# Sharp self-improving properties of generalized Orlicz-Poincaré inequalities in connected metric measure spaces

Toni Heikkinen*

**Abstract**

We study the self-improving properties of (generalized) $\Phi$-Poincaré inequalities in connected metric spaces equipped with a doubling measure. Our results are optimal and generalize some of the results of Cianchi [1, 2], Hajłasz and Koskela [5, 6], and MacManus and Pérez [12].

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## 1 Introduction and main results

Let $X = (X, d, \mu)$ be a metric measure space with $\mu$ a Borel regular outer measure satisfying $0 < \mu(U) < \infty$, whenever $U$ is nonempty, open and bounded. Suppose further that $\mu$ is doubling, that is, there exists a constant $C_d$ such that

$$\mu(2B) \leq C_d \mu(B), \tag{1}$$

whenever $B$ is a ball. It is easy to see that the doubling property is equivalent to the existence of constants $s$ and $C_s$ such that

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \geq C_s^{-1} \left(\frac{r}{r_0}\right)^s \tag{2}$$

holds, whenever $x \in B(x_0, r_0)$ and $r \leq r_0$.

A pair $(u, g)$ of measurable functions, $g \geq 0$, satisfies the $p$-Poincaré inequality, if there exist constants $C_P$ and $\tau \geq 1$ such that

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \left(\fint_{\tau B} g^p \, d\mu\right)^{1/p} \tag{3}$$

for every ball $B = B(x, r) \subset X$. Hajłasz and Koskela [5, 6] proved the following self-improving properties of (3): Assume that $\mu$ satisfies (2), and

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that a pair $(u,g)$, where $g \in L^p_{\mathrm{loc}}(X)$ satisfies the $p$-Poincaré inequality (3). Let $\delta > 0$ and $\hat{B} = (1+\delta)\tau B$. There exists a constant $C = C(C_s, s, C_P, \tau, \delta)$ such that the following holds.

1) If $p < s$, then

$$\sup_{t>0} t \left( \frac{\mu(\{x : |u(x) - u_B| > t\})}{\mu(B)} \right)^{1/p_s} \leq C r_B \left( \fint_{\hat{B}} g^p \, d\mu \right)^{1/p}, \tag{4}$$

where $p_s = \frac{sp}{s-p}$. Consequently, for $q < p_s$, we have

$$\left( \fint_B |u - u_B|^q \, d\mu \right)^{1/q} \leq C' r_B \left( \fint_{\tau' B} g^p \, d\mu \right)^{1/p}, \tag{5}$$

where $C'$ depends on $C$ and $q$. In general, (3) does not yield (5) with $q = p_s$. However, if a pair $(u,g)$ has the truncation property, which means that for every $b \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$ and $\varepsilon \in \{-1, 1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1 < v \leq t_2\}})$, where $v = \varepsilon(u - b)$ and $v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}$, satisfies the $p$-Poincaré inequality, then we have (5) with $q = p_s$.

2) If $p = s > 1$ and $X$ is connected, then

$$\|u - u_B\|_{Ⱡ^{\Phi}(B)} \leq C r_B \|g\|_{Ⱡ^s(\hat{B})}, \tag{6}$$

where $\|\cdot\|_{Ⱡ^{\Phi}(B)}$ is the normalized Luxemburg norm generated by the function $\Phi(t) = \exp(t^{s'}) - 1$ (see Section 2) and $s' = \frac{s}{s-1}$.

3) If $p > s$, then $u$ has a locally Hölder continuous representative, for which

$$|u(x) - u(y)| \leq C r_B^{s/p} d(x,y)^{1-s/p} \left( \fint_{\hat{B}} g^p \, d\mu \right)^{1/p} \tag{7}$$

for $x, y \in B$.

Franchi, Pérez and Wheeden [4] and MacManus and Pérez [11, 12] studied the self-improving properties of inequalities of type

$$\fint_B |u - u_B| \, d\mu \leq \|u\|_a a(\tau B), \tag{8}$$

where $\|u\|_a > 0$, $\tau \geq 1$ and $a : \{B \subset X : B \text{ is a ball}\} \to [0, \infty)$ is a functional that satisfies certain discrete summability conditions. In [11] MacManus and Pérez showed that if $\delta > 0$ is fixed, and the functional $a$ satisfies condition

$$\sum a(B_i)^r \mu(B_i) \leq c^r a(B)^r \mu(B), \tag{9}$$

whenever the balls $B_i$ are disjoint and contained in the ball $B$, then the Poincaré-type inequality (8) improves to

$$\sup_{\lambda>0} \lambda \left( \frac{\mu(\{x \in B : |u(x) - u_B| > \lambda\})}{\mu(B)} \right)^{1/r} \leq C\|a\| \|u\|_a a(\hat{B}), \tag{10}$$

where $\|a\|$ is the minimum of the constants $c$ so that (9) holds and $\hat{B} = (1+\delta)\tau B$. In [12], they proved that if $X$ is connected, $r > 1$, and $a$ satisfies the stronger condition

$$\sum a(B_i)^r \leq c^r a(B)^r, \tag{11}$$

whenever the balls $B_i$ are disjoint and contained in the ball $B$, then

$$\|u - u_B\|_{\text{Ł}^\Phi(B)} \leq Ca(\hat{B}), \tag{12}$$

where $\Phi(t) = \exp(t^{r'}) - 1$ and $r' = \frac{r}{r-1}$.

To see that the results of MacManus and Pérez generalize those of Hajłasz and Koskela, simply note that if $\mu$ satisfies (2), then the functional

$$a(B) = r_B \left( \fint_B g^p \, d\mu \right)^{1/p},$$

where $0 \leq g \in L^p_{\text{loc}}(X)$, satisfies condition (9) with $r = sp/(s-p)$, if $p < s$, and condition (11) with $r = s$, if $p = s$.

In this paper we are interested in the self-improving properties of the following $\Phi$-Poincaré inequality, introduced recently in [14]. For the definition and properties of Young functions and Orlicz spaces, see Section 2.

**Definition 1.1** *Let $\Phi$ be a Young function. A pair $(u, g)$ of measurable functions, $u \in L^1_{loc}(X)$ and $g \geq 0$, satisfies the $\Phi$-Poincaré inequality (in an open set $U$), if there are constants $C_P$ and $\tau$ such that*

$$\fint_B |u - u_B| \, d\mu \leq C_P r_B \Phi^{-1} \left( \fint_{\tau B} \Phi(g) \, d\mu \right) \tag{13}$$

*for every ball $B \subset X$ (such that $\tau B \subset U$).*

Assuming that the underlying space is connected, we obtain results which are sharp in the sense that they reproduce a version of Cianchi's optimal embedding theorem for Orlicz-Sobolev spaces on $\mathbb{R}^n$ [1, 2]. Notice that a pair $(u, |\nabla u|)$ of a weakly differentiable function and the length of its weak gradient satisfies the 1-Poincaré inequality, and so, by Jensen's inequality, the $\Phi$-Poincaré inequality for every Young function $\Phi$.

Let $s > 1$. For a Young function $\Phi$ satisfying

$$\int_0^1 \left( \frac{t}{\Phi(t)} \right)^{s'-1} dt < \infty \quad \text{and} \quad \int_0^\infty \left( \frac{t}{\Phi(t)} \right)^{s'-1} dt = \infty, \tag{14}$$

define

$$\Phi_s = \Phi \circ \Psi_s^{-1}, \tag{15}$$

where

$$\Psi_s(r) = \left( \int_0^r \left( \frac{t}{\Phi(t)} \right)^{s'-1} dt \right)^{1/s'}. \tag{16}$$

If

$$\int^{\infty}\left(\frac{t}{\Phi(t)}\right)^{s'-1}dt<\infty, \tag{17}$$

define

$$\omega_s(t)=(t\Theta^{-1}(t^{s'}))^{s'}, \tag{18}$$

where $\Theta^{-1}$ is the left-continuous inverse of the function given by

$$\Theta(r)=s'\int_r^{\infty}\frac{\hat{\Phi}(t)}{t^{1+s'}}\,dt \tag{19}$$

and $\hat{\Phi}$ is the conjugate of $\Phi$. We wish to point out that, under (14), functions $\Phi,\Psi_s$ and $\Phi_s$ are bijections. Notice also that one can modify any Young function $\Phi$ near zero so that the condition

$$\int_0^1\left(\frac{t}{\tilde{\Phi}(t)}\right)^{s'-1}dt<\infty$$

is satisfied for the modified function $\tilde{\Phi}$ and that $L^{\tilde{\Phi}}_{\mathrm{loc}}(X)=L^{\Phi}_{\mathrm{loc}}(X)$.

