Self-improving properties of generalized Orlicz-Poincaré inequalities in metric measure spaces

Toni Heikkinen*

Abstract

The sharp self-improving properties of generalized Φ-Poincaré inequalities in *connected* metric measure spaces were recently obtained in [6]. In this paper we investigate the general setting. We also include the case where Φ increases essentially more slowly than the function $t \mapsto t$. Our results generalize some results of Hajłasz and Koskela [4, 5] and MacManus and Pérez [8].

Mathematics Subject Classification (2000): 46E35

1 Introduction and main results

Let $X = (X, d, \mu)$ be a metric measure space with μ a Borel regular outer measure satisfying $0 < \mu(U) < \infty$, whenever U is nonempty, open and bounded. Suppose further that μ is doubling, that is, there exists a constant C_d such that

$$\mu(2B) \leq C_d \mu(B), \tag{1}$$

whenever B is a ball. It is easy to see that the doubling property is equivalent to the existence of constants s and C_s such that

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \geq C_s^{-1} \left(\frac{r}{r_0}\right)^s \tag{2}$$

holds, whenever $x \in B(x_0, r_0)$ and $r \leq r_0$.

Definition 1.1 ([10]) *Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing bijection. A pair (u, g) of measurable functions, $u \in L^1_{loc}(X)$ and $g \geq 0$, satisfies the Φ-Poincaré inequality (in an open set U), if there are constants C_P and τ such that*

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \Phi^{-1} \left(\fint_{\tau B} \Phi(g) \, d\mu \right) \tag{3}$$

for every ball $B \subset X$ (such that $\tau B \subset U$).

*The author was supported by Vilho, Yrjö and Kalle Väisälä Foundation

The following sharp self-improving result for the Φ-Poincaré inequality was recently proved in [6].

Theorem A *Assume that Φ is a Young function, X is connected, μ satisfies (2) with $1 < s < \infty$, $B \subset X$ is a ball, $\delta > 0$, $\tau \geq 1$, $\hat{B} = (1+\delta)\tau B$, and that a pair $(\hat{u}, \hat{g})$, where $\hat{u} = \|g\|^{-1}_{L^\Phi(\hat{B})} u$ and $\hat{g} = \|g\|^{-1}_{L^\Phi(\hat{B})} g$, satisfies the Φ-Poincaré inequality in $\hat{B}$.*

1) If

$$\int_0^1 \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty \quad \text{and} \quad \int_0^\infty \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt = \infty, \tag{4}$$

then

$$\|u - u_B\|_{L^{\Phi_s}_w(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}, \tag{5}$$

where

$$\Phi_s = \Phi \circ \Psi_s^{-1}, \tag{6}$$

$$\Psi_s(r) = \left(\int_0^r \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt\right)^{1/s'} \tag{7}$$

and $s' = s/(s-1)$.

2) If

$$\int^\infty \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty, \tag{8}$$

then, for Lebesgue points $x, y \in B$ of u,

$$|u(x) - u(y)| \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \omega_s(\mu(B)^{-1} r_B^s d(x,y)^{-s}), \tag{9}$$

where

$$\omega_s^{-1}(t) = (t\Theta^{-1}(t^{s'}))^{s'} \tag{10}$$

and Θ^{-1} is the left-continuous inverse of the function given by

$$\Theta(r) = s' \int_r^\infty \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt. \tag{11}$$

Here, $C = C(C_s, s, C_P, \tau, \delta)$.

Let $U \subset X$ be open, $0 < s \leq \infty$ and $\tau \geq 1$. Denote

$$\mathcal{B}_\tau(U) = \{\{B_i\} : \text{balls } \tau B_i \text{ are disjoint and contained in } U\},$$

$$\|u\|_{A^{\Phi,s}_\tau(U)} = \sup_{\mathcal{B} \in \mathcal{B}_\tau(U)} \| \sum_{B \in \mathcal{B}} \left(\mu(B)^{-1/s} \fint_B |u - u_B|\, d\mu\right) \chi_B \|_{L^\Phi(X)}$$

and

$$A^{\Phi,s}_\tau(U) = \{u \in L^1(U) : \|u\|_{A^{\Phi,s}_\tau(U)} < \infty\}.$$

It is easy to see that $\|u\|_{A_\tau^{\Phi,s}(U)} \le \lambda$, if and only if there is a functional $\nu : \{B \subset U : \ B \text{ is a ball }\} \to [0,\infty)$ such that

$$\sum \nu(B_i) \le 1, \tag{12}$$

whenever the balls B_i are disjoint, and that

$$\fint_B |u - u_B|\, d\mu \le \lambda \mu(B)^{1/s} \Phi^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right), \tag{13}$$

whenever $\tau B \subset U$. The self-improving properties of abstract Poincaré-type inequalities similar to (13), for $\Phi(t) = t^p$, were studied by Franchi, Pérez and Wheeden [2, 3], and MacManus and Pérez [8, 9].

If μ satisfies (2) and a pair $(\hat{u}, \hat{g})$, where $\hat{u} = \|g\|_{L^\Phi(B)}^{-1} u$ and $\hat{g} = \|g\|_{L^\Phi(B)}^{-1} g$, satisfies the Φ-Poincaré inequality in a ball B, then

$$\|u\|_{A_\tau^{\Phi,s}(B)} \le C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(B)}. \tag{14}$$

Thus, the first case of Theorem A is a consequence of the following embedding theorem for the space $A_\tau^{\Phi,s}(U)$.

Theorem B [6, Theorem 1.9] *Let X be connected, μ doubling, Φ a Young function, $B \subset X$ a ball, $1 < s < \infty$, $\tau \ge 1$ and $\delta > 0$. Denote $\hat{B} = (1+\delta)\tau B$.*

1) If (4) holds, then

$$\|u - u_B\|_{L_w^{\Phi_s}(B)} \le C \|u\|_{A_\tau^{\Phi,s}(\hat{B})},$$

where Φ_s is defined by (6)-(7).

