WEIGHTED POINTWISE HARDY INEQUALITIES

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Abstract. We show that the weighted Hardy inequality $\int_\Omega |u|^p d_\Omega{}^{\beta-p} \leq C\int_\Omega |\nabla u|^p d_\Omega{}^\beta$, where $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, holds for all $u \in C_0^\infty(\Omega)$ even for certain (sharp) exponents $\beta > p-1$ when the visual boundary of the domain $\Omega \subset \mathbb{R}^n$ is sufficiently big. In the case of the usual von Koch snowflake domain the sharp bound is shown to be $\beta < p - 2 + \lambda$, where $\lambda = \log 4/\log 3$. These results are based on new pointwise Hardy inequalities.

1. Introduction

The classical Hardy inequality

$$(1) \qquad \int_\Omega |u(x)|^p\, d(x,\partial\Omega)^{\beta-p}\, dx \leq C \int_\Omega |\nabla u(x)|^p\, d(x,\partial\Omega)^\beta\, dx\,,$$

where $1 < p < \infty$, was first considered by G. H. Hardy [6], [7] in the one-dimensional, unweighted ($\beta = 0$) case with $\Omega = (0,\infty) \subset \mathbb{R}$. It was later proved by Hardy et al. (cf. [8, Section 9.8] and references therein) that if u is an absolutely continuous function and $u(0) = 0$, the weighted inequality (1) holds in $(0,\infty)$ with a constant $C = C(p,\beta) > 0$ whenever $\beta < p-1$. Since then, it has been a question of considerable interest to find conditions which guarantee that the (p,β)-Hardy inequality (1) is valid in some more general settings. The situation we are interested in this paper is the one where Ω is a domain in $\mathbb{R}^n$, $n \geq 2$, and (1) holds for all functions $u \in C_0^\infty(\Omega)$ with a constant $C_\Omega = C_\Omega(p,\beta) > 0$. If this is the case, we say that the domain Ω admits the (p,β)-Hardy inequality.

Let us briefly discuss the know results for (p,β)-Hardy inequalities in this setting for a fixed $1 < p < \infty$. In the unweighted case $\beta = 0$, it is well known by results of Ancona [1] (for $p = n = 2$), Lewis [16], and Wannebo [26], that $\Omega \subset \mathbb{R}^n$ admits the p-Hardy inequality provided that the boundary of Ω is sufficiently big everywhere, namely

$$(2) \qquad \mathcal{H}^\lambda_\infty\big(\partial\Omega \cap B(w,r)\big) \geq Cr^\lambda \quad \text{for all } w \in \partial\Omega,\ 0 < r < \operatorname{diam}(\Omega),$$

where $\lambda > n-p$. Moreover, Wannebo [26] proved in fact that, under condition (2), there exists some small positive $\beta_0 = \beta_0(p,n,\Omega)$ so that Ω admits the weighted Hardy inequality (1) for all $\beta < \beta_0$. If $\Omega \subsetneq \mathbb{R}^n$, that is, Ω is a proper subdomain of $\mathbb{R}^n$, then Ω satisfies (2) with $\lambda = 0$, and hence Ω admits the p-Hardy inequality for every $p > n$.

Note that in all the results above the density condition was given in terms of the local p-capacity of the complement of Ω, but such a condition is always

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true if (2) holds, see e.g. calculations in the proof of [12, Theorem 5.9]. Conversely, positive p-capacity implies positive $(n-p)$-Hausdorff content with estimates, cf. [11, Theorem 2.27]. Since the p-capacity condition has a self-improving property [16, Theorem 1], it is in fact equivalent to (2).

On the other hand, Nečas proved in [20] that if $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain, then Ω admits (p,β)-Hardy inequalities for all $\beta < \beta_0$, where $\beta_0 = p-1$; see also Kufner [15]. In this case, Ω satisfies the condition (2) with $\lambda = n-1$. The bound $\beta < p-1$ for Lipschitz domains is also sharp, since the (p,β)-Hardy inequality fails e.g. in the unit ball $B(0,1) \subset \mathbb{R}^n$ for every $\beta \geq p-1$.

Based on the above considerations, one could ask if (2) with $\lambda = n-1$ is enough to guarantee (p,β)-Hardy inequalities for all $\beta < \beta_0 = p-1$, or even if (2) with some $\lambda > n-1$ would imply (p,β)-Hardy inequalities for all $\beta < \beta_0$, where $\beta_0 > p-1$. For example, if Ω is the von Koch snowflake domain in the plane, then (2) holds with $\lambda = \dim \partial\Omega = \log 4/\log 3$, and direct calculations indicate that when $\beta < \beta_0$, where $\beta_0 = p-2+\lambda > p-1$, the (p,β)-Hardy inequality should hold for all $u \in C_0^\infty(\Omega)$.

It turns out, however, that this is not true in general; in this paper we give a construction which proves the next theorem.

Theorem 1.1. *For every $1 < \lambda < 2$ there exists a simply connected domain $\Omega = \Omega_\lambda \subset \mathbb{R}^2$ so that Ω satisfies the condition (2) with the exponent λ, but Ω fails to admit the $(p,p-1)$-Hardy inequality.*

See Example 6.2 for the proof of Theorem 1.1. Actually, in Example 6.3 we construct for any $1 \leq \sigma < \lambda < 2$ a domain $\Omega = \Omega_{\lambda,\sigma}$ such that Ω satisfies the condition (2) with the given exponent λ, and admits the (p,β)-Hardy inequality for all $\beta < p-2+\lambda$ *but* for $\beta(\sigma) = p-2+\sigma$. So the thickness of the boundary $\partial\Omega$, in the sense of (2) with $\lambda > n-1$, is not sufficient to guarantee that Ω admits (p,β)-Hardy inequalities for all $\beta < \beta_0(p,n,\Omega)$, where $\beta_0 > p-1$. Nevertheless, the von Koch snowfake example leads one to ask if it is possible to obtain results for $\beta \geq p-1$ under additional conditions on the domain. For instance, it is well-known that snowflake type domains are John domains, which implies that all the boundary points are "visible" or "easily accessible" from the points inside the domain. This motivates the definition of the *visual boundary $v_x(c)$–$\partial\Omega$ near $x \in \Omega$* (see Section 2.5). The next theorem, which is the main theorem of this paper, states that if the visual part of the boundary is big enough near every $x \in \Omega$, then Ω admits the desired Hardy inequalities.

Theorem 1.2. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that there exist $0 \leq \lambda \leq n$, $c \geq 1$, and $C_\Omega > 0$ so that*

$$\mathcal{H}^\lambda_\infty\big(v_x(c)\text{–}\partial\Omega\big) \geq C_\Omega d(x,\partial\Omega)^\lambda \qquad \text{for every } x \in \Omega.$$

Then Ω admits the (p,β)-Hardy inequality whenever $\beta < \beta_0$, where $\beta_0 = \beta_0(p,n,\Omega) = p-n+\lambda$.

As a direct consequence of Theorem 1.2, we obtain the fact that if $\Omega \subset \mathbb{R}^2$ is a von Koch snowflake type domain with $\dim \partial\Omega = \lambda \in (1,2)$, then Ω indeed admits (p,β)-Hardy inequalities for all $1 < p < \infty$ and $\beta < \beta_0 =$

$p-2+\lambda$. Furthermore, this β_0 is critical, since Ω fails to admit the (p,β)-Hardy inequality whenever $\beta \geq \beta_0$.

Even though the visual boundary condition in Theorem 1.2 is closely related to John domains, it is not true that Theorem 1.2 holds for all John domains with uniformly big boundary, in the sense of (2); the domain constructed in Example 6.2 works as a counterexample. However, we prove that each simply connected John domain in the plane admits (p,β)-Hardy inequalities for all $\beta < p-1$. This improves on the result of Nečas, where one assumes that Ω is a bounded Lipschitz domain; each Lipschitz domain is in fact a John domain. More generally, if $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a John domain and if in addition Ω is quasiconformally equivalent to the unit ball of $\mathbb{R}^n$, then Ω admits (p,β)-Hardy inequalities for all $\beta < p-1$. See Section 5 for the proofs.

In spite of these results on John domains, we presume that, in the case $\beta < p-1$, the visibility of the boundary plays no essential role. We state this as a conjecture.

Conjecture 1.3. *Let $1 < p < \infty$ and $\beta < p-1$. If $\Omega \subset \mathbb{R}^n$ satisfies the density condition* (2) *with $\lambda = n-1$, then Ω admits the (p,β)-Hardy inequality. In particular, if Ω is a simply connected domain in the plane, then Ω admits the (p,β)-Hardy inequality for all $\beta < p-1$.*

In order to prove the (p,β)-Hardy inequalities of Theorem 1.2, we in fact establish as a tool stronger inequalities, *pointwise (p,β)-Hardy inequalities*, which are also of their own independent interest. In the unweighted case $\beta = 0$, pointwise Hardy inequalities were introduced by Hajłasz [5], and also Kinnunen and Martio considered similar inequalities independently in [14]. Generalizing the approach of [5] we call the inequality

$$|u(x)| \leq C d_\Omega(x)^{1-\frac{\beta}{p}} M_{L d_\Omega(x),q}\big(|\nabla u| {d_\Omega}^{\beta/p}\big)(x), \tag{3}$$

where $u \in C_0^\infty(\Omega)$, $1 < q < p$, and $L \geq 1$, the pointwise (p,β)-Hardy inequality; see Proposition 3.1 for the justification of this notation. In (3) we denote $M_{R,q} f = (M_R f^q)^{1/q}$, where $M_R f$ is the usual restricted Hardy-Littlewood maximal function of f, and $d_\Omega(x) = d(x,\partial\Omega)$. We say that a domain $\Omega \subset \mathbb{R}^n$ admits the pointwise (p,β)-Hardy inequality if there exist some $1 < q < p$ and constants $L \geq 1$, $C_\Omega = C_\Omega(p,\beta,q) > 0$, so that the inequality (3) holds for every $u \in C_0^\infty(\Omega)$ with these q, L and C_Ω. It was proved in [5] (see also [14]) that in all unweighted cases considered by Ancona, Lewis and Wannebo, one obtains in fact pointwise p-Hardy inequalities. However, the pointwise (p,β)-Hardy inequality is not equivalent to the usual (p,β)-Hardy inequality, since there are domains which admit the latter for some p and β, but the corresponding pointwise inequality fails; see Examples 6.2 and 6.3. In particular, when $\Omega \subset \mathbb{R}^n$ and $1 < p < \infty$ is fixed, the set of β's for which Ω admits the pointwise (p,β)-Hardy inequality is always an interval — see Lemma 3.2 — but this is not necessarily the case with the usual Hardy inequality, as can be seen for instance in Examples 6.2 and 6.3.

