(\widetilde{Z}_\beta)\right| \le 3\varepsilon + \left|\mathbb{E}\varphi(Y^{T,n}) - \mathbb{E}\varphi(\widetilde{Z}_\beta^T)\right|.$$

Defining the bounded and continuous function $\varphi_T : C[0,T] \to \mathbb{R}$ by $\varphi_T(g) := \varphi(f)$ with $f(t) := g(t\wedge T)$, we get

$$\lim_n \mathbb{E}\varphi(Y^{T,n}) = \lim_n \mathbb{E}\varphi_T((Y_t^n)_{t\in[0,T]}) = \mathbb{E}\varphi_T((\widetilde{Z}_\beta(t))_{t\in[0,T]}) = \mathbb{E}\varphi(\widetilde{Z}_\beta^T),$$

where we used (13), and

$$\limsup_n \left|\mathbb{E}\varphi(Y^n) - \mathbb{E}\varphi(\widetilde{Z}_\beta)\right| \le 3\varepsilon.$$

Since this is true for all $\varepsilon > 0$ we are done. □

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