We will state Cianchi's result only for balls, but it actually holds for much more general domains (see [1, 2, 3]): *Let $s\geq 2$, let $B\subset\mathbb{R}^s$ be a ball, and let $u$ be a weakly differentiable function such that $|\nabla u|\in L^{\Phi}(B)$. Then there is a constant $C$ depending only on $s$ such that*

1) *If* (14) *holds, then*

$$\|u-u_B\|_{L^{\Phi_s}(B)}\leq C\||\nabla u|\|_{L^{\Phi}(B)}.$$

*Moreover, $L^{\Phi_s}(B)$ is the smallest Orlicz space into which $W^{1,\Phi}(B)$ can be continuously embedded.*

2) *If* (17) *holds, then $u$ has a continuous representative for which*

$$|u(x)-u(y)|\leq C\||\nabla u|\|_{L^{\Phi}(B)}\omega_s^{-1}(|x-y|^{-s}),$$

*for $x,y\in B$.*

Theorems 1.2 and 1.4 below generalize the result of Cianchi.

**Theorem 1.2** *Assume that $X$ is connected, $\mu$ satisfies* (2) *with $1<s<\infty$, $B\subset X$ is a ball, $\delta>0$, $\hat{B}=(1+\delta)\tau B$, $g\in L^{\Phi}(\hat{B})$, and that a pair $(\hat{u},\hat{g})$, where $\hat{u}=\|g\|^{-1}_{L^{\Phi}(\hat{B})}u$ and $\hat{g}=\|g\|^{-1}_{L^{\Phi}(\hat{B})}g$, satisfies the $\Phi$-Poincaré inequality in $\hat{B}$.*

1) *If* (14) *holds, then*

$$\|u-u_B\|_{L^{\Phi_s}_w(B)}\leq Cr_B\mu(B)^{-1/s}\|g\|_{L^{\Phi}(\hat{B})}, \tag{20}$$

*where $\Phi_s$ is defined by* (15)-(16).

*2) If* (17) *holds, then, for Lebesgue points $x, y \in B$ of $u$,*

$$|u(x) - u(y)| \le C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \omega_s^{-1}(r_B^s \mu(B)^{-1} d(x,y)^{-s}), \quad (21)$$

*where $\omega_s$ is defined by* (18)-(19).

*Here, $C = C(C_s, s, C_P, \tau, \delta)$.*

If the $\Phi$-Poincaré inequality is stable under truncations, the weak estimate (20) turns into a strong one.

**Definition 1.3** *A pair $(u, g)$ has the truncation property, if for every $b \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$ and $\varepsilon \in \{-1, 1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1 < v \le t_2\}})$, where $v = \varepsilon(u - b)$ and*

$$v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\},$$

*satisfies the $\Phi$-Poincaré inequality (with fixed constants).*

A weakly differentiable function $u$ on $\mathbb{R}^n$ satisfies $|\nabla v_{t_1}^{t_2}| = |\nabla u|\chi_{\{t_1 < v \le t_2\}}$, which implies that the pair $(u, |\nabla u|)$ has the truncation property.

**Theorem 1.4** *Suppose that the assumptions of Theorem 1.2 are in force,* (14) *holds, and that the pair $(\hat{u}, \hat{g})$ has the truncation property. Then*

$$\|u - u_B\|_{L^{\Phi_s}(B)} \le C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})}, \quad (22)$$

*where $\Phi_s$ is defined by* (15)-(16) *and $C = C(C_s, s, C_P, \tau, \delta)$.*

The following example gives concrete expressions for the "Sobolev conjugate" $\Phi_s$.

**Example 1.5** *Let $\Phi$ be equivalent to the function $t^p \log^q t$ near infinity, where either $p = 1$ and $q \ge 0$ or $p > 1$ and $q \in \mathbb{R}$. Then $\Phi_s$ is equivalent near infinity to*

$$\begin{cases} t^{sp/(s-p)}(\log t)^{sq/(s-p)} & \text{if } 1 \le p < s \\ \exp(t^{s/(s-1-q)}) & \text{if } p = s,\ q < s-1 \\ \exp(\exp(t^{s/(s-1)})) & \text{if } p = s,\ q = s-1. \end{cases}$$

In a general metric space we cannot talk about partial derivatives, but the concept of an upper gradient has turned out to be a useful substitute for the length of a gradient.

**Definition 1.6 ([10])** *A Borel function $g : X \to [0, \infty]$ is an upper gradient of a function $u : X \to \overline{\mathbb{R}}$, if for all rectifiable curves $\gamma : [0, l] \to X$,*

$$|u(\gamma(0)) - u(\gamma(l))| \le \int_\gamma g\, ds \quad (23)$$

*whenever both $u(\gamma(0))$ and $u(\gamma(l))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise.*

More generally, $g$ is a $\Phi$-weak upper gradient of $u$, if the family of rectifiable curves for which (23) does not hold has zero $\Phi$-modulus (see Section 2). The Orlicz-Sobolev space $N^{1,\Phi}(X)$ consisting of functions $u \in L^{\Phi}(X)$ having a $\Phi$-weak upper gradient $g \in L^{\Phi}(X)$ was recently studied by Tuominen [14]. We say that $X$ supports the $\Phi$-Poincaré inequality, if the $\Phi$-Poincaré inequality holds for all locally integrable functions and their upper gradients. If $X$ supports the $\Phi$-Poincaré inequality, then any pair $(u, g)$ of a locally integrable function and its $\Phi$-weak upper gradient $g \in L^{\Phi}(X)$ has the truncation property (Lemma 2.4). Thus, we obtain an optimal embedding theorem for the space $N^{1,\Phi}(X)$.

**Theorem 1.7** *Assume that $(X, d, \mu)$ is a doubling metric measure space that supports the $\Phi$-Poincaré inequality and satisfies* (2) *with $s > 1$. Let $B$ be a ball, $\delta > 0$ and $\hat{B} = (1+\delta)\tau B$.*

1) *If $\Phi$ satisfies* (14)*, then $N^{1,\Phi}(\hat{B}) \subset L^{\Phi_s}(B)$, where $\Phi_s$ is defined by* (15)-(16)*. Moreover, for every $u \in N^{1,\Phi}(\hat{B})$ and for every $\Phi$-weak upper gradient $g$ of $u$, we have*

$$\|u - u_B\|_{L^{\Phi_s}(B)} \leq C r_B \mu(B)^{-1/s} \|g\|_{L^{\Phi}(\hat{B})}.$$

2) *If $\Phi$ satisfies* (17)*, then every $u \in N^{1,\Phi}(\hat{B})$ has a locally uniformly continuous representative. Moreover, for every $\Phi$-weak upper gradient $g$ of $u$, we have*

$$|u(x) - u(y)| \leq C r_B \mu(B)^{-1/s} \|g\|_{L^{\Phi}(\hat{B})} \omega_s^{-1}(r_B^s \mu(B)^{-1} d(x,y)^{-s}),$$

*for $x, y \in B$, where $\omega_s$ is defined by* (18)-(19)*.*

*Here, $C = C(C_s, s, C_P, \tau, \delta)$.*

Apart from the case $X = \mathbb{R}^n$, theorems 1.2, 1.4 and 1.7 seem to be new even if the $\Phi$-Poincaré inequality in the assumptions is replaced by the 1-Poincaré inequality. The spaces supporting the 1-Poincaré inequality include Riemannian manifolds with nonnegative Ricci curvature, $Q$-regular orientable topological manifolds satisfying the local linear contractability condition, Carnot groups and more general Carnot-Carathéodory spaces associated with a system of vector fields satisfying Hörmander's condition, as well as more exotic spaces constructed by Bourdon and Pajot, Laakso, and Hanson and Heinonen, see [6] and the references therein.

Our next result is an embedding theorem for the space $A_\tau^{\Phi,s}(U)$ defined as follows.

