Boundary behavior of conformal deformations

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Abstract

We study conformal deformations of the Euclidean metric in the unit ball $\mathbb{B}^n$. We assume that the density associated with the deformation satisfies a Harnack inequality and an arbitrary volume growth condition on the isodiametric profile. We establish a Hausdorff (gauge) dimension estimate for the set $E \subset \partial\mathbb{B}^n$ where such a deformation mapping can "blow up". We also prove a generalization of Gerasch's theorem in our setting.

1 Introduction

We consider conformal deformations of type $f := Id : (\mathbb{B}^n, g_0) \to (\mathbb{B}^n, d_\rho)$ where g_0 is the canonical metric of the Euclidean unit ball $\mathbb{B}^n$ and d_ρ is a conformal metric derived from the continuous density $\rho : \mathbb{B}^n \to \mathbb{R}_+$ in the usual way:

$$d_\rho(x, y) = \inf_\gamma \int_\gamma \rho(z)|dz| \quad \text{for } x, y \in \mathbb{B}^n,$$

where the infimum is taken over all rectifiable curves joining x and y in $\mathbb{B}^n$. We also define a measure μ_ρ by setting

$$\mu_\rho(E) = \mathrm{Vol}_\rho(E) = \int_E \rho^n dm_n \quad \text{for a Borel set } E \subset \mathbb{B}^n,$$

where m_n denotes the n-dimensional Lebesgue measure. Deformations of this kind are originally motivated by the theory of (quasi)conformal mappings. We refer the reader to [1], [2] and [5] for more information and concrete examples of conformal metrics. Further, we say that the deformation mapping f *blows up* at a point $z \in \partial\mathbb{B}^n$ if

$$\lim_{x \to z} d_\rho(0, x) = \infty.$$

In our setting we assume that the density ρ satisfies a Harnack inequality, i.e., there exists a constant $A \geq 1$ so that

$$\frac{1}{A} \leq \frac{\rho(x)}{\rho(y)} \leq A \tag{1.1}$$

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whenever $y \in B(x, \frac{1}{2}d(x, \partial\mathbb{B}^n))$ for some $x \in \mathbb{B}^n$. This is equivalent to assuming that the identity mapping f above is uniformly quasi-symmetric in each ball $B(x, \frac{1}{2}(1-|x|))$. Note that in (1.1) the point x is the center of the Whitney-type ball instead of an arbitrary point. We prefer this formulation for technical reasons.

We also assume a growth condition on the *isodiametric profile* of $(\mathbb{B}^n, d_\rho)$ which we, following [5], define as a function $\eta_\rho : [0, \mathrm{diam}_\rho(\mathbb{B}^n)] \to [0, \infty]$,

$$\eta_\rho(r) = \sup\{\mu_\rho(D) \ : \ D \subset \mathbb{B}^n \text{ and } \mathrm{diam}_\rho(D) \leq r\}.$$

Notice that the condition $\eta_\rho(r) \leq Cr^n$ for all $r > 0$ is equivalent to assuming the so called *volume growth condition*

$$\mu_\rho(B(x, r)) \leq Cr^n \quad \text{for all } x \in \mathbb{B}^n \text{ and } r > 0. \tag{1.2}$$

It was shown in [2, Theorem 4.4] that if a continuous density ρ satisfies the Harnack inequality and the volume growth condition (1.2), then there is a set $E \subset \partial\mathbb{B}^n$ of n-capacity zero so that

$$\mathrm{length}_\rho([0, \xi)) < \infty \quad \text{for all } \xi \in \partial\mathbb{B}^n \setminus E,$$

and thus f cannot blow up on a set $E \subset \partial\mathbb{B}^n$ of positive n-capacity. This classical result was obtained by relying on the Gehring-Hayman inequality.

In this paper we shall establish a more general relationship between the growth of the isodiametric profile and the size of the set $E \subset \partial\mathbb{B}^n$ where the deformation mapping f can blow up. In our more general setting the Gehring-Hayman theorem is no longer available and, thus, a different approach is needed. Previously it was shown in [5, Theorem 5B] that the condition $\eta_\rho(r) \leq Cr^{n+\varepsilon}$ with $\varepsilon = \varepsilon(n, A) > 0$ together with the Harnack inequality is enough to guarantee that f cannot blow up on a set $E \subset \partial\mathbb{B}^n$ of positive $(n-1)$-Hausdorff measure. We shall extend this result.

