Characterizations of Orlicz-Sobolev spaces in terms of generalized Orlicz-Poincaré inequalities

Toni Heikkinen

Abstract

We show that the Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^n)$ can be characterized in terms of the (generalized) $\Phi$-Poincaré inequality. We also prove similar results in the general metric space setting.

Mathematics Subject Classification (2000): 46E35

1 Introduction

Let $\Phi$ be a Young function and let $\Omega \subset \mathbb{R}^n$ be open. A pair $(u, g)$ of measurable functions, $u \in L^{1}_{\text{loc}}(\Omega)$ and $g \geq 0$, satisfies the $\Phi$-Poincaré inequality in $\Omega$, if there are constants $C_{P} \geq 1$ and $\tau \geq 1$ such that

$$\int_{B} |u-u_{B}| \, d\mu \leq C_{P} r_{B} \Phi^{-1}\left(\int_{\tau B} \Phi(g) \, d\mu\right)$$

for every ball $B = B(x, r_{B})$ such that $\tau B \subset \Omega$. Here, $u_{B} = \frac{1}{\mu(B)} \int_{B} u \, d\mu$ and $\tau B = B(x, \tau r_{B})$. It is well known that $u \in W^{1,1}_{\text{loc}}(\Omega)$ satisfies the 1-Poincaré inequality

$$\int_{B} |u-u_{B}| \, d\mu \leq C_{P} r_{B} \int_{B} |\nabla u| \, d\mu$$

for every ball $B \subset \Omega$. Thus, by Jensen’s inequality, (1) holds with $\tau = 1$ and $g = |\nabla u|$. Our first result says that also the converse holds: If $u \in L^{\Phi}(\Omega)$ and there exists $g \in L^{\Phi}(\Omega)$ such that (1) holds (for the normalized pair), then $u$ belongs to the Sobolev class $W^{1,\Phi}(\Omega)$.

Theorem 1.1 Suppose that $\Phi$ is an $N$-function, $\Omega \subset \mathbb{R}^n$ is open, $u, g \in L^{\Phi}(\Omega)$ and that the pair $(u/\|g\|_{L^{\Phi}(\Omega)}, g/\|g\|_{L^{\Phi}(\Omega)})$ satisfies the $\Phi$-Poincaré inequality in $\Omega$. Then $u \in W^{1,\Phi}(\Omega)$ and $\|\nabla u\|_{L^{\Phi}(\Omega)} \leq C(C_{P}, \tau, n)\|g\|_{L^{\Phi}(\Omega)}$.

*The author was supported by Vilho, Yrjö and Kalle Väisälä Foundation
For the definitions of Young and $N$-functions, see Section 2.1 below. Theorem 1.1 was proven by Hajłasz in [4] for $\Phi(t) = t^p$, $p \geq 1$, and by Tuominen in [13] for a doubling $\Phi$ whose conjugate is also doubling.

Our second result is a counterpart of Theorem 1.1 in the general metric setting. Let $X = (X, d, \mu)$ be a metric measure space with $\mu$ a Borel regular outer measure satisfying $0 < \mu(U) < \infty$, whenever $U$ is nonempty, open and bounded. Suppose further that $\mu$ is doubling, that is, there exists a constant $C_d$ such that

$$\mu(2B) \leq C_d \mu(B),$$

whenever $B$ is a ball.

Our substitute for the usual Sobolev class $W^{1,\Phi}$ is based on upper gradients. We call a Borel function $g : X \to [0, \infty]$ an upper gradient of a function $u : X \to \mathbb{R}$, if

$$|u(\gamma(0)) - u(\gamma(l))| \leq \int_{\gamma} g \, ds$$

for all rectifiable curves $\gamma : [0, l] \to X$. The concept of an upper gradient was introduced in [8]; also see [9]. Further, $g$ as above is called a $\Phi$-weak upper gradient if (3) holds for all curves $\gamma$ except for a family of $\Phi$-modulus zero, see Section 2.2 below. The Sobolev space $N^{1,\Phi}(X)$ consists of all functions in $L^{\Phi}(X)$ that have a ($\Phi$-weak) upper gradient that belongs to $L^{\Phi}(X)$.