**Definition 1.8** *Let $U$ be an open set, $\Phi$ a Young function, $\tau \geq 1$ and $0 < s \leq \infty$. Denote*

$$\mathcal{B}_\tau(U) = \{\{B_i\} : \text{balls } \tau B_i \text{ are disjoint and contained in } U\}$$

*and*

$$\|u\|_{A_\tau^{\Phi,s}(U)} = \sup_{\mathcal{B}\in\mathcal{B}_\tau(U)} \| \sum_{B\in\mathcal{B}} \left( \mu(B)^{-1/s} \fint_B |u - u_B|\, d\mu \right) \chi_B \|_{L^\Phi(U;\mu)}.$$

*Then* $A_\tau^{\Phi,s}(U)$ *consists of all locally integrable functions* $u$ *for which the number* $\|u\|_{A_\tau^{\Phi,s}(U)}$ *is finite.*

Notice that below $1 < s < \infty$ is *any* number and need not have anything to do with (2).

**Theorem 1.9** *Let* $X$ *be connected,* $\mu$ *doubling,* $\Phi$ *a Young function,* $B \subset X$ *a ball,* $1 < s < \infty$*,* $\tau \geq 1$ *and* $\delta > 0$*. Denote* $\hat{B} = (1+\delta)\tau B$*.*

*1) If* (14) *holds, then*

$$\|u - u_B\|_{L_w^{\Phi_s}(B)} \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}, \tag{24}$$

*where* $\Phi_s$ *is defined by* (15)-(16)*.*

*2) If* (17) *holds, then, for Lebesgue points* $x, y \in B$ *of* $u$*,*

$$|u(x) - u(y)| \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \omega_s^{-1}(\mu(B_{xy})^{-1}), \tag{25}$$

*where* $B_{xy} = B(x, 2d(x,y))$*, and* $\omega_s$ *is defined by* (18)-(19)*.*

*Here,* $C = C(C_d, \tau, \delta)$*.*

It is easy to see that the first part of Theorem 1.2 is a consequence of inequality (24). In Section 4 we will show that it also implies the generalized Trudinger inequality (12) of MacManus and Peréz.

The results in this paper deal with connected spaces. The setting of a disconnected space will be investigated in the forthcoming paper [8].

# 2 Preliminaries

## 2.1 Metric measure spaces

Throughout this paper $X = (X, d, \mu)$ is a metric space equipped with a measure $\mu$. By a measure we mean Borel regular outer measure satisfying $0 < \mu(U) < \infty$ whenever $U$ is open and bounded.

Open and closed balls of radius $r$ centered at $x$ will be denoted by $B(x,r)$ and $\overline{B}(x,r)$. Sometimes we denote the radius of a ball $B$ by $r_B$. For a positive number $\lambda$, we define $\lambda B(x,r) := B(x, \lambda r)$.

Recall from the introduction that the doubling property of a measure implies a lower decay estimate (2) for the measure of a ball. In connected spaces we can estimate the measure of a ball also from above.

**Lemma 2.1** *Let $X$ be connected and $\mu$ doubling. Then there are constants $\alpha > 0$ and $C \geq 1$ depending only on $C_d$ such that*

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \leq C \left(\frac{r}{r_0}\right)^{\alpha}, \tag{26}$$

*whenever $x \in B(x_0, r_0)$ and $r \leq r_0$.*

For a proof, see for example [12].

## 2.2 Young functions and Orlicz spaces

In this subsection we give a brief review of Young functions and Orlicz spaces. A more detailed treatment of the subject can be found for example in [13].

A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if it has the form

$$\Phi(t) = \int_0^t \phi(s)\, ds,$$

where $\phi : [0,\infty) \to [0,\infty]$ is increasing, left-continuous function, which is neither identically zero nor identically infinite on $(0,\infty)$. A Young function is convex and, in particular, satisfies

$$\Phi(\varepsilon t) \leq \varepsilon \Phi(t) \tag{27}$$

for $0 < \varepsilon \leq 1$ and $0 \leq t < \infty$.

The right-continuous generalized inverse of a Young function $\Phi$ is

$$\Phi^{-1}(t) = \inf\{s : \Phi(s) > t\}.$$

We have that

$$\Phi(\Phi^{-1}(t)) \leq t \leq \Phi^{-1}(\Phi(t))$$

for $t \geq 0$.

The conjugate of a Young function $\Phi$ is the Young function defined by

$$\hat{\Phi}(t) = \sup\{ts - \Phi(s) : s > 0\}$$

for $t \geq 0$.

Let $\Phi$ be a Young function. The Orlicz space $L^{\Phi}(X)$ is the set of all measurable functions $u$ for which there exists $\lambda > 0$ such that

$$\int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) < \infty.$$

The Luxemburg norm of $u \in L^{\Phi}(X)$ is

$$\|u\|_{L^{\Phi}(X)} = \|u\|_{L^{\Phi}(X;\mu)} = \inf\{\lambda > 0 : \int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) \leq 1\}.$$

If $\|u\|_{L^\Phi(X)} \neq 0$, we have that

$$\int_X \Phi\left(\frac{|u(x)|}{\|u\|_{L^\Phi(X)}}\right) d\mu(x) \leq 1.$$

The following generalized Hölder inequality holds for Luxemburg norms:

$$\int_X u(x)v(x)\, d\mu(x) \leq 2\|u\|_{L^\Phi(X)}\|v\|_{L^{\hat\Phi}(X)}.$$

The weak Orlicz space $L^\Phi_w(X)$ is defined to be the set of all those measurable functions for which the weak Luxemburg norm

$$\|u\|_{L^\Phi_w(X)} = \inf\{\lambda > 0 : \sup_{t>0} \Phi(t)\mu(\{x \in X : \frac{|u(x)|}{\lambda} > t\}) \leq 1\}$$

is finite. If $\|u\|_{L^\Phi_w(X)} \neq 0$, it follows that

$$\sup_{t>0} \Phi(t)\mu(\{x \in X : \frac{|u(x)|}{\|u\|_{L^\Phi_w(X)}} > t\}) \leq 1.$$

The normalized (weak) Luxemburg norm, that is, the (weak) Luxemburg norm taken with respect to measure $\mu(X)^{-1}\mu$, will be denoted by $\|\cdot\|_{\text{Ł}^\Phi(X)}$ ($\|\cdot\|_{\text{Ł}^\Phi_w(X)}$).

A function $\Phi$ dominates a function $\Psi$ globally (resp. near infinity), if there is a constant $C$ such that

$$\Psi(t) \leq \Phi(Ct)$$

for all $t \geq 0$ (resp. for $t$ larger than some $t_0$).

Functions $\Phi$ and $\Psi$ are equivalent globally (near infinity), if each dominates the other globally (near infinity).

If $\mu(X) < \infty$ and $\Phi$ dominates $\Psi$ near infinity, we have that

$$\|u\|_{\text{Ł}^\Psi(X)} \leq C(\Phi, \Psi)\|u\|_{\text{Ł}^\Phi(X)}. \tag{28}$$

## 2.3 $\Phi$-weak upper gradients

Let $\Phi$ be a Young function. The $\Phi$-modulus of a curve family $\Gamma$ is

$$\mathrm{Mod}_\Phi(\Gamma) = \inf\left\{\|g\|_{L^\Phi(X)} : \int_\gamma g\, ds \geq 1 \text{ for all } \gamma \in \Gamma\right\}.$$

If $X$ supports the $\Phi$-Poincaré inequality, then (13) holds for functions and their $\Phi$-weak upper gradients. This is an immediate consequence of the following lemma ([14], Lemma 4.3).

**Lemma 2.2** *Let $\Phi$ be a Young function and let $g \in L^\Phi(X)$ be a $\Phi$-weak upper gradient of a function $u$. Then there is a decreasing sequence $(g_i)$ of upper gradients of $u$ such that $g_i \to g$ in $L^\Phi(X)$.*

An important property of $\Phi$-weak upper gradients is the following ([14],Lemma 4.11).

**Lemma 2.3** *Let $\Phi$ be a Young function. Assume that $u \in ACC_\Phi(X)$ and that the functions $v$ and $w$ have $\Phi$-weak upper gradients $g_v, g_w \in L^\Phi(X)$. If $E$ is a Borel set such that $u|_E = v$ and $u|_{X\setminus E} = w$, then the function*

$$g = g_v \chi_E + g_w \chi_{X\setminus E}$$

*is a $\Phi$-weak upper gradient of $u$.*

Here "$u \in ACC_\Phi(X)$" means that the family $\Gamma$ of rectifiable curves for which $u \circ \gamma$ is not absolutely continuous on $[0, l(\gamma)]$ has zero $\Phi$-modulus.