First, observe that if the density ρ satisfies the Harnack inequality with a constant $A < 2$, then f cannot blow up anywhere on the boundary of $\mathbb{B}^n$, regardless of the growth of the isodiametric profile. Namely, in this case it follows immediately from the Harnack inequality that there is a constant $a < 1$ so that $\rho(z(1-t)) \leq Ct^{-a}$ for all $z \in \partial\mathbb{B}^n$ and $0 < t < 1$, see [5, Proposition 1]. Consequently, the integral $\int_{[0,z)} \rho(x)|dx|$ converges for all $z \in \partial\mathbb{B}^n$.

If $A \geq 2$, then the situation is no longer trivial. However, our first theorem will imply, for instance, that if $A > 2$ and $\eta_\rho(r) = o(r^n(\log r)^p)$ as $r \to \infty$ for some $p > 0$, then f cannot blow up on a subset $E \subset \partial\mathbb{B}^n$ of positive h-measure of gauge

$$h(t) = \frac{1}{\log(\frac{1}{t})^{p+n-1}}.$$

On the other hand, if $A = 2$, a much weaker growth condition on the isodiametric profile is sufficient for the previous conclusion. Indeed, our second theorem will show that it suffices to assume that $\eta_\rho(r) = o(r^{n+p})$ as $r \to \infty$.

We are now ready to state our results. The first theorem covers the case $A > 2$:

Theorem 1.1. *Let $A > 2$ and $c = 2\log_2 A - 2$. Let $\psi(r)$ be an increasing, differentiable and doubling function such that*

$$h(t) = \frac{\psi'(t^{-c})^{n-1} t^{(-c)(n-1)}}{\psi(t^{-c})^n}$$

is increasing, continuous and doubling so that $h(t) \to 0$ as $t \to 0$. Suppose that $\eta_\rho(r) = o(r^n \psi(r))$ as $r \to \infty$. Then f cannot blow up on a subset $E \subset \partial\mathbb{B}^n$ of positive h-measure.

Recall that a function $h(t)$ is doubling if there exists a constant $\beta > 0$ such that $h(2t) \leq \beta h(t)$ for all $t > 0$. Notice also that the qualitative properties of monotonity, differentiability or continuity for the functions ψ and h are only needed to guarantee that h is a proper "gauge function". Recall that the *generalized Hausdorff h-measure*, or simply *h-measure*, is defined by

$$H^h(E) = \lim_{r \to 0} \left(\inf \left\{ \sum h(\operatorname{diam} B_i) : E \subset \bigcup B_i,\ \operatorname{diam}(B_i) \leq r \right\} \right),$$

where the dimension gauge function h is required to be continuous and increasing with $h(0) = 0$. In particular, if $h(t) = t^\alpha$ with some $\alpha > 0$, then H^h is the usual *α-dimensional Hausdorff measure*, denoted also by H^α. See [8] or [3] for more information on the generalized Hausdorff measure.

The next theorem covers the case $A = 2$. Instead of assuming the doubling condition for ψ, it now suffices to assume that, for some $\beta > 0$, the function ψ satisfies

$$\psi(r+1) \leq \beta \psi(r) \tag{1.3}$$

for all $r > 0$. Observe that even the function $\psi(r) = \exp(r)$ satisfies this weaker condition.

Theorem 1.2. *Let $A = 2$ and $c = 4\rho(0)/\log 2$ and let $\psi(r)$ be an increasing, differentiable function satisfying* (1.3) *such that*

$$h(t) = \frac{\psi'(c \log \frac{1}{t})^{n-1}}{\psi(c \log \frac{1}{t})^n}$$

is increasing, continuous and doubling so that $h(t) \to 0$ as $t \to 0$. Suppose that $\eta_\rho(r) = o(r^n \psi(r))$ as $r \to \infty$. Then f cannot blow up on a subset $E \subset \partial\mathbb{B}^n$ of positive h-measure.

Let us now consider some concrete examples that show how our results parallel previous recent results. This discussion also demonstrates the essential sharpness of our results. For instance, suppose that the density ρ satisfies, in addition to the Harnack inequality, the growth condition $\eta_\rho(r) = o(r^{n+p})$ as $r \to \infty$ with some $p > 0$. Then Theorem 1.1 implies that f cannot blow up on a subset $E \subset \partial\mathbb{B}^n$ of positive α-dimensional Hausdorff-measure where $\alpha = cp > 0$ depends only on A and p. In particular, if $p = (n-1)/c$, then f cannot blow up on a set E of positive $(n-1)$-Hausdorff measure. Thus we recover [5, Theorem 5B] as a special case of Theorem 1.1.

