SHARP EXPONENTIAL INTEGRABILITY FOR TRACES OF MONOTONE SOBOLEV FUNCTIONS

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Abstract. We answer a question posed in [12] on exponential integrability of functions of restricted n-energy. We use geometric methods to obtain a sharp exponential integrability result for boundary traces of monotone Sobolev functions defined on the unit ball.

1. Introduction

The following result answered a problem of A. Beurling, mentioned by J. Moser in a famous paper [10].

Theorem A (Chang-Marshall (1985), [1]). There is a universal constant $C < \infty$ so that if $f$ is analytic in $\mathbb{D}$, $f(0) = 0$, and

$$
(1.1) \quad \int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi} \leq 1,
$$

then

$$
\int_0^{2\pi} \exp \left( |f^*(e^{i\theta})|^2 \right) \, d\theta \leq C,
$$

where $f^*$ is the trace of $f$ on $\partial \mathbb{D}$, i.e., $f^*(\zeta) = \lim_{t \uparrow 1} f(t\zeta)$ for $\mathcal{H}^1$-a.e. $\zeta \in \partial \mathbb{D}$.

This result is moreover “sharp” in the following sense: the Beurling functions,

$$
B_a(z) := \left( \log \frac{1}{1 - az} \right) \left( \log \frac{1}{1 - a^2} \right)^{-\frac{1}{2}} \quad 0 < a < 1
$$

are analytic in $\mathbb{D}$, satisfy $B_a(0) = 0$ and (1.1), and have the property that for any given $\alpha > 1$, one can choose $a$ so that the integral

$$
\int_0^{2\pi} \exp \left( \alpha |B_a(e^{i\theta})|^2 \right) \, d\theta
$$

is as large as desired.

The following is an easy corollary of the Chang-Marshall Theorem.

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Corollary A. There is a universal constant $C < \infty$ so that if $u : \mathbb{D} \to \mathbb{R}$ is harmonic with $u(0) = 0$ and

$$\int_{\mathbb{D}} |\nabla u(z)|^2 \frac{dA(z)}{\pi} \leq 1,$$

then

$$\int_0^{2\pi} \exp \left( u^* (e^{i\theta})^2 \right) d\theta \leq C,$$

where $u^*$ is the trace of $u$ on $\partial \mathbb{D}$, i.e., $u^*(\zeta) = \lim_{t \uparrow 1} u(t\zeta)$ for $\mathcal{H}^1$-a.e. $\zeta \in \partial \mathbb{D}$.

This can also be shown to be sharp by considering the real parts of the Beurling functions.

In [12] the last two authors generalized the Chang-Marshall theorem to quasiregular mappings in all dimensions. They also asked in [12] whether Corollary A also generalizes, perhaps substituting “harmonic” with “$n$-harmonic”. In this note we show that this is indeed possible.

The key concept is that of a monotone Sobolev function, whose definition we recall below, and which is quite general, and includes for instance $n$-harmonic functions.

2. Main results

For a continuous function $u : \Omega \to \mathbb{R}$, we define the oscillation of $u$ on a compact set $K \subset \Omega$ by

$$\text{osc}_K u = \max_{x,y \in K} |u(x) - u(y)|.$$ 

We say that $u : \Omega \to \mathbb{R}$ is monotone if $\text{osc}_{\partial B} u = \text{osc}_{\bar{B}} u$ for all $n$-balls $B$ compactly contained in $\Omega$.

Given $u : B^n \to \mathbb{R}$ in the Sobolev space $W^{1,n}(B^n)$, the radial limit

$$\tilde{u}(y) = \lim_{r \to 1} u(ry)$$

exists at $\mathcal{H}^{n-1}$-a.e. point $y \in S^{n-1}$. We denote by $\tilde{u}$ the almost everywhere defined trace of $u$. Moreover, we denote the $L^p$-norm of a $p$-integrable $g : \Omega \to \mathbb{R}^n$ by $\|g\|_p = \|g\|_{\Omega,p}$. The surface measure $\mathcal{H}^{n-1}(S^{n-1})$ of the unit sphere $S^{n-1}$ is $\omega_{n-1}$. The notations $B^n(r) = B^n(0,r)$, $B^n = B^n(1)$ for $n$-dimensional balls will be used.

