DYNAMICS OF SECTIONAL MAPPINGS

TAPANI TOLONEN

ABSTRACT. For a given mapping f we define the concept of a sectional mapping f_Λ and show by examples that the dynamics of the latter can differ considerably from the dynamics of f. This is done both in two- and infinite dimensional settings.

1. Introduction

For any dynamical system in a linear space of dimension at least two, one can define sectional mappings or sectional dynamical systems by fixing some of the directions and applying the dynamics only on the remaining directions while keeping the boundary values intact. Then one may ask whether the dynamics of the original mapping corresponds in some sense with the dynamics of these sectional mappings. Of course, in order to do that, one has to describe the dynamics in some qualitative way. The obvious problem here is the fact that the sectional mappings are defined in a lower dimensional space than the actual mapping, so one cannot directly compare them. This problem is solved by studying the type of the corresponding observable invariant measures.

The idea of sectional mappings is applicable both in finite and infinite dimensional spaces. The results in the present paper may have relevance especially in the understanding of coupled map lattices, for which the definition of the SRB-measure (Sinai, Ruelle, Bowen) is not quite settled yet. (See [4] or [5].) In fact, our results indicate that a proper definition must deal with the whole infinite system, not only with finite dimensional approximations. In a finite dimensional setting, the best alternative for SRB-measure, i.e. for the "physically relevant" invariant measure, seems to be the so called observable measure, defined loosely as follows. Consider a space X and a mapping T from X to itself. If there exists an open set $U \subset X$ such that for Lebesgue-almost every $x \in U$ the weak* limit of the sum $\frac{1}{n}\sum_{k=0}^{n-1} T_*^k \delta_x$ converges to a unique limit measure μ as n tends to infinity, then this μ is observable, or SRB-measure. The exact definition will be given below. There are also other candidates for SRB-measures, some of them coinciding in some settings with observable measures. (See [1], [5] and [6].) One way to evaluate the physical relevance of an invariant measure is to

2000 *Mathematics Subject Classification.* 37A05, 28D05.

Key words and phrases. sectional mappings, SRB measure, ergodicity.

find out, whether it is stochastically stable, or robust under random perturbations. The idea behind this is that we perturb the action of the mapping slightly, take the invariant measure of this perturbed process, and see what happens, when the perturbation is diminished to zero. If the aforementioned invariant measures converge to a certain invariant measure μ, we say that this μ is stochastically stable. (In fact, we can construct invariant measures in this way. It could also be a good candidate for SRB-measure in terms of physical relevance, as mentioned in [12, Thm 1].)

However, in the infinite dimensional setting, for instance in the theory of coupled map lattices, the situation is more complicated, as is shown in [5]. One of the problems lies in the physical relevance. In any real situation we cannot study but finite number of quantities. Therefore one is tempted to define the relevant measure for the mapping as a limit (in some sense) of the SRB-measures of the sectional finite-dimensional functions, as the dimension increases. This method does not seem to be satisfactory, since, as we show below, (almost) all the sections may have SRB-measures that are qualitatively quite different from any invariant measure of the infinite dimensional mapping.

One way of introducing a physically reasonable invariant measure, is by the equilibrium states. For example, in the statistical mechanics of lattice gases we know that the set of equilibrium states of the whole system is the closed convex hull of limits of the equilibrium states of finite subsystems, with different boundary conditions (see for example [11, Thm III.2.6, p. 251]). Therefore, in a sense the sectional systems determine the behaviour of the whole system. While these equilibrium states are not given as invariant measures of some dynamical systems, one can nevertheless think that there is some “dynamics of nature” behind them. One purpose of this article is to show that this dynamics must be of a rather special kind, since no standard smoothness assumptions made in the theory of dynamical systems imply this kind of behaviour. Indeed, one can find examples of quite the contrary: We construct respectable mappings for which the invariant measures of the finite subsystems with fixed boundary values do not have anything in common with the respective measure of the whole system.

The paper is organized as follows: Chapter 2 is devoted to the definitions that are in use throughout the whole article, both in finite and infinite dimensional settings. In Chapter 3 we define the two-dimensional sectional mappings and give examples of mappings with dynamics diametrically opposed to the sectional dynamics. In Chapter 4 we do the same in the space $\mathbb{S}^{\mathbb{Z}}$ to show that the infinite-dimensional systems are not different from this point of view.