**Theorem 1.2** Suppose that $\Phi$ is a doubling Young function, $\Omega \subset X$ is open, $u, g \in L^{\Phi}(\Omega)$, and that the pair $(u/\|g\|_{L^{\Phi}(\Omega)}, g/\|g\|_{L^{\Phi}(\Omega)})$ satisfies the $\Phi$-Poincaré inequality in $\Omega$. Then a representative of $u$ has a $\Phi$-weak upper gradient $g_u$ such that $\|g_u\|_{L^{\Phi}(\Omega)} \leq C(C_d, C_P, \tau)\|g\|_{L^{\Phi}(\Omega)}$.

In the case $\Phi(t) = t^p$, $p \geq 1$, the result was essentially proven in [3], see [5].

In many important settings, including Riemannian manifolds with non-negative Ricci curvature and Carnot-Carathéodory spaces associated with a system of vector fields satisfying Hörmander’s condition, the $\Phi$-Poincaré inequality holds for pairs $(u, g)$, where $u \in N^{1,\Phi}(X)$ and $g$ is an upper gradient of $u$, see [6]. In these settings Theorem 1.2 gives a characterization for $N^{1,\Phi}(X)$.

If both $\Phi$ and its conjugate are doubling, then the assumptions of Theorem 1.2 can be relaxed. In order to conclude that a representative of $u \in L^{\Phi}(\Omega)$ is in $N^{1,\Phi}(\Omega)$, it suffices to assume that the number

$$\|u\|_{A^{1,\Phi}_\tau(\Omega)} = \sup_{B \in \mathcal{B}_\tau(\Omega)} \| \sum_{B \in \mathcal{B}} (\tau_B^{-1}) \int_B |u - u_B| \, d\mu \|_{L^{\Phi}(\Omega)},$$

where

$$\mathcal{B}_\tau(\Omega) = \{ \{B_i\} : \text{balls } \tau B_i \text{ are disjoint and contained in } \Omega \},$$

is finite. Notice that $\|u\|_{A^{1,\Phi}_\tau(\Omega)} \leq \lambda$ if and only if there is a functional $\nu : \{ B \subset \Omega : B \text{ is a ball} \} \to [0, \infty)$ such that

$$\sum_i \nu(B_i) \leq 1,$$
whenever the balls $B_i$ are disjoint, and that the generalized $\Phi$-Poincaré inequality
\[
\int_B |u - u_B| \, d\mu \leq \lambda r_B \Phi^{-1} \left( \frac{\nu(\tau B)}{\mu(B)} \right)
\]
holds whenever $\tau B \subset \Omega$. In particular, if a pair $(u/\|u\|_{L^\Phi(\Omega)}, g/\|g\|_{L^\Phi(\Omega)})$ satisfies the $\Phi$-Poincaré inequality in $\Omega$, then
\[
\|u\|_{A^1,\Phi(\tau(\Omega))} \leq C_{P} \|g\|_{L^\Phi(\Omega)}.
\]
(6)

The spaces $A^1,\Phi(\Omega) = \{u \in L^1_{\text{loc}}(\Omega) : \|u\|_{A^1,\Phi(\Omega)} < \infty\}$, for $\Phi(t) = t^p$, were studied in [7]. Theorem 1.3 below is a generalization of [7, Theorem 1.1].

Theorem 1.3 Let $\Omega \subset X$ be an open set and let $\Phi$ be a doubling Young function whose conjugate is doubling. Then a representative of $u \in A^1,\Phi(\Omega) \cap L^\Phi(\Omega)$ has a $\Phi$-weak upper gradient $g$ with $\|g\|_{L^\Phi(\Omega)} \leq C(C_d, \tau)\|u\|_{A^1,\Phi(\Omega)}$.

If the assumptions of Theorem 1.3 are in force and the space $X$ supports the $\Phi$-Poincaré inequality (that is, (1) holds for pairs $(u, g)$, where $u \in N^{1,\Phi}(X)$ and $g$ is an upper gradient of $u$), then $A^1,\Phi(\Omega) \cap L^\Phi(\Omega)$ is isomorphic to $N^{1,\Phi}(\Omega)$ and the norms $\| \cdot \|_{L^\Phi(\Omega)} + \| \cdot \|_{A^1,\Phi(\Omega)}$ and $\| \cdot \|_{N^{1,\Phi}(\Omega)}$ are equivalent.

