SHARP EXPONENTIAL INTEGRABILITY FOR TRACES OF MONOTONE SOBOLEV FUNCTIONS

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ABSTRACT. We answer a question posed in [12] on exponential integrability of functions of restricted n-energy. We use geometric methods to obtain a sharp exponential integrability result for boundary traces of monotone Sobolev functions defined on the unit ball.

1. INTRODUCTION

The following result answered a problem of A. Beurling, mentioned by J. Moser in a famous paper [10].

Theorem A (Chang-Marshall (1985), [1]). *There is a universal constant $C < \infty$ so that if f is analytic in $\mathbb{D}$, $f(0) = 0$, and*

$$\int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi} \leq 1, \tag{1.1}$$

then

$$\int_0^{2\pi} \exp\left(|f^\star(e^{i\theta})|^2\right) d\theta \leq C,$$

where $f^\star$ is the trace of f on $\partial\mathbb{D}$, i.e., $f^\star(\zeta) = \lim_{t\uparrow 1} f(t\zeta)$ for $\mathcal{H}^1$-a.e. $\zeta \in \partial\mathbb{D}$.

This result is moreover "sharp" in the following sense: the Beurling functions,

$$B_a(z) := \left(\log\frac{1}{1-az}\right)\left(\log\frac{1}{1-a^2}\right)^{\frac{-1}{2}} \qquad 0 < a < 1$$

are analytic in $\mathbb{D}$, satisfy $B_a(0) = 0$ and (1.1), and have the property that for any given $\alpha > 1$, one can choose a so that the integral

$$\int_0^{2\pi} \exp\left(\alpha|B_a(e^{i\theta})|^2\right) d\theta$$

is as large as desired.

The following is an easy corollary of the Chang-Marshall Theorem.

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Corollary A. *There is a universal constant $C < \infty$ so that if $u : \mathbb{D} \to \mathbb{R}$ is harmonic with $u(0) = 0$ and*

$$\int_{\mathbb{D}} |\nabla u(z)|^2 \, \frac{dA(z)}{\pi} \leq 1,$$

then

$$\int_0^{2\pi} \exp\left(u^\star(e^{i\theta})^2\right) \, d\theta \leq C,$$

where $u^\star$ is the trace of u on $\partial\mathbb{D}$, i.e., $u^\star(\zeta) = \lim_{t\uparrow 1} u(t\zeta)$ for $\mathcal{H}^1$-a.e. $\zeta \in \partial\mathbb{D}$.

This can also be shown to be sharp by considering the real parts of the Beurling functions.

In [12] the last two authors generalized the Chang-Marshall theorem to quasiregular mappings in all dimensions. They also asked in [12] whether Corollary A also generalizes, perhaps substituting "harmonic" with "n-harmonic". In this note we show that this is indeed possible. The key concept is that of a monotone Sobolev function, whose definition we recall below, and which is quite general, and includes for instance n-harmonic functions.

2. Main results

For a continuous function $u \colon \Omega \to \mathbb{R}$, we define the oscillation of u on a compact set $K \subset \Omega$ by

$$\operatorname*{osc}_K u = \max_{x,y \in K} |u(x) - u(y)|.$$

We say that $u \colon \Omega \to \mathbb{R}$ is *monotone* if $\operatorname{osc}_{\partial B} u = \operatorname{osc}_{\bar{B}} u$ for all n-balls B compactly contained in Ω.

Given $u \colon B^n \to \mathbb{R}$ in the Sobolev space $W^{1,n}(B^n)$, the radial limit

$$\tilde{u}(y) = \lim_{r \to 1} u(ry)$$

exists at $\mathcal{H}^{n-1}$-a.e. point $y \in S^{n-1}$. We denote by $\tilde{u}$ the almost everywhere defined trace of u. Moreover, we denote the L^p-norm of a p-integrable $g : \Omega \to \mathbb{R}^n$ by $\|g\|_p = \|g\|_{\Omega,p}$. The surface measure $\mathcal{H}^{n-1}(S^{n-1})$ of the unit sphere S^{n-1} is ω_{n-1}. The notations $B^n(r) = B^n(0,r)$, $B^n = B^n(1)$ for n-dimensional balls will be used.