2. Basic definitions

Let X be a topological space equipped with some reasonable background measure λ that gives positive measure to open sets and let $f : X \to X$ be a mapping. For any measure μ we denote the image measure μf^{-1} by $f_*\mu$. A measure μ is said to be *observable*, with respect to f, if there exists an open $U \subset X$ such that

$$\frac{1}{n}\sum_{k=0}^{n-1} f_*^k \delta_x \to \mu.$$

for λ-almost all points $x \in U$. Here the convergence is in the normal weak*-topology. In this paper we require that $U = X$ and take λ to be the suitable dimensional Lebesgue measure $\mathcal{L}$. In the infinite dimensional space we will use the product measure $\mathcal{L}^\infty$, though it is not as obvious choice for the background as it is in the finite dimensional space. In fact, there exists measures with all the finite dimensional marginals absolutely continuous with respect to the Lebesgue measure, but which are nevertheless singular with it. For instance, if one modifies $\mathcal{L}$ a bit and takes the product measure of this modified measure, the result is typically singular with $\mathcal{L}^\infty$.

Let Q_x^ε be a family of Borel probability measures on X, defined for every $x \in X$ and $\varepsilon > 0$ such that the mapping $x \mapsto Q_x^\varepsilon$ is continuous. Moreover, let

$$\sup_{x\in X} |\int g(y) dQ_x^\varepsilon(y) - g(x)| \to 0$$

as $\varepsilon \to 0$ for every continuous $g : X \to \mathbb{R}$. We say that a probability measure μ^ε is an invariant measure of the *random ε-perturbation* of $f : X \to X$, if

$$\mu^\varepsilon(B) = \int Q_{f(x)}^\varepsilon(B) d\mu^\varepsilon(x)$$

for every Borel set $B \subset X$. These invariant measures are not necessarily unique. (See [9] for details. The setting there is somewhat more general than ours.) Let μ be an invariant measure of the unperturbed mapping f. We say that μ is *stochastically stable*, if measures μ^ε exist and

$$\mu^\varepsilon \to \mu$$

weakly* as $\varepsilon \to 0$. The stochastic stability makes sense in any metric space, so it is applicable also in the infinite dimensional case.

3. Two-dimensional sectional mappings

3.1. **Definitions.** As an illustration we first define the sections of a mapping in a two-dimensional space. Of course one could define the concept right away for any multi-dimensional space, but it is easier to see the main idea involved here by starting from the simplest case.

Let $\mathbb{S}$ be the unit circle and $\mathbb{T} = \mathbb{S} \times \mathbb{S}$ the torus, which we often identify with the unit square. Consider a mapping $f : \mathbb{T} \to \mathbb{T}$, that is $f(x, y) = (f_1(x, y), f_2(x, y))$, where f_1 and f_2 are from $\mathbb{T}$ to $\mathbb{S}$. For a fixed $a \in \mathbb{S}$ this f defines two *sectional mappings*. The vertical section is $V_a(y) = f_2(a, y)$ and the horizontal one $H_a(x) = f_1(x, a)$. Both of them are from the unit circle to itself.

In the following we require that the mapping f as well as the sections V_a and H_a are *non-singular* for every $a \in \mathbb{S}$. This means that they map every set with positive Lebesgue measure (with the dimension of the domain space) to a set of positive measure.

We say that dynamics is *contracting* if its observable measure is a Dirac measure, and *chaotic* if the observable measure is absolutely continuous with respect to the Lebesgue measure. These wide terms describe here two qualitatively opposite types of dynamics. The sectional dynamics is called chaotic or contracting, if all the sectional mappings (both horizontal and vertical) have such dynamics, with the respective measures in their domain spaces. This definition can be easily generalized to any finite-dimensional space.

3.2. **A function with chaotic dynamics and contracting sectional dynamics.** Let $f : \mathbb{T} \to \mathbb{T}$ be a function defined as

$$f(x, y) = (3y + \varphi(x), 3x + \varphi(y)) \pmod 1 \ =: (f_1(x, y), f_2(x, y)),$$

where $\varphi(\cdot)$ is a smooth function with $0 < |\frac{d\varphi(x)}{dx}| < 1$ everywhere. (For instance take $\frac{x(x-1)}{4}$.) The Jacobian determinant is $\neq 0$ everywhere, so it is non-singular. Also its sectional mappings are non-singular. All the mappings considered are smooth as well. Obviously f is basically $(3y, 3x) \pmod 1$, but the small additional terms are needed to secure the non-singularity of the sectional mappings.

