Characterizations of Orlicz-Sobolev spaces in terms of generalized Orlicz-Poincaré inequalities

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Abstract

We show that the Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^n)$ can be characterized in terms of the (generalized) Φ-Poincaré inequality. We also prove similar results in the general metric space setting.

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1 Introduction

Let Φ be a Young function and let $\Omega \subset \mathbb{R}^n$ be open. A pair (u, g) of measurable functions, $u \in L^1_{\mathrm{loc}}(\Omega)$ and $g \geq 0$, satisfies the Φ-Poincaré inequality in Ω, if there are constants $C_P \geq 1$ and $\tau \geq 1$ such that

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \Phi^{-1} \left(\fint_{\tau B} \Phi(g) \, d\mu \right) \tag{1}$$

for every ball $B = B(x, r_B)$ such that $\tau B \subset \Omega$. Here, $u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u$ and $\tau B = B(x, \tau r_B)$. It is well known that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies the 1-Poincaré inequality

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \fint_B |\nabla u| \, d\mu$$

for every ball $B \subset \Omega$. Thus, by Jensen's inequality, (1) holds with $\tau = 1$ and $g = |\nabla u|$. Our first result says that also the converse holds: If $u \in L^\Phi(\Omega)$ and there exists $g \in L^\Phi(\Omega)$ such that (1) holds (for the normalized pair), then u belongs to the Sobolev class $W^{1,\Phi}(\Omega)$.

Theorem 1.1 *Suppose that Φ is an N-function, $\Omega \subset \mathbb{R}^n$ is open, $u, g \in L^\Phi(\Omega)$ and that the pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ-Poincaré inequality in Ω. Then $u \in W^{1,\Phi}(\Omega)$ and $\||\nabla u|\|_{L^\Phi(\Omega)} \leq C(C_P, \tau, n)\|g\|_{L^\Phi(\Omega)}$.*

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For the definitions of Young and N-functions, see Section 2.1 below. Theorem 1.1 was proven by Hajłasz in [4] for $\Phi(t) = t^p$, $p \geq 1$, and by Tuominen in [13] for a doubling Φ whose conjugate is also doubling.

Our second result is a counterpart of Theorem 1.1 in the general metric setting. Let $X = (X, d, \mu)$ be a metric measure space with μ a Borel regular outer measure satisfying $0 < \mu(U) < \infty$, whenever U is nonempty, open and bounded. Suppose further that μ is doubling, that is, there exists a constant C_d such that

$$\mu(2B) \leq C_d \mu(B), \tag{2}$$

whenever B is a ball.

Our substitute for the usual Sobolev class $W^{1,\Phi}$ is based on upper gradients. We call a Borel function $g : X \to [0, \infty]$ an upper gradient of a function $u : X \to \overline{\mathbb{R}}$, if

$$|u(\gamma(0)) - u(\gamma(l))| \leq \int_\gamma g \, ds \tag{3}$$

for all rectifiable curves $\gamma : [0, l] \to X$. The concept of an upper gradient was introduced in [8]; also see [9]. Further, g as above is called a Φ-weak upper gradient if (3) holds for all curves γ except for a family of Φ-modulus zero, see Section 2.2 below. The Sobolev space $N^{1,\Phi}(X)$ consists of all functions in $L^\Phi(X)$ that have a (Φ-weak) upper gradient that belongs to $L^\Phi(X)$.

Theorem 1.2 *Suppose that Φ is a doubling Young function, $\Omega \subset X$ is open, $u, g \in L^\Phi(\Omega)$, and that the pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ-Poincaré inequality in Ω. Then a representative of u has a Φ-weak upper gradient g_u such that $\|g_u\|_{L^\Phi(\Omega)} \leq C(C_d, C_P, \tau)\|g\|_{L^\Phi(\Omega)}$.*

In the case $\Phi(t) = t^p$, $p \geq 1$, the result was essentially proven in [3], see [5].

In many important settings, including Riemannian manifolds with non-negative Ricci curvature and Carnot-Carathéodory spaces associated with a system of vector fields satisfying Hörmander's condition, the Φ-Poincaré inequality holds for pairs (u, g), where $u \in N^{1,\Phi}(X)$ and g is an upper gradient of u, see [6]. In these settings Theorem 1.2 gives a characterization for $N^{1,\Phi}(X)$.