The outline of this paper is as follows: In Section 2 we go through some basic notation and definitions used in the rest of the paper, especially we give the exact definition of the visual boundary. Section 3 is devoted to some preliminary results on the pointwise Hardy inequalities and technical lemmas that we use in the proof of our main theorem in Section 4. Next, in Section 5, we prove results on John domains, and finally, in Section 6 we give some examples which show that Condition 2.1 is essentially the weakest possible condition that guarantees a domain Ω to admit the (pointwise) (p,β)-Hardy inequality for all $1 < p < \infty$ and $\beta < n - p + \lambda$. As noted before, these examples also shed some light to many other questions concerning usual and pointwise Hardy inequalities and their relations.

2. Definitions

2.1. **Notation.** Let A be a subset of the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$. Then ∂A denotes the boundary of A, $\overline{A}$ is the closure of A, and the n-dimensional Lebesgue measure of A is denoted $|A|$, provided that A is measurable. The characteristic function of A is χ_A, and $\operatorname{diam}(A)$ is the usual Euclidean diameter of A. For $A \subset \mathbb{R}^n$, $b \in \mathbb{R}^n$, and $\kappa \in \mathbb{R}$ (we allow also $\kappa \in \mathbb{C}$ if $n = 2$) we denote $\kappa A + b = \{\kappa x + b : x \in A\}$, unless A is a cube, cf. 2.2. Depending on the situation, $d(\cdot,\cdot)$ denotes either the Euclidean distance between two points, two sets, or a point and a set. We use also the notation $|x|$ for the Euclidean norm of $x \in \mathbb{R}^n$; then $|x - y| = d(x,y)$. An open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted $B(x,r)$. An open and connected set $\Omega \subset \mathbb{R}^n$ is called a *domain*. As in the introduction, we denote $d_\Omega(x) = d(x, \partial\Omega)$ for $x \in \Omega$.

Let $U \subset \mathbb{R}^n$ and let $f\colon U \to \mathbb{R}^m$ be a mapping. If $A \subset U$, $f|_A$ denotes the restriction of f to A. The *support* of f, $\operatorname{spt}(f)$, is the closure of the set where f is non-zero. For $f \in L^1_{\text{loc}}(U)$ and a measurable $A \subset U$ with $0 < |A| < \infty$ we denote

$$f_A = \fint_A f(x)\,dx = \frac{1}{|A|}\int_A f(x)\,dx\,.$$

All the integrals in this paper are taken with respect to the n-dimensional Lebesgue measure, if not stated otherwise.

A continuous mapping $\gamma\colon [a,b] \to \mathbb{R}^n$, $a,b \in \mathbb{R}$, $n \geq 2$, as well as the image $\gamma = \gamma([a,b]) \subset \mathbb{R}^n$, is called a *curve*. The Euclidean length of a curve γ is denoted $l(\gamma)$. A curve γ is *rectifiable* if $l(\gamma) < \infty$. Every rectifiable curve γ can be parameterized by arc length, i.e. $\gamma = \gamma\colon [0,l] \to \mathbb{R}^n$ so that $l(\gamma|_{[0,t]}) = t$ for all $t \in [0,l]$. We say that a curve $\gamma\colon [a,b] \to \mathbb{R}^n$ *joins x to y* (in $A \subset \mathbb{R}^n$), if $\gamma(a) = x$ and $\gamma(b) = y$ (and $\gamma \subset A$). When $x,y \in \mathbb{R}^n$, $[x,y]$ is the line segment with endpoints x and y.

We use the letter C to denote various positive constants, which may vary from expression to expression. If g and h are some quantities, we write $g \lesssim h$ if there exists a constant $C > 0$ so that $g \leq Ch$. When F is a finite set, $\#F$ denotes the cardinality of F.

2.2. **Whitney decomposition.** Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$, be a proper subdomain. Then $\mathcal{W} = \mathcal{W}(\Omega)$ denotes a *Whitney decomposition* of Ω, i.e. a collection of cubes $Q \subset \Omega$ with pairwise disjoint interiors and having edges parallel to the coordinate axes. Also, the diameters of $Q \in \mathcal{W}$ are in the set $\{2^{-j} : j \in \mathbb{Z}\}$ and satisfy the condition

$$\operatorname{diam}(Q) \leq d(Q, \partial\Omega) \leq 4\operatorname{diam}(Q).$$

We refer to [24] for the existence and further properties of Whitney decompositions. For $j \in \mathbb{Z}$ we define

$$\mathcal{W}_j = \{Q \in \mathcal{W} : \operatorname{diam}(Q) = 2^{-j}\}.$$

When Q is a cube, c_Q denotes the center point of Q, and if $L > 0$, then LQ is the cube with the center point c_Q and diameter $L \operatorname{diam}(Q)$.

2.3. **Maximal functions.** The classical *restricted Hardy-Littlewood maximal function* of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$M_R f(x) = \sup_{0<r<R} \fint_{B(x,r)} |f(y)|\, dy\,,$$

where $0 < R \le \infty$ may depend on x. In the case $R = \infty$ we denote $M_\infty f = Mf$. The well-known maximal function theorem of Hardy, Littlewood and Wiener (see e.g. [24]) states that if $1 < p < \infty$, we have $||M_R f||_p \le C(n,p)||f||_p$ for all $0 < R \le \infty$.

However, in this paper it will be more convenient to consider maximal functions where the integrals are taken over cubes instead of balls. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we define

$$M^c f(x) = \sup_{Q \ni x} \fint_Q |f(y)|\, dy\,,$$

where the supremum is taken over all (closed) cubes Q with $x \in Q$. We need also a restricted version of M^c, and therefore we fix for each x a cube $Q(x) \ni x$ and define

$$M^c_{Q(x)} f(x) = \sup_{x \in \tilde{Q} \subset Q(x)} \fint_{\tilde{Q}} |f(y)|\, dy.$$

Maximal functions Mf and $M^c f$ are equivalent in the sense that there exist constants $0 < c_1 < c_2 < \infty$ so that $c_1 Mf \le M^c f \le c_2 Mf$ for each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, so especially a version of the maximal function theorem holds also for $M^c f$.

When $1 < q < \infty$, we define $M_q f = \big(M f^q\big)^{1/q}$; $M_{R,q}$, M^c_q and $M^c_{Q(x),q}$ are then defined similarly. From the maximal function theorem it follows that M_q is bounded on L^p for each $q < p < \infty$.

2.4. **John domains and uniform domains.** Let $\Omega \subset \mathbb{R}^n$ be a domain and let $c \ge 1$. We say that Ω is a *c-John domain* with center point x_0 if for every $x \in \Omega$ there exists a curve (called a John curve) $\gamma\colon [0,l] \to \Omega$, parameterized by arc length, so that $\gamma(0) = x$, $\gamma(l) = x_0$, and

$$d(\gamma(t), \partial\Omega) \ge \tfrac{1}{c} t \tag{4}$$

for each $t \in [0,l]$. Geometrically this means that each point in Ω can be joined to the central point by a "twisted cone", which is sometimes called also a "carrot". If Ω is a c-John domain with center point x_0, then $\Omega \subset B\big(x_0, c\, d(x_0, \partial\Omega)\big)$, so in particular Ω is bounded. Also, if Ω is a c-John domain, then for each $w \in \partial\Omega$ there is a curve $\gamma\colon [0,l] \to \Omega \cup \{w\}$ joining w to x_0 and satisfying (4). We say also in this case that γ joins w to x_0 *in* Ω. John domains were introduced in [22] and named in [17] after F. John who had considered a similar class of domains earlier (cf. [13]). There are also several other ways to define John domains, see [21]. However, for bounded domains $\Omega \subset \mathbb{R}^n$ these definitions are equivalent, but with possibly different constants.

A domain $\Omega \subset \mathbb{R}^n$ is *uniform* if there is a constant $C \geq 1$ so that each pair of points $x, y \in \Omega$ can be joined by a curve $\gamma\colon [0,l] \to \Omega$, parameterized by arc length, so that $l \leq Cd(x,y)$ and $d(z,\partial\Omega) \geq \frac{1}{C}\min\{d(z,x), d(z,y)\}$ for each $z \in \gamma$. Such a curve γ is called a "double cone" or a "cigar" arc. Every bounded uniform domain is also a c-John domain for some $c \geq 1$.

Let Ω be a c-John domain with center point x_0 and let $w \in \partial\Omega$. Let $\mathcal{J}_c(w, x_0)$ denote the collection of all c-John curves joining w to x_0 in Ω. We then define

$$P(w) = \big\{Q \in \mathcal{W} : Q \cap \gamma \neq \emptyset \text{ for some } \gamma \in \mathcal{J}_c(w, x_0)\big\}.$$

When $E \subset \partial\Omega$, we also denote

$$P(E) = \bigcup_{w \in E} P(w).$$

The *(John-)shadow* $S(Q)$ of a cube $Q \in \mathcal{W}$ on the boundary $\partial\Omega$ is now defined by

$$S(Q) = \{w \in \partial\Omega : Q \in P(w)\}.$$

Then $S(Q)$ is a closed set for each $Q \in \mathcal{W}$. Indeed, if $w_j \in S(Q)$ and $w_j \to w$ as $j \to \infty$, we have for each $j \in \mathbb{N}$ a curve $\gamma_j \in \mathcal{J}_c(w_j, x_0)$ so that $\gamma_j \cap Q \neq \emptyset$. It follows from the Arzelà-Ascoli theorem that there exists a subsequence of (γ_j) converging uniformly to a curve γ. It is then easy to show that γ is a c-John curve joining w to x_0 and intersecting Q.