Fix $x \in [0, 1]$. Then $V_x(y) = 3x + \varphi(y)$ is a contracting function in y, since $|\frac{dV_x(y)}{dy}| < 1$ everywhere, and it has an attracting fixed point, say z, that attracts all points. Its observable measure is thus δ_z. The horizontal sections have similar dynamics and the same kind of observable measures. Therefore the sectional dynamics is contracting. However, the absolute values of both eigenvalues of the Jacobian matrix of f are

greater than 1, so the function is expanding. By [3] it has a unique ergodic invariant measure μ that is equivalent with the Lebesgue measure. Since the measure is invariant and ergodic, it is also observable by the Birkhoff ergodic theorem. Thus the dynamics is chaotic.
In order to use the result in [3], we must identify the torus with $[0,1]^2$. Taken as a function operating in the unit square, f is C^∞ wherever it is continuous. Therefore it is enough to check the pieces bounded by the points of discontinuity, that is, the points in $f_1^{-1}(\{0,1\}) \cup f_2^{-1}(\{0,1\})$. It is quite easy to see that these pieces do not contain cusps, which is required in [3].

Remark 1. The above example is robust under random perturbations. Indeed, we know, by [9, Thm 1.1, Prop. 1.4, Thm. 4.2 (p.154)], that the random ε-perturbations of f have invariant measures μ^ε, and if μ is the weak limit of some sequence of measures μ^ε as $\varepsilon \to 0$, then μ is absolutely continuous with respect to the Lebesgue measure, since f is expanding. We know also, again by the aforementioned results, that the measure μ is invariant. By the uniqueness the observable measure is stochastically stable. It is easy to see that the non-wandering set of any sectional mapping consists of the fixed point z and therefore δ_z is also stochastically stable. (For this result and for the definition of non-wandering set, see [9, Thm.4.4, p.50].)

3.3. **A function with contracting dynamics and chaotic sectional dynamics.** Let $f : \mathbb{T} \to \mathbb{T}$ (the torus taken as the unit square) be a mapping defined as $f(x,y) = (f_1(x,y), f_2(x,y))$, where

$$\begin{cases} f_1(x,y) = f_2(x,y) + \epsilon(x,y) \\ f_2(x,y) = 3x - 2y \,(\mathrm{mod}\ 1). \end{cases}$$

Set $A = \{(x,y) \in \mathbb{T} \mid x \geq y, 3x - 2y \leq 1\}$. We define the function ε such that f will be smooth and $f(\mathbb{T}) \subset A$.

The function $f_2 : \mathbb{T} \to [0,1]$ is affine in sets that are bounded by lines $3x - 2y = m$, where $m = 0, \pm 1, \pm 2$ The same holds for the function $\varphi(x,y) = \frac{1 - f_2(x,y)}{3}$. Take ρ to be a C^∞ function $\mathbb{T} \to [0,1]$ such that on the lines $3x - 2y = m$ and on the set $\{(x,y) \mid x = 0 \text{ or } y = 0\,(\mathrm{mod} 1)\}$ both ρ and the derivatives $\frac{d\rho}{dx}$ and $\frac{d\rho}{dy}$ vanish. (One can build this function for instance by using suitable polynomials.) We require also that $\rho \neq 0$ outside these lines. Since the derivatives $\frac{d\varphi}{dx}$ and $\frac{d\varphi}{dy}$ are constants where they exist, and φ is piecewise affine, one can rescale ρ such that $0 \leq \rho \leq \varphi$. By rescaling it again, if needed, and modifying it in a suitable manner, we get a function ε which has the properties of ρ mentioned above and in addition $0 \leq |\frac{d\varepsilon}{dx}| \leq 1$ and $2\frac{d\varepsilon}{dx} + 3\frac{d\varepsilon}{dy} \neq 0$ almost everywhere. This makes the function f non-singular, because its Jacobian determinant is $-(2\frac{d\varepsilon}{dx}+3\frac{d\varepsilon}{dy})\neq 0$ almost everywhere. Also the sections are non-singular.