It follows from the lemma above that if $g \in L^\Phi(X)$ is a $\Phi$-weak upper gradient of a measurable function $v$, then $g\chi_{\{t_1 < v \le t_2\}}$ is a $\Phi$-weak upper gradient of the function $v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}$. Thus, we have the following.

**Lemma 2.4** *If $X$ supports the $\Phi$-Poincaré inequality, then every pair $(u, g)$ of a locally integrable function and its $\Phi$-weak upper gradient $g \in L^\Phi(X)$ has the truncation property.*

# 3 Proofs of main theorems

The proof of Theorem 1.9 requires several lemmas. In the first three lemmas equivalent representations of conditions (14) and (17) and of functions $\Phi_s$ and $\omega_s$ are given. The proofs of lemmas 3.1 and 3.2 can be found in [3], and the proof of 3.3 in [1].

**Lemma 3.1** *Let $\Phi$ be a Young function. We have*

$$\int_0 \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt < \infty \quad \text{if and only if} \quad \int_0 \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty \tag{29}$$

*and*

$$\int^\infty \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt < \infty \quad \text{if and only if} \quad \int^\infty \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty. \tag{30}$$

*Moreover, the function $\Phi_s$ is globally equivalent to the function $D_s$ given by*

$$D_s(t) = (tJ^{-1}(t^{s'}))^{s'} \tag{31}$$

*for $t \ge 0$, where $J^{-1}$ is the left-continuous inverse of the function given by*

$$J(r) = s' \int_0^r \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt. \tag{32}$$

**Lemma 3.2** *Let $\Phi$ be a Young function. Then $\|r^{-1/s'}\|_{L^{\hat{\Phi}}(t,\infty)} < \infty$ for every $t > 0$, if and only if*

$$\int_0 \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt < \infty. \tag{33}$$

*Moreover,*

$$\|r^{-1/s'}\|_{L^{\hat{\Phi}}(t,\infty)} = D_s^{-1}(1/t) \tag{34}$$

*for $t > 0$, where $D_s^{-1}$ is the right-continuous inverse of $D_s$.*

$$\tag{35}$$

**Lemma 3.3** *Let $\Phi$ be a Young function. Then $\|r^{-1/s'}\|_{L^{\hat{\Phi}}(0,t)} < \infty$ for every $t > 0$, if and only if*

$$\int^{\infty} \frac{\hat{\Phi}(t)}{t^{1+s'}}\, dt < \infty. \tag{36}$$

*Moreover,*

$$\|r^{-1/s'}\|_{L^{\hat{\Phi}}(0,t)} = \omega_s^{-1}(1/t) \tag{37}$$

*for $t > 0$, where $\omega_s^{-1}$ is the right-continuous inverse of $\omega_s$.*

It is easy to see that, for $C \geq 1$,

$$D_s^{-1}(Ct) \leq C D_s^{-1}(t) \tag{38}$$

and

$$\omega_s^{-1}(C^{-1}t) \leq C\omega_s^{-1}(t). \tag{39}$$

**Lemma 3.4** *Let $\Phi$ be a Young function. Then*

$$\Phi(r)^{-1/s} r \leq \Phi_s^{-1}(\Phi(r))$$

*for $r \geq 0$.*

**Proof** Since $\Phi$ is convex, the function $t \mapsto t/\Phi(t)$ is decreasing. Hence

$$\Phi_s^{-1}(\Phi(r)) = \left(\int_0^r \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt\right)^{1/s'} \geq \left(r\left(\frac{r}{\Phi(r)}\right)^{s'-1}\right)^{1/s'} = \Phi(r)^{-1/s} r.$$

□

The next lemma is the part of the proofs of theorems 1.9 and 1.2, where the connectedness of the space comes into play.

**Lemma 3.5** *Assume that $X$ is connected, $\mu$ doubling, $\tau \geq 1$ and $\delta > 0$. Let $B$ be a ball, $x \in B$ and $0 < r < \delta r_B$. Then there is a sequence $\{B_0, \ldots, B_k\}$ of balls contained in $(1+\delta)B$ such that $\mu(B_0)$ is comparable to $\mu(B)$, $\mu(B_k)$ is comparable to $\mu(B(x,r))$, $\{B_1, \ldots, B_k\} \in \mathcal{B}_\tau(\hat{B})$,*

$$2\mu(B_{i+1}) \leq \mu(B_i) \leq C\mu(B_{i+1}), \tag{40}$$

*for* $1 \le i < k$*, and*

$$|u_{B(x,r)} - u_{B_0}| \le C \sum_{i=1}^{k} \fint_{B_i} |u - u_{B_i}| \, d\mu, \tag{41}$$

*where* $C = C(C_d, \tau, \delta)$*.*

**Proof** Fix $x \in B$ and $0 < r < \delta r_B$. Let $\mathcal{C}_j$ be a cover of $A_j = B(x, 2^{-j}\delta r_B) \setminus B(x, 2^{-j-1}\delta r_B)$ by balls of radius $(20\tau)^{-1}2^{-j}\delta r_B$ centered at $A_j$ such that the balls $\frac{1}{2}D$, $D \in \mathcal{C}_j$, are disjoint. It follows easily from the doubling property of $\mu$ that $\#\mathcal{C}_j \le C$. Since $X$ is connected, there must be a sequence $\{B'_0, \dots, B'_{k-1}\} \subset \cup_{j=1}^{m} \mathcal{C}_j$ so that $B'_0 \in \mathcal{C}_1$, $B'_i \cap B'_{i+1} \ne \emptyset$ for all $i$, $B'_{k-1} \subset B(x,r)$ and $\mu(B'_{k-1})$ is comparable to $\mu(B(x,r))$. Denote $B_0 = B'_0$, $B_k = B'_k = B(x,r)$ and $B_i := 5B'_i$ for $1 \le i < k$. Then $B'_i \subset B_{i+1}$, and so

$$|u_{B'_i} - u_{B'_{i+1}}| \le |u_{B'_i} - u_{B_{i+1}}| + |u_{B_{i+1}} - u_{B'_{i+1}}| \le C \fint_{B_{i+1}} |u - u_{B_{i+1}}| \, d\mu.$$

Thus

$$|u_{B(x,r)} - u_{B_0}| \le \sum_{i=0}^{k-1} |u_{B'_i} - u_{B'_{i+1}}| \le C \sum_{i=1}^{k} \fint_{B_i} |u - u_{B_i}| \, d\mu.$$

We will show that $\{B_i\}$ has a subsequence that belongs to $\mathcal{B}_\tau(\hat{B})$ and satisfies (40) and (41). For $1 \le j \le m$, choose $D_j \in \{B_i\}$ centered at $A_j$ such that

$$\fint_{D_j} |u - u_{D_j}| \, d\mu = \max \left\{ \fint_{B_i} |u - u_{B_i}| \, d\mu : x_{B_i} \in A_j \right\},$$

where $x_{B_i}$ denotes the center of $B_i$. Then

$$|u_{B(x,r)} - u_{B_0}| \le C \sum_{j=1}^{m} \fint_{D_j} |u - u_{D_j}| \, d\mu.$$

If $|i - j| \ge 2$, then $\tau D_i \cap \tau D_j = \emptyset$.

By (2) and (26) there are constants $\alpha > 0$ and $\beta > 0$ depending on $C_d$ such that

$$C^{-1}2^{-\beta n} \le \frac{\mu(D_{j+n})}{\mu(D_j)} \le C2^{-\alpha n} \tag{42}$$

for $j, n \ge 1$. Let $n \ge 2$ be such that $C2^{-\alpha n} \le 2^{-1}$. For $p + (i-1)n \le m$, denote $B_i^p = D_{p+(i-1)n}$ . Then the sequence $\{B_1^p, B_2^p \dots\}$ satisfies (40) and belongs to $\mathcal{B}_\tau(\hat{B})$. By choosing $1 \le p < n$ such that

$$\sum_i \fint_{B_i^p} |u - u_{B_i^p}| \, d\mu = \max_{1 \le q < n} \sum_i \fint_{B_i^q} |u - u_{B_i^q}| \, d\mu,$$

we obtain

$$|u(x) - u_{B_0}| \le C \sum_i \fint_{B_i^p} |u - u_{B_i^p}| \, d\mu.$$

The proof is complete. $\square$

We need one more lemma, a weak-type estimate for a sharp fractional maximal function defined by

$$M^{\#}_{s,B_0}u(x) = \sup_{x\in B\subset B_0} \mu(B)^{-1/s} \fint_B |u - u_B|\, d\mu, \tag{43}$$

for a ball $B_0 \subset X$, $u \in L^1(B_0)$ and $0 < s \le \infty$.