2 Preliminaries

Throughout this paper $C$ will denote a positive constant whose value is not necessarily the same at each occurrence. By writing $C = C(\lambda_1, \ldots, \lambda_n)$ we indicate that the constant depends only on $\lambda_1, \ldots, \lambda_n$.

2.1 Young functions and Orlicz spaces

In this subsection we recall the basic facts about Young functions and Orlicz spaces. An exhaustive treatment of the subject is [11].

A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if it has the form
\[
\Phi(t) = \int_0^t \phi(s) \, ds,
\]
where $\phi : [0, \infty) \to [0, \infty]$ is an increasing, left-continuous function, which is neither identically zero nor identically infinite on $(0, \infty)$.

If, in addition, $\phi(0) = 0$, $0 < \phi(t) < \infty$ for $t > 0$ and $\lim_{t \to \infty} \phi(t) = \infty$, then $\Phi$ is called an $N$-function.

A Young function is convex and, in particular, satisfies
\[
\Phi(\varepsilon t) \leq \varepsilon \Phi(t)
\]
for $0 < \varepsilon \leq 1$ and $0 \leq t < \infty$. 

3
If $\Phi$ is a Young function and $\mu(X) < \infty$, then Jensen’s inequality

$$\Phi\left(\int_X u \, d\mu\right) \leq \int_X \Phi(u) \, d\mu$$

holds for $0 \leq u \in L^1(X)$.

The right-continuous generalized inverse of a Young function $\Phi$ is

$$\Phi^{-1}(t) = \inf\{s : \Phi(s) > t\}.$$

We have that

$$\Phi(\Phi^{-1}(t)) \leq t \leq \Phi^{-1}(\Phi(t))$$

for $t \geq 0$.

The conjugate of a Young function $\Phi$ is the Young function defined by

$$\hat{\Phi}(t) = \sup\{ts - \Phi(s) : s > 0\}$$

for $t \geq 0$.

Let $\Phi$ be a Young function. The Orlicz space $L^\Phi(X)$ is the set of all measurable functions $u$ for which there exists $\lambda > 0$ such that

$$\int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) \, d\mu(x) < \infty.$$

The Luxemburg norm of $u \in L^\Phi(X)$ is

$$\|u\|_{L^\Phi(X)} = \inf\{\lambda > 0 : \int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) \, d\mu(x) \leq 1\}.$$ 

If $\|u\|_{L^\Phi(X)} \neq 0$, we have that

$$\int_X \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(X)}}\right) \, d\mu(x) \leq 1.$$

The following generalized Hölder inequality holds for Luxemburg norms:

$$\int_X u(x)v(x) \, d\mu(x) \leq 2\|u\|_{L^\Phi(X)}\|v\|_{L^\hat{\Phi}(X)}.$$

Let $E^\Phi(X)$ denote the closure of the space of bounded, boundedly supported functions in $L^\Phi(X)$.

**Lemma 2.1** Let $\Phi$ be an $N$-function.

(a) The dual of $E^\Phi(X)$ is isomorphic to $L^\hat{\Phi}(X)$; For every $F \in (E^\Phi(X))^*$, there exists $v \in L^\hat{\Phi}(X)$ such that

$$F(u) = \int uv \, d\mu.$$

Moreover,

$$\|v\|_{L^\Phi(X)} \leq \|F\| \leq 2\|v\|_{L^\Phi(X)}.$$
(b) If $\Omega \subset \mathbb{R}^n$ is open, then $C_0^\infty(\Omega)$ is dense in $E^\Phi(\Omega)$.

A Young function $\Phi$ is doubling, if there exists a constant $C_\Phi \geq 1$ such that

$$\Phi(2t) \leq C_\Phi \Phi(t)$$

for $t \geq 0$.

Lemma 2.2  Let $\Phi$ be doubling a Young function.

1) The space $C_0^\infty(X)$ of bounded, boundedly supported continuous functions is dense in $L^\Phi(X)$.