Estimates for the sizes of these shadows will provide us with one key element in the proof of our main theorem.

2.5. **Visual boundary and the main condition.** Let $\Omega \subset \mathbb{R}^n$ be a domain. When $x \in \Omega$ and $c \geq 1$ is a constant we define a subdomain $\Omega_x(c)$ by

$$\Omega_x(c) = \bigcup\{U \subset \Omega : U \text{ is a } c\text{-John domain with center point } x\}.$$

Then clearly $\emptyset \neq \Omega_x(c) \subset \Omega$ and $\Omega_x(c)$ is also a c-John domain with center point x. We say that the set

$$v_x(c)\text{–}\partial\Omega = \partial\Omega \cap \partial\Omega_x(c)$$

is the *c-visual boundary of* Ω *near* x. In our main theorem, as well as in the corresponding pointwise result, we assume that the visual boundary of Ω near each point $x \in \Omega$ is uniformly big, in the sense of the following density condition.

Condition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain. We assume that there exist constants $c \geq 1$, $C_0 > 0$, and $\lambda > 0$ so that for each $x \in \Omega$*

$$\mathcal{H}^\lambda_\infty\big(v_x(c)\text{–}\partial\Omega\big) \geq C_0 d(x,\partial\Omega)^\lambda\,. \tag{5}$$

Here $\mathcal{H}^\lambda_\infty$ is the λ-*Hausdorff content* of a set, defined by

$$\mathcal{H}^\lambda_\infty(A) = \inf\left\{\sum_{k=1}^\infty \operatorname{diam}(E_k)^\lambda : A \subset \bigcup_{k=1}^\infty E_k\right\}$$

for $A \subset \mathbb{R}^n$. The *Hausdorff dimension* of a set $A \subset \mathbb{R}^n$ is then

$$\dim(A) = \inf\{\lambda > 0 : \mathcal{H}^\lambda_\infty(A) = 0\}.$$

Notice that a domain $\Omega \subset \mathbb{R}^n$ satisfies Condition 2.1 if and only if for each $x \in \Omega$ there exists *some* c-John domain $U_x \subset \Omega$ with center point x so that

$$\mathcal{H}^\lambda_\infty(\partial U_x \cap \partial\Omega) \geq C_0 d(x, \partial\Omega)^\lambda .$$

If $\Omega \subset \mathbb{R}^n$ is a uniform domain, then it is sufficient to assume merely that the usual boundary $\partial\Omega$ satisfies a density condition similar to (5), since then we conclude using the uniformity of Ω that Ω satisfies Condition 2.1. Let us state this as a proposition:

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain and let $1 < p < \infty$. Assume that there exist constants $\tau \geq 1$, $C_0 > 0$, and $\lambda > 0$ so that*

$$\mathcal{H}^\lambda_\infty\big(\partial\Omega \cap B\big(x, \tau d(x, \partial\Omega)\big)\big) \geq C_0 d(x, \partial\Omega)^\lambda$$

for each $x \in \Omega$. Then Ω satisfies Condition 2.1.

Proof. Let C_U be the constant from the uniformity condition for Ω. Let $x_0 \in \Omega$ and let $B_0 = B\big(x_0, \tau d_\Omega(x_0)\big)$. Then it is easy to show that if $x \in B_0 \cap \Omega$, the double cone arc γ_x joining x to x_0 is also a c-John arc, with a constant $c = c(C_U, \tau) > 0$. Denote for $x \in B_0 \cap \Omega$

$$\mathrm{Cig}(x) = \bigcup_{z \in \gamma_x} B\big(z, C_U{}^{-1} \min\big\{d(z, x), d(z, x_0)\big\}\big).$$

Then $U_{x_0} = \bigcup_{x \in B_0 \cap \Omega} \mathrm{Cig}(x)$ is a c-John domain satisfying $B_0 \cap \partial\Omega \subset \partial U_{x_0} \cap \partial\Omega$. This, together with the density assumption, implies that Condition 2.1 holds in Ω. $\square$

It follows that a uniform domain Ω satisfying the density condition in Proposition 2.2 admits the Hardy inequalities of Theorem 1.2. In fact, such a domain admits also the corresponding pointwise inequalities; see Theorem 4.1.

3. Preliminary results

We begin by recording some basic properties of the weighted pointwise Hardy inequalities. First, let us justify the notation of the pointwise (p, β)-Hardy inequality, i.e. that the pointwise inequality always implies the usual (p, β)-Hardy inequality. We give the proof, which uses some well-known arguments, for the sake of the completeness.

Proposition 3.1. *Suppose that the pointwise (p, β)-Hardy inequality* (3) *holds for a function $u \in C_0^\infty(\Omega)$ with a constant $C_1 > 0$. Then u satisfies the (p, β)-Hardy inequality* (1) *with a constant $C = C(C_1, p, n) > 0$.*

Proof. Denote $R = R(x) = L d_\Omega(x)$. Divide the inequality (3) by $d_\Omega(x)^{1-\frac{\beta}{p}}$, integrate to power p over Ω, and use the fact that $M_{R,q}$ is bounded on L^p

to obtain

$$
\begin{aligned}
\int_\Omega |u(x)|^p d_\Omega(x)^{\beta-p}\,dx &\le C_1^p \int_\Omega \Big(M_{R,q}\big(|\nabla u| {d_\Omega}^{\beta/p}\big)(x)\Big)^p\,dx \\
&\le C \int_\Omega \Big(|\nabla u(x)|^q d_\Omega(x)^{\beta(q/p)}\Big)^{p/q}\,dx \\
&= C \int_\Omega |\nabla u(x)|^p d_\Omega(x)^\beta\,dx\,.
\end{aligned}
$$

Here the constant $C > 0$ depends only on C_1, p, and the constant from the maximal function theorem, so that $C = C(C_1, n, p)$. $\square$

From the next lemma we obtain the fact that the pointwise (p, β_0)-Hardy inequality implies pointwise (p, β)-Hardy inequalities for all $\beta < \beta_0$.

Lemma 3.2. *Let $u \in C_0^\infty(\Omega)$ and let $1 < p < \infty$, $\beta_0 \in \mathbb{R}$. If u satisfies the pointwise (p, β_0)-Hardy inequality* (3) *with $1 < q < p$, $L \ge 1$, and $C_1 > 0$, then u satisfies the pointwise (p, β)-Hardy inequality for all $\beta < \beta_0$ with q, L and a constant $C = C(C_1, L, p, \beta_0, \beta) > 0$.*

Proof. Let $\beta < \beta_0$ and denote $\alpha = \beta_0 - \beta > 0$. If $0 < r \le L d_\Omega(x)$ and $y \in B(x, r)$, we have that $d_\Omega(y) \le (L + 1)d_\Omega(x)$. Thus we obtain from the pointwise (p, β)-Hardy inequality that

$$
\begin{aligned}
|u(x)| &\le C_1 d_\Omega(x)^{1-\frac{\beta_0}{p}} M_{L d_\Omega(x),q}\big(|\nabla u| {d_\Omega}^{\beta_0/p}\big)(x) \\
&\le C_1 d_\Omega(x)^{1-\frac{\beta_0}{p}} (L+1)^{\frac{\alpha}{p}} d_\Omega(x)^{\frac{\alpha}{p}} \\
&\qquad \cdot \left(\sup_{0<r<L d_\Omega(x)} \fint_{B(x,r)} |\nabla u(y)|^q d_\Omega(y)^{\beta_0 \frac{q}{p} - \alpha\frac{q}{p}}\,dy \right)^{1/q} \\
&\le C d_\Omega(x)^{1-\frac{\beta}{p}} M_{L d_\Omega(x),q}\big(|\nabla u| {d_\Omega}^{\beta/p}\big)(x),
\end{aligned}
$$

where $C = C(C_1, L, p, \beta_0, \beta) > 0$. $\square$

Next we prove some simple results on John-domains that we need in the proof of our main theorem. First of all, the diameter of the shadow of a Whitney cube is bounded, up to a constant, by the diameter of the cube itself.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a c-John domain. Then there exists a constant $C = C(c) > 0$ so that*

$$\operatorname{diam}(S(Q)) \le C \operatorname{diam}(Q)$$

for each $Q \in \mathcal{W}$.

Proof. If $S(Q) = \emptyset$ there is nothing to prove, so we may assume that $S(Q) \ne \emptyset$. Let $w \in S(Q)$. Then there exists, by definition, a c-John curve γ joining w to x_0 in Ω so that $\gamma(t_Q) \in Q$ for some $t_Q \in [0, l(\gamma)]$. It follows that

$$
\begin{aligned}
d(w, Q) &\le d(w, \gamma(t_Q)) \le l\big(\gamma|_{[0,t_Q]}\big) = t_Q \\
&\le c\, d(\gamma(t_Q), \partial\Omega) \le 5c \operatorname{diam}(Q),
\end{aligned}
\tag{6}
$$

and hence, by the triangle inequality, $\operatorname{diam}(S(Q)) \le (10c + 1) \operatorname{diam}(Q)$. $\square$

In the last lemma of this section we show that the shadows of the Whitney cubes of a given size have bounded overlap, and hence we obtain a bound for the sum of the measures of these shadows as well.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a c-John domain. Then there exists a constant $C = C(n, c) > 0$ so that*
(i) for each $j \in \mathbb{Z}$ and each $w \in \partial\Omega$,

$$\#\{Q \in \mathcal{W}_j : w \in S(Q)\} \leq C.$$

(ii) if μ is a Borel measure on $\partial\Omega$, we have for every measurable subset $E \subset \partial\Omega$ and each $j \in \mathbb{Z}$ that

$$\sum_{Q \in \mathcal{W}_j} \mu(S(Q) \cap E) \leq C\mu(E).$$

Proof. (i) When $w \in S(Q)$ (i.e. $Q \in P(w)$) we have, by (6), that

$$Q \subset B\big(w, (5c + 1)\operatorname{diam}(Q)\big).$$

Now, let us fix $w \in \partial\Omega$ and $j \in \mathbb{Z}$. Also, denote

$$a_j = \#\{Q \in \mathcal{W}_j : w \in S(Q)\} = \#\{Q \in \mathcal{W}_j : Q \in P(w)\},$$

and let $d_j = 2^{-j}$. Since the cubes $Q \in \mathcal{W}_j$ are essentially disjoint, we obtain that

$$\begin{aligned} a_j\, {d_j}^n &= C(n) \sum_{Q \in \mathcal{W}_j \cap P(w)} |Q| \leq C(n) \left|B\big(w, (5c + 1)d_j\big)\right| \\ &\leq C(n, c){d_j}^n . \end{aligned}$$

Thus $a_j \leq C(n, c)$.