**Lemma 3.6** *Let $\Phi$ be a Young function. Then*

$$\|M^{\#}_{s,B}u\|_{L^{\Phi}_w(B)} \le C(C_d,\tau)\|u\|_{A^{\Phi,s}_\tau(\tau B)}.$$

**Proof** We may assume that $\|u\|_{A^{\Phi,s}_\tau(\tau B)} = 1$. Let $x \in B$ such that $M^{\#}_{s,B}u(x) > \lambda$. By the definition of $M^{\#}_{s,B}u$, there is a ball $B_x \subset B$ containing $x$ such that

$$\mu(B_x)^{-1/s} \fint_{B_x} |u - u_{B_x}|\, d\mu > \lambda.$$

So,

$$\mu(B_x) \le \Phi(\lambda)^{-1}\Phi\Big(\mu(B_x)^{-1/s}\fint_{B_x} |u - u_{B_x}|\, d\mu\Big)\mu(B_x). \tag{44}$$

By the standard $5r$-covering lemma ([9, Theorem 1.16]), we can cover the set

$$\{x \in B : M^{\#}_{s,B}(x) > \lambda\}$$

by balls $5\tau B_i$ such that the balls $\tau B_i$ are disjoint and that each $B_i$ is contained in $B$ and satisfies (44). Using the doubling property of $\mu$, estimate (44), inequality (27), and the fact that $\{B_i\} \in \mathcal{B}_\tau(\tau B)$, we obtain

$$\begin{aligned}
&\mu(\{x \in B : M^{\#}_{s,B}u(x) > \lambda\}) \le \sum_i \mu(5\tau B_i) \le C(C_d,\tau)\sum_i \mu(B_i)\\
&\le C(C_d,\tau)\Phi\,(\lambda)^{-1}\sum_i \Phi\Big(\mu(B_i)^{-1/s}\fint_{B_i} |u - u_{B_i}|\, d\mu\Big)\mu(B_i)\\
&\le \Phi\left(\frac{\lambda}{C(C_d,\tau)}\right)^{-1}\sum_i \Phi\Big(\mu(B_i)^{-1/s}\fint_{B_i} |u - u_{B_i}|\, d\mu\Big)\mu(B_i)\\
&\le \Phi\left(\frac{\lambda}{C(C_d,\tau)}\right)^{-1}.
\end{aligned}$$

The claim follows by the definition of $\|\cdot\|_{L^{\Phi}_w}$. $\square$

**Proof of Theorem 1.9.** 1) Denote $B' = (1+\delta)B$. It suffices to show that the pointwise inequality

$$|u(x) - u_B| \le C\|u\|_{A^{\Phi,s}_\tau(\hat{B})}\Phi_s^{-1}\left(\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right)\right) \tag{45}$$

holds for Lebesgue points $x \in B$. Indeed, if (45) holds, then

$$\begin{aligned}
\mu\left(x \in B : \frac{|u(x)-u_B|}{C\|u\|_{A^{\Phi,s}_\tau(\hat{B})}} > t\right) &\le \mu\left(x \in B : \Phi_s^{-1} \circ \Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}}\right) > t\right)\\
&\le \mu\left(x \in B : \frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^{\Phi}_w(B')}} > \Phi^{-1} \circ \Phi_s(t)\right)\\
&\le \Phi_s(t)^{-1}.
\end{aligned}$$

Fix a Lebesgue point $x \in B$ of $u$ and $0 < r \le \delta r_B$. Let $\{B_0, \dots B_k\}$ be the chain from Lemma 3.5 corresponding to $x$ and $r$. Since the balls $B_i$, $i \ge 1$, are disjoint, we have that

$$\sum_{i=1}^k \fint_{B_i} |u - u_{B_i}|\, d\mu = \|\sum_{i=1}^k \fint_{B_i} |u - u_{B_i}|\, d\mu \frac{\chi_{B_i}}{\mu(B_i)}\|_{L^1(X)}$$

and

$$\sum_{i=1}^k \fint_{B_i} |u-u_{B_i}|\, d\mu \frac{\chi_{B_i}}{\mu(B_i)} = \sum_{i=1}^k \mu(B_i)^{-1/s} \fint_{B_i} |u-u_{B_i}|\, d\mu \chi_{B_i} \cdot \sum_{i=1}^k \mu(B_i)^{-1/s'} \chi_{B_i}.$$

Hence, by the Hölder inequality,

$$\begin{aligned}
&\sum_{i=1}^k \fint_{B_i} |u - u_{B_i}|\, d\mu\\
&\le 2\|\sum_{i=1}^k \mu(B_i)^{-1/s} \fint_{B_i} |u - u_{B_i}|\, d\mu \chi_{B_i}\|_{L^{\Phi}(X)} \cdot \|\sum_{i=1}^k \mu(B_i)^{-1/s'} \chi_{B_i}\|_{L^{\hat{\Phi}}(X)}\\
&\le 2\|u\|_{A^{\Phi,s}_\tau(\hat{B})} \cdot \|\sum_{i=1}^k \mu(B_i)^{-1/s'} \chi_{B_i}\|_{L^{\hat{\Phi}}(X)}.
\end{aligned}$$

By the definition of Luxemburg norm

$$\|\sum_{i=1}^k \mu(B_i)^{-1/s'} \chi_{B_i}\|_{L^{\hat{\Phi}}(X)} = \inf\{\lambda > 0 : \sum_{i=1}^k \hat{\Phi}\left(\frac{\mu(B_i)^{-1/s'}}{\lambda}\right)\mu(B_i) \le 1\}.$$

For each $i$, we have that

$$\begin{aligned}
\hat{\Phi}\left(\frac{\mu(B_i)^{-1/s'}}{\lambda}\right)\mu(B_i) &\le 2\int_{\frac{\mu(B_i)}{2}}^{\mu(B_i)} \hat{\Phi}\left(\frac{t^{-1/s'}}{\lambda}\right) dt\\
&\le \int_{\frac{\mu(B_i)}{2}}^{\mu(B_i)} \hat{\Phi}\left(\frac{2t^{-1/s'}}{\lambda}\right) dt,
\end{aligned}$$

where the first inequality follows from the fact that the function

$$t \mapsto \hat{\Phi}(t^{-1/s'}/\lambda)$$

is decreasing, and the second from (27). Since

$$\mu(B_{i+1}) \leq \frac{\mu(B_i)}{2},$$

we obtain

$$\sum_{i=1}^{k} \hat{\Phi}\left(\frac{\mu(B_i)^{-1/s'}}{\lambda}\right)\mu(B_i) \leq \int_{\frac{\mu(B_k)}{2}}^{\mu(B_1)} \hat{\Phi}\left(\frac{2t^{-1/s'}}{\lambda}\right) dt,$$

which implies that

$$\begin{aligned}\|\sum_{i=1}^{k}\mu(B_i)^{-1/s'}\chi_{B_i}\|_{L^{\hat{\Phi}}(X)} &\leq \inf\{\lambda > 0 : \int_{\frac{\mu(B_k)}{2}}^{\mu(B_1)} \hat{\Phi}\left(\frac{2t^{-1/s'}}{\lambda}\right) dt \leq 1\} \\ &= 2\|t^{-1/s'}\|_{L^{\hat{\Phi}}(\frac{\mu(B_k)}{2},\mu(B_1))}.\end{aligned}$$

Thus

$$\sum_{i=1}^{k} \fint_{B_i} |u - u_{B_i}|\, d\mu \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\|t^{-1/s'}\|_{L^{\hat{\Phi}}(\frac{\mu(B_k)}{2},\mu(B_1))}. \tag{46}$$

By similar reasoning,

$$\begin{aligned}|u_{B_0} - u_B| &\leq |u_{B_0} - u_{B'}| + |u_{B'} - u_B| \leq C\fint_{B'} |u - u_{B'}|\, d\mu \\ &\leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\|t^{-1/s'}\|_{L^{\hat{\Phi}}(\frac{\mu(B')}{2},\mu(B'))}.\end{aligned} \tag{47}$$

It follows from estimates (46) and (47) that

$$|u_{B(x,r)} - u_B| \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\|t^{-1/s'}\|_{L^{\hat{\Phi}}(C^{-1}\mu(B(x,r)),C\mu(B))}. \tag{48}$$