2) The modular convergence and the norm convergence are equivalent, that is,

$$\|f_j - f\|_{L^\Phi(X)} \to 0,$$

if and only if

$$\int_X \Phi(|f_j - f|) \, d\mu \to 0.$$

Lemma 2.3  Suppose that $\Phi$ is a doubling Young function and that $\{g_i\} \subset L^\Phi(X)$ satisfies

$$\sup_i \|g_i\|_{L^\Phi(X)} < \infty$$

and

$$\lim_{\mu(A) \to 0} \sup_i \int_A \Phi(g_i) \, d\mu = 0.$$

Then there exists a subsequence $(g_{ij})$ of $(g_i)$ and $g \in L^\Phi(X)$ such that $g_{ij} \to g$ weakly in $L^\Phi(X)$.

Lemma 2.3 easily follows from [11, p.144, Corollary 2].

If both $\Phi$ and $\hat{\Phi}$ are doubling, then $L^\Phi(X)$ is reflexive, and so every bounded sequence in $L^\Phi(X)$ admits a weakly converging subsequence.

2.2 Sobolev spaces on metric measure spaces

The $\Phi$-modulus of a curve family $\Gamma$ is

$$\text{Mod}_\Phi(\Gamma) = \inf \left\{ \|g\|_{L^\Phi(X)} : \int_\gamma g \, ds \geq 1 \text{ for all } \gamma \in \Gamma \right\}. \tag{10}$$

The Sobolev space $N^{1,\Phi}(X)$, defined by Tuominen in [12], consists of the functions $u \in L^\Phi(X)$ having a $\Phi$-weak upper gradient $g \in L^\Phi(X)$. The space $N^{1,\Phi}(X)$ is a Banach space with the norm

$$\|u\|_{N^{1,\Phi}(X)} = \|u\|_{L^\Phi(X)} + \inf \|g\|_{L^\Phi(X)},$$

where the infimum is taken over $\Phi$-weak upper gradients $g \in L^\Phi(X)$ of $u$.

We need the following lemma from [12].
Lemma 2.4 ([12], Theorem 4.17) Suppose that \( u_i \to u \in L^\Phi(X) \) and \( g_i \to g \in L^\Phi(X) \) weakly in \( L^\Phi(X) \) and that \( g_i \) is a \( \Phi \)-weak upper gradient of \( u_i \). Then \( g \) is a \( \Phi \)-weak upper gradient of a representative of \( u \).

If \( \Phi \) is doubling and \( \Omega \subset \mathbb{R}^n \) is an open set, then \( N^{1,\Phi}(\Omega) \) is isomorphic to \( W^{1,\Phi}(\Omega) \) [12, Theorem 6.19]. As usual, \( W^{1,\Phi}(\Omega) \) is the space of functions \( u \in L^\Phi(\Omega) \) having weak partial derivatives in \( L^\Phi(\Omega) \). A function \( \partial u / \partial x_i \in L^1_{\text{loc}}(\Omega) \) is a weak partial derivative of \( u \) (w.r.t. \( x_i \)) if
\[
\int u \frac{\partial \varphi}{\partial x_i} = -\int \frac{\partial u}{\partial x_i} \varphi
\]
for all \( \varphi \in C_0^\infty(\Omega) \).

### 2.3 Lipschitz functions

A function \( u : X \to \mathbb{R} \) is \( L \)-Lipschitz if \( |u(x) - u(y)| \leq L d(x, y) \) for all \( x, y \in X \). The lower and upper pointwise Lipschitz constants of a locally Lipschitz function \( u \) are
\[
\text{lip } u(x) = \liminf_{r \to 0} \frac{L(u, x, r)}{r} \quad \text{and} \quad \text{Lip } u(x) = \limsup_{r \to 0} \frac{L(u, x, r)}{r},
\]
where
\[
L(u, x, r) = \sup_{d(x, y) \leq r} |u(x) - u(y)|.
\]
The lower Lipschitz constant \( \text{lip } u \), and hence also \( \text{Lip } u \), is an upper gradient of a locally Lipschitz function \( u \) (cf. [1]).