(ii) Let μ be a Borel measure on $\partial\Omega$ and let $E \subset \partial\Omega$ be a μ-measurable set. By the the first part of the lemma, $\#\{Q \in \mathcal{W}_j : w \in S(Q)\}$ is uniformly bounded on E by a constant $C = C(n, c) > 0$, independent of j. Since $S(Q)$ is closed, it is μ-measurable, and hence

$$\sum_{Q \in \mathcal{W}_j} \mu(S(Q) \cap E) = \int_E \sum_{Q \in \mathcal{W}_j} \chi_{S(Q)}(w)\, d\mu(w) \leq C\mu(E).$$

This proves the lemma. □

4. The proof of the Main Theorem

In this section, we give the proof of our main result, Theorem 1.2. In fact, we prove the next theorem, which is the corresponding result for pointwise Hardy inequalities.

Theorem 4.1. *Let $1 < p < \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain satisfying Condition 2.1 with exponent λ, and let $\beta < p - n + \lambda$. Then Ω admits the pointwise (p, β)-Hardy inequality, i.e. there exist $1 < q < p$, $L \geq 1$, and $C > 0$ so that*

$$|u(x)| \leq C d_\Omega(x)^{1-\frac{\beta}{p}} M_{L d_\Omega(x), q}\big(|\nabla u| {d_\Omega}^{\beta/p}\big)(x)$$

whenever $u \in C_0^\infty(\Omega)$ and $x \in \Omega$.

By Proposition 3.1, Theorem 1.2 is a direct consequence of Theorem 4.1. The proof of Theorem 4.1 relies on the following lemma.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying Condition 2.1 with exponent λ. Then, if $1 < p < \infty$ and $\beta < p - n + \lambda$, there exist $1 \le q_0 < p$ and $L \ge 1$ with the following property: For every $q_0 < q < p$ there is a constant $C > 0$ such that the inequality*

$$|u_Q| \le C \operatorname{diam}(Q)^{1-\frac{\beta}{p}} \left(\fint_{LQ} |\nabla u(y)|^q \, d_\Omega(y)^{\frac{\beta}{p}q} \, dy \right)^{1/q}$$

holds for every $Q \in \mathcal{W}(\Omega)$ and every $u \in C_0^\infty(\Omega)$. Here the constants q_0 and L depend only on n, p, β, and the constants in Condition 2.1; C may depend also on q.

Proof. Fix a cube $Q_0 \in \mathcal{W}(\Omega)$ and take $j_0 \in \mathbb{Z}$ so that $\operatorname{diam}(Q_0) = 2^{-j_0}$. Let $\Omega_0 = \Omega_{c_{Q_0}}$ be the c-John domain with center point c_{Q_0} from the definition of the visual boundary in Section 2.5. Then there exists $L \ge 1$, depending only on n and c, so that $\Omega_0 \subset LQ_0$. From now on, let $\mathcal{W}$ denote a Whitney decomposition of Ω_0. We may assume that also $Q_0 \in \mathcal{W}$, since from the definition of Ω_0 it follows that $d(Q_0, \partial\Omega_0) = d(Q_0, \partial\Omega)$. In addition, we may assume, for simplicity, that $\operatorname{diam}(Q) \le \operatorname{diam}(Q_0)$ for all $Q \in \mathcal{W} = \mathcal{W}(\Omega_0)$. This last claim can be justified by the nature of our calculations in the proof and the fact that $\operatorname{diam}(\Omega_0) \le L \operatorname{diam}(Q_0)$.

Now let $1 < p < \infty$ and $\beta < p - n + \lambda$, and define

$$q_0 = \max\Big(1, p\,\frac{n-\lambda}{p-\beta}\Big).$$

Since $0 \le n - \lambda < p - \beta$, we have that $1 \le q_0 < p$. Let $q_0 < q < p$ and denote $\beta' = \frac{q}{p}\beta$. Then $q/p > (n-\lambda)/(p-\beta)$, and thus we obtain

$$\lambda + q - \beta' - n = \lambda + \tfrac{q}{p}(p-\beta) - n > \lambda + (n-\lambda) - n = 0.$$

Denote $E = \partial\Omega \cap \partial\Omega_0$ (i.e. E is the c-visual boundary of Ω near c_{Q_0}), let $w \in E$, and let γ be a c-John curve joining w to c_{Q_0} in Ω_0. We apply a chaining argument involving the Poincaré inequality on cubes, similar to the one in [23, Lemma 8], for the cubes $Q \in \mathcal{W}$ intersecting γ, and we obtain

$$|u_{Q_0}| = |u_{Q_0} - u(w)| \le C \sum_{Q \in P(w)} \operatorname{diam}(Q) \fint_Q |\nabla u(y)| \, dy\,,$$

where the constant $C > 0$ is independent of w. A simple use of Hölder's inequality leads us to

$$|u_{Q_0}| \le C \sum_{Q \in P(w)} \operatorname{diam}(Q)^{1-\frac{\beta}{p}} \left(\fint_Q |\nabla u(y)|^q \, d_\Omega(y)^{\frac{\beta}{p}q} \, dy \right)^{1/q}. \tag{7}$$

Note that here we have to use different sides of the inequality

$$\operatorname{diam}(Q) \le d_\Omega(y) \le 5 \operatorname{diam}(Q) \qquad \text{for all } y \in Q \in \mathcal{W}$$

depending whether $\beta \ge 0$ or $\beta < 0$.

From now on, let us denote $g(y) = |\nabla u(y)|\, d_\Omega(y)^{\beta/p}$. We apply Frostman's lemma (see e.g. [18, Theorem 8.8]) and choose a Radon measure μ such that μ is supported on E, $\mu(B(x,r)) \le r^\lambda$ for all $x \in \mathbb{R}^n$ and $r > 0$, and

$\mu(E) \geq C\mathcal{H}^\lambda_\infty(E)$. Integration of (7) over E with respect to the measure μ yields

$$|u_{Q_0}|\,\mu(E) \leq C \int_E \sum_{Q\in P(w)} \operatorname{diam}(Q)^{1-\frac{\beta}{p}} \left(\fint_Q g(y)^q\,dy \right)^{1/q} d\mu(w). \tag{8}$$

Then interchange the order of summation and integration in (8) and use Hölder's inequality for sums to obtain

$$\begin{aligned} |u_{Q_0}| &\leq C\mu(E)^{-1} \sum_{Q\in P(E)} \mu(S(Q))\operatorname{diam}(Q)^{1-\frac{\beta}{p}-\frac{n}{q}} \left(\int_Q g(y)^q\,dy \right)^{1/q} \\ &\leq C\mu(E)^{-1} \Bigg(\sum_{Q\in P(E)} \mu(S(Q))^{\frac{q}{q-1}} \operatorname{diam}(Q)^{\frac{q-\beta'-n}{q-1}} \Bigg)^{\frac{q-1}{q}} \\ &\qquad \cdot \Bigg(\sum_{Q\in P(E)} \int_Q g(y)^q\,dy \Bigg)^{1/q}. \end{aligned} \tag{9}$$

In the next step we estimate the sum in (9):

$$\begin{aligned} &\sum_{Q\in P(E)} \mu(S(Q))^{\frac{q}{q-1}} \operatorname{diam}(Q)^{\frac{q-\beta'-n}{q-1}} \\ &\qquad \leq \sum_{j=j_0}^{\infty} \max_{Q\in\mathcal{W}_j} \left(\mu(S(Q))^{\frac{1}{q-1}} \operatorname{diam}(Q)^{\frac{q-\beta'-n}{q-1}} \right) \sum_{Q\in\mathcal{W}_j} \mu(S(Q))\,, \end{aligned} \tag{10}$$

where by Lemma 3.4(ii)

$$\sum_{Q\in\mathcal{W}_j} \mu(S(Q)) \leq C\mu(E). \tag{11}$$

For the cubes $Q \in \mathcal{W}_j$ we have by definition that $\operatorname{diam}(Q) = 2^{-j}$, and further, by the properties of μ and Lemma 3.3, we obtain

$$\mu(S(Q)) \leq C\operatorname{diam}(S(Q))^\lambda \leq C\operatorname{diam}(Q)^\lambda \leq C2^{-j\lambda}\,.$$

Recall that by the choice of q and β' we have $\lambda + q - \beta' - n > 0$, so that

$$\begin{aligned} \sum_{j=j_0}^{\infty} \max_{Q\in\mathcal{W}_j} \left(\mu(S(Q))^{\frac{1}{q-1}} \operatorname{diam}(Q)^{\frac{q-\beta'-n}{q-1}} \right) &\leq \sum_{j=j_0}^{\infty} C2^{-j\,\frac{\lambda+q-\beta'-n}{q-1}} \\ &\leq C\,2^{-j_0\,\frac{\lambda+q-\beta'-n}{q-1}}\,. \end{aligned}$$

Combining this with equations (9), (10), and (11) yields

$$\begin{aligned} |u_{Q_0}| &\leq C\mu(E)^{-1+\frac{q-1}{q}} \left(2^{-j_0\,\frac{\lambda+q-\beta'-n}{q-1}} \right)^{\frac{q-1}{q}} \left(\int_{LQ_0} g(y)^q\,dy \right)^{1/q} \\ &\leq C\mu(E)^{-\frac{1}{q}} \operatorname{diam}(Q_0)^{\frac{\lambda+q-\beta'-n}{q}} \left(\int_{LQ_0} g(y)^q\,dy \right)^{1/q}. \end{aligned} \tag{12}$$

Finally, the properties of the Frostman measure μ and Condition 2.1 together imply that

$$\operatorname{diam}(Q_0)^\lambda \leq C\mathcal{H}^\lambda_\infty(E) \leq C\mu(E),$$

and hence we obtain from (12) that

$$|u_{Q_0}| \le C \operatorname{diam}(Q_0)^{1-\frac{\beta}{p}} \left(\fint_{LQ_0} |\nabla u(y)|^q \, d_\Omega(y)^{\beta'} \, dy \right)^{1/q}.$$

This proves the lemma. □

We are now ready to prove the pointwise (p, β)-Hardy inequality for the domains in question.

