Haar type and Carleson Constants

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Abstract

For a collection $\mathcal{E}$ of dyadic intervals, a Banach space X, and $p \in (1,2]$ we assume the upper ℓ^p estimates

$$\left\| \sum_{I \in \mathcal{E}} x_I h_I / |I|^{1/p} \right\|_{L^p_X}^p \leq c^p \sum_{I \in \mathcal{E}} \|x_I\|_X^p,$$

where $x_I \in X$ and h_I denotes the L^∞ normalized Haar function supported on I. We determine the minimal requirement on the size of $\mathcal{E}$ so that these estimates imply that X is of Haar type p. The characterization is given in terms of the Carleson constant of $\mathcal{E}$.

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1 Introduction

Let X be a Banach space. We fix a non-empty collection of dyadic intervals $\mathcal{E}$ and assume the upper ℓ^p estimates

$$\left\| \sum_{I \in \mathcal{E}} x_I \frac{h_I}{|I|^{1/p}} \right\|_{L^p_X} \leq c \left(\sum_{I \in \mathcal{E}} \|x_I\|_X^p \right)^{\frac{1}{p}} \tag{1}$$

for finitely supported $(X_I)_{I \in \mathcal{E}} \subset X$ and some $p \in (1,2]$, where h_I is the L^∞ normalized Haar function supported on I. The consequences for X, one may draw from (1), depend on the size of the collection $\mathcal{E}$. For instance, if

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$\mathcal{E}$ is a collection of pairwise disjoint dyadic intervals, then any Banach space satisfies (1), hence it does not impose any restriction on X. On the other hand, if $\mathcal{E}$ is the collection of all dyadic intervals, then the upper ℓ^p estimates (1) simply state that X is of Haar type p, a condition which is known – due to important work of G. Pisier [6] – to be equivalent to certain renorming properties of the Banach space X.

In this paper we ask how massive a collection $\mathcal{E}$ has to be so that (1) implies that X is of Haar type p. We answer this question in terms of the Carleson constant $[\![\mathcal{E}]\!]$ of $\mathcal{E}$ defined by

$$[\![\mathcal{E}]\!] := \sup_{I \in \mathcal{E}} \frac{1}{|I|} \sum_{J \in I \cap \mathcal{E}} |J|. \tag{2}$$

The proof relies on a well-known dichotomy for $\mathcal{E}$ saying that one has either some kind of disjointification or some kind of condensation of the dyadic intervals from $\mathcal{E}$.

Initially we encountered the problem treated here in connection with our efforts to obtain a vector valued version of E. M. Semenov's characterization of bounded operators rearranging the Haar system. See [7] and [3].

2 Preliminaries

In the following we equip the unit interval $[0, 1)$ with the Lebesgue measure denoted by $|\cdot|$. Let $\mathcal{D}$ denote the collection of dyadic intervals in $[0, 1)$, i.e. $I \in \mathcal{D}$ provided that there exist $m \geq 0$ and $1 \leq k \leq 2^m$ such that

$$I = [(k-1)/2^m, k/2^m),$$

and let

$$\mathcal{D}_n := \{I \in \mathcal{D} : |J| \geq 2^{-n}\} \quad \text{where} \quad n \geq 0.$$

The L^∞ normalized Haar function supported on $I \in \mathcal{D}$ is denoted by h_I, i.e. $h_I = -1$ in the left half of I and $h_I = 1$ on the right half of I. By L^p_X, $p \in [1, \infty)$, we denote the space of Radon random variables $f : [0, 1) \to X$ such that

$$\|f\|_{L^p_X} = \left(\int_0^1 \|f(t)\|^p_X dt \right)^{1/p} < \infty.$$

Haar type. Given $p \in (1, 2]$, a Banach space X is of Haar type p provided that there exists a constant $c > 0$ such that

$$\Big\| \sum_{I \in \mathcal{D}} x_I \frac{h_I}{|I|^{1/p}} \Big\|_{L^p_X} \le c \Big(\sum_{I \in \mathcal{D}} \|x_I\|_X^p \Big)^{\frac{1}{p}}$$

for all finitely supported families $(x_I)_{I \in \mathcal{D}} \subset X$. We let $HT_p(X)$ be the infimum of all possible $c > 0$ as above. The central result result concerning Haar type is due to G. Pisier [6] and asserts that Haar type p is equivalent to the fact that X can be equivalently renormed such that the new norm has a modulus of smoothness of power type p. For additional information see [1, 5] and the references therein.