Theorem 1. *There exists a constant $C = C(n) > 0$ so that if $u \in W^{1,n}(B^n)$ is a non-constant continuous monotone function such that $u(0) = 0$, then*

$$\int_{S^{n-1}} \exp\left(\alpha(|\tilde{u}(y)|/\|\nabla u\|_n)^{n/(n-1)}\right) \, d\mathcal{H}^{n-1}(y) \leq C, \tag{2.2}$$

where

$$\alpha = (n-1)\left(\frac{\omega_{n-1}}{2}\right)^{\frac{1}{n-1}}. \tag{2.3}$$

The continuity assumption in Theorem 1 is of technical nature. By a theorem of Manfredi [8], so-called weakly monotone functions in $W^{1,n}$ are always continuous and monotone in the above sense. In general, $W^{1,n}$-functions need not be continuous.

Theorem 1 is not true without the monotonicity assumption. Indeed, since the n-capacity of a point is zero, one can construct Sobolev functions $u_i \in W^{1,n}(B^n)$ so that $u_i(0) = 0$, $\|\nabla u_i\|_n \leq 1$, and $\tilde{u}_i(y) \geq i$ for every $y \in S^{n-1}$.

Our method of proof for Theorem 1 has a similar geometric flavor as in [9] and in [12], and the end-game is again to appeal to Moser's original one-dimensional proof. However, the so-called "egg-yolk" property, which was the hardest part to establish in the two papers cited above, can be quickly established in our present case. It might come as a surprise then that Theorem 1 is sharp, as we will see in Theorem 2 below, as opposed to the situation in [12].

A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called *p-harmonic*, $1 < p < \infty$, if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \,\mathrm{d}x = 0$$

for every C^∞-smooth test function ϕ with compact support in Ω, see [6]. Since p-harmonic functions satisfy the maximum principle ([6, 6.5]), they are, in particular, monotone.

The next result shows that the constant α in Theorem 1 is sharp.

Theorem 2. *Let α be as in Theorem 1. There exists a sequence of n-harmonic functions $u_i \in W^{1,n}(B^n)$ satisfying $\|\nabla u_i\|_n \leq 1$ and $u_i(0) = 0$, so that*

$$\int_{S^{n-1}} \exp\left(\beta |\tilde{u}_i(y)|^{n/(n-1)}\right) \,\mathrm{d}\mathcal{H}^{n-1}(y) \to \infty \quad \text{as } i \to \infty$$

whenever $\beta > \alpha$.

3. Proof of Theorem 1

In this section we assume that u satisfies the assumptions of Theorem 1. Moreover, by considering balls $B^n(0, 1-1/j)$, for j large, and using Fatou's lemma, we may assume that the function u in Theorem 1 is defined in a neighborhood of the unit ball.

Lemma 3. *There exists a constant $r_0 = r_0(n) > 0$ so that if $M_0 := \max_{\bar{B}^n(r_0)} |u|$, then*

$$\int_{\{|u| \leq M_0\}} |\nabla u|^n \,\mathrm{d}x \geq M_0^n.$$

Proof. For $0 < r < 1$ let $m := \max_{\bar{B}^n(r)} |u|$ and set $v := \min\{|u|, m\}$. By monotonicity, and since $u(0) = 0$, $\operatorname{osc}_{S^{n-1}(t)} v = m$ for every $t \geq r$.