### 3 Proofs

**Proof of Theorem 1.1** We may assume that \( \|g\|_{L^\Phi(\Omega)} = 1 \). By Lemma 2.1, it suffices to show that the functional \( \frac{\partial u}{\partial x_i} : C_0^\infty(\Omega) \to \mathbb{R}; \)
\[
\frac{\partial u}{\partial x_i}[\varphi] := -\int u \frac{\partial \varphi}{\partial x_i}
\]
is bounded with respect to the norm \( \| \cdot \|_{L^\Phi(\Omega)} \) and satisfies \( \| \frac{\partial u}{\partial x_i} \| \leq C \). Choose 
\( 0 \leq \psi \in C_0^\infty(B(0, 1)) \) such that \( \int \psi = 1 \) and let \( \psi_\varepsilon(x) = \varepsilon^{-n} \psi(x/\varepsilon) \) for \( \varepsilon > 0 \). Then
\[
\frac{\partial u}{\partial x_i}[\varphi] = -\lim_{\varepsilon \to 0} \int (u * \psi_\varepsilon) \frac{\partial \varphi}{\partial x_i} = \lim_{\varepsilon \to 0} \int \left( u * \frac{\partial \psi_\varepsilon}{\partial x_i} \right) \varphi.
\]
By the Hölder inequality,
\[
\left| \frac{\partial u}{\partial x_i}[\varphi] \right| \leq 2 \liminf_{\varepsilon \to 0} \left\| u * \frac{\partial \psi_\varepsilon}{\partial x_i} \right\|_{L^\Phi(\text{supp } \varphi)} \| \varphi \|_{L^\Phi(\text{supp } \varphi)}.
\]
Since \( \int \frac{\partial \psi}{\partial x_i} = 0 \), we have that
\[
(u * \frac{\partial \psi}{\partial x_i})(x) = \left( (u - u_{B(x,\varepsilon)}) * \frac{\partial \psi}{\partial x_i} \right)(x).
\]
Thus
\[
\left| u * \frac{\partial \psi}{\partial x_i} \right|(x) \leq C \varepsilon^{-n-1} \int_{B(x,\varepsilon)} |u(y) - u_{B(x,\varepsilon)}| \, dy \leq C \Phi^{-1} \left( \int_{B(x,\tau \varepsilon)} \Phi(g(y)) \, dy \right).
\]
Let \( K = \text{supp } \varphi \) and let \( \varepsilon > 0 \) be such that \( K_{\tau \varepsilon} = \{ x \in \mathbb{R}^n : d(x, K) < \tau \varepsilon \} \subset \Omega \). Then, by Fubini’s theorem,
\[
\int_K \Phi \left( C^{-1} \left| u * \frac{\partial \psi}{\partial x_i} \right|(x) \right) \, dx \leq \int_K \Phi(g(y)) \, dy \, dx
\[
= \int_{K_{\tau \varepsilon}} \Phi(g(y)) \int_{B(y,\tau \varepsilon) \cap K} |B(x,\tau \varepsilon)|^{-1} \, dx \, dy.
\]
\[
\leq \int_{K_{\tau \varepsilon}} \Phi(g(y)) \, dy \leq 1.
\]
Thus
\[
\liminf_{\varepsilon \to 0} \left\| u * \frac{\partial \psi}{\partial x_i} \right\|_{L^\Phi(\text{supp } \varphi)} \leq C,
\]
which completes the proof.

For the proofs of Theorem 1.2 and Theorem 1.3, which are based on approximation by discrete convolutions, we need a couple of lemmas. Lemma 3.1 follows from a Whitney type covering result for doubling metric measure spaces, see [2, Theorem III.1.3], [10, Lemma 2.9]. For the proof of Lemma 3.2, we refer to [10, Lemma 2.16].

**Lemma 3.1** Let \( \Omega \subset X \) be open. Given \( \varepsilon > 0 \), \( \lambda \geq 1 \), there is a cover \( \{ B_i = B(x_i, r_i) \} \) of \( \Omega \) with the following properties:

1. \( r_i \leq \varepsilon \) for all \( i \),
2. \( \lambda B_i \subset \Omega \) for all \( i \),
3. if \( \lambda B_i \) meets \( \lambda B_j \), then \( r_i \leq 2r_j \),
4. each ball \( \lambda B_i \) meets at most \( C = C(d, \lambda) \) balls \( \lambda B_j \).

