Homeomorphisms of finite distortion: Discrete length of radial images

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Abstract

We study homeomorphisms of finite exponentially integrable distortion of the unit ball B^n onto a domain Ω of finite volume. We show that under such a mapping the images of almost all radii (in terms of a gauge dimension) have finite discrete length. We also show that our dimension estimate is essentially sharp.

1 Introduction

We continue the study of mappings $f : B^n \to \Omega \subset \mathbb{R}^n$ of *finite distortion* defined in the unit ball B^n of the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Thus f belongs to the Sobolev space $W^{1,1}_{\text{loc}}(B^n, \mathbb{R}^n)$, the Jacobian determinant J_f is locally integrable in B^n, and there is a measurable function $K \geq 1$ so that K is finite almost everywhere in B^n and that f satisfies the distortion inequality

$$|Df(x)|^n \leq K(x)J_f(x) \quad \text{for almost every } x \in B^n. \tag{1.1}$$

Here $|Df(x)|$ denotes the operator norm of the differential Df. In our setting we also assume that f is a homeomorphism. Thus, if we were to require that K be bounded, then f would be a quasiconformal mapping. However, instead of boundedness we only require that K is exponentially integrable, i.e. there exists a constant $\lambda > 0$ such that $\exp(\lambda K) \in L^1(B^n)$. The mappings of this kind have been studied extensively in the recent years. See e.g. [1], [5], [6], [8], [9], [11], [12] for some important properties of (possibly non-homeomorphic) mappings of finite distortion. Also see [13], [15] and [17] for boundary behavior properties.

In this paper we show that if $f(B^n)$ has finite volume, then the images of "most" radii under a homeomorphism f of finite exponentially integrable distortion have finite discrete length. This is an analog of Beurling's theorem about the existence of radial limits for conformal homeomorphisms, see [2]. More precisely, our main result is the following:

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Theorem 1.1. *Let $f : B^n \to f(B^n) \subset \mathbb{R}^n$ be a homeomorphism of finite distortion so that $f(B^n)$ has finite volume and $\exp(\lambda K) \in L^1(B^n)$ for some $\lambda > 0$. Let h be a doubling weight function such that*

$$\int_0 \Big(h(t) \log \frac{1}{t} \Big)^{1/(n-1)} \frac{dt}{t} < \infty. \tag{1.2}$$

Then there exists a set $E \subset \partial B^n$ so that $H^h(E) = 0$ and $\operatorname{length}_d(f(I_\xi)) < \infty$ for all $\xi \in \partial B^n \setminus E$.

Here $I_\xi \subset B^n$ denotes the radius corresponding to the point $\xi \in \partial B^n$ and the *discrete length* of the image of I_ξ is defined by

$$\operatorname{length}_d(f(I_\xi)) = \sum_{Q \in \mathcal{W}:\ Q \cap I_\xi \neq \emptyset} \operatorname{diam}(f(Q)),$$

where $\mathcal{W}$ is a fixed Whitney decomposition of B^n, i.e. $\mathcal{W}$ is a collection of closed dyadic cubes $Q \subset B^n$ with pairwise disjoint interiors such that

$$\bigcup_{Q \in \mathcal{W}} Q = B^n$$

and that $\operatorname{diam}(Q) \leq \operatorname{dist}(Q, \partial B^n) \leq 4 \operatorname{diam}(Q)$. Recall that the *generalized Hausdorff h-measure*, or simply *h-measure*, is defined by

$$H^h(E) = \lim_{r \to 0} \Big(\inf \Big\{ \sum h(\operatorname{diam} B_i) : E \subset \bigcup B_i,\ \operatorname{diam}(B_i) \leq r \Big\} \Big),$$

where the dimension gauge function h is required to be continuous and increasing with $h(0) = 0$. In particular, if $h(t) = t^\alpha$ with some $\alpha > 0$, then H^h is the usual *α-dimensional Hausdorff measure*, denoted also by H^α. See [16] for more information on the generalized Hausdorff measure.

The notion of discrete length was used already by Heinonen and Rohde in [7] as a tool for proving the Gehring-Hayman inequality for quasihyperbolic geodesics. The use of the discrete length in Theorem 1.1 instead of the ordinary length is essential. Indeed, there exists even a bounded quasiconformal mapping of B^n so that $f(I_\xi)$ has infinite length for all $\xi \in E \subset \partial B^n$, where E has Hausdorff dimension $n - 1$, see [7].