If both Φ and its conjugate are doubling, then the assumptions of Theorem 1.2 can be relaxed. In order to conclude that a representative of $u \in L^\Phi(\Omega)$ is in $N^{1,\Phi}(\Omega)$, it suffices to assume that the number

$$\|u\|_{\mathcal{A}^{1,\Phi}_\tau(\Omega)} = \sup_{\mathcal{B} \in \mathcal{B}_\tau(\Omega)} \Big\| \sum_{B \in \mathcal{B}} \big(r_B^{-1} \fint_B |u - u_B| \, d\mu\big) \chi_B \Big\|_{L^\Phi(\Omega)}, \tag{4}$$

where

$$\mathcal{B}_\tau(\Omega) = \big\{\{B_i\} : \text{balls } \tau B_i \text{ are disjoint and contained in } \Omega\big\},$$

is finite. Notice that $\|u\|_{\mathcal{A}^{1,\Phi}_\tau(\Omega)} \leq \lambda$ if and only if there is a functional $\nu : \{B \subset \Omega : B \text{ is a ball}\} \to [0, \infty)$ such that

$$\sum_i \nu(B_i) \leq 1, \tag{5}$$

whenever the balls B_i are disjoint, and that the generalized Φ-Poincaré inequality

$$\fint_B |u - u_B|\, d\mu \le \lambda r_B \Phi^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right) \tag{6}$$

holds whenever $\tau B \subset \Omega$. In particular, if a pair $(u/\|g\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the Φ-Poincaré inequality in Ω, then

$$\|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)} \le C_P \|g\|_{L^\Phi(\Omega)}. \tag{7}$$

The spaces $\mathcal{A}_\tau^{1,\Phi}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) : \|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)} < \infty\}$, for $\Phi(t) = t^p$, were studied in [7]. Theorem 1.3 below is a generalization of [7, Theorem 1.1].

Theorem 1.3 *Let $\Omega \subset X$ be an open set and let Φ be a doubling Young function whose conjugate is doubling. Then a representative of $u \in \mathcal{A}_\tau^{1,\Phi}(\Omega) \cap L^\Phi(\Omega)$ has a Φ-weak upper gradient g with $\|g\|_{L^\Phi(\Omega)} \le C(C_d, \tau)\|u\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)}$.*

If the assumptions of Theorem 1.3 are in force and the space X supports the Φ-Poincaré inequality (that is, (1) holds for pairs (u, g), where $u \in N^{1,\Phi}(X)$ and g is an upper gradient of u), then $A_\tau^{1,\Phi}(\Omega) \cap L^\Phi(\Omega)$ is isomorphic to $N^{1,\Phi}(\Omega)$ and the norms $\|\cdot\|_{L^\Phi(\Omega)} + \|\cdot\|_{\mathcal{A}_\tau^{1,\Phi}(\Omega)}$ and $\|\cdot\|_{N^{1,\Phi}(\Omega)}$ are equivalent.

2 Preliminaries

Throughout this paper C will denote a positive constant whose value is not necessarily the same at each occurrence. By writing $C = C(\lambda_1, \ldots, \lambda_n)$ we indicate that the constant depends only on $\lambda_1, \ldots, \lambda_n$.

2.1 Young functions and Orlicz spaces

In this subsection we recall the basic facts about Young functions and Orlicz spaces. An exhaustive treatment of the subject is [11].

A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if it has the form

$$\Phi(t) = \int_0^t \phi(s)\, ds,$$

where $\phi : [0, \infty) \to [0, \infty]$ is an increasing, left-continuous function, which is neither identically zero nor identically infinite on $(0, \infty)$.

If, in addition, $\phi(0) = 0$, $0 < \phi(t) < \infty$ for $t > 0$ and $\lim_{t\to\infty} \phi(t) = \infty$, then Φ is called an N-function.

A Young function is convex and, in particular, satisfies

$$\Phi(\varepsilon t) \le \varepsilon \Phi(t) \tag{8}$$

for $0 < \varepsilon \le 1$ and $0 \le t < \infty$.

If Φ is a Young function and $\mu(X) < \infty$, then Jensen's inequality

$$\Phi\left(\fint_X u\,d\mu\right) \le \fint_X \Phi(u)\,d\mu \tag{9}$$

holds for $0 \le u \in L^1(X)$.