2) If (8) holds, then, for Lebesgue points $x, y \in B$ of u,

$$|u(x) - u(y)| \le C \|u\|_{A_\tau^{\Phi,s}(\hat{B})} \omega_s(\mu(B_{xy})^{-1}),$$

where $B_{xy} = B(x, 2d(x,y))$, and ω_s is defined by (10)-(11).

Here, $C = C(C_d, \tau, \delta)$.

It is essential in the above theorems that the underlying space X is connected. In this paper we investigate the general case. Instead of assuming that Φ is a Young function, we assume the following:

(Φ-1) $\Phi : [0,\infty) \to [0,\infty)$ is an increasing bijection.

(Φ-2) The function $t \mapsto \frac{\Phi(t)}{t^{s/(s+1)}}$ is increasing.

Notice that (Φ-2) allows Φ to increase essentially more slowly than any Young function. The results concerning such Φ are new also for connected spaces.

Our first result is a counterpart of Theorem A in the general setting. It extends the results of Hajłasz and Koskela [4, 5].

Theorem 1.2 *Assume that Φ satisfies (Φ-1) and (Φ-2), μ satisfies* (2) *with $0 < s < \infty$, $B \subset X$ is a ball, $\delta > 0$, $\tau \geq 1$, $\hat{B} = (1+\delta)\tau B$, and that a pair $(\hat{u}, \hat{g})$, where $\hat{u} = \|g\|^{-1}_{L^\Phi(\hat{B})} u$ and $\hat{g} = \|g\|^{-1}_{L^\Phi(\hat{B})} g$, satisfies the Φ-Poincaré inequality in $\hat{B}$.*

1) If

$$\int_0^1 \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt < \infty \quad \text{and} \quad \int_0^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt = \infty, \tag{15}$$

then

$$\|u - u_B\|_{L_w^{\tilde{\Phi}_s}(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}, \tag{16}$$

where

$$\tilde{\Phi}_s^{-1}(r) = \int_0^r \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt. \tag{17}$$

2) If

$$\int^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt < \infty, \tag{18}$$

then, for Lebesgue points $x, y \in B$ of u,

$$|u(x) - u(y)| \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \tilde{\omega}_s(\mu(B)^{-1} r_B^s d(x,y)^{-s}),$$

where

$$\tilde{\omega}_s(r) = \int_r^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt. \tag{19}$$

Here, $C = C(C_s, s, C_P, \tau, \delta)$.

If the Φ-Poincaré inequality is stable under truncations, the weak estimate (5) turns into a strong one. We say that a pair (u, g) has the truncation property, if for every $b \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$ and $\varepsilon \in \{-1, 1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1 < v \leq t_2\}})$, where $v = \varepsilon(u - b)$ and

$$v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\},$$

satisfies the Φ-Poincaré inequality (with fixed constants).

Theorem 1.3 *Suppose that the assumptions of Theorem 1.2 are in force,* (15) *holds, and that the pair $(\hat{u}, \hat{g})$ has the truncation property. Then*

$$\|u - u_B\|_{L^{\tilde{\Phi}_s}(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}, \tag{20}$$

where Φ_s is defined by (6)-(7) *and $C = C(C_s, s, C_P, \tau, \delta)$.*

How good is Theorem 1.2 compared to Theorem A? If Φ is "close" to the function $t \mapsto t^s$, then $\tilde{\Phi}_s$ increases essentially more slowly than Φ_s.

Example 1.4 *Let Φ be equivalent near infinity to the function $t^s \log^q t$. Then the function Φ_s is equivalent near infinity to*

$$\begin{cases} \exp(t^{s/(s-1-q)}) & \text{if } q < s-1 \\ \exp(\exp(t^{s/(s-1)})) & \text{if } q = s-1, \end{cases}$$

and the function $\tilde{\Phi}_s$ is equivalent near infinity to

$$\begin{cases} \exp(t^{s/(s-q)}) & \text{if } q < s \\ \exp(\exp(t)) & \text{if } q = s. \end{cases}$$

If Φ is a Young function such that the function $t \mapsto \Phi(t)/t^p$ is either decreasing for some $p < s$, or increasing for some $p > s$, then Theorem 1.2 gives the same result as Theorem A. In these cases the Sobolev conjugate Φ_s and the function ω_s can be represented in a very simple form.

Theorem 1.5 *(1) Suppose that Φ satisfies (Φ-1) and (Φ-2) and that the function $t \mapsto \Phi(t)/t^p$ is decreasing for some $p < s$. Then $\tilde{\Phi}_s$ is globally equivalent to the function Φ_s^* whose inverse is given by*

$$(\Phi_s^*)^{-1}(r) = \Phi^{-1}(r) r^{-1/s}.$$

If Φ is a Young function, then also Φ_s is globally equivalent to Φ_s^*.

(2) If Φ is a Young function such that $\Phi(t)/t^p$ is increasing for some $p > s$, then both ω_s and $\tilde{\omega}_s$ are comparable the function ω_s^* given by

$$\omega_s^*(r) = \Phi^{-1}(r) r^{-1/s}.$$

Let us now turn to the results concerning the embeddings of spaces $A_\tau^{\Phi,s}(U)$. We begin with the case $s = \infty$. Theorem 1.6 below extends (the non-weighted version of) the result of MacManus and Pérez [8].

Theorem 1.6 *Assume that μ is doubling, $s = \infty$, Φ is doubling and satisfies (Φ-1) and (Φ-2), $B \subset X$ is a ball, $\tau \geq 1$ and $\delta > 0$. Then*

$$\|u - u_B\|_{L_w^\Phi(B)} \leq C \|u\|_{A_\tau^{\Phi,\infty}(\hat{B})},$$

where $\hat{B} = (1+\delta)\tau B$ and $C = C(C_d, \tau, \delta, \Phi)$.