Theorem 1. There exists a constant $C = C(n) > 0$ so that if $u \in W^{1,n}(B^n)$ is a non-constant continuous monotone function such that $u(0) = 0$, then

$$(2.2) \quad \int_{S^{n-1}} \exp \left( \alpha (|\tilde{u}(y)|/\|\nabla u\|_n)^{(n-1)/(n-1)} \right) d\mathcal{H}^{n-1}(y) \leq C,$$

where

$$(2.3) \quad \alpha = (n-1) \left( \frac{\omega_{n-1}}{2} \right)^{\frac{1}{n-1}}.$$
The continuity assumption in Theorem 1 is of technical nature. By a theorem of Manfredi [8], so-called weakly monotone functions in $W^{1,n}$ are always continuous and monotone in the above sense. In general, $W^{1,n}$-functions need not be continuous.

Theorem 1 is not true without the monotonicity assumption. Indeed, since the $n$-capacity of a point is zero, one can construct Sobolev functions $u_i \in W^{1,n}(B^n)$ so that $u_i(0) = 0$, $\|\nabla u_i\|_n \leq 1$, and $\bar{\partial}_i(y) \geq i$ for every $y \in S^{n-1}$.

Our method of proof for Theorem 1 has a similar geometric flavor as in [9] and in [12], and the end-game is again to appeal to Moser’s original one-dimensional proof. However, the so-called “egg-yolk” property, which was the hardest part to establish in the two papers cited above, can be quickly established in our present case. It might come as a surprise then that Theorem 1 is sharp, as we will see in Theorem 2 below, as opposed to the situation in [12].

A function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is called $p$-harmonic, $1 < p < \infty$, if

$$
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = 0
$$

for every $C^\infty$-smooth test function $\phi$ with compact support in $\Omega$, see [6]. Since $p$-harmonic functions satisfy the maximum principle ([6, 6.5]), they are, in particular, monotone.

The next result shows that the constant $\alpha$ in Theorem 1 is sharp.

**Theorem 2.** Let $\alpha$ be as in Theorem 1. There exists a sequence of $n$-harmonic functions $u_i \in W^{1,n}(B^n)$ satisfying $\|\nabla u_i\|_n \leq 1$ and $u_i(0) = 0$, so that

$$
\int_{S^{n-1}} \exp \left( \beta |\bar{\partial}_i(y)|^{n/(n-1)} \right) \, d\mathcal{H}^{n-1}(y) \to \infty \quad \text{as } i \to \infty
$$

whenever $\beta > \alpha$.

### 3. Proof of Theorem 1

In this section we assume that $u$ satisfies the assumptions of Theorem 1. Moreover, by considering balls $B^n(0, 1 - 1/j)$, for $j$ large, and using Fatou’s lemma, we may assume that the function $u$ in Theorem 1 is defined in a neighborhood of the unit ball.

**Lemma 3.** There exists a constant $r_0 = r_0(n) > 0$ so that if $M_0 := \max_{B^n(r_0)} |u|$, then

$$
\int_{\{|u| \leq M_0\}} |\nabla u|^n \, dx \geq M_0^n.
$$

**Proof.** For $0 < r < 1$ let $m := \max_{B^n(r)} |u|$ and set $v := \min\{|u|, m\}$. By monotonicity, and since $u(0) = 0$, $\text{osc}_{S^{n-1}(t)} v = m$ for every $t \geq r$. 

By the Sobolev embedding theorem on spheres, see e.g. [5, Lemma 1] or [11], there exists a constant $a$ depending only on $n$ such that

$$\int_{B^n \setminus B^n(r)} |\nabla v|^n \, dx = \int_r^1 \left( \int_{S^{n-1}(t)} |\nabla v|^n \, d\mathcal{H}^{n-1} \right) \, dt \geq \int_r^1 \frac{(\text{osc}_{S^{n-1}(t)} v)^n}{at} \, dt = \frac{m^n}{a} \log \frac{1}{r}.$$  