If ρ satisfies the Harnack inequality with the constant $A = 2$, then Theorem 1.2 implies even a stronger result. Namely, it suffices to assume that $\eta_\rho(r) = o(\exp(pr))$ as $r \to \infty$ with a sufficiently small constant $p > 0$ depending only on n and $\rho(0)$ in order to conclude that f cannot blow up on a subset $E \subset \partial\mathbb{B}^n$ of positive $(n-1)$-dimensional Hausdorff measure. Note that this estimate is essentially sharp in the following sense: There exists a density ρ so that $A = 2$, the growth of η_ρ is exponential and f blows up on the entire boundary $\partial\mathbb{B}^n$. To see this, simply consider the radial density $\rho(x) = (1 - |x|)^{-1}$.

In the classical setting, where $\eta_\rho(r) \le Cr^n$ for all $r > 0$, we essentially recover the result of [2, Theorem 4.4], as the next remark shows. As a matter of fact, this even can be considered as a slight generalization, since we only require that $\eta_\rho(r) = O(r^n)$ as $r \to \infty$.

Remark 1.3. *Let $A \ge 1$ and let h be a proper gauge function satisfying*

$$\int_0 \frac{h(t)^{1/(n-1)}}{t} dt < \infty. \tag{1.4}$$

Suppose that $\eta_\rho(r) = O(r^n)$ as $r \to \infty$. Then f cannot blow up on a subset $E \subset \partial\mathbb{B}^n$ of positive h-measure.

In particular, the set E above cannot be of positive h-measure of gauge $1/(\log \frac{1}{t})^{n-1+\varepsilon}$, where $\varepsilon > 0$ is arbitrary. Thus we essentially (in terms of a gauge dimension) recover the sharp result of [2, Theorem 4.4], which states that E has n-capacity zero.

As a consequence for Theorems 1.1 and 1.2 we shall establish the following corollaries. They provide us a generalization of results of [7, Theorem 1] and [2, Lemma 7.5], which in turn are extensions of a theorem originally due to Gerasch [4], on the broadly accessibility of the boundary points of domains quasiconformally equivalent to a ball.

Corollary 1.4. *Let A and c be as in Theorem 1.1 and write $\alpha = cp$ for $0 < p \le (n-1)/c$. Suppose that $\eta_\rho(r) = o(r^{n+p})$ as $r \to \infty$. Then outside an α-dimensional set E for all $z \in \partial\mathbb{B}^n$ there is a sequence $(t_k) \to 1$ such that*

$$z \in B_\rho\big(t_k z, C\rho(t_k z)(1 - |t_k z|)\big)$$

for all $k \in \mathbb{N}$. Here $C > 0$ depends only on A, p and n.

Corollary 1.5. *Let $A \leq 2$ and let c be as in Theorem 1.2 and write $\alpha = cp$ for $0 < p \leq (n-1)/c$. Suppose that $\eta_\rho(r) = o(r^n \exp(pr))$ as $r \to \infty$. Then outside an α-dimensional set E for all $z \in \partial\mathbb{B}^n$ there is a sequence $(t_k) \to 1$ such that*

$$z \in B_\rho\big(t_k z, C\rho(t_k z)(1 - |t_k z|)\big)$$

for all $k \in \mathbb{N}$. Here $C > 0$ depends only on A, p and n.

2 Proofs of the results

The proof of Theorem 1.1. Let $E \subset \partial\mathbb{B}^n$ consist of all points where f blows up. Then $E \subset \{z \in \partial\mathbb{B}^n : \int_{[0,z)} \rho(x)|dx| = \infty\}$. Fix $r > 0$ and put

$$E_r = \{z \in \partial\mathbb{B}^n : \int_{[0,z)} \rho(x)|dx| \geq r\}.$$

Assume towards a contradiction that $H^h(E) > 0$, whence also $H^h(E_r) > 0$ since $E \subset E_r$. Then, by Frostman's lemma [6, Theorem 8.8], there exists a Radon measure μ supported in E_r such that $\mu(B(x, \hat{r})) \leq h(\hat{r})$ for all $x \in \partial\mathbb{B}^n$ and $\hat{r} > 0$ and that

$$\mu(E_r) \geq CH^h_\infty(E_r) \geq CH^h_\infty(E) > 0, \tag{2.1}$$

where $H^h_\infty(E) = \inf\{\sum_i h(r_i) : \ E \subset \bigcup_i B(x_i, r_i)\}$ is the usual Hausdorff h-content of E and the constant $C > 0$ depends only on n.