Proof of Theorem 4.1. Let $x \in \Omega$ and let $u \in C_0^\infty(\Omega)$. Choose $Q \in \mathcal{W}$ so that $x \in Q$. By Lemma 4.2, there exist some $1 < q < p$ and $L \ge 1$ so that

$$\begin{aligned} |u_Q| &\le C \operatorname{diam}(Q)^{1-\frac{\beta}{p}} M^c_{LQ,q} g(x) \\ &\le C d_\Omega(x)^{1-\frac{\beta}{p}} M^c_{LQ,q} g(x), \end{aligned}$$

where we denote as before $g(x) = |\nabla u(x)| \, d_\Omega(x)^{\beta/p}$. Using variations of the well-known inequalities [4, Lemma 7.16] and [9, Lemma (a)] we obtain that

$$|u(x) - u_Q| \le C \operatorname{diam}(Q) M^c_Q \nabla u(x),$$

where $C = C(n) > 0$. Therefore we have for *each* $1 < q < \infty$ that

$$\begin{aligned} |u(x) - u_Q| &\le C d_\Omega(x)^{1-\frac{\beta}{p}} \sup_{x \in \tilde{Q} \subset Q} \fint_{\tilde{Q}} |\nabla u(y)| d_\Omega(y)^{\beta/p} \, dy \\ &\le C d_\Omega(x)^{1-\frac{\beta}{p}} M^c_{LQ,q} g(x), \end{aligned}$$

where the second inequality follows from Hölder's inequality. Hence

$$|u(x)| \le |u(x) - u_Q| + |u_Q| \le C d_\Omega(x)^{1-\frac{\beta}{p}} M^c_{LQ,q} g(x).$$

We may now choose $L' = L'(L, n) \ge 1$ so that the pointwise (p, β)-Hardy inequality, i.e. the above inequality with $M^c_{LQ,q}$ replaced by $M_{L' d_\Omega(x), q}$, holds with a constant $C > 0$, independent of $x \in \Omega$ and $u \in C_0^\infty(\Omega)$. □

Condition 2.1 is not very meaningful when $\lambda = 0$; in fact, this kind of a condition is satisfied whenever $\Omega \subsetneq \mathbb{R}^n$ is a proper subdomain, since then $\partial\Omega \ne \emptyset$, and thus $v_x(1)$–$\partial\Omega \ne \emptyset$ for each $x \in \Omega$. Hence we obtain that

$$\mathcal{H}^0_\infty\big(v_x(1)\text{–}\partial\Omega\big) = 1 = d(x, \partial\Omega)^0$$

for every $x \in \Omega$. Lemma 4.2 holds also in this case — the proof is just a simplified version of the proof above — and we obtain the following corollary which generalizes the result known for $\beta = 0$.

Corollary 4.3. *Let $1 < p < \infty$. Then each subdomain $\Omega \subsetneq \mathbb{R}^n$ admits the pointwise (p, β)-Hardy inequality for every $\beta < p - n$.*

We note that the bound $\beta < p - n$ in Corollary 4.3 is sharp, since the $(p, p-n)$-Hardy inequality fails e.g. in $\mathbb{R}^n \setminus \{0\}$ and in $B(0,1) \setminus \{0\} \subset \mathbb{R}^n$, as can be seen with elementary calculations.

5. John domains quasiconformally equivalent to the unit ball

Recall that a homeomorphism $f\colon \Omega \to \Omega'$ between domains Ω, $\Omega' \subset \mathbb{R}^n$, $n \geq 2$, is called a (K-)*quasiconformal (qc) mapping* if f belongs to the Sobolev class $W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ and there is a constant $K \geq 1$ so that

$$||Df(x)||^n \leq K\, J_f(x) \quad \text{for a.e. } x \in \Omega.$$

Here $||\cdot||$ denotes the operator norm and J_f is the Jacobian determinant of f. Domains Ω and Ω' are said to be *quasiconformally equivalent* if there exists a qc mapping $f\colon \Omega \to \Omega'$. We refer to [25] for the basic theory of qc mappings.

John domains which are in addition quasiconformally equivalent to the unit ball have some special properties among all John domains (see [10] and references therein). In the proof of the next theorem we need to use one of those properties together with general results on qc mappings and John domains.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a John domain that is quasiconformally equivalent to the unit ball $B(0,1) \subset \mathbb{R}^n$, and let $1 < p < \infty$. Then Ω admits the pointwise (p,β)-Hardy inequality for each $\beta < p-1$.*

Proof. Denote $B = B(0,1) \subset \mathbb{R}^n$ and let $f\colon B \to \Omega$ be a K-qc mapping. Fix a point $y \in \Omega$ and take $x \in B$ so that $f(x) = y$. Since Ω is a John domain, f extends continuously to $\partial\Omega$ (cf. [25, Corollary 17.14] and [21, 2.17]). It follows from [25, Theorem 18.1] that there is a constant $\alpha = \alpha(n,K) > 0$ so that $f\big(B\big(x, \alpha\, d(x,\partial B)\big)\big) \subset B\big(y, \frac{1}{2}\, d(y,\partial\Omega)\big)$. By [2, Corollary 6.4], we have that

$$\mathcal{H}^{n-1}_\infty\big(f(S_x)\big) \geq C\, d\big(y,\partial\Omega\big)^{n-1},$$

where S_x is the radial projection of the ball $B\big(x,\alpha\, d(x,\partial B)\big)$ on ∂B and $C = C(n,K,\alpha) > 0$. Note that in [2] they have $\alpha = \frac{1}{2}$, but the results hold for any fixed $0 < \alpha < 1$ as well.

In order to prove the theorem it is now enough to show that for each $w \in f(S_x) \subset \partial\Omega$ there is a John curve γ joining w to y, with a John constant independent of w and y, since then Ω satisfies the visual boundary Condition 2.1 with $\lambda = n-1$, and Theorem 4.1 gives the claim. To this end, let $w \in f(S_x)$ and let $w' \in S_x$ be a preimage of w. Choose $x' \in B\big(x,\alpha\, d(x,\partial B)\big)$ so that $x' \in [0,w']$. We now define a curve $\gamma_1\colon [0,1] \to \Omega$ by $\gamma_1(t) = f\big(w' + t(x'-w')\big)$ for all $t \in [0,1]$. Note that γ_1 need not to be rectifiable. However, by [10, Theorem 3.1], there exists a constant $b = b(n,K,\Omega) \geq 1$ so that if $z = w' + t(x'-w') \in [w',x']$ for $t \in [0,1]$, we have

$$\operatorname{diam}\big(\gamma_1([0,t])\big) = \operatorname{diam}\big(f[w',z]\big) \leq b\, d\big(f(z),\partial\Omega\big) = b\, d\big(\gamma_1(t),\partial\Omega\big).$$

Now, by [17, Lemma 2.7] (see also [21, Section 2]), there is a constant $c = c(b,n) \geq 1$ and a c-John curve γ_2 joining w to $f(x')$ in Ω.

If $f(x') = y$, we take $\gamma = \gamma_2$ and the proof is complete. Otherwise we define our curve γ in parts:

$$\gamma(t) = \begin{cases} \gamma_2(t)\,, & 0 \leq t \leq l(\gamma_2) \\ f(x') + \big(t - l(\gamma_2)\big)\dfrac{y - f(x')}{|y-f(x')|}\,, & l(\gamma_2) < t \leq l(\gamma_2) + |y - f(x')|\,, \end{cases}$$

and it easily follows that γ is a c_1-John curve joining w to y in Ω, with a constant $c_1 = c_1(n, K, \Omega) > 0$. $\square$

The following planar result is an immediate consequence of Theorem 5.1, since by the Riemann mapping theorem each simply connected proper subdomain $\Omega \subsetneq \mathbb{R}^2$ is conformally, and thus especially quasiconformally equivalent to the unit ball $B(0,1) \subset \mathbb{R}^2$.

Corollary 5.2. *Let Ω be a simply connected John domain in the plane and let $1 < p < \infty$. Then Ω admits the pointwise (p, β)-Hardy inequality for each $\beta < p - 1$.*

6. Examples

In this section, we give various planar examples which prove the essential sharpness of our theorems; higher dimensional examples can be constructed along same lines. The first brief example, however, shows that, at least in the case of $\beta < p - 1$, Condition 2.1 is not very restrictive. Also, we record for von Koch -type snowflake domains the Hardy inequalities mentioned in the Introduction.

Example 6.1. (a) It is not necessary for a domain to be John, or even bounded, in order to satisfy Condition 2.1: Let $\Omega \subset \mathbb{R}^n$ be a strip,

$$\Omega = \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon 0 < x_n < 1\}.$$

Then Ω is unbounded, but it satisfies Condition 2.1 with $\lambda = n - 1$, and thus Ω admits the pointwise (p, β)-Hardy inequality for all $\beta < p - 1$.