Since, by Lemma 3.3,

$$\|u\|_{A_\tau^{\tilde{\Phi}_s,\infty}(U)} \leq \|u\|_{A_\tau^{\Phi,s}(U)}, \tag{21}$$

we have the following.

Theorem 1.7 *Assume that μ is doubling, Φ satisfies (Φ-1) and (Φ-2),* (15) *holds and that $\tilde{\Phi}_s$ is doubling. Let $B \subset X$ a ball, $\tau \geq 1$ and $\delta > 0$. Then*

$$\|u - u_B\|_{L_w^{\tilde{\Phi}_s}(B)} \leq C \|u\|_{A_\tau^{\Phi,s}(\hat{B})}, \tag{22}$$

where $\hat{B} = (1+\delta)\tau B$ and $C = C(C_d, \tau, \delta, s, \Phi)$.

Under the extra assumption that singletons have zero measure, (22) holds also for non-doubling $\tilde{\Phi}_s$.

Theorem 1.8 *Assume that μ is doubling, $\mu(\{x\}) = 0$ for $x \in X$, $0 < s < \infty$, and that Φ satisfies (Φ-1) and (Φ-2). Let $B \subset X$ be a ball, $\tau \geq 1$ and $\delta > 0$. Denote $\hat{B} = (1+\delta)\tau B$.*

1) If (15) *holds, then*

$$\|u - u_B\|_{L_w^{\tilde{\Phi}_s}(B)} \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})},$$

where $\tilde{\Phi}_s$ is defined by (17).

2) If (18) *holds, then, for Lebesgue points $x, y \in B$ of u,*

$$|u(x) - u(y)| \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\tilde{\omega}_s(\mu(B_{xy})^{-1}),$$

where $B_{xy} = B(x, 2d(x,y))$ and $\tilde{\omega}_s$ is defined by (19).

Here, $C = C(C_d, \tau, \delta, s)$.

2 Preliminaries

Throughout this paper $X = (X, d, \mu)$ is a metric space equipped with a measure μ. By a measure we mean a Borel regular outer measure satisfying $0 < \mu(U) < \infty$ whenever U is open and bounded.

Open and closed balls of radius r centered at x will be denoted by $B(x,r)$ and $\overline{B}(x,r)$. Sometimes we denote the radius of a ball B by r_B. For a positive number λ we define $\lambda B(x,r) := B(x, \lambda r)$.

Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing bijection. Denote by $L^\Phi(X)$ the set of all measurable functions u for which there exists $\lambda > 0$ such that

$$\int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) < \infty.$$

For $u \in L^\Phi(X)$, define

$$\|u\|_{L^\Phi(X)} = \inf\{\lambda > 0 : \int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) \leq 1\}.$$

If Φ is convex, the functional $\|\cdot\|_{L^\Phi(X)}$ is a norm on $L^\Phi(X)$.

If $\|u\|_{L^\Phi(X)} \neq 0$, we have that

$$\int_X \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(X)}}\right) d\mu(x) \leq 1.$$

Denote by $L_w^\Phi(X)$ the set of all measurable functions for which the number

$$\|u\|_{L_w^\Phi(X)} = \inf\{\lambda > 0 : \sup_{t>0} \Phi(t)\mu(\{x \in X : \frac{|u(x)|}{\lambda} > t\}) \leq 1\}$$

is finite. If $\|u\|_{L^\Phi_w(X)} \neq 0$, it follows that

$$\sup_{t>0} \Phi(t)\mu(\{x \in X : \frac{|u(x)|}{\|u\|_{L^\Phi_w(X)}} > t\}) \leq 1.$$

A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if it has the form

$$\Phi(t) = \int_0^t \phi(s)\, ds,$$

where $\phi : [0,\infty) \to [0,\infty]$ is increasing, left-continuous function, which is neither identically zero nor identically infinite on $(0,\infty)$. A Young function is convex and, in particular, satisfies

$$\Phi(\varepsilon t) \leq \varepsilon \Phi(t) \tag{23}$$

for $0 < \varepsilon \leq 1$ and $0 \leq t < \infty$.

The right-continuous generalized inverse of a Young function Φ is

$$\Phi^{-1}(t) = \inf\{s : \Phi(s) > t\}.$$

We have that

$$\Phi(\Phi^{-1}(t)) \leq t \leq \Phi^{-1}(\Phi(t))$$

for $t \geq 0$.

The conjugate of a Young function Φ is the Young function defined by

$$\hat{\Phi}(t) = \sup\{ts - \Phi(s) : s > 0\}$$

for $t \geq 0$.

We have that

$$t \leq \Phi^{-1}(t)\hat{\Phi}^{-1}(t) \leq 2t \tag{24}$$

for $t \geq 0$.

A function Φ dominates a function Ψ globally (resp. near infinity), if there is a constant C such that

$$\Psi(t) \leq \Phi(Ct)$$

for all $t \geq 0$ (resp. for t larger than some t_0).

Functions Φ and Ψ are equivalent globally (near infinity), if each dominates the other globally (near infinity).

If Φ dominates Ψ near infinity and Φ and Ψ are not equivalent near infinity, then Ψ increases essentially more slowly that Φ.

Φ is doubling, if there is a constant C such that

$$\Phi(2t) \leq C\Phi(t)$$

for all t.