By the Sobolev embedding theorem on spheres, see e.g. [5, Lemma 1] or [11], there exists a constant a depending only on n such that

$$\begin{aligned}\int_{B^n\setminus \bar{B}^n(r)} |\nabla v|^n \,\mathrm{d}x &= \int_r^1 \left(\int_{S^{n-1}(t)} |\nabla v|^n \,\mathrm{d}\mathcal{H}^{n-1}\right) \mathrm{d}t \\ &\geq \int_r^1 \frac{\left(\operatorname{osc}_{S^{n-1}(t)} v\right)^n}{at} \,\mathrm{d}t = \frac{m^n}{a}\log\frac{1}{r}.\end{aligned}$$

The claim follows by choosing $r_0 := \exp(-a)$. □

Let Γ be a family of paths in a domain Ω. The *n-modulus* $\mathsf{M}_n(\Gamma)$ of Γ is defined as follows:

$$\mathsf{M}_n(\Gamma) = \inf_\rho \int_\Omega \rho^n \,\mathrm{d}x,$$

where $\rho\colon \Omega \to [0,\infty]$ is *an admissible function* for Γ, i.e. a Borel function satisfying

$$\int_\gamma \rho \,\mathrm{d}s \geq 1 \tag{3.4}$$

for every locally rectifiable $\gamma \in \Gamma$. The family of all paths joining two sets $A, B \subset \bar{\Omega}$ in Ω is denoted by $\Delta(A,B;\Omega)$. We say that a given property holds for n-almost every path in a path family Γ if the property holds for all paths in $\Gamma \setminus \Gamma_0$, where Γ_0 is a subfamily of Γ having n-modulus zero.

Lemma 4. *For every $r \in (0,1)$, there exists a constant $c = c(n,r)$, so that*

$$\mathcal{H}^{n-1}\left(\{y \in S^{n-1}\colon |u(y)| \geq s\}\right) \leq c\exp\left(-\alpha I_M^s(u)\right) \tag{3.5}$$

for $s \geq M$. Here α is as in (2.3), $M = M(r,u) = \max_{S^{n-1}(r)} |u|$, and

$$I_M^s(u) = \int_M^s \frac{\mathrm{d}t}{\left(\int_{\{|u|=t\}} |\nabla u|^{n-1} \,\mathrm{d}\mathcal{H}^{n-1}\right)^{1/(n-1)}}.$$

Proof. Fix $r \in (0,1)$ and $s > M = M(r,u)$. Write

$$E = E_s := \{y \in S^{n-1}\colon |u(y)| \geq s\}$$

and

$$U_M := \{x \in B^n\colon |u(x)| \geq M\}.$$

Also, here and in what follows we write

$$A_t := \int_{\{|u|=t\}} |\nabla u|^{n-1} \,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.6}$$

The fact that A_t is a Borel function of t is standard, see for instance [2] p. 117.

We construct an admissible function ρ for $\Delta(\partial U_M, E; B^n)$ as follows: Let $I = I^s_M(u)$, and set

$$\rho(x) := \frac{|\nabla u(x)|}{I A^{1/(n-1)}_{|u(x)|}} \chi_{U_M}(x).$$

Since every path in $\Delta(\partial U_M, E; B^n)$ has a subpath in $\Delta(\partial U_M, E; U_M)$, it suffices to show that ρ is admissible for n-almost every path in $\Delta(\partial U_M, E; U_M)$, i.e. that the set of paths where (3.4) fails has n-modulus zero. Recall that, by Fuglede's theorem [3, Theorem 3], u is absolutely continuous on n-almost every path. So, for n-almost every $\gamma \in \Delta(\partial U_M, E; U_M)$ parameterized by arc length $\ell(\gamma)$, we have, by change of variables

$$\begin{aligned}
\int_\gamma \rho \,\mathrm{d}s &= \int_0^{\ell(\gamma)} \frac{|\nabla u(\gamma(t))|}{I A^{1/(n-1)}_{|u(\gamma(t))|}} \,\mathrm{d}t \geq I^{-1} \int_0^{\ell(\gamma)} \frac{|(u \circ \gamma)'(t)|}{A^{1/(n-1)}_{|(u\circ\gamma)(t)|}} \,\mathrm{d}t \\
&\geq I^{-1} \int_M^s \frac{\mathrm{d}t}{A_t^{1/(n-1)}} = 1.
\end{aligned}$$