Since ε vanishes in the set $\{(x,y)\mid x=0 \text{ or } y=0\,(\text{mod}1)\}$, the function f is continuous on the torus. In fact it is C^∞. Moreover, for every $(x,y)\in\mathbb{T}$ we have

$$3f_1(x,y)-2f_2(x,y)=f_2(x,y)+3\varepsilon(x,y)\leq f_2(x,y)+3\varphi(x,y)=1,$$

and $f_1(x,y)\geq f_2(x,y)$, which means $f(\mathbb{T})\subset A$.

Let $(x,y)\in A$, define $x_0=x, y_0=y$ and $x_{n+1}=f_1(x_n,y_n)$, $y_{n+1}=f_2(x_n,y_n)$. Then

$$\begin{cases} x_{n+1}=3x_n-2y_n+\epsilon(x_n,y_n)\\ y_{n+1}=3x_n-2y_n. \end{cases} \tag{3.1}$$

Now $x_n\geq y_n$, when $n\geq 1$, and $y_n=3x_{n-1}-2y_{n-1}\geq x_{n-1}$, when $n\geq 2$. Thus the sequences (x_n) and (y_n) are increasing and dominated by 1. Therefore the limit $x'=\lim_{n\to\infty}x_n$ exists and is equal to $\lim_{n\to\infty}y_n$, since $x_n\geq y_n\geq x_{n-1}$. From the equations (3.1) and the definition of f we see that (x',x') is a fixed point of f and $x'=3x'-2x'+\varepsilon(x',x')$, which is equivalent to $\varepsilon(x',x')=0$. Since that happens only on the boundary of A and the only fixed point on this boundary is $(1,1)$, we have $\lim_{n\to\infty}f^n(x,y)=(1,1)$.

We have seen that $f^n(x,y)\to(1,1)$ when $n\to\infty$ for every $(x,y)\in X$. Therefore $\delta_{(1,1)}$ is observable for f. Also $|\frac{df_1}{dx}|\geq 3-1=2$, because $|\frac{d\varepsilon}{dx}|\leq 1$. Obviously $\frac{df_2}{dy}=-2$. Therefore, the sectional mappings are expanding, mixing, and by [10] have a unique Lebesgue-equivalent, ergodic invariant measure, which is again observable by the same arguments as in the previous example. Therefore the sectional dynamics is chaotic.

Remark 2. To see the stochastic stability of these measures it suffices to point out that the non-wandering set for the mapping f is its fixed point $(1,1)$ and therefore the Dirac-measure in this point is stochastically stable. The Lebesgue-equivalent invariant measures of the sectional mappings are also stochastically stable, since the sections are expanding.

4. Infinite dimensional case

4.1. **The definition and the setting.** Let $\mathbb{X}=\mathbb{S}^{\mathbb{Z}}$ and recall that we think $\mathbb{S}$ as $[0,1]$ with the endpoints identified. We equip $\mathbb{X}$ with the product topology. The symbol x_n means the nth coordinate of x. Let $\Lambda\subset\mathbb{Z}$ be finite and $a\in\mathbb{S}^{\mathbb{Z}\setminus\Lambda}$. These Λ and a define a sectional mapping from $\mathbb{S}^\Lambda$ to itself, which we denote by $f_{\Lambda,a}$. Take $x \in \mathbb{S}^\Lambda$ and define $x \vee a \in \mathbb{X}$ as

$$(x \vee a)_n = \begin{cases} x_n, & n \in \Lambda \\ a_n, & n \in \mathbb{Z} \setminus \Lambda \end{cases}.$$

Then

$$f_{\Lambda,a}(x)_n = f(x \vee a)_n,$$

where $n \in \Lambda$.

The idea in the following is the same as in the two-dimensional case. The sectional dynamics is defined in the same manner. Recall that the domain of f_Λ is a finite dimensional space $\mathbb{S}^\Lambda$. We say that sectional dynamics is contracting (resp. chaotic) if the dynamics of $f_{\Lambda,a}$ for every finite Λ and $\mathcal{L}^\infty$-almost all a is contracting (chaotic) in the finite sense. We say that f has chaotic dynamics, if it has an invariant measure whose finite dimensional marginals are absolutely continuous with the corresponding Lebesgue measure. The dynamics is contracting if the observable measure with $\mathcal{L}^\infty$ as the background is a Dirac-measure.