We write

$$A_r = \{x \in \mathbb{B}^n : \ x/|x| \in E_r \wedge \frac{r}{2} \leq d_\rho(0, x) \leq r\}.$$

Let $\mathcal{W}$ be a Whitney decomposition of $\mathbb{B}^n$ and let $\mathcal{W}_r$ be the collection of all the cubes $Q \in \mathcal{W}$ for which $Q \cap A_r \neq \emptyset$. Further, we denote the union of all the cubes $Q \in \mathcal{W}_r$ by D_r. Then it follows from the Harnack inequality that $\operatorname{diam}_\rho(D_r) \leq C(A)r$ for some constant $C(A) \geq 1$.

For a point $z \in \partial\mathbb{B}^n$, we define $t_z(\hat{r})$ by

$$\int_{t_z(\hat{r})}^1 \rho(\phi(z, t))dt = \hat{r},$$

where $\phi(z, t) = z(1 - t) \in \mathbb{B}^n$. Now, by the inequalities of Harnack and

Hölder, we have

$$\begin{aligned}
\frac{r}{2}\mu(E_r) &\le \int_{\partial\mathbb{B}^n}\int_{[t_z(r),t_z(r/2)]}\rho(\phi(z,t))dtd\mu \\
&\le \sum_{Q\in\mathcal{W}_r}\mu(S(Q))\operatorname{diam}_\rho(Q) \\
&\le c_0\sum_{Q\in\mathcal{W}_r}\mu(S(Q))\Big(\int_Q\rho^n dm\Big)^{1/n} \\
&\le c_0\Big(\sum_{Q\in\mathcal{W}_r}\int_Q\rho^n dm\Big)^{1/n}\Big(\sum_{Q\in\mathcal{W}_r}\mu(S(Q))^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \\
&= c_0\operatorname{Vol}_\rho(D_r)^{1/n}\Big(\sum_{Q\in\mathcal{W}_r}\mu(S(Q))^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}. \qquad (2.2)
\end{aligned}$$

Here and throughout the proof c_i denotes constants depending at most on $A, n, \rho(0)$ and the doubling constants of ψ and h. Also, we write $S(Q)\subset\partial\mathbb{B}^n$ for the "shadow" of the cube Q, i.e., $S(Q)$ consists of all points $z\in\partial\mathbb{B}^n$ for which the radius $[0,z)$ intersects the cube Q.

The Harnack inequality guarantees a polynomial growth behavior for the density ρ. More precisely, $\rho(\phi(z,t))\le A\rho(0)t^{-a}$ with $a=\log_2 A>1$ for all $z\in\partial\mathbb{B}^n$ and all $0<t<1$ whenever the Harnack inequality is satisfied by ρ. This in turn implies that

$$t_z(r/2)\le c_2 r^{-\frac{1}{a-1}}$$

and hence, for sufficiently large r, we obtain

$$\begin{aligned}
\Big(\sum_{Q\in\mathcal{W}_r}\mu(S(Q))^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} &\le \Big(\sum_{i\ge c_3\log_2 r}\sum_{Q\in\mathcal{W}_i}\mu(S(Q))^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \\
&\le \Big(\sum_{i\ge c_3\log_2 r}\max_{Q\in\mathcal{W}_i}\mu(S(Q))^{\frac{1}{n-1}}\sum_{Q\in\mathcal{W}_i}\mu(S(Q))\Big)^{\frac{n-1}{n}} \\
&\le \Big(\sum_{i\ge c_3\log_2 r}\max_{Q\in\mathcal{W}_i}\mu(S(Q))^{\frac{1}{n-1}}\mu(E_r)\Big)^{\frac{n-1}{n}}, \qquad (2.3)
\end{aligned}$$

where $\mathcal{W}_i$ denotes the ith generation of Whitney cubes, i.e., all the cubes $Q\in\mathcal{W}$ with sidelength 2^{-i}. Here we can take $c_3=\frac{1}{2(a-1)}$. Since $\operatorname{diam}(S(Q))\le C2^{-i}$ for each $Q\in\mathcal{W}_i$ and some constant C depending on n, it follows from the doubling property of h that, for each such Q,

$$\mu(S(Q))\le h(C2^{-i})\le c_4 h(2^{-i}).$$

By combining this with (2.2) and (2.3) we arrive at

$$r^n\mu(E_r)\le c_5\operatorname{Vol}_\rho(D_r)\Big(\sum_{i\ge c_3\log_2 r}h(2^{-i})^{\frac{1}{n-1}}\Big)^{n-1}. \qquad (2.4)$$