Thus ρ is an admissible function for $\Delta(\partial U_M, E; U_M)$, and so also for $\Delta(B^n(r), E; U_M)$, by the definition of n-modulus. By the coarea formula, cf. [7], we have

$$\begin{aligned}
\mathsf{M}_n(\Delta(B^n(r), E; B^n)) &\leq \int_{U_M} \rho^n \,\mathrm{d}x = I^{-n} \int_{U_M} \frac{|\nabla u(x)|^n}{A^{n/(n-1)}_{|u(x)|}} \,\mathrm{d}x \\
&= I^{-n} \int_M^s \int_{\{|u|=t\}} \frac{|\nabla u(y)|^{n-1}}{A_t^{n/(n-1)}} \,\mathrm{d}\mathcal{H}^{n-1}(y) \,\mathrm{d}t \\
&= I^{-n} \int_M^s \frac{A_t}{A_t^{n/(n-1)}} \,\mathrm{d}t = I^{1-n}.
\end{aligned}$$

By the conformal invariance of n-modulus, taking inversion with respect to the unit sphere yields

$$2\mathsf{M}_n(\Delta(B^n(r), E; B^n)) \geq \mathsf{M}_n(\Delta(S^{n-1}(r) \cup S^{n-1}(1/r), E; \mathbb{R}^n)).$$

By spherical symmetrization and [4, Theorem 4],

$$\begin{aligned}
2I^{1-n} &\geq \mathsf{M}_n(\Delta(S^{n-1}(r) \cup S^{n-1}(1/r), E; \mathbb{R}^n)) \\
&\geq \omega_{n-1} \left(\log \frac{c_2}{\mathcal{H}^{n-1}(E)^{1/(n-1)}} \right)^{1-n},
\end{aligned}$$

where c_2 depends only on n and r. See [12] for further details. This implies (3.5). □

Proof of Theorem 1. We will use the following result of Moser [10, Equation (6)]: If $\omega : [0, \infty) \to [0, \infty)$ is absolutely continuous and satisfies $\omega(0)=0$, $\omega'\geq 0$ almost everywhere, and

$$\int_0^\infty (\omega'(t))^n \,\mathrm{d}t \leq 1,$$

then

$$\int_0^\infty \exp(\omega(t)^{n/(n-1)}-t)\,\mathrm{d}t \leq C, \tag{3.7}$$

where $C>0$ depends only on n. By scaling invariance of (2.2), we may assume that

$$\int_{B^n} |\nabla u|^n \,\mathrm{d}x = 1. \tag{3.8}$$

Moreover, we fix $r=r_0$ and $M=M_0$ as in Lemma 3. Then, in particular, $M<1$.

By the Cavalieri principle,

$$\begin{aligned}\int_{S^{n-1}} &\exp\left(\alpha|u(x)|^{n/(n-1)}\right)\,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &= \omega_{n-1}+\frac{\alpha n}{n-1}\int_0^\infty s^{1/(n-1)}\mathcal{H}^{n-1}(E_s)\exp\left(\alpha s^{n/(n-1)}\right)\,\mathrm{d}s,\end{aligned}$$

where

$$E_s=\{y\in S^{n-1}:|u(y)|\geq s\}.$$

Then, by Lemma 4, it suffices to bound

$$\int_0^{||u||_\infty} s^{1/(n-1)}\exp\left(\alpha(s^{n/(n-1)}-I^s_M(u))\right)\,\mathrm{d}s, \tag{3.9}$$

where $||u||_\infty=\max_{y\in S^{n-1}}|u(y)|$, and $I^s_M(u)=0$ for $0<s<M$. We define a function $\psi:[0,\infty)\to[0,\infty)$,

$$\psi(s)=\begin{cases}\mu s, & 0<s<M\\ \alpha I^s_M(u)+\mu M, & M\leq s\leq ||u||_\infty\\ \alpha I^{||u||_\infty}_M(u)+\mu M, & s>||u||_\infty,\end{cases}$$

where

$$\mu=\alpha\Big(\frac{M}{\int_{\{|u|\leq M\}}|\nabla u|^n\,\mathrm{d}x}\Big)^{1/(n-1)}. \tag{3.10}$$