4.2. **A function with chaotic dynamics and contracting sectional dynamics.** We define a function $f : \mathbb{X} \to \mathbb{X}$ as a coupled map $f = g_\varepsilon \circ h$, where $h(x)_n = 4x_n \,(\text{mod}1)$. The coupling g_ε is given by $g_\varepsilon(x)_n = x_n + \varepsilon \cdot \lambda(x_{n-1}) \,(\text{mod}1)$, where λ is a contracting C^∞-function $\mathbb{S} \to \mathbb{S}$. For some ε small enough, this function f has an invariant measure μ with finite dimensional marginals absolutely continuous with respect to the Lebesgue measure, by [7]. This μ is also unique in the set of sufficiently regular measures $\mathcal{B}_1$ defined in [7].

Let σ be a shift in $\mathbb{X}$, defined by $\sigma(x)_n = x_{n+1}$. Let us define

$$q(x)_n = f \circ \sigma(x)_n = 4x_{n+1} + \varepsilon \cdot \lambda(4x_n) = \sigma \circ f(x)_n.$$

This q is obviously non-singular and C^∞. Since $\sigma \circ f = f \circ \sigma$, we see that $f_*\sigma_*\mu = \sigma_* f_* \mu = \sigma_*\mu$, and thus $\sigma_*\mu$ is also invariant measure of f. It is easy to see that the measure $\sigma_*\mu$ also belongs to the set $\mathcal{B}_1$ and therefore by the uniqueness we have $\sigma_*\mu = \mu$. Moreover $q_*\mu = f_*\sigma_*\mu = \mu$, which shows μ to be q-invariant, and therefore, the dynamics of q is chaotic.

Take arbitrary Λ and $a \in \mathbb{X}$ as above and let m be the maximal integer of Λ. Without loss of generality, we can assume that $\Lambda = \{m, m-1, m-2, \ldots, m-l\}$, so that it is "a box". (Remark that in any case it is a finite union of boxes.) The function $q_{\Lambda,a}$ is smooth and non-singular.

We proceed as follows. First we show that $q^k_{\Lambda,a}(x)_m$ tends to the same constant for every $x \in \mathbb{S}^\Lambda$ as $k \to \infty$. This is obvious, because $q^k_{\Lambda,a}(x)_m$ depends essentially only on the constant a_{m+1}. Then we proceed by showing that also $q^k_{\Lambda,a}(x)_{m-1}$ tends to some constant for all x, as $k \to \infty$. This is also easy, since $q^k_{\Lambda,a}(x)_{m-1}$ depends on $q^{k-1}_{\Lambda,a}(x)_m$ and this we have shown to be approximately constant for large k. The rest of the argument follows by induction.

Let us denote the lifts of the corresponding functions by upper-case letters. Define $b := 4a_{m+1}$ and $P : \mathbb{R} \to \mathbb{R}$ to be $P(z) = b + L(z)$, where L is the lift of $\varepsilon\lambda$. We use here z as a real variable, leaving x to be in $\mathbb{S}^\mathbb{Z}$. Let $Q_{\Lambda,a}$ be the lift of $q_{\Lambda,a}$ to $\mathbb{R}^\Lambda$. We see that $Q^k_{\Lambda,a}(x)_m = P^k(x_m)$ has the same limit for all x, as $k \to \infty$, since P is a contracting mapping in $\mathbb{R}$ as well as L. (They have the same derivative.)

We know already that $Q^k_{\Lambda,a}(x)_m = P^k(x_m)$ tends to a constant for all x, but we still have to show this for $Q^k_{\Lambda,a}(x)_i$, where $m > i \in \Lambda$. Let us write $4Q^{k-1}_{\Lambda,a}(x)_{i+1} = b_k(x, i+1) = b_k$. Now

$$Q^k_{\Lambda,a}(x)_i = b_k + L(b_{k-1} + L(b_{k-2} + L(\ldots + L(x_i)))) = K_k \circ K_{k-1} \circ \cdots \circ K_1(x_i),$$