The right-continuous generalized inverse of a Young function Φ is

$$\Phi^{-1}(t) = \inf\{s : \Phi(s) > t\}.$$

We have that

$$\Phi(\Phi^{-1}(t)) \le t \le \Phi^{-1}(\Phi(t))$$

for $t \ge 0$.

The conjugate of a Young function Φ is the Young function defined by

$$\hat{\Phi}(t) = \sup\{ts - \Phi(s) : s > 0\}$$

for $t \ge 0$.

Let Φ be a Young function. The Orlicz space $L^\Phi(X)$ is the set of all measurable functions u for which there exists $\lambda > 0$ such that

$$\int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) < \infty.$$

The Luxemburg norm of $u \in L^\Phi(X)$ is

$$\|u\|_{L^\Phi(X)} = \inf\{\lambda > 0 : \int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) \le 1\}.$$

If $\|u\|_{L^\Phi(X)} \ne 0$, we have that

$$\int_X \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(X)}}\right) d\mu(x) \le 1.$$

The following generalized Hölder inequality holds for Luxemburg norms:

$$\int_X u(x)v(x)\,d\mu(x) \le 2\|u\|_{L^\Phi(X)}\|v\|_{L^{\hat{\Phi}}(X)}.$$

Let $E^\Phi(X)$ denote the closure of the space of bounded, boundedly supported functions in $L^\Phi(X)$.

Lemma 2.1 *Let Φ be an N-function.*

(a) The dual of $E^\Phi(X)$ is isomorphic to $L^{\hat{\Phi}}(X)$; For every $F \in (E^\Phi(X))^*$, there exists $v \in L^{\hat{\Phi}}(X)$ such that

$$F(u) = \int uv\,d\mu.$$

Moreover,

$$\|v\|_{L^{\hat{\Phi}}(X)} \le \|F\| \le 2\|v\|_{L^{\hat{\Phi}}(X)}.$$

(b) If $\Omega \subset \mathbb{R}^n$ is open, then $C_0^\infty(\Omega)$ is dense in $E^\Phi(\Omega)$.

A Young function Φ is doubling, if there exists a constant $C_\Phi \geq 1$ such that

$$\Phi(2t) \leq C_\Phi \Phi(t)$$

for $t \geq 0$.

Lemma 2.2 *Let Φ be doubling a Young function.*

1) The space $C_0(X)$ of bounded, boundedly supported continuous functions is dense in $L^\Phi(X)$.

2) The modular convergence and the norm convergence are equivalent, that is,

$$\|f_j - f\|_{L^\Phi(X)} \to 0,$$

if and only if

$$\int_X \Phi(|f_j - f|)\, d\mu \to 0.$$

Lemma 2.3 *Suppose that Φ is a doubling Young function and that $\{g_i\} \subset L^\Phi(X)$ satisfies*

$$\sup_i \|g_i\|_{L^\Phi(X)} < \infty$$

and

$$\lim_{\mu(A)\to 0} \sup_i \int_A \Phi(g_i)\, d\mu = 0.$$

Then there exists a subsequence (g_{i_j}) of (g_i) and $g \in L^\Phi(X)$ such that $g_{i_j} \to g$ weakly in $L^\Phi(X)$.

Lemma 2.3 easily follows from [11, p.144, Corollary 2].

If both Φ and $\hat{\Phi}$ are doubling, then $L^\Phi(X)$ is reflexive, and so every bounded sequence in $L^\Phi(X)$ admits a weakly converging subsequence.

2.2 Sobolev spaces on metric measure spaces

The Φ-modulus of a curve family Γ is

$$\mathrm{Mod}_\Phi(\Gamma) = \inf\Big\{\|g\|_{L^\Phi(X)} : \int_\gamma g\, ds \geq 1 \text{ for all } \gamma \in \Gamma\Big\}. \tag{10}$$

The Sobolev space $N^{1,\Phi}(X)$, defined by Tuominen in [12], consists of the functions $u \in L^\Phi(X)$ having a Φ-weak upper gradient $g \in L^\Phi(X)$. The space $N^{1,\Phi}(X)$ is a Banach space with the norm

$$\|u\|_{N^{1,\Phi}(X)} = \|u\|_{L^\Phi(X)} + \inf \|g\|_{L^\Phi(X)},$$

where the infimum is taken over Φ-weak upper gradients $g \in L^\Phi(X)$ of u.