3 Proofs

Lemma 3.1 *Suppose that Φ satisfies (Φ-1) and (Φ-2). Then, for $0 \le \varepsilon \le 1$ and $t \ge 0$,*

$$\Phi(\varepsilon t) \le \varepsilon^{s/(s+1)}\Phi(t), \tag{25}$$

$$\tilde{\Phi}_s(\varepsilon t) \le \varepsilon \tilde{\Phi}_s(t), \tag{26}$$

and

$$\tilde{\omega}_s(\varepsilon t) \le \varepsilon^{-1/s}\tilde{\omega}_s(t), \tag{27}$$

Proof. We have

$$\Phi(\varepsilon t) = \frac{\Phi(\varepsilon t)}{(\varepsilon t)^{s/(s+1)}}(\varepsilon t)^{s/(s+1)} \le \frac{\Phi(t)}{t^{s/(s+1)}}(\varepsilon t)^{s/(s+1)} = \varepsilon^{s/(s+1)}\Phi(t).$$

By (Φ-2), the function $\Phi^{-1}(t)/t^{1+1/s}$ is decreasing. Hence

$$\begin{aligned}\tilde{\Phi}_s^{-1}(\varepsilon^{-1}r) &= \int_0^{\varepsilon^{-1}r} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt = \varepsilon^{-1}\int_0^r \frac{\Phi^{-1}(\varepsilon^{-1}t)}{(\varepsilon^{-1}t)^{1+1/s}}\,dt \\ &\le \varepsilon^{-1}\int_0^r \frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt = \varepsilon^{-1}\tilde{\Phi}_s^{-1}(r),\end{aligned}$$

which is equivalent to (26). Since Φ^{-1} is increasing, we have

$$\tilde{\omega}_s(\varepsilon r) = \int_{\varepsilon r}^{\infty} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt = \varepsilon\int_r^{\infty} \frac{\Phi^{-1}(\varepsilon t)}{(\varepsilon t)^{1+1/s}}\,dt \le \varepsilon^{-1/s}\tilde{\omega}_s(r).$$

□

Let $U \subset X$ be open and let $v \in L^1(U)$. The maximal function of v is

$$M_U v(x) = \sup_{x \in B \subset U} \fint_B |v|\,d\mu.$$

It is well known that there is a constant $C = C(C_d)$ such that

$$\|M_U v\|_{L^1_w(U)} \le C\|v\|_{L^1(U)}. \tag{28}$$

Proof of Theorem 1.2. We may assume that $\|g\|_{L^\Phi(\hat{B})} = 1$.

1) It suffices to show that the pointwise inequality

$$|u(x) - u_B| \le C r_B \mu(B)^{-1/s}\tilde{\Phi}_s^{-1}\Big(M_{\hat{B}}\Phi(g)(x)\Big), \tag{29}$$

holds for Lebesgue points $x \in B$ of u. Indeed, if (29) holds, then by (28),

$$\left\|\tilde{\Phi}_s\left(\frac{|u-u_B|}{C r_B \mu(B)^{-1/s}}\right)\right\|_{L^1_w(B)} \le \left\|M_{\hat{B}}\Phi(g)\right\|_{L^1_w(B)} \le C\left\|\Phi(g)\right\|_{L^1(\hat{B})} \le C,$$

and the claim follows by (26).

Fix a Lebesgue point x of u. For $i \geq 1$, let $B_i = B(x, 2^{-i/s}\delta)$. By the Lebesgue differentiation theorem, $\lim_{i\to\infty} u_{B_i} = u(x)$. So

$$|u(x) - u_{B_1}| \leq \sum_{i=1}^{\infty} |u_{B_i} - u_{B_{i+1}}| \leq C \sum_{i=1}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu.$$

By denoting $B_0 = (1+\delta)B$,

$$|u_B - u_{B_1}| \leq |u_B - u_{B_0}| + |u_{B_0} - u_{B_1}| \leq C \fint_{B_0} |u - u_{B_0}|\, d\mu.$$

Thus

$$|u(x) - u_B| \leq C \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu.$$

By (3) and (2),

$$\begin{aligned}\fint_{B_i} |u - u_{B_i}|\, d\mu &\leq C r_i \Phi^{-1}(\mu(B_i)^{-1}) \\ &\leq C r_i \Phi^{-1}((C_B r_i)^{-s}),\end{aligned}$$

where $C_B = r_B^{-1}\mu(B)^{1/s}$. Hence, by denoting $t_i = (C_B r_i)^{-s}$,

$$\begin{aligned}\sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu &\leq C C_B^{-1} \sum_{i=0}^{k} t_i^{-1/s} \Phi^{-1}(t_i) \\ &= C r_B \mu(B)^{-1/s} \sum_{i=0}^{k} t_i \frac{\Phi^{-1}(t_i)}{t_i^{1+1/s}}.\end{aligned}$$

Since the function $t \mapsto \frac{\Phi^{-1}(t)}{t^{1+1/s}}$ is decreasing, we have that

$$t_i \frac{\Phi^{-1}(t_i)}{t_i^{1+1/s}} \leq 2 \int_{\frac{1}{2}t_i}^{t_i} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt,$$

for $i \geq 0$. So, by summing and noting that $t_i \leq \frac{1}{2} t_{i+1}$, we obtain

$$\sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu \leq C r_B \mu(B)^{-1/s} \int_{\frac{1}{2}t_0}^{t_k} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt. \tag{30}$$

Thus

$$\sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu \leq C r_B \mu(B)^{-1/s} \tilde{\Phi}_s^{-1}(t_k). \tag{31}$$

The remaining part of the series will be estimated in terms of $M\Phi(g)$:

$$\begin{aligned}\sum_{i=k}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu &\leq C \sum_{i=k}^{\infty} r_i \Phi^{-1}\Big(M_{\hat{B}}\Phi(g)(x)\Big) \\ &\leq C r_k \Phi^{-1}\Big(M_{\hat{B}}\Phi(g)(x)\Big) \\ &= C r_B \mu(B)^{-1/s} t_k^{-1/s} \Phi^{-1}\Big(M_{\hat{B}}\Phi(g)(x)\Big).\end{aligned} \tag{32}$$

By combining (31) and (32), we obtain

$$|u(x)-u_B| \le Cr_B\mu(B)^{-1/s}\left(\tilde{\Phi}_s^{-1}(t_k)+t_k^{-1/s}\Phi^{-1}\Big(M_{\hat{B}}\Phi(g)(x)\Big)\right). \tag{33}$$

Since, by (Φ-2), the function $\Phi^{-1}(t)/t^{1+1/s}$ is decreasing, we have that

$$\tilde{\Phi}_s^{-1}(r)=\int_0^r \frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt \ge r\frac{\Phi^{-1}(r)}{r^{1+1/s}}=r^{-1/s}\Phi^{-1}(r). \tag{34}$$

If $M_{\hat{B}}\Phi(g)(x)\ge t_0$, we choose k such that

$$t_k \le M_{\hat{B}}\Phi(g)(x)\le Ct_k.$$

Then, by (33) and (34), we obtain (29). If $M_{\hat{B}}\Phi(g)(x)<t_0$, it suffices to use (32), with $k=0$, and (34).