Then, by Lemma 3, $\mu M\leq\alpha$, and thus we may consider

$$\int_0^{||u||_\infty} s^{1/(n-1)}\exp(\alpha s^{n/(n-1)}-\psi(s))\,\mathrm{d}s \tag{3.11}$$

instead of (3.9). We define ϕ by $\phi(y)=\psi^{-1}(y)$ for $0<y<||\psi||_\infty$, and $\phi(y)=||u||_\infty$ for $y\geq||\psi||_\infty$. Then, changing variables $y=\psi(s)$ in (3.11) yields

$$\int_0^\infty \exp(\alpha\phi(y)^{n/(n-1)}-y)\phi'(y)\phi(y)^{1/(n-1)}\,\mathrm{d}y. \tag{3.12}$$

Integrating by parts, we then have that (3.12) equals $C_1(n) + C_2(n)T$,

$$T = \int_0^\infty \exp((\alpha^{(n-1)/n}\phi(y))^{n/(n-1)} - y)\,\mathrm{d}y.$$

Now, since ϕ is absolutely continuous and increasing, and $\phi(0) = 0$, Theorem 1 follows from Moser's result (3.7) if we can show that

$$\int_0^\infty (\alpha^{(n-1)/n}\phi'(y))^n\,dy \le 1. \tag{3.13}$$

We have

$$\alpha^{(n-1)/n}\phi'(y) = \begin{cases} \alpha^{(n-1)/n}\mu^{-1}, & 0 < y < \mu M \\ \alpha^{-1/n}A_{\phi(y)}^{1/(n-1)}, & \mu M < y < \|\psi\|_\infty \\ 0, & y > \|\psi\|_\infty, \end{cases}$$

where $A_{\phi(y)}$ as in (3.6). Hence,

$$\alpha^{n-1}\int_0^\infty \phi'(y)^n\,\mathrm{d}y = \alpha^{n-1}\mu^{1-n}M + \alpha^{-1}\int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)}^{n/(n-1)}\,\mathrm{d}y. \tag{3.14}$$

By our choice of μ, the first term equals $\int_{\{|u|\le M\}} |\nabla u|^n\,\mathrm{d}x$. Also, by changing variables $\phi(y) = s$ in the right hand integral, and using the coarea formula, we have

$$\begin{aligned} \alpha^{-1}\int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)}^{n/(n-1)}\,\mathrm{d}y &= \int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)}\phi'(y)\,\mathrm{d}y \\ &= \int_M^{\|u\|_\infty} A_s\,\mathrm{d}s = \int_{\{|u|\ge M\}} |\nabla u|^n\,\mathrm{d}x. \end{aligned} \tag{3.15}$$

Combining (3.14), (3.15), (3.10) and (3.8) then yields (3.13). The proof is complete.

□

4. Proof of Theorem 2

Fix $\beta > \alpha$. For notational convenience, we consider first functions in $B^n(e_n, 1)$ instead of B^n. Fix $2 \le i \in \mathbb{N}$, and denote $\varepsilon = \varepsilon_i = i^{-1}$. Define $v = v_i\colon B^n(-\varepsilon e_n, 2+\varepsilon) \to \mathbb{R}$,

$$v(x) = -\log|x + \varepsilon e_n|.$$

Then v is n-harmonic in $B^n(-\varepsilon e_n, 2+\varepsilon) \setminus \{-\varepsilon e_n\}$. We first show that

$$\int_{B^n(e_n,1)} |\nabla v|^n\,\mathrm{d}x \le \frac{\omega_{n-1}}{2}\log\frac{2+\varepsilon}{\varepsilon}. \tag{4.16}$$