Carleson Constants. Let $\mathcal{E} \subseteq \mathcal{D}$ be a non-empty collection of dyadic intervals. Recall, that the Carleson constant of $\mathcal{E}$ is given by equation (2). Next we define consecutive generations of $\mathcal{E}$ and, using $[\![\mathcal{E}]\!]$, describe a dichotomy for $\mathcal{E}$ known as the almost disjointification and condensation properties.

We define $G_0(\mathcal{E})$ to be the maximal dyadic intervals of $\mathcal{E}$ where *maximal* refers to inclusion. Suppose, we have already defined $G_0(\mathcal{E})$,..., $G_p(\mathcal{E})$, we form

$$G_{p+1}(\mathcal{E}) := G_0(\mathcal{E} \setminus \{G_0(\mathcal{E}) \cup \cdots \cup G_p(\mathcal{E})\}).$$

Moreover, given $I \in \mathcal{D}$, we let

$$G_k(I, \mathcal{E}) := G_k(I \cap \mathcal{E}) \quad \text{for} \quad k \ge 1.$$

Assume that $[\![\mathcal{E}]\!] < \infty$ and that M is the largest integer smaller than $4[\![\mathcal{E}]\!]+1$. Then

$$\mathcal{E}_i := \bigcup_{k=0}^{\infty} G_{Mk+i}(\mathcal{E}), \quad 0 \le i \le M-1, \tag{3}$$

satisfies

$$\sum_{J \in G_1(I, \mathcal{E}_i)} |J| \le \frac{|I|}{2} \quad \text{for all} \quad I \in \mathcal{E}_i.$$

Conversely, if $[\![\mathcal{E}]\!] = \infty$, then for all $n \ge 1$ and $\varepsilon \in (0, 1)$ there exists a $K_0 \in \mathcal{E}$ such that

$$\sum_{J \in G_n(K_0, \mathcal{E})} |J| \ge (1 - \varepsilon)|K_0|.$$

Based on this, one can remodel in distribution the Haar system by the help of $(h_I)_{I \in \mathcal{E}}$ which will be used in Lemma 3.4. The proof of this basic dichotomy and some of its applications can be found in [4, Chapter 3].

3 Haar Type and Carleson Constants

The main results of this note are Theorems 3.1 and 3.2 below which complement each other.

Theorem 3.1. *Let $p \in (1,2]$ and $\mathcal{E} \subseteq \mathcal{D}$ be a non-empty collection of dyadic intervals. Then the following statements are equivalent:*

(1) *For any Banach space X the existence of a constant $c > 0$ such that for all finitely supported families $(x_I)_{I\in\mathcal{D}} \subset X$ one has*

$$\left\| \sum_{I\in\mathcal{E}} x_I \frac{h_I}{|I|^{1/p}} \right\|_{L^p_X} \le c \left(\sum_{I\in\mathcal{E}} \|x_I\|_X^p \right)^{\frac{1}{p}}$$

implies that X is of Haar type p.

(2) $[\![\mathcal{E}]\!] = \infty$.

Theorem 3.2. *Let $p \in (1,2]$, $\mathcal{E} \subseteq \mathcal{D}$ be a non-empty collection of dyadic intervals, and X be a Banach space which is not of Haar type p. Then the following statements are equivalent:*

(1) *There exists a constant $c > 0$ such that for all finitely supported families $(x_I)_{I\in\mathcal{D}} \subset X$ one has*

$$\left\| \sum_{I\in\mathcal{E}} x_I \frac{h_I}{|I|^{1/p}} \right\|_{L^p_X} \le c \left(\sum_{I\in\mathcal{E}} \|x_I\|_X^p \right)^{\frac{1}{p}}.$$

(2) $[\![\mathcal{E}]\!] < \infty$.

Theorem 3.1 and Theorem 3.2 follow immediately from the following two lemmas (and the obvious fact that there are Banach without Haar type p if $p \in (1,2]$).

Lemma 3.3. *Let $p \in (1,\infty)$, $[\![\mathcal{E}]\!] < \infty$, and X be a Banach space. Then there is a constant $c_p > 0$, depending at most on p, such that one has*

$$\left\| \sum_{I\in\mathcal{E}} x_I \frac{h_I}{|I|^{1/p}} \right\|_{L^p_X} \le c_p [\![\mathcal{E}]\!]^{1-\frac{1}{p}} \left(\sum_{I\in\mathcal{E}} ||x_I||_X^p \right)^{\frac{1}{p}}$$

for all finitely supported $(x_I)_{I\in\mathcal{E}} \subset X$.