2) We may assume that $\delta<1/2$. Let D be a ball centered at B so that $\hat{D}=(1+\delta)\tau D\subset\hat{B}$. Fix a Lebesgue point $x\in D$ of u. Let $B_0=(1+\delta)D$ and $B_i=B(x,2^{-i/s}\delta)$ for $i\ge 1$. By the same argument that led to (30), we have that

$$\sum_{i=0}^{\infty}\fint_{B_i}|u-u_{B_i}|\,d\mu \le Cr_B\mu(B)^{-1/s}\int_{C^{-1}\mu(B)^{-1}r_B^s r_D^{-s}}^{\infty}\frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt.$$

Thus

$$|u(x)-u_D|\le Cr_B\mu(B)^{-1/s}\tilde{\omega}_s(C^{-1}\mu(B)^{-1}r_B^s r_D^{-s}). \tag{35}$$

Let $x,y\in B$ be Lebesgue points of u. If $d(x,y)>\frac{1}{3}\delta r_B$, then (35), with $D=B$, yields

$$\begin{aligned}|u(x)-u(y)| &\le |u(x)-u_B|+|u(y)-u_B|\\ &\le Cr_B\mu(B)^{-1/s}\tilde{\omega}_s(C^{-1}\mu(B)^{-1}r_B^s d(x,y)^{-s}).\end{aligned}$$

If $d(x,y)\le\frac{1}{3}\delta r_B$, then $\hat{D}\subset\hat{B}$, for the ball $D=B(x,2d(x,y))$, and so, by (35),

$$\begin{aligned}|u(x)-u(y)| &\le |u(x)-u_D|+|u(y)-u_D|\\ &\le Cr_B\mu(B)^{-1/s}\tilde{\omega}_s(C^{-1}\mu(B)^{-1}r_B^s d(x,y)^{-s}).\end{aligned}$$

Thus, the claim follows by (27). □

The proof of Theorem 1.3 is completely analogous to the proof of Theorem 1.4 in [6]. We will not repeat the details.

Proof of Theorem 1.5 (1) By (34),

$$\tilde{\Phi}_s^{-1}(r)\ge r^{-1/s}\Phi^{-1}(r).$$

If $t\mapsto\Phi(t)/t^p$ is decreasing, then $t\mapsto\Phi^{-1}(t)/t^{1/p}$ is increasing. Hence, if $p<s$,

$$\tilde{\Phi}_s^{-1}(r)=\int_0^r\frac{\Phi^{-1}(t)}{t^{1+1/s}}\,dt\le\frac{\Phi^{-1}(r)}{r^{1/p}}\int_0^r t^{1/p-1/s-1}\,dt=(1/p-1/s)^{-1}\frac{\Phi^{-1}(r)}{r^{1/s}}.$$

Thus $\tilde{\Phi}_s$ and Φ_s^* are globally equivalent. Let Φ be a Young function. Then the function $t \mapsto t/\Phi(t)$ is decreasing and so

$$\Psi_s(r) = \left(\int_0^r \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt\right)^{1/s'} \geq \left(r\left(\frac{r}{\Phi(r)}\right)^{s'-1}\right)^{1/s'} = \Phi(r)^{-1/s} r.$$

On the other hand, if $t \mapsto \Phi(t)/t^p$ is decreasing, with $p < s$, then

$$\begin{aligned}\Psi_s(r) &= \left(\int_0^r \left(\frac{t}{\Phi(t)}\right)^{\frac{1}{s-1}} dt\right)^{\frac{s-1}{s}} \leq \frac{r^{p/s}}{\Phi(r)^{1/s}}\left(\int_0^r t^{\frac{s-p}{s-1}-1} dt\right)^{\frac{s-1}{s}} \\ &= \frac{r^{p/s}}{\Phi(r)^{1/s}}\left(\frac{s-1}{s-p} r^{\frac{s-p}{s-1}}\right)^{\frac{s-1}{s}} = \left(\frac{s-1}{s-p}\right)^{\frac{s-1}{s}} \frac{r}{\Phi(r)^{1/s}}.\end{aligned}$$

Since $\Phi_s^{-1} = \Psi_s \circ \Phi^{-1}$, we have that

$$\frac{\Phi^{-1}(r)}{r^{1/s}} \leq \Phi_s^{-1}(r) \leq \left(\frac{s-1}{s-p}\right)^{\frac{s-1}{s}} \frac{\Phi^{-1}(r)}{r^{1/s}}.$$

(2) Let $p > s$ and let $\Phi(t)/t^p$ be increasing. Then $\Phi^{-1}(t)/t^{1/p}$ is decreasing and so

$$\tilde{\omega}_s(r) = \int_r^\infty \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt \leq \frac{\Phi^{-1}(r)}{r^{1/p}} \int_r^\infty t^{1/p-1/s-1}\, dt = (1/s - 1/p)^{-1} \frac{\Phi^{-1}(r)}{r^{1/s}}.$$

On the other hand,

$$\tilde{\omega}_s(r) \geq \int_r^{2r} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt \geq r \frac{\Phi^{-1}(2r)}{(2r)^{1+1/s}} \geq 2^{-1-1/s} \frac{\Phi^{-1}(r)}{r^{1/s}}.$$