We need the following lemma from [12].

Lemma 2.4 ([12], Theorem 4.17) *Suppose that $u_i \to u \in L^\Phi(X)$ and $g_i \to g \in L^\Phi(X)$ weakly in $L^\Phi(X)$ and that g_i is a Φ-weak upper gradient of u_i. Then g is a Φ-weak upper gradient of a representative of u.*

If Φ is doubling and $\Omega \subset \mathbb{R}^n$ is an open set, then $N^{1,\Phi}(\Omega)$ is isomorphic to $W^{1,\Phi}(\Omega)$ [12, Theorem 6.19]. As usual, $W^{1,\Phi}(\Omega)$ is the space of functions $u \in L^\Phi(\Omega)$ having weak partial derivatives in $L^\Phi(\Omega)$. A function $\partial u/\partial x_i \in L^1_{\mathrm{loc}}(\Omega)$ is a weak partial derivative of u (w.r.t. x_i) if

$$\int u \frac{\partial \varphi}{\partial x_i} = -\int \frac{\partial u}{\partial x_i} \varphi$$

for all $\varphi \in C_0^\infty(\Omega)$.

2.3 Lipschitz functions

A function $u : X \to \mathbb{R}$ is L-Lipschitz if $|u(x) - u(y)| \le L\,\mathrm{d}(x,y)$ for all $x, y \in X$. The lower and upper pointwise Lipschitz constants of a locally Lipschitz function u are

$$\operatorname{lip} u(x) = \liminf_{r \to 0} \frac{L(u,x,r)}{r} \quad \text{and} \quad \operatorname{Lip} u(x) = \limsup_{r \to 0} \frac{L(u,x,r)}{r},$$

where

$$L(u,x,r) = \sup_{\mathrm{d}(x,y) \le r} |u(x) - u(y)|.$$

The lower Lipschitz constant $\operatorname{lip} u$, and hence also $\operatorname{Lip} u$, is an upper gradient of a locally Lipschitz function u (cf. [1]).

3 Proofs

Proof of Theorem 1.1 We may assume that $\|g\|_{L^\Phi(\Omega)} = 1$. By Lemma 2.1, it suffices to show that the functional $\frac{\partial u}{\partial x_i} : C_0^\infty(\Omega) \to \mathbb{R}$;

$$\frac{\partial u}{\partial x_i}[\varphi] := -\int u \frac{\partial \varphi}{\partial x_i}$$

is bounded with respect to the norm $\|\cdot\|_{L^{\hat\Phi}(\Omega)}$ and satisfies $\|\frac{\partial u}{\partial x_i}\| \le C$. Choose $0 \le \psi \in C_0^\infty(B(0,1))$ such that $\int \psi = 1$ and let $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$ for $\varepsilon > 0$. Then

$$\frac{\partial u}{\partial x_i}[\varphi] = -\lim_{\varepsilon \to 0} \int (u * \psi_\varepsilon) \frac{\partial \varphi}{\partial x_i} = \lim_{\varepsilon \to 0} \int \left(u * \frac{\partial \psi_\varepsilon}{\partial x_i} \right) \varphi.$$

By the Hölder inequality,

$$\left| \frac{\partial u}{\partial x_i}[\varphi] \right| \le 2 \liminf_{\varepsilon \to 0} \left\| u * \frac{\partial \psi_\varepsilon}{\partial x_i} \right\|_{L^\Phi(\mathrm{supp}\ \varphi)} \|\varphi\|_{L^{\hat\Phi}(\mathrm{supp}\ \varphi)}.$$

Since $\int \frac{\partial \psi_\varepsilon}{\partial x_i} = 0$, we have that

$$\left(u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right)(x) = \left((u - u_{B(x,\varepsilon)}) * \frac{\partial \psi_\varepsilon}{\partial x_i}\right)(x).$$

Thus

$$\left|u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right|(x) \leq C\varepsilon^{-n-1} \int_{B(x,\varepsilon)} |u(y) - u_{B(x,\varepsilon)}|\, dy \leq C\Phi^{-1}\left(\fint_{B(x,\tau\varepsilon)} \Phi(g(y))\, dy\right).$$