Proof. Using (3), we write $\mathcal{E} = \mathcal{E}_0 \cup \cdots \cup \mathcal{E}_{M-1}$ with $M < 4[\![\mathcal{E}]\!] + 1$ such that the collections $\{A_I : I \in \mathcal{E}_i\}$ of pairwise disjoint and measurable sets defined by

$$A_I := I \setminus \bigcup_{J \in G_1(I, \mathcal{E}_i)} J, \quad I \in \mathcal{E}_i,$$

satisfy

$$\frac{1}{2}|I| \leq |A_I| \leq |I|.$$

Because

$$\begin{aligned} \left\| \sum_{I \in \mathcal{E}} x_I \frac{h_I}{|I|^{1/p}} \right\|_{L^p_X} &\leq \sum_{i=0}^{M-1} \left\| \sum_{I \in \mathcal{E}_i} x_I \frac{h_I}{|I|^{1/p}} \right\|_{L^p_X} \\ &\leq M^{1-\frac{1}{p}} \left(\sum_{i=0}^{M-1} \left\| \sum_{I \in \mathcal{E}_i} x_I \frac{h_I}{|I|^{1/p}} \right\|^p_{L^p_X} \right)^{1/p} \end{aligned}$$

it is sufficient to prove

$$\left\| \sum_{I \in \mathcal{E}_i} x_I \frac{h_I}{|I|^{1/p}} \right\|_{L^p_X} \leq c_p \left(\sum_{I \in \mathcal{E}_i} \|x_I\|^p_X \right)^{\frac{1}{p}}$$

for fixed i. But here we get that

$$\begin{aligned} & \left\| \sum_{I \in \mathcal{E}_i} x_I \frac{h_I}{|I|^{\frac{1}{p}}} \right\|_{L^p_X} \\ = & \left(\sum_{K \in \mathcal{E}_i} \int_{A_K} \left\| \sum_{I \in \mathcal{E}_i} x_I \frac{h_I(t)}{|I|^{\frac{1}{p}}} \right\|^p_X dt \right)^{\frac{1}{p}} \\ = & \left(\sum_{K \in \mathcal{E}_i} \frac{|A_K|}{|K|} \int_{A_K} \left\| \sum_{I \in \mathcal{E}_i} x_I \left(\frac{|K|}{|I|} \right)^{\frac{1}{p}} h_I(t) \right\|^p_X \frac{dt}{|A_K|} \right)^{\frac{1}{p}} \\ \leq & \left(\sum_{K \in \mathcal{E}_i} \int_{A_K} \left\| \sum_{I \in \mathcal{E}_i} x_I \left(\frac{|K|}{|I|} \right)^{\frac{1}{p}} h_I(t) \right\|^p_X \frac{dt}{|A_K|} \right)^{\frac{1}{p}} \\ = & \left(\sum_{K \in \mathcal{E}_i} \int_{A_K} \left\| \sum_{K \subseteq I \in \mathcal{E}_i} x_I \left(\frac{|K|}{|I|} \right)^{\frac{1}{p}} h_I(t) \right\|^p_X \frac{dt}{|A_K|} \right)^{\frac{1}{p}} \end{aligned}$$

$$= \left(\sum_{K\in\mathcal{E}_i}\int_{A_K}\left\|\sum_{l=0}^{n(K)} x_{G_{-l}(K,\mathcal{E}_i)}\left(\frac{|K|}{|G_{-l}(K,\mathcal{E}_i)|}\right)^{\frac{1}{p}} h_{G_{-l}(K,\mathcal{E}_i)}(t)\right\|_X^p \frac{dt}{|A_K|}\right)^{\frac{1}{p}}$$

$$= \left(\sum_{K\in\mathcal{E}_i}\int_{A_K}\left\|\sum_{l=0}^{\infty} x_{G_{-l}(K,\mathcal{E}_i)}\chi_{\{l\le n(K)\}}\left(\frac{|K|}{|G_{-l}(K,\mathcal{E}_i)|}\right)^{\frac{1}{p}} h_{G_{-l}(K,\mathcal{E}_i)}(t)\right\|_X^p \frac{dt}{|A_K|}\right)^{\frac{1}{p}}$$