Choosing $c = \frac{1}{c_3} = 2\log_2 A - 2$ and integrating with a change of variable we deduce that

$$\begin{aligned}\sum_{i\geq c_3 \log_2 r} h(2^{-i})^{\frac{1}{n-1}} &\leq c_6 \int_0^{r^{-c_3}} h(t)^{\frac{1}{n-1}} \frac{dt}{t} \\ &= c_6 \int_0^{r^{-c_3}} \frac{\psi'(t^{-c})t^{-c-1}}{\psi(t^{-c})^{\frac{n}{n-1}}} dt \\ &\leq c_7 \psi(r)^{-\frac{1}{n-1}}. \qquad (2.5)\end{aligned}$$

Now we conclude by the definition of the isodiametric profile and (2.4) and (2.5) and the doubling property of ψ (recall that $\operatorname{diam}_\rho(D_r) \leq C(A)r$) that

$$\begin{aligned}\frac{\eta_\rho(\operatorname{diam}_\rho(D_r))}{\operatorname{diam}_\rho(D_r)^n \psi(\operatorname{diam}_\rho(D_r))} &\geq \frac{\operatorname{Vol}_\rho(D_r)}{\operatorname{diam}_\rho(D_r)^n \psi(\operatorname{diam}_\rho(D_r))} \\ &\geq \frac{c_8 \mu(E_r) r^n \psi(r)}{\operatorname{diam}_\rho(D_r)^n \psi(\operatorname{diam}_\rho(D_r))} \\ &\geq \frac{c_8 \mu(E_r) r^n \psi(r)}{c_9 r^n \psi(r)}.\end{aligned}$$

Furthermore, by the assumption on $\eta_\rho(r)$, this quantity tends to zero as $r \to \infty$. It follows that

$$\mu(E_r) \to 0 \text{ as } r \to \infty$$

which is a contradiction with (2.1). Hence the proof is complete. □

The proof of Theorem 1.2 The proof of Theorem 1.2 is similar to the one of Theorem 1.1, and thus we only indicate the important modifications needed. Notice first that the Harnack inequality with the constant $A = 2$ implies the growth condition $\rho(\phi(z,t)) \leq 2\rho(0)t^{-1}$ for all $z \in \partial\mathbb{B}^n$ and $0 < t < 1$. Consequently, we have the estimates

$$\operatorname{diam}_\rho(D_r) \leq r + c_0 \qquad (2.6)$$

and

$$t_z(r/2) \leq \exp(-\frac{r}{c_1}),$$

where c_0 depends only on n and $c_1 = 4\rho(0)$. Hence the inequality corresponding to (2.4) takes the form

$$r^n \mu(E_r) \leq c_2 \operatorname{Vol}_\rho(D_r)\Big(\sum_{i\geq r/c_1} h(2^{-i})^{\frac{1}{n-1}}\Big)^{n-1}.$$

The integration now implies

$$\begin{aligned}\sum_{i\geq r/c_1} h(2^{-i})^{\frac{1}{n-1}} &\leq c_2 \int_0^{2^{-r/c_1}} h(t)^{\frac{1}{n-1}} \frac{dt}{t} \\ &= c_2 \int_0^{2^{-r/c_1}} \frac{\psi'(c\log\frac{1}{t})}{\psi(c\log\frac{1}{t})^{\frac{n}{n-1}}} \frac{dt}{t} \\ &\leq c_3 \psi(c\log(2^{r/c_1}))^{-\frac{1}{n-1}} \\ &\leq c_3 \psi(r)^{-\frac{1}{n-1}},\end{aligned}$$

when we choose $c = c_1/\log 2 = 4\rho(0)/\log 2$. The final conclusions then follow in the same way as in the proof of Theorem 1.1, except that the inequality

$$\operatorname{diam}_\rho(D_r)^n \psi(\operatorname{diam}_\rho(D_r)) \leq c_9 r^n \psi(r)$$

follows from (2.6) and (1.3) instead of the doubling property of ψ. $\square$

The proof of Remark 1.3 We modify the proof of Theorem 1.1 in the following way. It follows from (1.4) that the sum

$$\Big(\sum_{i\geq c_3 \log_2 r} h(2^{-i})^{\frac{1}{n-1}}\Big)^{n-1}$$

in (2.4) tends to zero as $r \to \infty$. Consequently, the inequality (2.4) becomes