Let $K = \operatorname{supp} \varphi$ and let $\varepsilon > 0$ be such that $K_{\tau\varepsilon} = \{x \in \mathbb{R}^n : d(x,K) < \tau\varepsilon\} \subset \Omega$. Then, by Fubini's theorem,

$$\begin{aligned}
\int_K \Phi\left(C^{-1}\left|u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right|(x)\, dx\right) &\leq \int_K \fint_{B(x,\tau\varepsilon)} \Phi(g(y))\, dy\, dx \\
&= \int_{K_{\tau\varepsilon}} \Phi(g(y)) \int_{B(y,\tau\varepsilon)\cap K} |B(x,\tau\varepsilon)|^{-1}\, dx\, dy. \\
&\leq \int_{K_{\tau\varepsilon}} \Phi(g(y))\, dy \\
&\leq 1.
\end{aligned}$$

Thus

$$\liminf_{\varepsilon \to 0} \left\|u * \frac{\partial \psi_\varepsilon}{\partial x_i}\right\|_{L^\Phi(\operatorname{supp} \varphi)} \leq C,$$

which completes the proof. □

For the proofs of Theorem 1.2 and Theorem 1.3, which are based on approximation by discrete convolutions, we need a couple of lemmas. Lemma 3.1 follows from a Whitney type covering result for doubling metric measure spaces, see [2, Theorem III.1.3], [10, Lemma 2.9]. For the proof of Lemma 3.2, we refer to [10, Lemma 2.16].

Lemma 3.1 *Let $\Omega \subset X$ be open. Given $\varepsilon > 0$, $\lambda \geq 1$, there is a cover $\{B_i = B(x_i, r_i)\}$ of Ω with the following properties:*

(1) $r_i \leq \varepsilon$ for all i,

(2) $\lambda B_i \subset \Omega$ for all i,

(3) if λB_i meets λB_j, then $r_i \leq 2r_j$,

(4) each ball λB_i meets at most $C = C(C_d, \lambda)$ balls λB_j.

A collection $\{B_i\}$ as above is called an (ε, λ)-covering of Ω. Clearly, an (ε, λ)-cover is an (ε', λ')-cover provided $\varepsilon' \geq \varepsilon$ and $\lambda' \leq \lambda$.

Lemma 3.2 *Let $\Omega \subset X$ be open, and let $\mathcal{B} = \{B_i = B(x_i, r_i)\}$ be an $(\infty, 2)$-cover of Ω. Then there is a collection $\{\varphi_i\}$ of functions $\Omega \to \mathbb{R}$ such that*

1) each φ_i is $C(C_d)r_i^{-1}$-Lipschitz.

2) $0 \leq \varphi_i \leq 1$ for all i, *3)* $\varphi_i(x) = 0$ *for* $x \in X \setminus 2B_i$ *for all* i*,*

4) $\sum_i \varphi_i(x) = 1$ *for all* $x \in \Omega$*.*

A collection $\{\varphi_i\}$ *as above is called a partition of unity with respect to* $\mathcal{B}$*.*

Let $\mathcal{B} = \{B_i\}$ be as in the lemma above, and let $\{\varphi_i\}$ be a partition of unity with respect to $\mathcal{B}$. For a locally integrable function u on Ω, define

$$u_{\mathcal{B}}(x) = \sum_i u_{B_i} \varphi_i(x). \tag{11}$$

The following lemma describes the most important properties of $u_{\mathcal{B}}$.

Lemma 3.3

1) The function $u_{\mathcal{B}}$ *is locally Lipschitz. Moreover, for each* $x \in B_i$*,*

$$\operatorname{Lip} u_{\mathcal{B}}(x) \leq C(C_d) r_{B_i}^{-1} \fint_{5B_i} |u - u_{5B_i}| \, d\mu.$$

2) Let Φ *be a doubling Young function and let* $u \in L^\Phi(\Omega)$*. If* $\mathcal{B}_k$ *is an* $(\varepsilon_k, 2)$*-cover of* Ω *and* $\varepsilon_k \to 0$ *as* $k \to \infty$*, then* $u_{\mathcal{B}_k} \to u$ *in* $L^\Phi(\Omega)$*.*