where $G_{-l}(K,\mathcal{E}_i)$ form the maximal sequence of dyadic intervals from $\mathcal{E}_i$ such that

$$K = G_0(K,\mathcal{E}_i) \subset G_{-1}(K,\mathcal{E}_i)\cdots \subset G_{-n(K)}(K,\mathcal{E}_i)$$

and $G_{-n(K)}(K,\mathcal{E}_i)$ is the unique maximal interval in $\mathcal{E}_i$ containing K. Now we can upper bound the last expression by

$$\sum_{l=0}^{\infty}\left(\sum_{K\in\mathcal{E}_i}\int_{A_K}\left\|x_{G_{-l}(K,\mathcal{E}_i)}\chi_{\{l\le n(K)\}}\left(\frac{|K|}{|G_{-l}(K,\mathcal{E}_i)|}\right)^{\frac{1}{p}} h_{G_{-l}(K,\mathcal{E}_i)}(t)\right\|_X^p \frac{dt}{|A_K|}\right)^{\frac{1}{p}}$$

$$= \sum_{l=0}^{\infty}\left(\sum_{K\in\mathcal{E}_i}\left\|x_{G_{-l}(K,\mathcal{E}_i)}\chi_{\{l\le n(K)\}}\right\|_X^p \frac{|K|}{|G_{-l}(K,\mathcal{E}_i)|}\right)^{\frac{1}{p}}$$

$$= \sum_{l=0}^{\infty}\left(\sum_{\substack{I,K\in\mathcal{E}_i\\ G_{-l}(K,\mathcal{E}_i)=I}}\|x_I\|_X^p\frac{|K|}{|I|}\right)^{\frac{1}{p}}$$

$$= \sum_{l=0}^{\infty}\left(\sum_{I\in\mathcal{E}_i}\|x_I\|_X^p\frac{\sum_{\substack{K\in\mathcal{E}_i\\ G_{-l}(K,\mathcal{E}_i)=I}}|K|}{|I|}\right)^{\frac{1}{p}}$$

$$\le \sum_{l=0}^{\infty}\left(\sum_{I\in\mathcal{E}_i}\|x_I\|_X^p\,2^{-l}\right)^{\frac{1}{p}}$$

$$= \left(\sum_{l=0}^{\infty}2^{-\frac{l}{p}}\right)\left(\sum_{I\in\mathcal{E}_i}\|x_I\|_X^p\right)^{\frac{1}{p}}.$$

■

Next we turn to the case $[\![\mathcal{E}]\!]=\infty$ for which it is known that the Gamlen-Gaudet construction yields an approximation of the Haar system by appropriate 'blocks' of $(h_I)_{I\in\mathcal{E}}$. The next lemma demonstrates that this construction perfectly fits with our Haar type inequalities. To avoid arguments which use the unconditionality of the Haar system (and therefore the UMD property of Banach spaces) we have exactly to remodel the Haar system rather to allow that the measures of the support of a Haar function and its approximation are related by uniformly bounded multiplicative constants.

Lemma 3.4. *Let X be a Banach space, $p \in (1,2]$, and $\mathcal{E}$ be a collection of dyadic intervals such that*

$$[\![\mathcal{E}]\!] = \infty.$$

If there is a constant $c > 0$ such that

$$\left\| \sum_{I \in \mathcal{E}} x_I \frac{h_I}{|I|^{1/p}} \right\|_{L_X^p} \leq c \left(\sum_{I \in \mathcal{E}} ||x_I||_X^p \right)^{\frac{1}{p}} \tag{4}$$

for all finitely supported families $(x_I)_{I \in \mathcal{E}} \subset X$, then X is of Haar type p with $HT_p(X) \leq c$.

Remark 3.5. *In Lemma 3.4 the range $p \in (2,\infty)$ does not make sense since already $X = \mathbb{R}$ does not have Rademacher type $p \in (2,\infty)$ and henceforth Haar type $p \in (2,\infty)$. In other words, for $[\![\mathcal{E}]\!] = \infty$ and $p \in (2,\infty)$ the inequality* (4) *fails to be true.*

Proof of Lemma 3.4. Let $n \geq 1$, $\delta \in (0,1)$, and $\varepsilon = 2^{-n-1}\delta$. Since $[\![\mathcal{E}]\!] = \infty$ the condensation property (cf. [4, Lemma 3.1.4]) yields the existence of a $K_0 \in \mathcal{E}$ such that

$$\sum_{J \in G_n(K_0,\mathcal{E})} |J| \geq (1-\varepsilon)|K_0|.$$

Examining the Gamlen-Gaudet construction [2] as (for example) presented in [4, Proposition 3.1.6], we obtain a family $(\mathcal{B}_I)_{I \in \mathcal{D}_n}$ of collections of dyadic intervals such that