A collection \( \{ B_i \} \) as above is called an \( (\varepsilon, \lambda) \)-covering of \( \Omega \). Clearly, an \( (\varepsilon, \lambda) \)-cover is an \( (\varepsilon', \lambda') \)-cover provided \( \varepsilon' \geq \varepsilon \) and \( \lambda' \leq \lambda \).

**Lemma 3.2** Let \( \Omega \subset X \) be open, and let \( \mathcal{B} = \{ B_i = B(x_i, r_i) \} \) be an \( (\infty, 2) \)-cover of \( \Omega \). Then there is a collection \( \{ \varphi_i \} \) of functions \( \Omega \to \mathbb{R} \) such that

1. \( \varphi_i \) is \( C(d) r_i^{-1} \)-Lipschitz,
2. \( 0 \leq \varphi_i \leq 1 \) for all \( i \),
3. \( \int_{\Omega} \Phi \left( \sum_{i} \left| \varphi_i * \frac{\partial \psi}{\partial x_i} \right|^2 \right) \, dx \leq \int_{\Omega} \Phi(g(y)) \, dx \).
3) \( \varphi_i(x) = 0 \) for \( x \in X \setminus 2B_i \) for all \( i \).
4) \( \sum_i \varphi_i(x) = 1 \) for all \( x \in \Omega \).

A collection \( \{ \varphi_i \} \) as above is called a partition of unity with respect to \( B \).

Let \( B = \{ B_i \} \) be as in the lemma above, and let \( \{ \varphi_i \} \) be a partition of unity with respect to \( B \). For a locally integrable function \( u \) on \( \Omega \), define

\[
u_B(x) = \sum_i u_{B_i} \varphi_i(x).
\]

(11)

The following lemma describes the most important properties of \( u_B \).

**Lemma 3.3**

1) The function \( u_B \) is locally Lipschitz. Moreover, for each \( x \in B_i \),

\[
\text{Lip } u_B(x) \leq C(C_d) r_{B_i}^{-1} \int_{5B_i} |u - u_{5B_i}| \, d\mu.
\]

2) Let \( \Phi \) be a doubling Young function and let \( u \in L^\Phi(\Omega) \). If \( B_k \) is an \((\varepsilon_k, 2)\)-cover of \( \Omega \) and \( \varepsilon_k \to 0 \) as \( k \to \infty \), then \( u_{B_k} \to u \) in \( L^\Phi(\Omega) \).

**Proof** 1) Let \( x, y \in B_i \), and let \( J = \{ j : 2B_j \cap 2B_i \neq \emptyset \} \). Then \#\( J \leq C(C_d) \) and \( B_j \subset 5B_i \) for each \( j \in J \). Using the properties of the functions \( \varphi_i \), we have that

\[
|u_B(x) - u_B(y)| = \left| \sum_{j \in J} (u_B_j - u_B_i)(\varphi_j(x) - \varphi_j(y)) \right|
\]

\[
\leq C(C_d) r_{B_i}^{-1} d(x, y) \max_{j \in J} |u_{B_j} - u_{B_i}|
\]

\[
\leq C(C_d) r_{B_i}^{-1} d(x, y) \int_{5B_i} |u - u_{5B_i}| \, d\mu,
\]

and the first claim follows.

2) We begin by showing that, for every \( w \in L^\Phi(\Omega) \),

\[
\|w_B\|_{L^\Phi(\Omega)} \leq C(C_d)\|w\|_{L^\Phi(\Omega)}.
\]

We may assume that \( \|w\|_{L^\Phi(\Omega)} = 1 \). By Jensen’s inequality \( \Phi(|w_B|) \leq (\Phi(|w|))_B \). Hence, by the properties of the functions \( \varphi_i \),

\[
\int_{\Omega} \Phi(|w_B|) \, d\mu \leq \int_{\Omega} (\Phi(|w|))_B \, d\mu \leq \sum_i \int_{\Omega} (\Phi(|w|))_{B_i} \varphi_i \, d\mu
\]

\[
\leq \sum_i \int_{2B_i} \Phi(|w|)_{B_i} \, d\mu \leq C_d \sum_i \int_{B_i} \Phi(|w|) \, d\mu
\]

\[
= C_d \int_{\Omega} \Phi(|w|) \sum_i \chi_{B_i} \, d\mu \leq C(C_d) \int_{\Omega} \Phi(|w|) \, d\mu
\]

\[
\leq C(C_d).
\]
Thus, by (8), we obtain (12).