(b) Let $\Omega \subset \mathbb{R}^n$ be a "room-and-corridor"-type domain which are widely used in the study of the Poincaré inequalities (see e.g. [23, Section 10] and references therein). Then Ω is not necessarily a John domain, but it satisfies clearly Condition 2.1 with $\lambda = n - 1$ and admits the pointwise (p, β)-Hardy inequality for all $\beta < p - 1$.

(c) Let $1 < \lambda < 2$ and let $\Omega \subset \mathbb{R}^2$ be a λ-snowflake domain, i.e. a "triangle" whose edges are copies of the von Koch -type snowflake curve K_λ with $\dim(K_\lambda) = \lambda$. Then Ω is a uniform domain, and by the self-similarity of the snowflake curve (cf. [3]) it is clear that the density assumption of Proposition 2.2 is satisfied. Hence Condition 2.1 holds in Ω with the exponent λ, and we conclude that Ω admits the pointwise (p, β)-Hardy inequality whenever $1 < p < \infty$ and $\beta < \beta_0 = p + \lambda - 2$. Furthermore, by considering functions $u_j \in C_0^\infty(\Omega)$ such that $u_j(x) = 1$ if $d_\Omega(x) \geq 2^{-j}$, and $|\nabla u| \lesssim 2^j$ if $d_\Omega(x) \leq 2^{-j}$, it is easy to see that the (p, β)-Hardy inequality fails whenever $\beta \geq \beta_0$.

In the following examples we show that a density condition of the type

$$\mathcal{H}^\lambda_\infty\big(\partial\Omega \cap B\big(x, \tau d_\Omega(x)\big)\big) \geq C d_\Omega(x)^\lambda \quad \text{for all } x \in \Omega, \tag{13}$$

for some constants $\tau \geq 1$ and $C > 0$, is not sufficient to guarantee (p, β)-Hardy inequalities in Ω for all $\beta < p - n + \lambda$. Therefore some kind of an accessibility condition similar to our visible boundary condition 2.1 is really needed.

We use both complex and vector notation in the following constructions, so, for instance, i denotes always the imaginary unit, and when $x \in \mathbb{R}^2 = \mathbb{C}$, we write $x = (x_1, x_2) = x_1 + ix_2$.

Example 6.2. Let $1 < p < \infty$ and $1 < \lambda < 2$. We construct a simply connected John domain $\Omega_\lambda \subset \mathbb{R}^2$ which satisfies the condition (13) with the exponent λ, but fails to admit the (p,β)-Hardy inequality for $\beta = p-1 < p-2+\lambda$. Furthermore, by Lemma 3.2, the pointwise (p,β)-Hardy inequality fails in Ω_λ for every $\beta \geq p-1$, too. Nevertheless, it turns out that Ω_λ admits the usual (p,β)-Hardy inequality also when $p-1 < \beta < p-2+\lambda$.

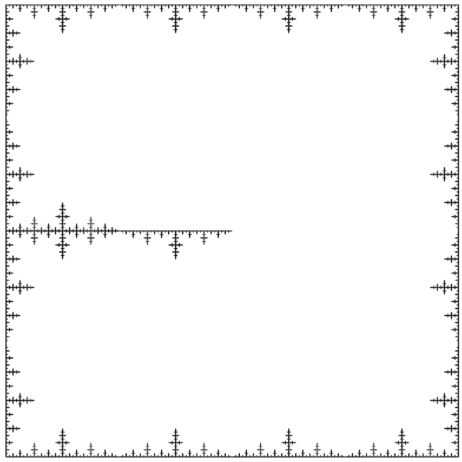

FIGURE 1. The domain Ω_λ of Example 6.2 for $\lambda = 1.45$

We begin by constructing a self-similar fractal called "the antenna set" in the (complex) plane. Let $0 < \alpha < \frac{1}{2}$ and let $F^\alpha = \{f_1, f_2, f_3, f_4\}$ be the iterated function system of similitudes

$$\begin{aligned} f_1(x) &= \tfrac{1}{2}x\,, & f_3(x) &= \alpha i x + \tfrac{1}{2}\,,\\ f_2(x) &= \tfrac{1}{2}x + \tfrac{1}{2}\,, & f_4(x) &= -\alpha i x + \tfrac{1}{2} + \alpha i\,. \end{aligned}$$

Then there exists a unique compact set $K = K^\alpha \subset \mathbb{R}^2$ which is invariant under F^α, i.e. $K = \bigcup_{j=1}^4 f_j(K)$. This K is the antenna set. It is easy to check that K satisfies the open set condition, and hence the Hausdorff dimension of K is $\lambda = \lambda(\alpha)$, where $1 < \lambda < 2$ is the solution of the equation

$$2 \cdot 2^{-\lambda} + 2\alpha^\lambda = 1\,,$$

and furthermore, we have that $0 < \mathcal{H}^\lambda(K) < \infty$. See e.g. [3, Chapter 9] for detailed information about iterated function systems, self-similar sets, and the open set condition. We now choose $0 < \alpha < \frac{1}{2}$ so that $\dim(K^\alpha) = \lambda$ for the fixed λ.

Take $\kappa = \frac{1}{4}$. Consider the unit square $[0,1]^2$ and replace each of the edges by four copies of κK, that is, K dilated by the factor $\frac{1}{4}$, oriented so that the "antennas" are inside the unit square. We call this domain Ω_λ^1. Notice that Ω_λ^1 satisfies Condition 2.1 for λ, even though it is not a uniform domain. Next we remove from Ω_λ^1 the sets $\kappa K + \frac{i}{2}$, $-\kappa K + \frac{i}{2} + \kappa$, and finally the set $A = -\kappa K + \frac{i}{2} + 2\kappa$. We have then constructed our domain $\Omega = \Omega_\lambda$ (see Fig. 1) which can quite easily be seen to be a simply connected John domain.

Let then $x \in \Omega$. Pick a point $w_x \in \partial\Omega$ so that $d(x, w_x) = d(x, \partial\Omega)$. Then $B\big(w_x, d_\Omega(x)\big) \subset B\big(x, 2d_\Omega(x)\big)$, and from the self-similarity of the antenna set we obtain

$$\mathcal{H}^\lambda_\infty\big(B(w_x, d_\Omega(x)) \cap \partial\Omega\big) \geq C d_\Omega(x)^\lambda\,,$$

where $C > 0$ is a constant independent of x. Hence Ω satisfies the condition (13), but it does *not* satisfy Condition 2.1 for any $\lambda > 1$.

Since Ω is a simply connected John domain, we know by Corollary 5.2 that Ω admits the (p,β)-Hardy inequality (even pointwise) whenever $1 < p < \infty$ and $\beta < p - 1$. Next we show that Ω fails to admit the $(p, p-1)$-Hardy inequality for every $1 < p < \infty$.

The failure happens above the "one-sided antenna" A. To see this, choose an open square $S \subset \Omega$ so that one edge of S is a subset of A and $d_\Omega(x) = d(x, A)$ for every $x \in S$. It is then enough to show that the $(p, p-1)$-Hardy inequality fails in the upper half plane $H_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ for functions in $C^\infty_0([0,3]^2)$, since then the case with $[0,3]^2$ and ∂H_+ replaced by S and A follows with a composition of a dilatation and a transformation.

We choose a sequence of functions $u_j \in C^\infty_0([0,3]^2)$ with the following properties: $\operatorname{spt}(u_j) \subset [0,3] \times [2^{-(j+1)}, 3]$, $u_j(x) = 1$ for all $x \in [1,2] \times [2^{-j}, 2]$, $|\nabla u_j(x)| \leq 2^{j+2}$ for all $x \in [0,3] \times [2^{-(j+1)}, 2^{-j}]$, and $|\nabla u_j(x)| \leq 2$ for all other $x \in \operatorname{spt}(|\nabla u_j|)$. Then, for $1 < p < \infty$ we have

$$\begin{aligned}\int_{H_+} |u_j(x)|^p\, d(x, \partial H_+)^{(p-1)-p}\, dx &\geq \sum_{k=0}^{j} \int_{[1,2]\times[2^{-k}, 2^{-k+1}]} d(x, \partial H_+)^{-1}\, dx \\ &\geq \sum_{k=0}^{j} 2^{-k}\, 2^{k-1} = \tfrac{1}{2}\,(j+1) \xrightarrow{j\to\infty} \infty\,,\end{aligned}$$

but

$$\begin{aligned}\int_{H_+} &|\nabla u_j(x)|^p\, d(x, \partial H_+)^{p-1}\, dx \\ &\leq \int_{[0,3]\times[2,3]} 2^p\, 3^{p-1}\, dx + \sum_{k=0}^{j} \int_{[0,3]\times[2^{-k}, 2^{-k+1}]} 2^p\, 2^{(-k+1)(p-1)}\, dx \\ &\qquad + \int_{[0,3]\times[2^{-(j+1)}, 2^{-j}]} 2^{(j+2)p}\, 2^{-j(p-1)}\, dx \\ &\leq C_1(p) + C_2(p) \sum_{k=0}^{j} 2^{-kp} + C_3(p) = C(p) < \infty\,.\end{aligned}$$

Thus the functions in $C^\infty_0([0,3]^2)$ do not satisfy the $(p, p-1)$-Hardy inequality with a universal constant in the upper half plane, and so we conclude that our domain Ω does not admit the $(p, p-1)$-Hardy inequality. This also means that Ω does not admit the pointwise $(p, p-1)$-Hardy inequality, and hence, by Lemma 3.2, the pointwise (p,β)-Hardy inequality fails in Ω for each $\beta \geq p - 1$.