Hence, by lemmas 3.1 and 3.2, and by (38),

$$|u_{B(x,r)} - u_B| \leq C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\Phi_s^{-1}(\mu(B(x,r))^{-1}). \tag{49}$$

Next, we will estimate $|u(x) - u_{B(x,r)}|$ in terms of maximal function (43). For $i \geq 0$, denote $B_i = B(x, 2^{-i}r)$. By the Lebesgue differentiation theorem ([9, Theorem 1.8]), $u_{B_i} \to u(x)$, as $i \to \infty$. Thus, by (1) and (26),

$$\begin{aligned}|u(x) - u_{B(x,r)}| &\leq \sum_{i\geq 0} |u_{B_i} - u_{B_{i+1}}| \\ &\leq C\sum_{i\geq 0}\fint_{B_i} |u - u_{B_i}|\, d\mu \\ &\leq C\sum_{i\geq 0}\mu(B_i)^{1/s}M^{\#}_{s,B'}u(x) \\ &\leq C\mu(B(x,r))^{1/s}M^{\#}_{s,B'}u(x).\end{aligned}$$

So, by Lemma 3.6,

$$|u(x) - u_{B(x,r)}| \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\mu(B(x,r))^{1/s}\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B')}}. \tag{50}$$

Combining the above estimates, we obtain

$$|u(x) - u_B| \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\left(\Phi_s^{-1}(\mu(B_r)^{-1}) + \mu(B_r)^{1/s}\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B')}}\right),$$

where $B_r = B(x,r)$. If $\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B')}}\right) \ge \mu(B_{\delta r_B})^{-1}$, we can choose $r \le \delta r_B$ such that

$$\mu(B_r)^{-1} \le \Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B')}}\right) \le C\mu(B_r)^{-1}.$$

Then

$$\begin{aligned}\mu(B_r)^{1/s}\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B')}} &\le C\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B')}}\right)^{-1/s}\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B')}} \\ &\le C\Phi_s^{-1}\left(\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B')}}\right)\right),\end{aligned} \tag{51}$$

where the last inequality comes from Lemma 3.4. Thus, we obtain (45).

If

$$\Phi\left(\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B')}}\right) < \mu(B_{\delta r_B})^{-1},$$

it suffices to combine estimate (50), where $r = \delta r_B$, with the estimate

$$\begin{aligned}|u_{B_{\delta r_B}} - u_B| &\le C\fint_{B'}|u - u_{B'}|\,d\mu \\ &\le C\mu(B')^{1/s}M^{\#}_{s,B'}u(x) \\ &\le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\mu(B_{\delta r_B})^{1/s}\frac{M^{\#}_{s,B'}u(x)}{\|M^{\#}_{s,B'}u\|_{L^\Phi_w(B')}}\end{aligned}$$

and argue as in (51).

2) Letting $r$ tend to zero in (48) and using Lemma 3.3 and (39), we obtain

$$|u(x) - u_B| \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\omega_s^{-1}(\mu(B)^{-1}). \tag{52}$$

Let $x, y \in B$ be Lebesgue points of $u$. Denote $B_{xy} = B(x, 2d(x,y))$. If $d(x,y) > \frac{1}{3}\delta r_B$, then $\mu(B) \le C\mu(B_{xy})$. So

$$\begin{aligned}|u(x) - u(y)| &\le |u(x) - u_B| + |u(y) - u_B| \\ &\le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\omega_s^{-1}(\mu(B)^{-1}) \\ &\le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\omega_s^{-1}(\mu(B_{xy})^{-1}).\end{aligned}$$

If $d(x,y) \le \frac{1}{3}\delta r_B$, then (52), applied to the ball $B_{xy}$, yields

$$\begin{aligned}|u(x)-u(y)| &\le |u(x)-u_{B_{xy}}| + |u(y)-u_{B_{xy}}| \\ &\le C\|u\|_{A_\tau^{\Phi,s}(\hat{B}_{xy})}\omega_s^{-1}(\mu(B_{xy})^{-1}).\end{aligned}$$

Since we may assume that $\delta < 1/2$, it follows that $\hat{B}_{xy} \subset \hat{B}$. Hence

$$|u(x)-u(y)| \le C\|u\|_{A_\tau^{\Phi,s}(\hat{B})}\omega_s^{-1}(\mu(B_{xy})^{-1}).$$

□

**Proof of Theorem 1.2.** 1) By Theorem 1.9, it suffices to show that

$$\|u\|_{A_\tau^{\Phi,s}(\hat{B})} \le C r_B \mu(B)^{-1/s}\|g\|_{L^\Phi(\hat{B})}. \tag{53}$$

We may assume that $\|g\|_{L^\Phi(\hat{B})} = 1$. Let $D$ be a ball such that $\tau D \subset \hat{B}$. Then, by (13) and (2),

$$\begin{aligned}\fint_D |u-u_D|\,d\mu &\le C_p r_D \Phi^{-1}\left(\fint_{\tau D}\Phi(g)\,d\mu\right) \\ &\le C r_B \mu(B)^{-1/s}\mu(D)^{1/s}\Phi^{-1}\left(\fint_{\tau D}\Phi(g)\,d\mu\right).\end{aligned}$$

Hence, for $\mathcal{D} \in \mathcal{B}_\tau(\hat{B})$,

$$\sum_{D\in\mathcal{D}}\Phi\left(\frac{\mu(D)^{-1/s}\fint_D|u-u_D|\,d\mu}{C r_B\mu(B)^{-1/s}}\right)\mu(D) \le \sum_{D\in\mathcal{D}}\int_{\tau D}\Phi(g)\,d\mu \le \int_{\hat{B}}\Phi(g)\,d\mu \le 1,$$

which implies that

$$\|\sum_{D\in\mathcal{D}}\left(\mu(D)^{-1/s}\fint_D|u-u_D|\,d\mu\right)\chi_D\|_{L^\Phi(\hat{B})} \le C r_B\mu(B)^{-1/s}.$$

By taking supremum over $\mathcal{B}_\tau(\hat{B})$, we obtain (53).

2) We may assume that $\delta < 1/2$. Let $D$ be a ball centered at $B$ so that $\hat{D} = (1+\delta)\tau D \subset \hat{B}$. Fix a Lebesgue point $x \in D$, $0 < r < \delta r_D$ and let $\{B_i\}$ be the chain from Lemma 3.5 corresponding to $D$, $x$ and $r$. Clearly, the chain can be chosen so that $r_{B_{i+1}} \le \frac{r_{B_i}}{2}$. Since the balls $B_i$, $i \ge 1$, are disjoint, we have that

$$\sum_{i=1}^k \fint_{B_i}|u-u_{B_i}|\,d\mu = \|\sum_{i=1}^k \mu(B_i)^{-1}\fint_{B_i}|u-u_{B_i}|\,d\mu\chi_{B_i}\|_{L^1(X)}$$

and

$$\sum_{i=1}^k \mu(B_i)^{-1}\fint_{B_i}|u-u_{B_i}|\,d\mu\chi_{B_i} = \sum_{i=1}^k r_i^{-1}\fint_{B_i}|u-u_{B_i}|\,d\mu\chi_{B_i}\cdot\sum_{i=1}^k r_i\mu(B_i)^{-1}\chi_{B_i}.$$

So, by the Hölder inequality,

$$\sum_{i=1}^{k} \fint_{B_i} |u-u_{B_i}|\,d\mu$$
$$\leq 2\|\sum_{i=1}^{k} r_i^{-1}\fint_{B_i} |u-u_{B_i}|\,d\mu\chi_{B_i}\|_{L^\Phi(X)} \cdot \|\sum_{i=1}^{k} r_i\mu(B_i)^{-1}\chi_{B_i}\|_{L^{\hat{\Phi}}(X)}.$$

Since the pair $\|g\|^{-1}_{L^\Phi(\hat{B})}(u,g)$ satisfies the $\Phi$-Poincaré inequality in $\hat{B}$ and $\{B_i\} \in \mathcal{B}_\tau(\hat{D}) \subset \mathcal{B}_\tau(\hat{B})$, we have that

$$\|\sum_{i=1}^{k} r_i^{-1}\fint_{B_i} |u-u_{B_i}|\,d\mu\chi_{B_i}\|_{L^\Phi(X)} \leq C\|g\|_{L^\Phi(\hat{B})}.$$