where $K_k(z) = b_k + L(z)$. Next we proceed inductively (with respect to i) and assume $b_k(x, i+1) \to b(i+1) = b'$ for all x, when $k \to \infty$. Define $P(z) = b' + L(z)$. As above, we see that $P^n(z)$ has the same limit, say y, for all $z \in \mathbb{R}$ as $n \to \infty$. We can require that $\sup_{z\in\mathbb{R}} |P^n(z) - y| \to 0$ as $n \to \infty$. (This can be required of $\varepsilon\lambda$.) Moreover $\sup_{z\in\mathbb{R}} |K_k(z) - P(z)| \to 0$ as $k \to \infty$. Hence we can take n and n' such that $\sup_{z\in\mathbb{R}} |P^n(z) - y| < \frac{\varepsilon}{2}$ and $\sup_{z\in\mathbb{R}} |K_k(z) - P(z)| < \frac{\varepsilon}{2n}$ for $k \geq n'$. Now

$$|K_{n+n'} \circ \cdots \circ K_1(z) - y| < |P^n(K_{n'} \circ \cdots \circ K_1(z)) - y| + \frac{\varepsilon}{2} < \varepsilon.$$

Therefore $Q^k_{\Lambda,a}(x)_i$ and also $q^k_{\Lambda,a}(x)_i$ have the same limit for all $x \in \mathbb{R}^\Lambda$ as $k \to \infty$. Thus its observable measure will be some Dirac measure and the dynamics of $q_{\Lambda,a}$ is contracting.

Remark 3. The stochastic stability of the invariant measure of the sectional mappings is seen in the same way here as in the finite dimensional case. For the mapping q, or more precisely for the coupled lattice map f, the stochastic stability can be proved by the results in [8, p. 150]. The definition of a random perturbation in [8] is different from ours in, but the stochastic stability turns out to be a special case of the one introduced in [9].

4.3. **A function with contracting dynamics and chaotic sectional dynamics.** Let $\mathbb{X} = \mathbb{S}^\mathbb{Z}$ and let $0 < \lambda < 1$. If one thinks

$\mathbb{S}$ as the interval $[0,1]$ with the endpoints identified, one can define $J = [\lambda, 1] \subset \mathbb{S}$. Set $g : [0,1] \to J$ to be the continuous mapping that is identity on J and an affine bijection on $[0,1] \setminus J$. Let $a : J \to [0,1]$ be the affine order-preserving bijection.

The idea below is that we define a mapping $T : \mathbb{X} \to J^{\mathbb{Z}}$ that is contracting, if x is in $J^{\mathbb{Z}}$ and expanding outside of it. Most of the sectional functions have such boundary values, that every iterate falls outside of the set $J^{\mathbb{Z}}$,under section, and thus they will be expanding. However the function itself will be contracting as the first iterate throws everything in $J^{\mathbb{Z}}$.

Define $G(x)_n = g(x_n)$ and $B : \mathbb{X} \to [0,1)$ as $B(x) = \inf\{x_i \mid x = (\ldots, x_{-1}, x_0, x_1, \ldots)\}$, where the representation $(\ldots, x_{-1}, x_0, x_1, \ldots)$ does not contain any ones. Take $h : [0,1] \to [0,1]$ to be the standard tent map

$$h(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}.$$

We know that Lebesgue measure is observable for h. Set $p : [0,1] \to [0,1]$ to be a piecewise affine bijection such that $p(0) = 1$ and $p(x) \le \frac{1}{4}$, when $\lambda \le x \le 1$. Finally define

$$T(x)_n = a^{-1} \circ (p(B(x)) \cdot h) \circ a \circ g(x_n)).$$

Now $T : \mathbb{X} \to J^{\mathbb{Z}}$, and T is thus singular with respect to $\mathcal{L}^{\infty}$, but $T_{\Lambda,a}$ is non-singular, continuous and smooth everywhere save for some linear subspaces.

It is easy to see that for almost every $x \in \mathbb{X}$, with respect to the measure $\mathcal{L}^{\infty}$, we have $B(x) = 0$ and on the other hand $B(x) \ge \lambda$ for every $x \in J^{\mathbb{Z}}$. Since $T(x)_n \subset J$, and $p(B(x)) \le \frac{1}{4}$ everywhere in $J^{\mathbb{Z}}$ we get that T is highly contracting in $J^{\mathbb{Z}}$, since $T^{m+1}(x)_n = a^{-1} \circ \beta h \circ a(T(x)_n)$, where $\beta < \frac{1}{4}$. Obviously for all x the sum $\frac{1}{n}\sum_{k=0}^{n-1} T_*^k \delta_x$ tends to $\delta_{\lambda^{\mathbb{Z}}}$.