Proof 1) Let $x, y \in B_i$, and let $J = \{j : 2B_j \cap 2B_i \neq \emptyset\}$. Then $\# J \leq C(C_d)$ and $B_j \subset 5B_i$ for each $j \in J$. Using the properties of the functions φ_i, we have that

$$\begin{aligned}
|u_{\mathcal{B}}(x) - u_{\mathcal{B}}(y)| &= \Big| \sum_{j \in J} (u_{B_j} - u_{B_i}) \big(\varphi_j(x) - \varphi_j(y) \big) \Big| \\
&\leq C(C_d) r_{B_i}^{-1} \, \mathrm{d}(x,y) \max_{j \in J} |u_{B_j} - u_{B_i}| \\
&\leq C(C_d) r_{B_i}^{-1} \, \mathrm{d}(x,y) \fint_{5B_i} |u - u_{5B_i}| \, d\mu,
\end{aligned}$$

and the first claim follows.

2) We begin by showing that, for every $w \in L^\Phi(\Omega)$,

$$\|w_{\mathcal{B}}\|_{L^\Phi(\Omega)} \leq C(C_d) \|w\|_{L^\Phi(\Omega)}. \tag{12}$$

We may assume that $\|w\|_{L^\Phi(\Omega)} = 1$. By Jensen's inequality $\Phi(|w_{\mathcal{B}}|) \leq (\Phi(|w|))_{\mathcal{B}}$. Hence, by the properties of the functions φ_i,

$$\begin{aligned}
\int_\Omega \Phi(|w_{\mathcal{B}}|) \, d\mu &\leq \int_\Omega (\Phi(|w|))_{\mathcal{B}} \, d\mu \leq \sum_i \int_\Omega (\Phi(|w|))_{B_i} \varphi_i \, d\mu \\
&\leq \sum_i \int_{2B_i} \Phi(|w|)_{B_i} \, d\mu \leq C_d \sum_i \int_{B_i} \Phi(|w|) \, d\mu \\
&= C_d \int_\Omega \Phi(|w|) \sum_i \chi_{B_i} \, d\mu \leq C(C_d) \int_\Omega \Phi(|w|) \, d\mu \\
&\leq C(C_d).
\end{aligned}$$

Thus, by (8), we obtain (12).

Let $u \in L^\Phi(\Omega)$ and $\varepsilon > 0$. By Lemma 2.2 (1), there exists $v \in C_0(\Omega)$ such that $\|u-v\|_{L^\Phi(\Omega)} < \varepsilon$. Then, by (12), we obtain

$$\|u_{\mathcal{B}} - v_{\mathcal{B}}\|_{L^\Phi(\Omega)} = \|(u-v)_{\mathcal{B}}\|_{L^\Phi(\Omega)} \le C(C_d)\|u-v\|_{L^\Phi(\Omega)} < C(C_d)\varepsilon,$$

and so

$$\begin{aligned}\|u_{\mathcal{B}} - u\|_{L^\Phi(\Omega)} &\le \|u_{\mathcal{B}} - v_{\mathcal{B}}\|_{L^\Phi(\Omega)} + \|v_{\mathcal{B}} - v\|_{L^\Phi(\Omega)} + \|v-u\|_{L^\Phi(\Omega)} \\ &< \|v_{\mathcal{B}} - v\|_{L^\Phi(\Omega)} + C(C_d)\varepsilon.\end{aligned}$$

Therefore it suffices to show that $\|v_{\mathcal{B}_k} - v\|_{L^\Phi(\Omega)} \to 0$ as $\varepsilon_k \to 0$. Now $|v_{\mathcal{B}_k} - v| \le 2\sup|v|$, and for all x we have that

$$|v_{\mathcal{B}_k}(x)-v(x)| \le \sum_{2B_i \ni x} \fint_{B_i} |v(y)-v(x)|\,d\mu(y) \le C(C_d)\fint_{B(x,5\varepsilon_k)} |v(y)-v(x)|\,d\mu(y),$$

which converges to 0 as $\varepsilon_k \to 0$ by the continuity of v. Thus, by the dominated convergence theorem,

$$\int_\Omega \Phi(|v_{\mathcal{B}_k} - v|)\,d\mu \to 0,$$

and so, by Lemma 2.2 (2), $\|v_{\mathcal{B}_k} - v\|_{L^\Phi(\Omega)} \to 0$. □

Proof of Theorem 1.3. Let $u \in \mathcal{A}^{1,\Phi}_\tau(\Omega) \cap L^\Phi(\Omega)$. For $j \in \mathbb{N}$, let $\mathcal{B}_j$ be a $(j^{-1}, 5\tau)$-cover (and hence also a $(j^{-1}, 2)$-cover) of Ω. Then, by Lemma 3.3 (2), $u_j := u_{\mathcal{B}_j} \to u$ in $L^\Phi(\Omega)$. Let us show that

$$\|\operatorname{Lip} u_j\|_{L^\Phi(\Omega)} \le C(C_d,\tau)\|u\|_{\mathcal{A}^{1,\Phi}_\tau(\Omega)}. \tag{13}$$