Clearly,

$$\int_{B^n(e_n,1)} |\nabla v|^n\,\mathrm{d}x \le \frac{1}{2}\int_A |\nabla v|^n\,\mathrm{d}x,$$

where

$$A = B^n(-\varepsilon e_n, 2+\varepsilon) \setminus \bar{B}^n(-\varepsilon e_n, \varepsilon).$$

Since
$$|\nabla v(x)|^n = |x+\varepsilon e_n|^{-n},$$
we have
$$\frac{1}{2}\int_A |\nabla v|^n \, \mathrm{d}x = \frac{1}{2}\int_{B^n(0,2+\varepsilon)\setminus \bar{B}^n(0,\varepsilon)} |x|^{-n}\, dx = \frac{\omega_{n-1}}{2}\log\frac{2+\varepsilon}{\varepsilon}.$$
Hence (4.16) holds.

To study exponential integrability of v, set
$$\gamma = \beta\left(\frac{\omega_{n-1}}{2}\log\frac{2+\varepsilon}{\varepsilon}\right)^{1/(1-n)} \tag{4.17}$$
and $\tau = \gamma/(n-1)$.

By the choice of γ, (4.16), and the Cavalieri principle,
$$\begin{aligned}&\int_{S^{n-1}(e_n,1)} \exp(\beta\,(|v|/\|\nabla v\|_n)^{n/(n-1)})\,\mathrm{d}\mathcal{H}^{n-1}\\ &\qquad \geq \omega_{n-1} + \frac{n\gamma}{n-1}\int_0^\infty \mathcal{H}^{n-1}(E_s)s^{1/(n-1)}\exp(\gamma s^{n/(n-1)})\,ds,\end{aligned} \tag{4.18}$$
where
$$E_s = \{x\in S^{n-1}(e_n,1)\colon |v(x)|\geq s\}.$$
Since
$$\begin{aligned}E_s &= S^{n-1}(e_n,1)\cap \bar{B}^n(-\varepsilon e_n, \exp(-s))\\ &\quad \cup S^{n-1}(e_n,1)\setminus B^n(-\varepsilon e_n,\exp(s)),\end{aligned}$$
we have
$$\mathcal{H}^{n-1}(E_s)\geq C(n)(\exp(-s))^{n-1} = C(n)\exp((1-n)s) \tag{4.19}$$
for $0\leq s\leq \log(1/(2\varepsilon))$.

Combining (4.18) and (4.19) yields
$$\begin{aligned}&\frac{1}{C(n)}\int_{S^{n-1}(e_n,1)} \exp(\gamma|v|^{n/(n-1)})\,\mathrm{d}\mathcal{H}^{n-1}\\
&\geq \frac{n\gamma}{n-1}\int_0^{\log(1/(2\varepsilon))} s^{1/(n-1)}\exp\left(\gamma s^{n/(n-1)}+(1-n)s\right)\,\mathrm{d}s\\
&= n\tau\int_0^{\log(1/(2\varepsilon))} s^{1/(n-1)}\exp\left((n-1)(\tau s^{n/(n-1)}-s)\right)\,\mathrm{d}s\\
&= \int_0^{\log(1/(2\varepsilon))} (n\tau s^{1/(n-1)}-(n-1))\exp\left((n-1)(\tau s^{n/(n-1)}-s)\right)\,\mathrm{d}s\\
&\quad +(n-1)\int_0^{\log(1/(2\varepsilon))}\exp\left((n-1)(\tau s^{n/(n-1)}-s)\right)\,\mathrm{d}s\\
&\geq \exp\left((n-1)\left(\tau\left(\log\frac{1}{2\varepsilon}\right)^{n/(n-1)} - \log\frac{1}{2\varepsilon}\right)\right)-1.\end{aligned}$$