It was shown in [1] that

$$\omega_s(r) = \|t^{-1/s'}\|_{L^{\hat{\Phi}}((0,1/r))}. \tag{36}$$

Since

$$\int_0^{1/r} \hat{\Phi}(t^{-1/s'}/\lambda)\, dt \geq r^{-1} \hat{\Phi}(r^{1/s'}/\lambda),$$

it follows that

$$\omega_s(r) \geq r^{1/s'} \hat{\Phi}^{-1}(r)^{-1}.$$

Hence, by (24),

$$\omega_s(r) \geq \frac{1}{2} \Phi^{-1}(r) r^{-1/s}.$$

Assume that $\Phi(t)/t^p$ is increasing. Let $t > t'$. By using (24), the fact that $\Phi^{-1}(t)/t^{1/p}$ is decreasing, and (24) again, we obtain

$$\hat{\Phi}(t) \leq \Phi^{-1}(\hat{\Phi}(t)) t \leq \Phi^{-1}(\hat{\Phi}(t')) \left(\frac{\hat{\Phi}(t)}{\hat{\Phi}(t')}\right)^{1/p} t \leq 2 \frac{\hat{\Phi}(t')}{t'} \left(\frac{\hat{\Phi}(t)}{\hat{\Phi}(t')}\right)^{1/p} t,$$

which implies that

$$\frac{\hat{\Phi}(t)}{t^{p'}} \leq 2^{p'} \frac{\hat{\Phi}(t')}{t'^{p'}}.$$

If $p > s$, then $p' < s'$. So,

$$\begin{aligned} \int_0^{1/r} \hat{\Phi}(t^{-1/s'}/\lambda)\, dt &\leq 2^{p'} \frac{\hat{\Phi}(r^{1/s'}/\lambda)}{r^{p'/s'}} \int_0^{1/r} t^{-p'/s'}\, dt \\ &= Cr^{-1}\hat{\Phi}(r^{1/s'}/\lambda) \\ &\leq r^{-1}\hat{\Phi}(Cr^{1/s'}/\lambda), \end{aligned}$$

where the last inequality comes from (23) and $C = 2^{p'}(1 - p'/s')^{-1}$. Hence, by (36) and (24),

$$\omega_s(r) \leq Cr^{1/s'}\hat{\Phi}^{-1}(r)^{-1} \leq C\Phi^{-1}(r)r^{-1/s}.$$

□

For a ball $B_0 \subset X$, $u \in L^1(B_0)$ and $0 < s \leq \infty$, define

$$M^{\#}_{s,B_0}u(x) = \sup_{x\in B\subset B_0} \mu(B)^{-1/s} \fint_B |u - u_B|\, d\mu. \tag{37}$$

Lemma 3.2 *Let Φ satisfy (Φ-1) and (Φ-2). Then*

$$\|M^{\#}_{s,B}u\|_{L^{\Phi}_w(B)} \leq C\|u\|_{A^{\Phi,s}_{\tau}(\tau B)},$$

where $C = C(C_d, \tau, s)$.

Proof. We may assume that $\|u\|_{A^{\Phi,s}_{\tau}(\tau B)} = 1$. Let $x \in B$ such that $M^{\#}_{s,B}u(x) > \lambda$. By the definition of $M^{\#}_{s,B}u$, there is a ball $B_x \subset B$ containing x such that

$$\mu(B_x)^{-1/s} \fint_{B_x} |u - u_{B_x}|\, d\mu > \lambda.$$

This implies that

$$\mu(B_x) \leq \Phi(\lambda)^{-1}\Phi\Big(\mu(B_x)^{-1/s} \fint_{B_x} |u - u_{B_x}|\, d\mu\Big)\mu(B_x). \tag{38}$$

By the standard $5r$-covering lemma ([7, Theorem 1.16]), we can cover the set $\{x \in B : M^{\#}_{s,B}(x) > \lambda\}$ by balls $5\tau B_i$ such that the balls τB_i are disjoint and that each B_i is contained in B and satisfies (38). Using the doubling property of μ, estimate (38), inequality (25), and the fact that $\{B_i\} \in \mathcal{B}_\tau(\tau B)$, we obtain

$$\mu(\{x \in B : M^{\#}_{s,B}u(x) > \lambda\}) \le \sum_i \mu(5\tau B_i) \le C(C_d,\tau) \sum_i \mu(B_i)$$

$$\le C(C_d,\tau)\Phi(\lambda)^{-1} \sum_i \Phi\Big(\mu(B_i)^{-1/s} \fint_{B_i} |u - u_{B_i}|\, d\mu\Big)\mu(B_i)$$

$$\le \Phi\left(\frac{\lambda}{C(C_d,\tau,s)}\right)^{-1} \sum_i \Phi\Big(\mu(B_i)^{-1/s} \fint_{B_i} |u - u_{B_i}|\, d\mu\Big)\mu(B_i)$$

$$\le \Phi\left(\frac{\lambda}{C(C_d,\tau,s)}\right)^{-1}.$$

The claim follows by the definition of $\|\cdot\|_{L^{\Phi}_w}$. $\square$

Proof of Theorem 1.6. Fix a ball $B_0 \subset X$. Denote $B_0' = (1+\delta)B_0$, $\mathcal{B} = \{B : x_B \in B_0 \text{ and } r_B \le \delta r_{B_0}\}$, $\mathcal{B}^+ = \mathcal{B} \cup \{B_0'\}$,

$$M_{\mathcal{B}}u(x) = \sup_{x \in B \in \mathcal{B}} \fint_B |u|\, d\mu \quad \text{and} \quad M^{\#}_{\mathcal{B}^+}u(x) = \sup_{x \in B \in \mathcal{B}^+} \fint_B |u - u_B|\, d\mu.$$