By Lemma 3.3 (1),

$$\operatorname{Lip} u_j \le C(C_d) \sum_{B \in \mathcal{B}_j} r_B^{-1} \fint_{5B} |u - u_{5B}|\,d\mu\,\chi_B.$$

It follows from Lemma 3.1 (4) that $\mathcal{B}_j$ can be divided into $k = C(C_d,\tau)$ subfamilies $\mathcal{B}_{j,1},\ldots,\mathcal{B}_{j,k}$ so that each of the families $5\tau\mathcal{B}_{j,l}$ consists of disjoint balls. Since the families $5\mathcal{B}_{j,1},\ldots,5\mathcal{B}_{j,k}$ belong to $\mathcal{B}_\tau(\Omega)$, we have that

$$\begin{aligned}\|\operatorname{Lip} u_j\|_{L^\Phi(\Omega)} &\le C(C_d)\sum_{l=1}^k \Big\| \sum_{B \in \mathcal{B}_{j,l}} r_B^{-1}\fint_{5B} |u-u_{5B}|\,d\mu\,\chi_B \Big\|_{L^\Phi(\Omega)} \\ &\le C(C_d)\sum_{l=1}^k \Big\| \sum_{B \in 5\mathcal{B}_{j,l}} r_B^{-1}\fint_{B} |u-u_{B}|\,d\mu\,\chi_B \Big\|_{L^\Phi(\Omega)} \\ &\le C(C_d,\tau)\|u\|_{\mathcal{A}^{1,\Phi}_\tau(\Omega)}.\end{aligned}$$

Since Φ and $\hat{\Phi}$ are doubling, $L^\Phi(\Omega)$ is reflexive. Thus the bounded sequence $(\operatorname{Lip} u_j)$ has a subsequence, also denoted by $(\operatorname{Lip} u_j)$, that converges weakly to some $g \in L^{\Phi}(\Omega)$. By Lemma 2.4, g is a Φ-weak upper gradient of a representative of u. As a weak limit g satisfies

$$\|g\|_{L^{\Phi}(\Omega)} \leq \liminf_{j\to\infty} \| \operatorname{Lip} u_j\|_{L^{\Phi}(\Omega)} \leq C(C_d,\tau)\|u\|_{\mathcal{A}^{1,\Phi}_{\tau}(\Omega)}.$$

Proof of Theorem 1.2 We may assume that $\|g\|_{L^{\Phi}(\Omega)} = 1$. Define the functions u_j as in the proof of Theorem 1.3. By (13) and (7), we have that

$$\|\operatorname{Lip} u_j\|_{L^{\Phi}(\Omega)} \leq C(C_d, C_P, \tau).$$

Let us show that

$$\lim_{\mu(E)\to 0} \sup_j \int_E \Phi(\operatorname{Lip} u_j)\, d\mu = 0. \tag{14}$$

By Lemma 3.3 (1) and by the Φ-Poincaré inequality,

$$\begin{aligned}\operatorname{Lip} u_j &\leq C(C_d) \sum_{B\in\mathcal{B}_j} r_B^{-1} \fint_{5B} |u-u_{5B}|\, d\mu \chi_B \\ &\leq C(C_d, C_P) \sum_{B\in\mathcal{B}_j} \Phi^{-1}\left(\fint_{5\tau B} \Phi(g)\, d\mu\right) \chi_B.\end{aligned}$$

Thus

$$\int_E \Phi(\operatorname{Lip} u_j)\, d\mu \leq C(C_d, C_P, C_\Phi) \sum_{B\in\mathcal{B}_j} \frac{\mu(E\cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g)\, d\mu.$$