The claim follows by choosing $r_0 := \exp(-a)$. \qed

Let $\Gamma$ be a family of paths in a domain $\Omega$. The $n$-modulus $M_n(\Gamma)$ of $\Gamma$ is defined as follows:

$$M_n(\Gamma) = \inf_\rho \int_\Omega \rho^n \, dx,$$

where $\rho: \Omega \to [0, \infty]$ is an admissible function for $\Gamma$, i.e. a Borel function satisfying

$$(3.4) \quad \int_\gamma \rho \, ds \geq 1$$

for every locally rectifiable $\gamma \in \Gamma$. The family of all paths joining two sets $A, B \subset \bar{\Omega}$ in $\Omega$ is denoted by $\Delta(A, B; \Omega)$. We say that a given property holds for $n$-almost every path in a path family $\Gamma$ if the property holds for all paths in $\Gamma \setminus \Gamma_0$, where $\Gamma_0$ is a subfamily of $\Gamma$ having $n$-modulus zero.

**Lemma 4.** For every $r \in (0, 1)$, there exists a constant $c = c(n, r)$, so that

$$(3.5) \quad \mathcal{H}^{n-1} \left( \{ y \in S^{n-1} : |u(y)| \geq s \} \right) \leq c \exp(-\alpha I_M^*(u))$$

for $s \geq M$. Here $\alpha$ is as in (2.3), $M = M(r, u) = \max_{S^{n-1}(r)} |u|$, and

$$I_M^*(u) = \int_M^s \frac{dt}{\left( \int_{\{|u| = t\}} |\nabla u|^{n-1} \, d\mathcal{H}^{n-1} \right)^{1/(n-1)}}.$$

**Proof.** Fix $r \in (0, 1)$ and $s > M = M(r, u)$. Write

$$E = E_s := \{ y \in S^{n-1} : |u(y)| \geq s \}$$

and

$$U_M := \{ x \in B^n : |u(x)| \geq M \}.$$ 

Also, here and in what follows we write

$$(3.6) \quad A_t := \int_{\{|u| = t\}} |\nabla u|^{n-1} \, d\mathcal{H}^{n-1}.$$ 

The fact that $A_t$ is a Borel function of $t$ is standard, see for instance [2] p. 117.
We construct an admissible function $\rho$ for $\Delta(\partial U_M, E; B^n)$ as follows: Let $I = I_M^s(u)$, and set
\[
\rho(x) := \frac{|\nabla u(x)|}{IA_{|u(x)|}^{1/(n-1)} x_M(x)}.
\]

Since every path in $\Delta(\partial U_M, E; B^n)$ has a subpath in $\Delta(\partial U_M, E; U_M)$, it suffices to show that $\rho$ is admissible for $n$-almost every path in $\Delta(\partial U_M, E; U_M)$, i.e. that the set of paths where (3.4) fails has $n$-modulus zero. Recall that, by Fuglede’s theorem [3, Theorem 3], it suffices to show that $\rho$ is an admissible function for $\Delta(\partial U_M, E; B^n)$ parameterized by arc length $\ell(\gamma)$, we have, by change of variables
\[
\int_\gamma \rho \, ds = \int_0^{\ell(\gamma)} \frac{|\nabla u(\gamma(t))|}{IA_{|u(\gamma(t))|}^{1/(n-1)}} |(u \circ \gamma)'(t)| \, dt 
\geq I^{-1} \int_0^s \frac{dt}{IA_t^{1/(n-1)}} = 1.
\]

Thus $\rho$ is an admissible function for $\Delta(\partial U_M, E; U_M)$, and so also for $\Delta(B^n(r), E; U_M)$, by the definition of $n$-modulus. By the coarea formula, cf. [7], we have
\[
M_n(\Delta(B^n(r), E; B^n)) \leq \int_{U_M} \rho^n \, dx = I^{-n} \int_{U_M} \frac{|\nabla u(x)|^n}{A_{|u(x)|}^{n/(n-1)}} \, dx
= I^{-n} \int_{M} \int_{\{u=t\}} \frac{|\nabla u(y)|^{n-1}}{A_t^{n/(n-1)}} \, dH^{n-1}(y) \, dt
= I^{-n} \int_{M} \frac{A_t}{A_t^{n/(n-1)}} \, dt = I^{1-n}.
\]

By the conformal invariance of $n$-modulus, taking inversion with respect to the unit sphere yields
\[
2M_n(\Delta(B^n(r), E; B^n)) \geq M_n(\Delta(S^{n-1}(r) \cup S^{n-1}(1/r), E; \mathbb{R}^n)).
\]

By spherical symmetrization and [4, Theorem 4],
\[
2I^{1-n} \geq M_n(\Delta(S^{n-1}(r) \cup S^{n-1}(1/r), E; \mathbb{R}^n)) \geq \omega_n^{-1} \left( \log \frac{c_2}{\mathcal{H}^{n-1}(E)^{1/(n-1)}} \right)^{1-n},
\]
where $c_2$ depends only on $n$ and $r$. See [12] for further details. This implies (3.5).