Since

$$\left(\log\frac{2+\varepsilon}{\varepsilon}\right)^{1/(1-n)}\left(\log\frac{1}{2\varepsilon}\right)^{1/(n-1)}\geq 1-\delta(\varepsilon),$$

where $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$, we have

$$\int_{S^{n-1}(e_n,1)}\exp(\gamma|v|^{n/(n-1)})\,\mathrm{d}\mathcal{H}^{n-1}\quad\geq C(n)\varepsilon^{-T}-C(n), \tag{4.20}$$

where

$$T=(\beta-\alpha)(2/\omega_{n-1})^{1/(n-1)}-\delta'(\varepsilon),$$

and $\delta'(\varepsilon)\to 0$ when $\varepsilon\to 0$.

To prove Theorem 2, we consider the sequence $u_i\colon \bar{B}^n\to\mathbb{R}$,

$$u_i(x)=v_i(x+e_n)-v_i(e_n),$$

where $v_i(e_n)=-\log(1-\varepsilon_i)\leq\log 2$ for all i. We fix M such that

$$\beta'=\beta\left(\frac{M-\log 2}{M}\right)^{n/(n-1)}>\alpha.$$

Set also $E_i=\{y\in S^{n-1}(e_n,1)\colon |v_i(y)|\geq M\}$. Then

$$\beta|v_i(y)-v_i(e_n)|^{n/(n-1)}\geq\beta'|v_i(y)|^{n/(n-1)}$$

on E_i for every i. Thus

$$\begin{aligned}
&\int_{S^{n-1}}\exp(\beta(|u_i|/\|\nabla u_i\|_n)^{n/(n-1)})\,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad=\int_{S^{n-1}(e_n,1)}\exp(\beta(|v_i(y)-v_i(e_n)|/\|\nabla v_i\|_n)^{n/(n-1)})\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
&\quad\geq\int_{E_i}\exp(\beta'(|v_i(y)|/\|\nabla v_i\|_n)^{n/(n-1)})\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
&\quad\geq\int_{S^{n-1}(e_n,1)}\exp(\beta'(|v_i(y)|/\|\nabla v_i\|_n)^{n/(n-1)})\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
&\qquad-\omega_{n-1}\exp(\beta'(M/\|\nabla v_i\|_n)^{n/(n-1)}).
\end{aligned}$$

Since $\beta'>\alpha$ and $\varepsilon_i=i^{-1}$ in (4.20), the claim now follows from (4.20).

References

[1] S.-Y. A. Chang and D. E. Marshall. On a sharp inequality concerning the Dirichlet integral. *Amer. J. Math.*, 107(5):1015–1033, 1985.

[2] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[3] B. Fuglede. Extremal length and functional completion. *Acta Math.*, 98:171–219, 1957.

[4] F. W. Gehring. Symmetrization of rings in space. *Trans. Amer. Math. Soc.*, 101:499–519, 1961.

[5] F. W. Gehring. Rings and quasiconformal mappings in space. *Trans. Amer. Math. Soc.*, 103:353–393, 1962.

[6] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1993.
[7] J. Malý, D. Swanson, and W. P. Ziemer. The co-area formula for Sobolev mappings. *Trans. Amer. Math. Soc.*, 355(2):477–492 (electronic), 2003.
[8] J. J. Manfredi. Weakly monotone functions. *J. Geom. Anal.*, 4(3):393–402, 1994.
[9] D. E. Marshall. A new proof of a sharp inequality concerning the Dirichlet integral. *Ark. Mat.*, 27(1):131–137, 1989.
[10] J. Moser. A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.*, 20:1077–1092, 1970/71.
[11] G. D. Mostow. Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. *Inst. Hautes Études Sci. Publ. Math.*, (34):53–104, 1968.
[12] P. Poggi-Corradini and K. Rajala. An egg-yolk principle and exponential integrability for quasiregular mappings. Preprint, 2006.

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