Let \( u \in L^\Phi(\Omega) \) and \( \varepsilon > 0. \) By Lemma 2.2 (1), there exists \( v \in C_0(\Omega) \) such that \( \| u - v \|_{L^\Phi(\Omega)} < \varepsilon. \) Then, by (12), we obtain

\[
\| u_B - v_B \|_{L^\Phi(\Omega)} = \|(u - v)B\|_{L^\Phi(\Omega)} \leq C(C_d)\| u - v \|_{L^\Phi(\Omega)} < C(C_d)\varepsilon,
\]

and so

\[
\| u_B - u \|_{L^\Phi(\Omega)} \leq \| u_B - v_B \|_{L^\Phi(\Omega)} + \| v_B - v \|_{L^\Phi(\Omega)} + \| v - u \|_{L^\Phi(\Omega)} < \| v_B - v \|_{L^\Phi(\Omega)} + C(C_d)\varepsilon.
\]

Therefore it suffices to show that \( \| v_B - v \|_{L^\Phi(\Omega)} \to 0 \) as \( \varepsilon_k \to 0. \) Now \( \| v_B - v \| \leq 2 \sup |v|, \) and for all \( x \) we have that

\[
|v_B(x) - v(x)| \leq \sum_{B_i \ni x} \int_{B_i} |v(y) - v(x)| \, d\mu(y) \leq C(C_d) \int_{B(x, \varepsilon_k)} |v(y) - v(x)| \, d\mu(y),
\]

which converges to 0 as \( \varepsilon_k \to 0 \) by the continuity of \( v. \) Thus, by the dominated convergence theorem,

\[
\int_\Omega \Phi(|v_B - v|) \, d\mu \to 0,
\]

and so, by Lemma 2.2 (2), \( \| v_{B_n} - v \|_{L^\Phi(\Omega)} \to 0. \)

\[\square\]

**Proof of Theorem 1.3.** Let \( u \in A_j^{\Phi,\Phi}(\Omega) \cap L^\Phi(\Omega). \) For \( j \in \mathbb{N}, \) let \( B_j \) be a \( (j^{-1}, 5\tau) \)-cover (and hence also a \( (j^{-1}, 2) \)-cover) of \( \Omega. \) Then, by Lemma 3.3 (2), \( u_j := u_{B_j} \to u \) in \( L^\Phi(\Omega). \) Let us show that

\[
\| \text{Lip} u_j \|_{L^\Phi(\Omega)} \leq C(C_d, \tau) \| u \|_{A_j^{\Phi,\Phi}(\Omega)}.
\]

By Lemma 3.3 (1),

\[
\text{Lip} u_j \leq C(C_d) \sum_{B \in B_j} r_B^{-1} \int_{5B} |u - u_{5B}| \, d\mu \chi_B.
\]

It follows from Lemma 3.1 (4) that \( B_j \) can be divided into \( k = C(C_d, \tau) \) subfamilies \( B_{j,1}, \ldots, B_{j,k}, \) so that each of the families \( 5\tau B_{j,l} \) consists of disjoint balls. Since the families \( 5B_{j,1}, \ldots, 5B_{j,k} \) belong to \( B_{\tau}(\Omega), \) we have that

\[
\| \text{Lip} u_j \|_{L^\Phi(\Omega)} \leq C(C_d) \sum_{l=1}^k \sum_{B \in B_{j,l}} r_B^{-1} \int_{5B} |u - u_{5B}| \, d\mu \chi_B \|_{L^\Phi(\Omega)}
\]

\[
\leq C(C_d) \sum_{l=1}^k \sum_{B \in 5B_{j,l}} r_B^{-1} \int_B |u - u_B| \, d\mu \chi_B \|_{L^\Phi(\Omega)}
\]

\[
\leq C(C_d, \tau) \| u \|_{A_j^{\Phi,\Phi}(\Omega)}.
\]