(i) $\mathcal{B}_I \subseteq K_0 \cap \mathcal{E}$,

(ii) the elements of $\mathcal{B}_I$ are pairwise disjoint,

(iii) for $B_I := \bigcup_{K \in \mathcal{B}_I} K$ one has that $B_I \cap B_J = \emptyset$ if and only if $I \cap J = \emptyset$, and $B_I \subseteq B_J$ if and only if $I \subseteq J$,

(iv) for

$$k_I := \sum_{K \in \mathcal{B}_I} h_K$$

and $I, I^-, I^+ \in \mathcal{D}_n$ such that I^- is the left half of I and I^+ the right half of I, one has $B_{I^-} \subseteq \{k_I = -1\}$ and $B_{I^+} \subseteq \{k_I = 1\}$,

(v) for $0 \le k \le n$ and $|I| = 2^{-k}$ one has

$$\frac{|K_0|}{2^k} - 2\varepsilon|K_0| \le |B_I| \le \frac{|K_0|}{2^k}.$$

As a consequence

$$\frac{1-\delta}{2^n}|K_0| \le |B_I| \le \frac{|K_0|}{2^n} \quad \text{for} \quad |I| = 2^{-n}$$

and

$$(1-\delta)|K_0| \le \sum_{|I|=2^{-n}} |B_I| \le |K_0|.$$

Choose measurable subsets $A_I \subseteq B_I$ for $|I| = 2^{-n}$ such that

(a) $|A_I| = (1-\delta)2^{-n}|K_0|$,

(b) the k_I restricted to A_I are symmetric.

Let $S := \bigcup_{|I|=2^{-n}} A_I$, so that $|S| = (1-\delta)|K_0|$, and

$$A_I := B_I \cap S \quad \text{for all (remaining)} \quad I \in \mathcal{D}_n.$$

Then $(k_I)_{I \in \mathcal{D}_n}$ forms, in distribution, a Haar system relative to S. Hence, as a consequence of the Gamlen-Gaudet construction, we obtain that

$$\begin{aligned}
\left\| \sum_{I \in \mathcal{D}_n} \frac{h_I}{|I|^{1/p}} x_I \right\|_{L^p_X} &= \left\| \sum_{I \in \mathcal{D}_n} \frac{k_I}{|I|^{1/p}} x_I \right\|_{L^p_X\left(S, \frac{dt}{|S|}\right)} \\
&= \left(\frac{1}{|S|}\right)^{\frac{1}{p}} \left\| \sum_{I \in \mathcal{D}_n} \frac{k_I}{|I|^{1/p}} x_I \right\|_{L^p_X(S,dt)} \\
&\le \left(\frac{1}{|S|}\right)^{\frac{1}{p}} \left\| \sum_{I \in \mathcal{D}_n} \sum_{K \in \mathcal{B}_I} \frac{h_K}{|K|^{1/p}} \left(\frac{|K|^{1/p}}{|I|^{1/p}} x_I \right) \right\|_{L^p_X}.
\end{aligned}$$

Recall that we selected the collection $\mathcal{B}_I$ as a sub-collection of $\mathcal{E}$. Using our hypothesis concerning $\mathcal{E}$ and X, we can upper bound the last term by

$$c\left(\frac{1}{|S|}\right)^{\frac{1}{p}} \left(\sum_{I \in \mathcal{D}_n} \sum_{K \in \mathcal{B}_I} \frac{|K|}{|I|} \|x_I\|^p \right)^{\frac{1}{p}} = c \left(\sum_{I \in \mathcal{D}_n} \frac{|B_I|}{|I||S|} \|x_I\|^p \right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&= c\left(\sum_{I\in\mathcal{D}_n}\frac{|B_I|}{|I|(1-\delta)|K_0|}\|x_I\|^p\right)^{\frac{1}{p}}\\
&\le \frac{c}{(1-\delta)^{\frac{1}{p}}}\left(\sum_{I\in\mathcal{D}_n}\|x_I\|^p\right)^{\frac{1}{p}}.
\end{aligned}
$$

Letting $\delta \downarrow 0$ yields our statement.

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