**Proof of Theorem 1.** We will use the following result of Moser [10, Equation (6)]: If $\omega : [0, \infty) \to [0, \infty)$ is absolutely continuous and
satisfies $\omega(0) = 0$, $\omega' \geq 0$ almost everywhere, and
\[ \int_0^\infty (\omega'(t))^n \, dt \leq 1, \]
then
\[ (3.7) \quad \int_0^\infty \exp(\omega(t)^{n/(n-1)} - t) \, dt \leq C, \]
where $C > 0$ depends only on $n$. By scaling invariance of (2.2), we may assume that
\[ (3.8) \quad \int_{B^n} |\nabla u|^n \, dx = 1. \]
Moreover, we fix $r = r_0$ and $M = M_0$ as in Lemma 3. Then, in particular, $M < 1$.

By the Cavalieri principle,
\[ \int_{S^{n-1}} \exp \left( \alpha |u(x)|^{n/(n-1)} \right) \, dH^{n-1}(x) = \omega_{n-1} + \frac{an}{n-1} \int_0^\infty s^{1/(n-1)}H^{n-1}(E_s) \exp \left( \alpha s^{n/(n-1)} \right) \, ds, \]
where
\[ E_s = \{ y \in S^{n-1} : |u(y)| \geq s \}. \]
Then, by Lemma 4, it suffices to bound
\[ (3.9) \quad \int_0^{||u||_\infty} s^{1/(n-1)} \exp \left( \alpha (s^{n/(n-1)} - I_M^s(u)) \right) \, ds, \]
where $||u||_\infty = \max_{y \in S^{n-1}} |u(y)|$, and $I_M^s(u) = 0$ for $0 < s < M$. We define a function $\psi : [0, \infty) \to [0, \infty)$,
\[ \psi(s) = \begin{cases} \mu s, & 0 < s < M \\ \alpha I_M^s(u) + \mu M, & M \leq s \leq ||u||_\infty \\ \alpha I_M^{||u||_\infty}(u) + \mu M, & s > ||u||_\infty \end{cases} \]
where
\[ (3.10) \quad \mu = \alpha \left( \frac{M}{\int_{\{|u| \leq M\}} |\nabla u|^n \, dx} \right)^{1/(n-1)}. \]
Then, by Lemma 3, $\mu M \leq \alpha$, and thus we may consider
\[ (3.11) \quad \int_0^{||u||_\infty} s^{1/(n-1)} \exp(\alpha s^{n/(n-1)} - \psi(s)) \, ds \]
instead of (3.9). We define $\phi$ by $\phi(y) = \psi^{-1}(y)$ for $0 < y < ||\psi||_\infty$, and $\phi(y) = ||u||_\infty$ for $y \geq ||\psi||_\infty$. Then, changing variables $y = \psi(s)$ in (3.11) yields
\[ (3.12) \quad \int_0^\infty \exp(\alpha\phi(y)^{n/(n-1)} - y)\phi'(y)\phi(y)^{1/(n-1)} \, dy. \]
Integrating by parts, we then have that (3.12) equals 

\[ T = \int_0^\infty \exp((\alpha^{(n-1)/n} \phi(y))^{n/(n-1)} - y) \, dy. \]

Now, since \( \phi \) is absolutely continuous and increasing, and \( \phi(0) = 0 \), Theorem 1 follows from Moser’s result (3.7) if we can show that

\[ \int_0^\infty (\alpha^{(n-1)/n} \phi'(y))^n \, dy \leq 1. \]