For $\lambda > 0$, let $\Omega_\lambda = \{x : M_{\mathcal{B}}u(x) > \lambda\}$ and $\Sigma_\lambda = \{x : M^{\#}_{\mathcal{B}^+}u(x) > \lambda\}$. The following *good λ inequality* was proved by MacManus and Pérez in [8]: There are constants C_0 and ε_0 such that for all $\lambda > 0$ and $0 < \varepsilon \le \varepsilon_0$,

$$\mu(\Omega_{C_0\lambda}) \le C_0\varepsilon\mu(\Omega_\lambda) + C_0\mu(\Sigma_{\varepsilon\lambda}). \tag{39}$$

We will show that (39) implies

$$\|M_{\mathcal{B}}u\|_{L^{\Phi}_w(B_0)} \le C\|M^{\#}_{\mathcal{B}^+}u\|_{L^{\Phi}_w(B_0)}. \tag{40}$$

We may assume that $\|M^{\#}_{\mathcal{B}^+}u\|_{L^{\Phi}_w(B)} = 1$. Then $\Phi(t)\mu(\Sigma_t) \le 1$ for $t > 0$. Since Φ is doubling, there is a constant C_1 such that $\Phi(C_0\lambda) \le C_1\Phi(\lambda)$ for $\lambda > 0$. By setting

$$\varepsilon = \min\{\varepsilon_0, (2C_0C_1)^{-1}\}$$

in (39), we obtain

$$\begin{aligned}\Phi(C_0\lambda)\mu(\Omega_{C_0\lambda}) &\le \frac{1}{2}\Phi(\lambda)\mu(\Omega_\lambda) + C\Phi(\varepsilon\lambda)\mu(\Sigma_{\varepsilon\lambda}) \\ &\le \frac{1}{2}\sup_{\lambda>0}\Phi(\lambda)\mu(\Omega_\lambda) + C\end{aligned}$$

for $\lambda > 0$. Hence

$$\sup_{\lambda>0}\Phi(\lambda)\mu(\Omega_\lambda) \le C.$$

By (25), we obtain (40). Denote $u_0 = u - u_{B_0}$. By the Lebesgue differentiation theorem

$$u_0(x) \le M_{\mathcal{B}}u_0(x)$$

for almost every $x \in B_0$. Hence

$$\|u - u_{B_0}\|_{L^\Phi_w(B_0)} \leq \|M_{\mathcal{B}} u_0\|_{L^\Phi_w(B_0)} \leq C\|M^{\#}_{\mathcal{B}^+} u_0\|_{L^\Phi_w(B_0)}.$$

Since $M^{\#}_{\mathcal{B}^+} u_0 = M^{\#}_{\mathcal{B}^+} u \leq M^{\#}_{\infty,B'_0} u$, the claim follows from Lemma 3.2. $\square$

Theorem 1.7 follows from Theorem 1.6 via the following lemma.

Lemma 3.3 *Assume that $U \subset X$ is open, Φ satisfies (Φ-1) and (Φ-2) and that* (15) *holds. Then*

$$\|u\|_{A^{\tilde{\Phi}_s,\infty}_\tau(U)} \leq \|u\|_{A^{\Phi,s}_\tau(U)}.$$

Proof. Since

$$\tilde{\Phi}_s^{-1}(t) \geq t^{-1/s}\Phi^{-1}(t),$$

we have, in (13), that

$$\begin{aligned}\mu(B)^{1/s}\Phi^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right) &= \nu(\tau B)^{1/s}\left(\frac{\nu(\tau B)}{\mu(B)}\right)^{-1/s}\Phi^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right)\\ &\leq \tilde{\Phi}_s^{-1}\left(\frac{\nu(\tau B)}{\mu(B)}\right).\end{aligned}$$

This implies the claim. $\square$

Proof of Theorem 1.8. 1) It suffices to show that the pointwise inequality

$$|u(x) - u_B| \leq C\|u\|_{A^{\Phi,s}_\tau(\hat{B})}\tilde{\Phi}_s^{-1}\left(\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B)}}\right)\right), \tag{41}$$

where $B' = (1+\delta)B$, holds for Lebesgue points $x \in B$. Fix such a point x and choose balls B_i, $i \geq 1$, centered at x, so that $B_1 \subset B'$, and

$$r_{B_{i+1}} = \sup\{r > 0 : \mu(B(x,r)) \leq \frac{1}{2}\mu(B_i)\}.$$

This is possible because $\lim_{r\to 0}\mu(B(x,r)) = \mu(\{x\}) = 0$. By (1), we have that

$$2\mu(B_{i+1}) \leq \mu(B_i) \leq C\mu(B_{i+1}) \tag{42}$$

for all i.

By the Lebesgue differentiation theorem, $\lim_{i\to\infty} u_{B_i} = u(x)$. So

$$|u(x) - u_{B_1}| \leq \sum_{i=1}^{\infty}|u_{B_i} - u_{B_{i+1}}| \leq C\sum_{i=1}^{\infty}\fint_{B_i}|u - u_{B_i}|\,d\mu.$$

By denoting $B_0 = B'$,

$$|u_B - u_{B_1}| \leq |u_B - u_{B_0}| + |u_{B_0} - u_{B_1}| \leq C\fint_{B_0}|u - u_{B_0}|\,d\mu.$$

Thus

$$|u(x) - u_B| \le C \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu.$$

For every i, we have

$$\fint_{B_i} |u - u_{B_i}|\, d\mu \le \|u\|_{A_\tau^{\Phi,s}(\hat{B})} \mu(B_i)^{1/s} \Phi^{-1}(\mu(B_i)^{-1}).$$

Hence

$$\begin{aligned}\sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu &\le \|u\|_{A_\tau^{\Phi,s}(\hat{B})} \sum_{i=0}^{k} \mu(B_i)^{1/s} \Phi^{-1}(\mu(B_i)^{-1}) \\ &= \|u\|_{A_\tau^{\Phi,s}(\hat{B})} \sum_{i=0}^{k} \mu(B_i)^{-1} \frac{\Phi^{-1}(\mu(B_i)^{-1})}{(\mu(B_i)^{-1})^{1+1/s}}.\end{aligned}$$