Since \( \Phi \) and \( \Phi \) are doubling, \( L^\Phi(\Omega) \) is reflexive. Thus the bounded sequence \( (\text{Lip} u_j) \) has a subsequence, also denoted by \( (\text{Lip} u_j), \) that converges
weakly to some $g \in L^\Phi(\Omega)$. By Lemma 2.4, $g$ is a $\Phi$-weak upper gradient of a representative of $u$. As a weak limit $g$ satisfies

$$
\|g\|_{L^\Phi(\Omega)} \leq \liminf_{j \to \infty} \|\text{Lip } u_j\|_{L^\Phi(\Omega)} \leq C(C_d, \tau)\|u\|_{A^1_\tau(\Omega)}.
$$

**Proof of Theorem 1.2** We may assume that $\|g\|_{L^\Phi(\Omega)} = 1$. Define the functions $u_j$ as in the proof of Theorem 1.3. By (13) and (7), we have that

$$
\|\text{Lip } u_j\|_{L^\Phi(\Omega)} \leq C(C_d, C_P, \tau).
$$

Let us show that

$$
\lim_{\mu(E) \to 0} \sup_j \int_E \Phi(\text{Lip } u_j) \, d\mu = 0. \tag{14}
$$

By Lemma 3.3 (1) and by the $\Phi$-Poincaré inequality,

$$
\text{Lip } u_j \leq C(C_d) \sum_{B \in B_j} r_B^{-1} \int_{5B} |u - u_{5B}| \, d\mu \chi_B 
\leq C(C_d, C_P) \sum_{B \in B_j} \Phi^{-1} \left( \int_{5\tau B} \Phi(g) \, d\mu \right) \chi_B.
$$

Thus

$$
\int_E \Phi(\text{Lip } u_j) \, d\mu \leq C(C_d, C_P, C\Phi) \sum_{B \in B_j} \frac{\mu(E \cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g) \, d\mu.
$$

Since $B_j$ can be divided into $k = C(C_d, \tau)$ subfamilies $B_{j,1}, \ldots, B_{j,k}$ so that each of the families $5\tau B_{j,l}$ consists of disjoint balls, it suffices to show that, for $1 \leq l \leq k$,

$$
\lim_{\mu(E) \to 0} \sum_{B \in B_{j,l}} \frac{\mu(E \cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g) \, d\mu = 0.
$$

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\int_A \Phi(g) < \varepsilon$ whenever $\mu(A) < \delta$. Denote by $B$ the family of those balls $B$ in $B_{j,l}$ for which

$$
\frac{\mu(E \cap B)}{\mu(5\tau B)} < \varepsilon.
$$

Also, let $B' = B_{j,l} \setminus B$. Now, if $\mu(E) < \varepsilon\delta$, we have that $\mu(\cup_{B \in B'} 5\tau B) \leq \varepsilon^{-1} \mu(E) < \delta$. Thus

$$
\sum_{B \in B_{j,l}} \frac{\mu(E \cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g) \, d\mu
= \sum_{B \in B} \frac{\mu(E \cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g) \, d\mu + \sum_{B \in B'} \frac{\mu(E \cap B)}{\mu(5\tau B)} \int_{5\tau B} \Phi(g) \, d\mu
\leq \varepsilon \int_{\Omega} \Phi(g) \, d\mu + \int_{\cup_{B \in B'} 5\tau B} \Phi(g) \, d\mu
\leq 2\varepsilon.
$$

10
This completes the proof of (14).

By Lemma 2.3, a subsequence of $(\operatorname{Lip} u_j)$ converges weakly to some $g_u \in L^\Phi(\Omega)$, which, by Lemma 2.4, is a $\Phi$-weak upper gradient of a representative of $u$. Moreover, as a weak limit, $g_u$ satisfies

$$
\|g_u\|_{L^\Phi(\Omega)} \leq \liminf_{j \to \infty} \|\operatorname{Lip} u_j\|_{L^\Phi(\Omega)} \leq C(C_d, C_P, \tau).
$$

\[\blacksquare\]  

References


University of Jyväskylä, Department of Mathematics and Statistics, P.O. Box 35, FI-40014 Jyväskylä, Finland

E-mail address: toheikki@maths.jyu.fi