Since B_j can be divided into $k = C(C_d,\tau)$ subfamilies $\mathcal{B}_{j,1},\ldots,\mathcal{B}_{j,k}$ so that each of the families $5\tau\mathcal{B}_{j,l}$ consists of disjoint balls, it suffices to show that, for $1 \leq l \leq k$,

$$\lim_{\mu(E)\to 0} \sum_{B\in\mathcal{B}_{j,l}} \frac{\mu(E\cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g)\, d\mu = 0.$$

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\int_A \Phi(g) < \varepsilon$ whenever $\mu(A) < \delta$. Denote by $\mathcal{B}$ the family of those balls B in $\mathcal{B}_{j,l}$ for which

$$\frac{\mu(E\cap B)}{\mu(5\tau B)} < \varepsilon.$$

Also, let $\mathcal{B}' = \mathcal{B}_{j,l} \setminus \mathcal{B}$. Now, if $\mu(E) < \varepsilon\delta$, we have that $\mu(\cup_{B\in\mathcal{B}'} 5\tau B) \leq \varepsilon^{-1}\mu(E) < \delta$. Thus

$$\begin{aligned}&\sum_{B\in\mathcal{B}_{j,l}} \frac{\mu(E\cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g)\, d\mu \\ &= \sum_{B\in\mathcal{B}} \frac{\mu(E\cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g)\, d\mu + \sum_{B\in\mathcal{B}'} \frac{\mu(E\cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g)\, d\mu \\ &\leq \varepsilon \int_\Omega \Phi(g)\, d\mu + \int_{\cup_{B\in\mathcal{B}'} 5\tau B} \Phi(g)\, d\mu \\ &\leq 2\varepsilon.\end{aligned}$$

This completes the proof of (14).

By Lemma 2.3, a subsequence of $(\operatorname{Lip} u_j)$ converges weakly to some $g_u \in L^{\Phi}(\Omega)$, which, by Lemma 2.4, is a Φ-weak upper gradient of a representative of u. Moreover, as a weak limit, g_u satisfies

$$\|g_u\|_{L^{\Phi}(\Omega)} \leq \liminf_{j\to\infty} \|\operatorname{Lip} u_j\|_{L^{\Phi}(\Omega)} \leq C(C_d, C_P, \tau).$$

□

References

[1] J. Cheeger: Differentiability of Lipschitz functions on metric measure spaces - Geom. Funct. Anal. 9 (1999), no.3, 428–517.

[2] R. R. Coifman, G. Weiss: Analyse harmonique non-commutative sur certains espaces homogènes - Lecture Notes in Mathematics, Vol.242. Springer-Verlag, Berlin-New York, 1971.

[3] B. Franchi, P. Hajłasz, and P. Koskela: Definitions of Sobolev classes on metric spaces. - Ann. Inst. Fourier (Grenoble), 49 (1999), no.6, 1903–1924.

[4] P. Hajłasz: A new characterization of the Sobolev space - Studia Math. 159 (2003), no.2, 263–275.

[5] P. Hajłasz: Sobolev spaces on metric-measure spaces. (Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)), 173–218, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003.

[6] P. Hajłasz, P. Koskela: Sobolev met Poincaré - Mem. Amer. Math. Soc. 145 (2000), no.688.

[7] T. Heikkinen, P. Koskela, H. Tuominen: Sobolev-type spaces from generalized Poincaré inequalities, preprint.

[8] J. Heinonen, P. Koskela: Quasiconformal maps on metric spaces with controlled geometry - Acta Math. 181 (1998), 1–61.

[9] P. Koskela, P. MacManus: Quasiconformal mappings and Sobolev spaces - Studia Math. 131 (1998), 1–17.

[10] R. A. Macías, C. Segovia: A decomposition into atoms of distributions on spaces of homogeneous type - Adv. in Math. 33 (1979), no.3, 271–309.

[11] M.M. Rao, Z.D. Ren: Theory of Orlicz spaces - Monographs and Textbooks in Pure and Applied Mathematics 146, Marcel Dekker, Inc., New York, 1991.

[12] H. Tuominen: Orlicz-Sobolev spaces on metric measure spaces. Ann. Acad. Sci. Fenn. Math. Diss. No. 135 (2004).

[13] H. Tuominen: Characterization of Orlicz-Sobolev space, to appear in Arkiv för Matematik.

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