We have

\[ \alpha^{(n-1)/n} \phi'(y) = \begin{cases} \alpha^{(n-1)/n} \mu^{-1}, & 0 < y < \mu M \\ \alpha^{-1/n} A^{1/(n-1)}_{\phi(y)}, & \mu M < y < \|\psi\|_\infty \\ 0, & y > \|\psi\|_\infty, \end{cases} \]

where \( A_{\phi(y)} \) as in (3.6). Hence,

\[ \alpha^{-1} \int_{\|u\|_\infty}^{\infty} \phi'(y)^n \, dy = \alpha^{n-1} \mu^{1-n} M + \alpha^{-1} \int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)}^{n/(n-1)} \, dy. \]

By our choice of \( \mu \), the first term equals \( \int_{\{|u|\leq M\}} |\nabla u|^n \, dx \). Also, by changing variables \( \phi(y) = s \) in the right hand integral, and using the coarea formula, we have

\[ \alpha^{-1} \int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)}^{n/(n-1)} \, dy = \int_{\mu M}^{\|\psi\|_\infty} A_{\phi(y)} \phi'(y) \, dy \]

\[ = \int_{\mu M}^{\|\psi\|_\infty} A_s \, ds = \int_{\{|u|\geq M\}} |\nabla u|^n \, dx. \]

Combining (3.14), (3.15), (3.10) and (3.8) then yields (3.13). The proof is complete.

\[ \square \]

4. Proof of Theorem 2

Fix \( \beta > \alpha \). For notational convenience, we consider first functions in \( B^n(e_n, 1) \) instead of \( B^n \). Fix \( 2 \leq i \in \mathbb{N} \), and denote \( \varepsilon = \varepsilon_i = i^{-1} \).

Define \( v = v_i : B^n(-\varepsilon e_n, 2 + \varepsilon) \to \mathbb{R} \),

\[ v(x) = -\log |x + \varepsilon e_n|. \]

Then \( v \) is \( n \)-harmonic in \( B^n(-\varepsilon e_n, 2 + \varepsilon) \setminus \{-\varepsilon e_n\} \). We first show that

\[ \int_{B^n(e_n, 1)} |\nabla v|^n \, dx \leq \frac{\omega_{n-1}}{2} \log \frac{2 + \varepsilon}{\varepsilon}. \]

Clearly,

\[ \int_{B^n(e_n, 1)} |\nabla v|^n \, dx \leq \frac{1}{2} \int_A |\nabla v|^n \, dx, \]

where

\[ A = B^n(-\varepsilon e_n, 2 + \varepsilon) \setminus \bar{B}^n(-\varepsilon e_n, \varepsilon). \]
Since
\[ |\nabla v(x)|^n = |x + \varepsilon e_n|^{-n}, \]
we have
\[ \frac{1}{2} \int_A |\nabla v|^n \, dx = \frac{1}{2} \int_{B^n(0,2+\varepsilon) \setminus B^n(0,\varepsilon)} |x|^{-n} \, dx = \frac{\omega_{n-1}}{2} \log \frac{2 + \varepsilon}{\varepsilon}. \]
Hence (4.16) holds.

To study exponential integrability of \( v \), set
\[ \gamma = \beta \left( \frac{\omega_{n-1}}{2} \log \frac{2 + \varepsilon}{\varepsilon} \right)^{1/(1-n)} \]
and \( \tau = \gamma / (n - 1) \).