Since the function $t \mapsto \frac{\Phi^{-1}(t)}{t^{1+1/s}}$ is decreasing, we have that

$$\mu(B_i)^{-1} \frac{\Phi^{-1}(\mu(B_i)^{-1})}{(\mu(B_i)^{-1})^{1+1/s}} \le 2 \int_{\frac{1}{2}\mu(B_i)^{-1}}^{\mu(B_i)^{-1}} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt,$$

for $i \ge 0$. So, by summing and noting that $\mu(B_i)^{-1} \le \frac{1}{2}\mu(B_{i+1})^{-1}$, we obtain

$$\sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \int_{\frac{1}{2}\mu(B_0)^{-1}}^{\mu(B_k)^{-1}} \frac{\Phi^{-1}(t)}{t^{1+1/s}}\, dt. \tag{43}$$

Thus

$$\sum_{i=0}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \tilde{\Phi}_s^{-1}(\mu(B_k)^{-1}). \tag{44}$$

The remaining part of the series will be estimated in terms of the sharp fractional maximal function (37). Using (42), we obtain

$$\begin{aligned}\sum_{i=k}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu &\le C \sum_{i=k}^{\infty} \mu(B_i)^{1/s} M^{\#}_{s,B'} u(x) \\ &\le C\mu(B_k)^{1/s} M^{\#}_{s,B'} u(x).\end{aligned}$$

So, by Lemma 3.2,

$$\sum_{i=k}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \mu(B_k)^{1/s} \frac{M^{\#}_{s,B'} u(x)}{\|M^{\#}_{s,B'} u\|_{L^{\Phi}_w(B')}}. \tag{45}$$

Combining the estimates (44) and (45), we obtain

$$|u(x) - u_B| \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \left(\tilde{\Phi}_s^{-1}(\mu(B_k)^{-1}) + \mu(B_k)^{1/s} \frac{M^{\#}_{s,B'} u(x)}{\|M^{\#}_{s,B'} u\|_{L^{\Phi}_w(B')}} \right).$$

If $\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_{w}(B')}}\right) \geq \mu(B_0)^{-1}$, we choose k such that

$$\mu(B_k)^{-1} \leq \Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_{w}(B')}}\right) \leq C\mu(B_k)^{-1}.$$

Since, by (34),

$$\Phi(r)^{-1/s}r \leq \tilde{\Phi}_s^{-1}(\Phi(r)), \tag{46}$$

we obtain (41).

If $\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_{w}(B')}}\right) < \mu(B_0)^{-1}$, it suffices use (45) and (46).

2) Letting k tend to infinity in (43), yields

$$|u(x) - u_B| \leq \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}|\, d\mu \leq C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B})} \tilde{\omega}_s(\mu(B)^{-1}). \tag{47}$$

Let $x, y \in B$ be Lebesgue points of u. Denote $B_{xy} = B(x, 2d(x,y))$. If $d(x,y) > \frac{1}{3}\delta r_B$, then $\mu(B) \leq C\mu(B_{xy})$. So, by (47) and (27),

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u_B| + |u(y) - u_B| \\ &\leq C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B})} \tilde{\omega}_s(\mu(B)^{-1}) \\ &\leq C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B})} \tilde{\omega}_s(\mu(B_{xy})^{-1}). \end{aligned}$$

If $d(x,y) \leq \frac{1}{3}\delta r_B$, then (47), applied to the ball B_{xy}, yields

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u_{B_{xy}}| + |u(y) - u_{B_{xy}}| \\ &\leq C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B}_{xy})} \tilde{\omega}_s(\mu(B_{xy})^{-1}). \end{aligned}$$

Since we may assume that $\delta < 1/2$, it follows that $\hat{B}_{xy} \subset \hat{B}$. Hence

$$|u(x) - u(y)| \leq C\|u\|_{A^{\Phi,s}_{\tau}(\hat{B})} \tilde{\omega}_s(\mu(B_{xy})^{-1}).$$

□

References

[1] A. Cianchi: Continuity properties of functions from Orlicz-Sobolev spaces and embedding theorems - Ann. Sc. Norm. Super. Pisa, Cl. Sci. IV 23 (1996), 576-608

[2] B. Franchi, C. Pérez, R.L. Wheeden: Self-improving properties of John-Nirenberg and Poincaré inequalities on spaces of homogeneous type - J. Funct. Anal. 153 (1998), no.1, 108–146.

[3] B. Franchi, C. Pérez, R.L. Wheeden: A sum operator with applications to self-improving properties of Poincaré inequalities in metric spaces - J. Fourier Anal. Appl. 9 (2003), no. 5, 511–540.

[4] P. Hajłasz, P. Koskela: Sobolev meets Poincaré. - C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), no.10, 1211–1215.

[5] P. Hajłasz, P. Koskela: Sobolev met Poincaré - Mem. Amer. Math. Soc., 145 (2000), no.688.

[6] T. Heikkinen: Sharp self-improving properties of generalized Orlicz-Poincaré inequalities in connected metric spaces, preprint

[7] J. Heinonen: Lectures on analysis on metric spaces - Universitext, Springer-Verlag, New York, 2001.

[8] P. MacManus, C. Pérez: Generalized Poincaré inequalities: sharp self-improving properties - Internat. Math. Res. Notices 1998, no.2, 101–116.

[9] P. MacManus, C. Pérez: Trudinger inequalities without derivatives - Trans. Amer. Math. Soc. 354 (2002), no.5, 1997–2012.

[10] H. Tuominen: Orlicz-Sobolev spaces on metric measure spaces, Ann. Acad. Sci. Fenn. Math. Diss. No. 135 (2004).

University of Jyväskylä, Department of Mathematics and Statistics, P.O. Box 35, FI-40014 Jyväskylä, Finland
E-mail address: **toheikki@maths.jyu.fi**