Next we show that Ω admits the usual (p,β)-Hardy inequality also when $p - 1 < \beta < p - 2 + \lambda$: Denote $S_b = \big(\frac{1}{4}, \frac{1}{2}\big) \times \big(\frac{1}{2}, \frac{3}{4}\big)$, so that S_b is a square above the antenna A, and let $\Omega_g = \Omega \setminus S_b$. Fix $1 < p < \infty$ and

$p-1<\beta<p-2+\lambda$. Let $u \in C_0^\infty(\Omega)$ and denote $\tilde{u}(s,t) = u\big(\frac{1}{4}+s, \frac{1}{2}+t\big)$, $\tilde{d_\Omega}(s,t) = d_\Omega\big(\frac{1}{4}+s, \frac{1}{2}+t\big)$. With an application of Fubini's theorem and integration by parts, we then calculate (recall that we denote $\kappa = \frac{1}{4}$)

$$
\begin{aligned}
\int_{S_b} |u|^p \, {d_\Omega}^{\beta-p} &\lesssim \int_0^\kappa \int_0^\kappa |\tilde{u}(s,t)|^p \, t^{\beta-p} \, dt \, ds \\
&\lesssim \int_0^\kappa \Big[|\tilde{u}(s,\kappa)|^p \, \kappa^{\beta-p+1} + \int_0^\kappa |\tilde{u}(s,t)|^{p-1} |\nabla \tilde{u}(s,t)| \, t^{\beta-p+1} \, dt \Big] ds \\
&\lesssim \int_0^\kappa v_s(\kappa) \, ds + \int_{S_b} |u|^{p-1} |\nabla u| \, {d_\Omega}^{\beta-p+1} \,,
\end{aligned} \tag{14}
$$

where we have denoted $v_s(t) = |\tilde{u}(s,t)|^p \, \tilde{d_\Omega}(s,t)^{\beta-p+1}$. Notice that in (14) we need to use the fact $\beta \neq p-1$. Now

$$
|v_s'(t)| \lesssim |\tilde{u}(s,t)|^{p-1} |\nabla \tilde{u}(s,t)| \, \tilde{d_\Omega}(s,t)^{\beta-p+1} + |\tilde{u}(s,t)|^p \, \tilde{d_\Omega}(s,t)^{\beta-p}
$$

since $|\nabla d_\Omega| \le 1$, and thus, by changing the integration to the square $\big(\frac{1}{4}, \frac{1}{2}\big) \times \big(\frac{3}{4}, 1\big) \cap \Omega \subset \Omega_g$ above S_b, we obtain

$$
\begin{aligned}
\int_0^\kappa v_s(\kappa) \, ds &\lesssim \int_0^\kappa \int_\kappa^{2\kappa} |v_s'(t)| \, dt \, ds \\
&\lesssim \int_{\Omega_g} |u|^{p-1} |\nabla u| \, {d_\Omega}^{\beta-p+1} + \int_{\Omega_g} |u|^p \, {d_\Omega}^{\beta-p}.
\end{aligned} \tag{15}
$$

The pointwise (p,β)-Hardy inequality (3) holds for all $x \in \Omega_g$ with a constant independent of x and u, since these points satisfy the visual boundary condition (5) with the exponent λ. The use of this fact and the maximal function theorem yields, together with (14), (15), and Hölder's inequality, that

$$
\begin{aligned}
\int_\Omega |u|^p \, {d_\Omega}^{\beta-p} &= \int_{S_b} |u|^p \, {d_\Omega}^{\beta-p} + \int_{\Omega_g} |u|^p \, {d_\Omega}^{\beta-p} \\
&\lesssim \int_{\Omega_g} |u|^{p-1} |\nabla u| \, {d_\Omega}^{\beta-p+1} + \int_{\Omega_g} |u|^p \, {d_\Omega}^{\beta-p} \\
&\qquad + \int_{S_b} |u|^{p-1} |\nabla u| \, {d_\Omega}^{\beta-p+1} + \int_{\Omega_g} |u|^p \, {d_\Omega}^{\beta-p} \\
&\lesssim \int_\Omega |\nabla u|^p \, {d_\Omega}^{\beta} + \int_\Omega |u|^{p-1} |\nabla u| \, {d_\Omega}^{\beta-p+1} \\
&\lesssim \int_\Omega |\nabla u|^p \, {d_\Omega}^{\beta} + \Big(\int_\Omega |u|^p \, {d_\Omega}^{\beta-p} \Big)^{\frac{p-1}{p}} \Big(\int_\Omega |\nabla u|^p \, {d_\Omega}^{\beta} \Big)^{\frac{1}{p}} .
\end{aligned} \tag{16}
$$

It is obvious that all the constants in the above calculations depend only on p, β, and Ω. We obtain the (p,β)-Hardy inequality from (16), since if $a, b, C_1 > 0$ and $a \le C_1(b + a^{1-1/p} \, b^{1/p})$, there exists $C = C(C_1, p) > 0$ so that $a \le Cb$.

Finally, it is clear (cf. Example 6.1(c)) that Ω does not admit the (p,β)-Hardy inequality when $\beta \ge p-2+\lambda$.

Remark. By the truncation technique of Maz'ja [19], the (p,β)-Hardy inequality could also be proven by showing that

$$\int_{\{x\in\Omega:|u(x)|\geq 1\}} d_\Omega{}^{\beta-p} \leq C\int_\Omega |\nabla u|^p\, d_\Omega{}^\beta$$

for each $u\in C_0^\infty(\Omega)$.

Example 6.3. Let $1<p<\infty$ and $1<\sigma<\lambda<2$. We construct a simply connected John domain $\Omega_{\lambda,\sigma}\subset\mathbb{R}^2$ which satisfies the density condition (13) with the exponent λ, admits the (p,β)-Hardy inequality for all $\beta<p-2+\sigma$ and $p-2+\sigma<\beta<p-2+\lambda$, but fails to admit the $(p,p-2+\sigma)$-Hardy inequality.

The idea of this construction is to modify the domain Ω_λ of Example 6.2 so that, instead of a "straight antenna" A, the boundary $\partial\Omega_{\lambda,\sigma}$ would contain a more complicated set A_σ, a "snowflake antenna" (see Fig. 2). This A_σ is constructed as follows:

Let F_σ be the standard von Koch snowflake curve of dimension σ, obtained as the invariant set of four similitudes $\varphi_1,\ldots,\varphi_4$ with contraction ratio $\frac14<\rho=\rho(\sigma)<\frac12$, ordered so that $\varphi_1(0)=0$, $\varphi_j(1)=\varphi_{j+1}(0)$ for $j=1,2,3$, and $\varphi_4(1)=1$. Denote $z_0=\varphi_2(1)$, so that z_0 is the "top" of F_σ, and let $z_j=\varphi_j(z_0)$ for $j\in\{1,2,3,4\}$. Denote $\delta=d(z_1,z_2)>0$ and $K'=K\cup(-K+1)$, where K is the basic antenna set of dimension λ. Then choose $0<\tau<\delta$, let $K_0=i\tau K'+z_0$, and define

$$K_{j_1,\ldots,j_k}=\varphi_{j_1}\circ\cdots\circ\varphi_{j_k}(K_0).$$

We then have, for example, that

$$d(K_1,K_2)\geq\delta-2\rho\,\operatorname{diam}(K_0)>0\,. \tag{17}$$

Let A'_σ be the union of F_σ and all the images of K_0 under iterations of $\varphi_1,\ldots,\varphi_4$:

$$A'_\sigma=F_\sigma\cup\bigcup_{k=1}^\infty\ \bigcup_{j_1,\ldots,j_k=1}^4 K_{j_1,\ldots,j_k}\,.$$

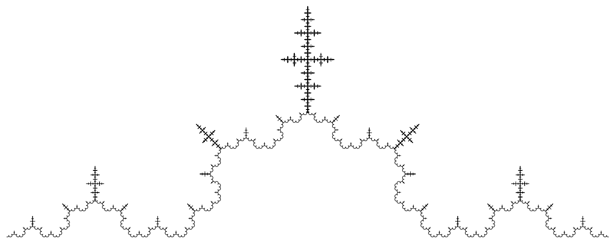

FIGURE 2. The set A'_σ in the construction of Example 6.3 for $\sigma=1.15$ (and $\lambda=1.45$)

Now, we construct the domain $\Omega_{\lambda,\sigma}$ in the same way as Ω_λ in Example 6.2, except that in the last stage we remove, instead of A, the set $A_\sigma=-\kappa A'_\sigma+\frac{i}{2}+2\kappa$. Using (17), the definition of A'_σ, and properties of F_σ and K, it is then straight-forward to verify that $\Omega_{\lambda,\sigma}$ is a simply connected John domain. Also, it is rather easy to see that $\Omega_{\lambda,\sigma}$ satisfies the visible boundary condition 2.1 with the exponent σ, and hence, by Theorem 4.1, $\Omega_{\lambda,\sigma}$ admits (pointwise) (p,β)-Hardy inequalities for all $\beta < p-2+\sigma$.

Let then $w \in \partial\Omega_{\lambda,\sigma}$ and let $0 < r < 1$. If $w \notin A_\sigma$, we obtain as before that $\mathcal{H}^\lambda_\infty\big(B(w,r)\cap\partial\Omega\big) \geq Cr^\lambda$. If $w \in A_\sigma$, we choose $k \in \mathbb{N}$ so that $\rho^k < r \leq \rho^{k-1}$; recall that ρ is the contraction ratio of the similitudes in the construction of the snowflake curve F_σ. Then it follows that there exist $j_1,\dots,j_k \in \{1,2,3,4\}$ so that $-\kappa K_{j_1,\dots,j_k} + \frac{i}{2} + 2\kappa \subset B(w,r)$, and thus

$$\mathcal{H}^\lambda_\infty\big(B(w,r)\cap\partial\Omega\big) \geq \kappa^\lambda\,\mathcal{H}^\lambda_\infty\big(K_{j_1,\dots,j_k}\big) = \kappa^\lambda(\rho^k)^\lambda\mathcal{H}^\lambda_\infty\big(K_0\big) \geq Cr^\lambda,$$

where $C = C(\lambda,\sigma) > 0$. This shows that $\Omega_{\lambda,\sigma}$ satisfies the density condition (13).