By the choice of \( \gamma \), (4.16), and the Cavalieri principle,
\[ \int_{S^{n-1}(e_n, 1)} \exp(\beta (|v|/\|\nabla v\|_n)^{n/(n-1)}) \, d\mathcal{H}^{n-1} \]
\[ \geq \omega_{n-1} + \frac{n \gamma}{n - 1} \int_0^\infty \mathcal{H}^{n-1}(E_s) s^{1/(n-1)} \exp(\gamma s^{n/(n-1)}) \, ds, \]
where
\[ E_s = \{ x \in S^{n-1}(e_n, 1): |v(x)| \geq s \}. \]
Since
\[ E_s = S^{n-1}(e_n, 1) \cap B^n(-\varepsilon e_n, \exp(-s)) \]
\[ \cup S^{n-1}(e_n, 1) \setminus B^n(-\varepsilon e_n, \exp(s)), \]
we have
\[ \mathcal{H}^{n-1}(E_s) \geq C(n)(\exp(-s))^{n-1} = C(n) \exp((1 - n)s) \]
for \( 0 \leq s \leq \log \left( \frac{1}{2\varepsilon} \right) \).
Combining (4.18) and (4.19) yields
\[ \frac{1}{C(n)} \int_{S^{n-1}(e_n, 1)} \exp(\gamma |v|^{n/(n-1)}) \, d\mathcal{H}^{n-1} \]
\[ \geq \frac{n \gamma}{n - 1} \int_0^{\log\left( \frac{1}{2\varepsilon} \right)} s^{1/(n-1)} \exp \left( \gamma s^{n/(n-1)} + (1 - n)s \right) \, ds \]
\[ = n \tau \int_0^{\log\left( \frac{1}{2\varepsilon} \right)} s^{1/(n-1)} \exp \left( (n - 1)(\tau s^{n/(n-1)} - s) \right) \, ds \]
\[ = \int_0^{\log\left( \frac{1}{2\varepsilon} \right)} (n \tau s^{1/(n-1)} - (n - 1)) \exp \left( (n - 1)(\tau s^{n/(n-1)} - s) \right) \, ds \]
\[ + (n - 1) \int_0^{\log\left( \frac{1}{2\varepsilon} \right)} \exp \left( (n - 1)(\tau s^{n/(n-1)} - s) \right) \, ds \]
\[ \geq \exp \left( (n - 1) \left( \tau \left( \log \frac{1}{2\varepsilon} \right)^{n/(n-1)} - \log \frac{1}{2\varepsilon} \right) \right) - 1. \]
Since
\[
\left( \log \frac{2 + \varepsilon}{\varepsilon} \right)^{1/(1-n)} \left( \log \frac{1}{2\varepsilon} \right)^{1/(n-1)} \geq 1 - \delta(\varepsilon),
\]
where \( \delta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), we have
\[
\int_{S^{n-1}(e_n,1)} \exp(\gamma |v|^{n/(n-1)}) \, d\mathcal{H}^{n-1} \geq C(n) \varepsilon^{-T} - C(n),
\]
where
\[
T = (\beta - \alpha)(2/\omega_{n-1})^{1/(n-1)} - \delta'(\varepsilon),
\]
and \( \delta'(\varepsilon) \to 0 \) when \( \varepsilon \to 0 \).

To prove Theorem 2, we consider the sequence \( u_i : \bar{B}^n \to \mathbb{R} \),
\[
u_i(x) = v_i(x + e_n) - v_i(e_n),
\]
where \( v_i(e_n) = -\log(1 - \varepsilon_i) \leq \log 2 \) for all \( i \). We fix \( M \) such that \( M - \log 2 \omega_{n-1} > \alpha \).

Set also \( E_i = \{ y \in S^{n-1}(e_n,1) : |v_i(y)| \geq M \} \). Then \( \beta |v_i(y) - v_i(e_n)|^{n/(n-1)} \geq \beta' |v_i(y)|^{n/(n-1)} \) on \( E_i \) for every \( i \). Thus
\[
\int_{S^{n-1}} \exp(\beta(\|u_i\|/\|\nabla v_i\|)^{n/(n-1)}) \, d\mathcal{H}^{n-1}
\]
\[
= \int_{S^{n-1}(e_n,1)} \exp(\beta(|v_i(y) - v_i(e_n)|/\|\nabla v_i\|)^{n/(n-1)}) \, d\mathcal{H}^{n-1}(y)
\]
\[
\geq \int_{E_i} \exp(\beta'(|v_i(y)|/\|\nabla v_i\|)^{n/(n-1)}) \, d\mathcal{H}^{n-1}(y)
\]
\[
\geq \int_{S^{n-1}(e_n,1)} \exp(\beta'(|v_i(y)|/\|\nabla v_i\|)^{n/(n-1)}) \, d\mathcal{H}^{n-1}(y)
\]
\[
- \omega_{n-1} \exp(\beta'(M/\|\nabla v_i\|)^{n/(n-1)}).
\]

Since \( \beta' > \alpha \) and \( \varepsilon_i = i^{-1} \) in (4.20), the claim now follows from (4.20).

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