However, the dynamics of the sectional mapping $T_{\Lambda,a}$ is different for almost all $a \in \mathbb{X}$. Namely for almost all points $(\ldots, a_{-1}, a_0, a_1, \ldots)$ we have $p(B(a)) = p(0) = 1$, and therefore

$$T^m_{\Lambda,a}(x)_n = (a^{-1} \circ h \circ a)^m (x_n),$$

where $n \in \Lambda$. The observable measure of $T_{\Lambda,a}$ is absolutely continuous with respect to $\mathcal{L}^{\Lambda}$ This is because h has Lebesgue-equivalent observable measure, and the conjugate function a preserves this, since it is affine. The mapping $T_{\Lambda,a}$ is basically a direct product of expanding functions in J and therefore itself expanding in J^{Λ}. One can again use the result in [3] to see that the sectional dynamics is chaotic.

Remark 4. It is easy to see that $\{\lambda^{\mathbb{Z}}\}$ is the non-wandering set of T so the stochastic stability of the measure $\delta_{\lambda^{\mathbb{Z}}}$ can be seen by [9, Thm 4.4]. (One needs the fact that $\mathbb{S}^{\mathbb{Z}}$ is separable.) Similarly the stochastic stability for the sectional dynamics can be established the same way as in the similar finite dimensional case.

Acknowledgements

I want to express my gratitude to my supervisor Esa Järvenpää for the invaluable help in writing this paper and Maarit Järvenpää for suggesting me its topic and for important comments. For details I have also received help from Jouni Parkkonen, Tuomas Puurtinen and Tapio Rajala. I am thankful for the hospitality of professor Jean-Pierre Eckmann and the Département de Physique Théorique, Université de Genève during my short, but most pleasant stay there in the summer 2005.

References

[1] Blank, Michael & Bunimovich, Leonid:
Multicomponent dynamical systems: SRB measures and phase transitions,
Nonlinearity 16 (2003), 387-401.

[2] Chazottes, J.-R. & Fernandez, B. (eds):
Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems, Lecture Notes in Physics 671
Springer, Berlin Heidelberg, 2005.

[3] Góra, P. & Boyarski, A.:
Absolutely continuous invariant measures for piecewise expanding C^2-transformations in $\mathbb{R}^n$,
Israel Journal of Mathematics, Vol. 67, No. 3, 1989.

[4] Järvenpää, Esa:
SRB-Measures for Coupled Map Lattices,
in [2],95-114.

[5] Järvenpää, Esa & Järvenpää, Maarit:
On the Definition of SRB-Measures for Coupled Map Lattices,
Communications in Mathematical Physics, 220, 1-12 (2001).

[6] Järvenpää, Esa & Tolonen, Tapani:
Relations between natural and observable measures,
Nonlinearity, 18 (2005),897-912.

[7] Keller, Gerhard & Liverani, Carlangelo.:
Coupled map lattices without cluster expansion,
Discrete and Continuous Dynamical Systems, 11 (2004), no 2-3, 325-335.

[8] Keller, Gerhard & Liverani, Carlangelo.:
A spectral Gap for a One-dimensional Lattice of Coupled Piecewise Expanding Interval Maps,
in [2], 115-151.

[9] Kifer, Y. :
Random perturbations of dynamical systems,
Birkhäuser, Boston, 1988.

[10] Lasota, A. & Yorke, J.:
On the existence of invariant measures for piecewise monotonic transformations,
Transactions of the American Mathematical Society, Vol. 186, December 1973.

[11] Simon, Barry:
The Statistical Mechanics of Lattice Gases, Vol I
Princeton University Press, New Jersey, 1993.

[12] Young, Lai-Sang:
What are SRB Measures and Which Dynamical Systems Have Them,
Journal of Statistical Physics, Vol. 108, Nos 5/6, September 2002.

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland
E-mail address: `taptolo@maths.jyu.fi`