By the definition of Luxemburg norm

$$\|\sum_{i=1}^{k} r_i\mu(B_i)^{-1}\chi_{B_i}\|_{L^{\hat{\Phi}}(X)} = \inf\{\lambda>0 : \sum_{i=1}^{k}\hat{\Phi}\left(\frac{r_i\mu(B_i)^{-1}}{\lambda}\right)\mu(B_i) \leq 1\}.$$

By (2),

$$\mu(B_i)^{-1} \leq (C_B r_i)^{-s},$$

where $C_B = Cr_B^{-1}\mu(B)^{1/s}$. Since the function $t \mapsto \hat{\Phi}(at)/t$ is increasing, for every $a>0$, we have

$$\hat{\Phi}\left(\frac{r_i\mu(B_i)^{-1}}{\lambda}\right)\mu(B_i) \leq \hat{\Phi}\left(\frac{r_i(C_B r_i)^{-s}}{\lambda}\right)(C_B r_i)^s = \hat{\Phi}\left(\frac{t_i^{-1/s'}}{C_B\lambda}\right)t_i,$$

where $t_i = (C_B r_i)^s$. It follows that

$$\begin{aligned}
\|\sum_{i=1}^{k} r_i\mu(B_i)^{-1}\chi_{B_i}\|_{L^{\hat{\Phi}}(X)} &\leq C_B^{-1}\inf\{\lambda>0 : \sum_{i=1}^{k}\hat{\Phi}\left(\frac{t_i^{-1/s'}}{\lambda}\right)t_i \leq 1\} \\
&\leq 2C_B^{-1}\inf\{\lambda>0 : \int_0^{t_1}\hat{\Phi}\left(\frac{t^{-1/s'}}{\lambda}\right) \leq 1\} \\
&= 2C_B^{-1}\|t^{-1/s'}\|_{L^{\hat{\Phi}}(0,t_1)} \\
&\leq Cr_B\mu(B)^{-1/s}\omega_s^{-1}(\mu(B)^{-1}r_B^s r_D^{-s}),
\end{aligned}$$

where the last inequality comes from Lemma 3.3 and from (39). Thus

$$\begin{aligned}
|u(x)-u_{B_0}| &= \lim_{r\to 0}|u_{B(x,r)}-u_{B_0}| \\
&\leq Cr_B\mu(B)^{-1/s}\|g\|_{L^\Phi(\hat{B})}\omega_s^{-1}(\mu(B)^{-1}r_B^s r_D^{-s}).
\end{aligned}$$

By similar reasoning,

$$\begin{aligned}
|u_{B_0}-u_D| &\leq |u_{D'}-u_{B_0}| + |u_{B_0}-u_{D'}| \leq C\fint_{D'}|u-u_{D'}|\,d\mu \\
&\leq Cr_B\mu(B)^{-1/s}\|g\|_{L^\Phi(\hat{B})}\omega_s^{-1}(\mu(B)^{-1}r_B^s r_D^{-s}).
\end{aligned}$$

So,

$$|u(x) - u_D| \leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \omega_s^{-1}(\mu(B)^{-1} r_B^s r_D^{-s}). \tag{54}$$

Let $x, y \in B$ be Lebesgue points of $u$. If $d(x,y) > \frac{1}{3}\delta r_B$, then (54) with $D = B$ yields

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_B| + |u(y) - u_B| \\ &\leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \omega_s^{-1}(\mu(B)^{-1} r_B^s d(x,y)^{-s}).\end{aligned}$$

If $d(x,y) \leq \frac{1}{3}\delta r_B$, then $\hat{D} \subset \hat{B}$, for the ball $D = B(x, 2d(x,y))$, and so by (54) and (39),

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_D| + |u(y) - u_D| \\ &\leq C r_B \mu(B)^{-1/s} \|g\|_{L^\Phi(\hat{B})} \omega_s^{-1}(\mu(B)^{-1} r_B^s d(x,y)^{-s}).\end{aligned}$$

□

**Remark 3.7** *As shown above, the first part of Theorem 1.2 is a consequence of Theorem 1.9. More generally, suppose that* (2) *holds, and that a function u satisfies an inequality of type*

$$\fint_D |u - u_D| \, d\mu \leq \|u\|_\nu r_D^\alpha \Phi^{-1}\left(\frac{\nu(\tau D)}{\mu(\tau D)}\right), \tag{55}$$

*where* $\alpha > 0$, *and* $\nu : \{B : B \text{ is a ball }\} \to [0,\infty)$ *satisfies* $\sum \nu(B_i) \leq 1$, *whenever the balls* $B_i$ *are disjoint and contained in* $\hat{B}$. *Then, an argument similar to the proof of* (53), *shows that*

$$\|u\|_{A_\tau^{\Phi, s/\alpha}(\hat{B})} \leq C r_B^\alpha \mu(B)^{-\alpha/s} \|u\|_\nu. \tag{56}$$

*Thus, if* (14) *holds, with* $s/\alpha$ *in place of* $s$, *Theorem 1.9 yields*

$$\|u - u_B\|_{L_w^{\Phi_{s/\alpha}}(B)} \leq C r_B^\alpha \mu(B)^{-\alpha/s} \|u\|_\nu.$$

*The properties of functions satisfying inequalities of type* (55) *with* $\Phi(t) = t^p$ *were studied in [7].*

**Remark 3.8** *Suppose that* (2) *and* (14) *hold, and that a pair* $(u,g)$, *where* $0 < \int_{\hat{B}} \Phi(g)\, d\mu < \infty$, *satisfies the* $\Phi$*-Poincaré inequality in* $\hat{B}$. *Then, for the measure* $\hat{\mu} = \left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1} \mu$, *we have that* $\|g\|_{L^\Phi(\hat{B};\hat{\mu})} = 1$. *Since* (2) *and* (13) *trivially hold for* $\hat{\mu}$ *with the same constants as they hold for* $\mu$, *Theorem 1.2, for the measure* $\hat{\mu}$, *yields*

$$\|u - u_B\|_{L_w^{\Phi_s}(B;\hat{\mu})} \leq C r_B \hat{\mu}(B)^{-1/s},$$

*which is equivalent to*

$$\sup_{t>0} \Phi_s\left(\frac{t}{C r \mu(B)^{-1/s}} \left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1/s}\right) \mu(\{|u - u_B| > t\}) \leq \int_{\hat{B}} \Phi(g)\, d\mu, \tag{57}$$

*where* $\{|u - u_B| > t\} = \{x \in B : |u(x) - u_B| > t\}$.

**Proof of Theorem 1.4.** Suppose that (2) and (14) hold, and that a pair $(u,g)$, where $0 < \int_{\hat{B}} \Phi(g)\, d\mu < \infty$, has the truncation property. Choose $b$ such that

$$\mu(\{u \geq b\}) \geq \mu(B)/2 \quad \text{and} \quad \mu(\{u \leq b\}) \geq \mu(B)/2.$$

Let $v_+ = \max\{u-b, 0\}$ and $v_- = -\min\{u-b, 0\}$. We need the following elementary lemma.

**Lemma 3.9** *Let $\nu$ be a finite measure on $Y$. If $w \geq 0$ is a $\nu$-measurable function such that $\nu(\{w=0\}) \geq \nu(Y)/2$, then, for $t > 0$,*

$$\nu(\{w > t\}) \leq 2 \inf_{c \in \mathbb{R}} \nu(\{|w-c| > t/2\}).$$

**Proof** If $|c| \leq t/2$, then $\{w > t\} \subset \{|w-c| > t/2\}$. Otherwise, $\{w = 0\} \subset \{|w-c| > t/2\}$, and so

$$\nu(\{w > t\}) \leq \nu(Y) \leq 2\nu(\{w=0\}) \leq 2\nu(\{|w-c| > t/2\}).$$

□

Let $v$ denote either $v_+$ or $v_-$. For $k \in \mathbb{Z}$, denote $v_k = v_{2^{k-1}}^{2^k}$ and $g_k = g\chi_{\{2^{k-1} < v \leq 2^k\}}$. Then

$$\mu(\{v > 2^k\}) \leq \mu(\{v_k > 2^{k-2}\}) \leq 2\mu(\{|v_k - (v_k)_B| > 2^{k-3}\}) \tag{58}$$

for $k \in \mathbb{Z}$. Let $C = 2^5 C_0$, where $C_0$ is the constant from inequality (57). Using (58) and (57) for the pair $(v_k, g_k)$ we obtain