$$r^n \mu(E_r) \leq C \operatorname{Vol}_\rho(D_r) \varepsilon(r),$$

where $\varepsilon(r) \to 0$ as $r \to \infty$. Hence, by the assumption on $\eta_\rho(r)$, the quantity

$$\begin{aligned}\frac{\eta_\rho(\operatorname{diam}_\rho(D_r))}{\operatorname{diam}_\rho(D_r)^n} &\geq \frac{\operatorname{Vol}_\rho(D_r)}{\operatorname{diam}_\rho(D_r)^n} \\ &\geq \frac{\mu(E_r) r^n}{C r^n \varepsilon(r)}\end{aligned}$$

stays bounded as r tends to infinity. Thus $\mu(E_r) \to 0$ as $r \to \infty$ and the claim follows. $\square$

The proof of Corollary 1.4 By Theorem 1.1, there exists a set E_∞ of α-Hausdorff measure zero so that we have $d_\rho(0, z) < \infty$ for all $z \in \partial\mathbb{B}^n \setminus E_\infty$. Let us denote the set of such z's by $\partial_\rho \mathbb{B}^n$.

Lemma 7.5 in [2] states that the claim of the corollary is valid for every $z \in \partial\mathbb{B}^n$ for which

$$\rho(\phi(z,t)) = O(t^{-a}) \quad \text{as } t \to 0, \tag{2.7}$$

where $a \in (0, 1)$ is a constant. Therefore, it is enough to show that there is a constant $a \in (0, 1)$ depending only on A and n and p so that outside a set E of α-Hausdorff measure zero the condition (2.7) holds. (Observe that now the growth condition implied by the Harnack inequality alone in the proof of Theorem 1.1 is not sharp enough.)

To that end, we shall next show that Theorem 5.2 in [2] remains valid also in our setting. More precisely, we shall find a set E with $H^\alpha(E) = 0$ so that for all $z \in \partial_\rho \mathbb{B}^n \backslash E$ we have

$$\rho(\phi(z,t)) = o(t^{-1+\alpha/n}) \quad \text{as } t \to 0.$$

Let us fix α and define, for $j \in \mathbb{N}$, sets $G_j = \{z \in \partial_\rho \mathbb{B}^n : \int_{[0,z)} \rho \, ds \ \leq j\}$ and

$$F_j = \bigcup_{z \in G_j} \bigcup \{B\big(tz, \frac{1}{2}(1-t)\big) : \ 0 \leq t < 1\}.$$

Then F_j is open. Moreover, it follows from Harnack inequality that

$$\operatorname{diam}_\rho(F_j) < \infty,$$

cf. (4.1) in [2]. Hence, by the assumption on $\eta_\rho(r)$, we also have $\operatorname{Vol}_\rho(F_j) < \infty$. This implies that the function u_j, defined as $u_j(x) = \rho(x)^n$ for $x \in F_j$ and $u_j(x) = 0$ elsewhere, belongs to $L^1(\mathbb{B}^n)$. Thus there exists a set $E_j \subset \partial \mathbb{B}^n$ with $H^\alpha(E_j) = 0$ such that, for all $z \in \partial \mathbb{B}^n \backslash E_j$,

$$\int_{B(z,r) \cap \mathbb{B}^n} u_j \, dm_n = o(r^\alpha) \quad \text{as } r \to 0,$$

cf. [9, p. 118]. In particular, for each $z \in G_j \backslash E_j$,

$$\int_{B((1-t)z, \frac{1}{2}t)} \rho^n \, dm_m \leq \int_{B(z, \frac{3}{2}t)} u_j \, dm_m = o\big(t^\alpha\big) \quad \text{as } t \to 0,$$

and so, by the Harnack inequality,

$$\rho(\phi(z,t))^n \leq 2^n A^2 t^{-n} \int_{B((1-t)z, \frac{1}{2}t)} \rho^n \, dm_m = o(t^{\alpha - n}) \quad \text{as } t \to 0.$$

Since $\bigcup_{j \in \mathbb{N}} G_j = \partial_\rho \mathbb{B}^n$, the desired growth condition holds for all $z \in \partial \mathbb{B}^n \backslash E$, where $E = E_\infty \cup \bigcup_{j \in \mathbb{N}} E_j$. Clearly, $H^\alpha(E) = 0$.

We now choose $a = -1 + \alpha/n$ in (2.7) and, hence, the constant a depends only on A and n and p. The conclusion of the corollary follows. □

The proof of Corollary 1.5 is very similar to the one of Corollary 1.4 and is left to the reader.

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