However, $\Omega_{\lambda,\sigma}$ does not admit the $(p,p-2+\sigma)$-Hardy inequality. This can be seen similarly as above for Ω_λ, by considering a suitable subdomain $S_\sigma \subset \Omega_{\lambda,\sigma}$ above A_σ, chosen so that $\dim(\partial S_\sigma \cap \partial\Omega_{\lambda,\sigma}) = \sigma$, $d(x,\partial\Omega_{\lambda,\sigma}) = d(x,A_\sigma)$ for all $x \in S_\sigma$, and $|S^k_\sigma| \gtrsim (4\rho^2)^k$ for all k greater than some $j_0 \in \mathbb{N}$, where

$$S^k_\sigma = \big\{x \in S_\sigma : \rho^{k+1} \leq d(x,\partial\Omega_{\lambda,\sigma}) \leq \rho^k\big\}.$$

We can then choose functions $u_j \in C^\infty_0(S_\sigma)$ in such a way that $|\nabla u_j| \lesssim \rho^{-j}$ in S^j_σ, $|\nabla u_j| \lesssim 1$ elsewhere in $\operatorname{spt}(|\nabla u_j|)$, and $\int_{S^k_\sigma}|u_j|^p \gtrsim |S^k_\sigma| \gtrsim (4\rho^2)^k$ for all j greater than j_0 and all $k \in \{j_0,\dots,j-1\}$. Then it follows with easy calculations and the use of the fact $\sigma = (\log 4)/(-\log\rho)$ that

$$\int_{\Omega_{\lambda,\sigma}} |u_j(x)|^p\, d(x,\partial\Omega_{\lambda,\sigma})^{(p-2+\sigma)-p}\,dx \gtrsim (j-j_0) \xrightarrow{j\to\infty} \infty\,,$$

but

$$\int_{\Omega_{\lambda,\sigma}} |\nabla u_j(x)|^p\, d(x,\Omega_{\lambda,\sigma})^{p-2+\sigma}\,dx \leq C(p,\sigma) < \infty\,.$$

Hence the $(p,p-2+\sigma)$-Hardy inequality fails in $\Omega_{\lambda,\sigma}$. Still, $\Omega_{\lambda,\sigma}$ admits the (p,β)-Hardy inequality also when $p-2+\sigma < \beta < p-2+\lambda$; the calculations are similar to those in Example 6.2, and we leave the details to the interested reader.

Examples 6.2 and 6.3 show that, for planar domains, the density condition (13) with $1 < \lambda < 2$ is not sufficient to guarantee (p,β)-Hardy inequalities for all $\beta < p-2+\lambda$. For instance, the $(p,p-1)$-Hardy inequality fails in the domain Ω_λ, since the dense part of the boundary is completely on the "wrong side" for points above A. Next we give yet another example in which a density condition even stronger than (13) is satisfied, but another kind of a phenomenon prevents the $(p,p-1)$-Hardy inequality.

Example 6.4. Let $1 < p < \infty$ and $1 < \lambda < 2$. We construct a simply connected domain $\Omega = \Omega_\lambda \subset \mathbb{R}^2$ which satisfies the condition (13) with the exponent λ, but fails to admit the $(p,p-1)$-Hardy inequality. Contrary to the previous examples, we have also for each $x \in \Omega$ that

$$\mathcal{H}^\lambda_\infty\big(\partial\Omega \cap \partial\Omega(x)\big) \geq C d_\Omega(x)^\lambda, \tag{18}$$

where $\Omega(x)$ is the x-component of the set $\Omega \cap B\big(x, 2d_\Omega(x)\big)$. Hence this example shows that $\Omega_x(c)$ being a c-*John* domain is essential in the definition of the visual boundary in Section 2.5.

Let K be the basic antenna set of dimension λ, as defined in Example 6.2. We now take the square $S = [-1,1] \times [0,2]$ and replace the edges, with the exception of the part $\big[-\frac{1}{2}, \frac{1}{2}\big] \times \{0\}$, by copies of $\frac{1}{2}K$, oriented so that the antennas are inside S.

Now choose an increasing sequence (n_j), $n_j \in \mathbb{N}$, and $n_1 \geq 2$, so that $2^{-jp} n_j \to \infty$ as $j \to \infty$. Let $K_j^R = i2^{-n_j}K$, $K_j^L = i2^{-n_j}(-K+1)$, and $K_j = K_j^R \cup K_j^L$ for $j \in \mathbb{N}$. Then define

$$T_j = \bigcup_{k=0}^{2^{n_j-1}-1} \big(K_j + ik2^{-n_j}\big),$$

so that T_j is a "stick", made of copies of K_j, with $\operatorname{diam}(T_j) = \frac{1}{2}$. Furthermore, T_j^R and T_j^L are defined analogically, with K_j replaced by K_j^R and K_j^L, respectively.

To complete the construction of Ω, we remove from the already modified square S the following union of sticks:

$$\begin{aligned}\bigcup_{j\in\mathbb{N}} \Bigg[&\Big(\big(T_j^L \cup 2^{-n_j}K\big) + 2^{-(j+1)}\Big) \\ &\cup \bigcup_{k=1}^{2^{-(j+1)+n_j}-1} \Big(\big(T_j \cup 2^{-n_j}K\big) + 2^{-(j+1)} + k2^{-n_j}\Big) \cup \Big(T_j^R + 2^{-j}\Big)\Bigg] \\ &\cup \Big(\tfrac{i}{2}(-K+1) + \tfrac{1}{2}\Big),\end{aligned}$$

as well as the reflection of the above set with respect to the line $i\mathbb{R} \subset \mathbb{C} = \mathbb{R}^2$. Finally, we have to remove also the line segment $[0, i2^{-1}]$ in order to obtain a simply connected domain Ω. It is quite obvious that Ω satisfies Condition 2.1 with exponent 1, and hence Ω admits (p,β)-Hardy inequalities for all $\beta < p-1$. Next we show that Ω does not admit the $(p, p-1)$-Hardy inequality.

Choose functions $u_j \in C_0^\infty(\Omega)$ with the following properties for each $j \geq 2$: $\operatorname{spt}(u_j) \subset [-2^{-j}, 2^{-j}] \times [2^{-1} + 2^{-n_j}, 1]$, $u_j = 1$ in $[-2^{-(j+1)}, 2^{-(j+1)}] \times [2^{-1} + 2^{-n_j+1}, 1 - 2^{-j}]$, $|\nabla u_j| \lesssim 2^{n_j}$ in $[-2^{-j}, 2^{-j}] \times [2^{-1} + 2^{-n_j}, 2^{-1} + 2^{-n_j+1}]$, and $|\nabla u_j| \lesssim 2^j$ elsewhere in $\operatorname{spt}(|\nabla u_j|)$. Then elementary calculations, similar to those in Example 6.2, give

$$\int_\Omega |u_j|^p \, d_\Omega{}^{-1} \gtrsim 2^{-j} n_j$$

and

$$\int_\Omega |\nabla u_j|^p \, d_\Omega{}^{p-1} \lesssim 2^{-j} 2^{jp}.$$

From these estimates we see that the $(p, p-1)$-Hardy inequality fails in Ω, since $2^{-jp} n_j \to \infty$ as $j \to \infty$.

It is left to show that the modified density condition (18) holds for each $x \in \Omega$. By the construction of Ω and the self-similarity of the antenna set, it is clear that this condition holds with some fixed constant $C > 0$ for all $x \in \Omega$ satisfying $d(x, \partial S) \leq \frac{1}{2}$. Hence we only need to consider points above the sticks, and, in fact, it is enough to consider points $x = it$ for $\frac{1}{2} < t < 1$, since other points can be treated similarly.

When $2^{-j+1} \leq t - \frac{1}{2} \leq 2^{-j+2}$ and $x = it$, the ball $B\big(x, 2d_\Omega(x)\big)$ intersects roughly $(2^{-j+n_j})^2$ copies of the scaled antenna K_j ; let $\mathcal{K}_j$ denote the union of these copies. Let $\mathcal{K}_j \subset \bigcup_{k\in\mathbb{N}} A_k$ so that $\sum_k \operatorname{diam}(A_k)^\lambda \leq 2\mathcal{H}^\lambda_\infty(\mathcal{K}_j)$. If $\operatorname{diam}(A_k) \leq 2^{-n_j}$ for all $k \in \mathbb{N}$, we obtain that

$$\begin{aligned}\sum_k \operatorname{diam}(A_k)^\lambda &\gtrsim (2^{-j+n_j})^2 \mathcal{H}^\lambda_\infty(K_j)\\ &\gtrsim (2^{-j+n_j})^2\, (2^{-n_j})^\lambda \mathcal{H}^\lambda_\infty(K) \gtrsim (2^{-j})^\lambda,\end{aligned}$$

and thus

$$\mathcal{H}^\lambda_\infty\big(\partial\Omega \cap \partial\Omega(x)\big) \geq C d_\Omega(x)^\lambda.$$

On the other hand, if $\operatorname{diam}(A_{k_0}) = \delta > 2^{-n_j}$ for some $k_0 \in \mathbb{N}$, we deduce, using the facts that the set $\mathcal{K}_j$ is made of similar sticks with constant distances and that $\{A_k\}$ is an “almost optimal” covering of $\mathcal{K}_j$ with respect to the λ-Hausdorff content, that there exists another “almost optimal” covering $\{\tilde{A}_k\}$ consisting of sets satisfying $\frac{1}{2}\,\delta \leq \operatorname{diam}(\tilde{A}_k) \leq 2\delta$. To cover the set $\mathcal{K}_j$, we need to have at least an amount of the order $(2^{-j})^2/\delta^2$ of such sets, and hence we obtain with simple calculations that (18) holds also in this case. This proves that the density condition (18) holds for all points in Ω, and hence such a condition is not sufficient to guarantee the (p, β)-Hardy inequality for all $\beta < p - n + \lambda$.

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