$$\begin{aligned}
&\int_B \Phi_s\left(\frac{v}{Cr\mu(B)^{-1/s}}\left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1/s}\right) d\mu \\
&\leq \sum_{k\in\mathbb{Z}} \int_{\{2^k < v \leq 2^{k+1}\}} \Phi_s\left(\frac{v}{Cr\mu(B)^{-1/s}}\left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1/s}\right) d\mu \\
&\leq \sum_{k\in\mathbb{Z}} \Phi_s\left(\frac{2^{k+1}}{Cr\mu(B)^{-1/s}}\left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1/s}\right) \mu(\{v > 2^k\}) \\
&\leq \sum_{k\in\mathbb{Z}} \Phi_s\left(\frac{2^{k-3}}{C_0 r\mu(B)^{-1/s}}\left(\int_{\hat{B}} \Phi(g_k)\, d\mu\right)^{-1/s}\right) \mu(\{|v_k - (v_k)_B| > 2^{k-3}\}) \\
&\leq \sum_{k\in\mathbb{Z}} \int_{\hat{B}} \Phi(g_k)\, d\mu \\
&\leq \int_{\hat{B}} \Phi(g)\, d\mu.
\end{aligned}$$

Thus

$$\inf_{b\in\mathbb{R}} \int_B \Phi_s\left(\frac{|u-b|}{Cr\mu(B)^{-1/s}}\left(\int_{\hat{B}} \Phi(g)\, d\mu\right)^{-1/s}\right) d\mu \leq \int_{\hat{B}} \Phi(g)\, d\mu. \tag{59}$$

This, for the pair $\|g\|^{-1}_{L^\Phi(\hat{B})}(u,g)$ in place of $(u,g)$, yields

$$\inf_{b\in\mathbb{R}} \|u-b\|_{L^{\Phi_s}(B)} \le Cr\mu(B)^{-1/s}\|g\|_{L^\Phi(\hat{B})}.$$

Since $\|u-u_B\|_{L^\Phi(A)} \le 2\inf_{b\in\mathbb{R}}\|u-b\|_{L^\Phi(A)}$ for any set $A$ of finite measure, the proof is complete. □

# 4 Strong inequalities without truncation

In this section we will show how the weak estimate (24) implies strong ones. We begin with an easy lemma.

**Lemma 4.1** *Let $\mu(X)<\infty$, and let $\Phi$ and $\Psi$ be Young functions such that*

$$\int_1^\infty \frac{\Psi'(t)}{\Phi(t)}\,dt < \infty. \tag{60}$$

*Then $L^\Phi_w(X)\subset L^\Psi(X)$ and there is a constant $C=C(\Psi,\Phi)$ such that*

$$\|u\|_{Ł^\Psi(X)} \le C\|u\|_{Ł^\Phi_w(X)}. \tag{61}$$

**Proof** Assume $\|u\|_{L^\Phi_w(X)}=1$. Denoting $\tilde{\mu}=\mu(X)^{-1}\mu$, we obtain

$$\begin{aligned}\int_X \Psi(|u|)\,d\tilde{\mu} &= \int_0^\infty \Psi'(t)\tilde{\mu}(\{x\in X: |u|>t\})\,dt \\ &\le \Psi(1)+\int_1^\infty \Psi'(t)\tilde{\mu}(\{x\in X: |u|>t\})\,dt \\ &\le \Psi(1)+\int_1^\infty \frac{\Psi'(t)}{\Phi(t)}\,dt =: C',\end{aligned}$$

which implies (61) with $C=\max\{C',1\}$. □

For a measure $\nu$ on $X$, denote

$$\|u\|_{A^{\Phi,s}_\tau(U;\nu)} = \sup_{\mathcal{B}\in\mathcal{B}_\tau(U)} \|\sum_{B\in\mathcal{B}}\left(\nu(B)^{-1/s}\fint_B |u-u_B|\,d\nu\right)\chi_B\|_{L^\Phi(U;\nu)}.$$

For a ball $B\subset X$, denote $\mu_B=\mu(B)^{-1}\mu$.

**Theorem 4.2** *Suppose that the assumptions of Theorem 1.9 are in force,* (14) *holds, and that $\Psi$ is a Young function satisfying*

$$\int_1^\infty \frac{\Psi'(t)}{\Phi_s(t)}\,dt < \infty. \tag{62}$$

*Then*

$$\|u-u_B\|_{Ł^\Psi(B)} \le C\|u\|_{A^{\Phi,s}_\tau(\hat{B};\mu_{\hat{B}})}, \tag{63}$$

*where $C = C(C_d, s, \tau, \delta, \Phi, \Psi)$. Moreover, if $\mu$ satisfies (2), $g \in L^{\Phi}(\hat{B})$, and a pair $\|g\|^{-1}_{L^{\Phi}(\hat{B})}(u, g)$ satisfies the $\Phi$-Poincaré inequality in $\hat{B}$, then*

$$\|u - u_B\|_{Ł^{\Psi}(B)} \le C r_B \|g\|_{Ł^{\Phi}(\hat{B})}, \tag{64}$$

*where $C = C(C_s, s, C_P, \tau, \delta, \Phi, \Psi)$.*

**Proof** Theorem 1.9, applied to the measure $\mu_B = \mu(B)^{-1}\mu$, yields

$$\|u - u_B\|_{Ł_w^{\Phi_s}(B)} \le C \|u\|_{A_\tau^{\Phi,s}(\hat{B};\mu_B)}.$$

So, by Lemma 4.1,

$$\|u - u_B\|_{Ł^{\Psi}(B)} \le C \|u\|_{A_\tau^{\Phi,s}(\hat{B};\mu_B)}.$$

Since

$$\|\cdot\|_{L^{\Phi}(\hat{B};\mu_B)} \le C \|\cdot\|_{L^{\Phi}(\hat{B};\mu_{\hat{B}})},$$

it follows that

$$\|u\|_{A_\tau^{\Phi,s}(\hat{B};\mu_B)} \le C \|u\|_{A_\tau^{\Phi,s}(\hat{B};\mu_{\hat{B}})}.$$

Inequality (64) follows from inequalities (63) and (53). □

Notice that if $\Phi_s$ increases quickly enough, condition (62) is satisfied with $\Psi(t) = \Phi_s(t/2)$, and we have

$$\|u - u_B\|_{Ł^{\Phi_s}(B)} \le C \|u\|_{A_\tau^{\Phi,s}(\hat{B};\mu_{\hat{B}})}. \tag{65}$$

In particular, this is the case when $\Phi$ is equivalent to $t \mapsto t^s$ near infinity.

Suppose now that (8) holds with a functional $a$ satisfying (11), and that $\Phi$ is equivalent to $t \mapsto t^s$ near infinity. Then

$$\begin{aligned}
\|u\|_{A_\tau^{\Phi,s}(\hat{B};\mu_{\hat{B}})} &= \sup_{\mathcal{B}\in\mathcal{B}_\tau(\hat{B})} \Big\| \sum_{B\in\mathcal{B}} \left( \mu_{\hat{B}}(B)^{-1/s} \fint_B |u - u_B|\, d\mu \right) \chi_B \Big\|_{Ł^{\Phi}(\hat{B})} \\
&\le C \sup_{\mathcal{B}\in\mathcal{B}_\tau(\hat{B})} \Big\| \sum_{B\in\mathcal{B}} \left( \mu_{\hat{B}}(B)^{-1/s} \fint_B |u - u_B|\, d\mu \right) \chi_B \Big\|_{Ł^{s}(\hat{B})} \\
&\le C \sup_{\mathcal{B}\in\mathcal{B}_\tau(\hat{B})} \left( \sum_{B\in\mathcal{B}} \left( \fint_B |u - u_B|\, d\mu \right)^s \right)^{1/s} \\
&\le C \|u\|_a \sup_{\mathcal{B}\in\mathcal{B}_\tau(\hat{B})} \left( \sum_{B\in\mathcal{B}} a(\tau B)^s \right)^{1/s} \\
&\le C\, \|u\|_a a(\hat{B}),
\end{aligned}$$

where the first inequality comes from (28). Thus (65) implies the generalized Trudinger inequality (12).

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University of Jyväskylä, Department of Mathematics and Statistics, P.O. Box 35, FI-40014 Jyväskylä, Finland
*E-mail address*: **toheikki@maths.jyu.fi**