In [4, Theorem 4.4] a quasiconformal version of Beurling's theorem was established by relying on the Gehring-Hayman Theorem. In our setting, however, the Gehring-Hayman Theorem is not available, and thus a different technique is applied in the proof of Theorem 1.1. This is also the reason why our estimate for the size of the exceptional set E is in terms of Hausdorff (gauge) dimension instead of capacity. Nevertheless, the dimension estimate of Theorem 1.1 is sharp at least in the plane:

Theorem 1.2. *For any dimension gauge h satisfying*

$$\int_0 h(t)\log(1/t)\frac{dt}{t}=\infty, \tag{1.3}$$

there exists a homeomorphism $f : B^2 \to f(B^2) \subset \mathbb{R}^2$ of finite distortion satisfying the assumptions of Theorem 1.1 and a set $E \subset \partial B^2$ so that $\mathrm{length}_d(f(I_\xi)) = \infty$ *for all $\xi \in E$ and that $H^h(E) > 0$.*

In particular, Theorem 1.1 implies that if a mapping $f : B^2 \to f(B^2) \subset \mathbb{R}^2$ satisfies the assumptions of Theorem 1.1, then the exceptional set $E = \{\xi \in \partial B^2 : \ \mathrm{length}_d(f(I_\xi)) = \infty\}$ has h-measure zero with the dimension gauge

$$h(t)=\frac{1}{(\log\frac{1}{t})^{2+\varepsilon}},$$

where $\varepsilon > 0$ is arbitrary. On the other hand, Theorem 1.2 implies that there exists a mapping f satisfying the assumptions of Theorem 1.1 such that the set E above has positive h-measure with the dimension gauge

$$h(t)=\frac{1}{(\log\frac{1}{t})^{2}}. \tag{1.4}$$

For a quasiconformal mapping (f as above but with a bounded distortion K) our proof implies that the exceptional set E has h-measure zero with any h satisfying

$$\int_0 \frac{h(t)^{1/(n-1)}}{t}dt<\infty.$$

In particular, $H^h(E) = 0$ with $h(t) = 1/(\log\frac{1}{t})^{n-1+\varepsilon}$, where $\varepsilon > 0$ is arbitrary. Thus, in this setting, we essentially (in terms of gauge dimension) recover the sharp classical result, which states that E has n-capacity zero.

The next corollary follows immediately from Theorem 1.1:

Corollary 1.3. *Let $f : B^n \to f(B^n) \subset \mathbb{R}^n$ be a homeomorphism of finite distortion so that $f(B^n)$ has finite volume and* $\exp(\lambda K) \in L^1(B^n)$ *for some* $\lambda > 0$. *Let h be a doubling weight function satisfying* (1.2). *Then the radial limit* $\lim_{t\to 1} f(t\xi)$ *exists for H^h-almost every $\xi \in \partial B^n$.*

2 Proof of the main result

Proof of Theorem 1.1. Let $E = \{x \in \partial B^n : \ \mathrm{length}_d(f(I_x)) = \infty\}$. Fix $k \in \mathbb{N}$ and put $E_k = \{x \in \partial B^n : \ \mathrm{length}_d(f(I_x)) \geq k\}$. Assume towards a contradiction that $H^h(E) > 0$, whence also $H^h(E_k) > 0$ because $E \subset E_k$. Then, by Frostman's lemma [14, Theorem 8.8], there exists a Radon measure μ supported in E_k so that $\mu(B(x, r)) \leq h(r)$ for all $x \in \partial B^n$ and $r > 0$ and that

$$\mu(E_k) \approx H^h_\infty(E_k) \geq H^h_\infty(E) > 0, \tag{2.1}$$

where $H^h_\infty(E) = \inf\{\sum_i h(r_i) : E \subset \bigcup_i B(x_i, r_i)\}$ is the usual Hausdorff h-content of E. Here and throughout the proof we denote by $\lesssim$, $\approx$ or $\gtrsim$ an inequality up to some positive constant depending only on n.

Define for each $Q \in \mathcal{W}$ a ball $B_Q = B(x_Q, \mathrm{diam}(Q)/2)$, where x_Q is the center of Q. Let $\mathcal{B}$ be the collection of all such balls.

Because $f \in W^{1,1}_{\mathrm{loc}}(B^n, \mathbb{R}^n)$ and $\exp(\lambda K) \in L^1(B^n)$ and $J_f \in L^1_{\mathrm{loc}}(B^n)$, the distortion inequality (1.1) together with Hölder's inequality implies that $f \in W^{1,p}_{\mathrm{loc}}(B^n, \mathbb{R}^n)$ for all $p < n$. Fix $n-1 < p < n$. Let x be the center of some ball $B = B(x,r) \in \mathcal{B}$ and let $c = c(n) > 1$ so that $\mathrm{dist}(cB, \partial B^n) \geq \frac{1}{2}\mathrm{dist}(B, \partial B^n)$. By Sobolev's inequality we deduce that (cf. [9, Lemma 4.10.1])

$$\mathrm{diam}(f(S^{n-1}(x,t))) \lesssim t^{1-\frac{n-1}{p}} \Big(\int_{S^{n-1}(x,t)} |Df|^p \Big)^{1/p} \tag{2.2}$$

for a.e. $t \in]r, cr[$. Here $S^{n-1}(x,t)$ denotes an x-centered sphere of radius t. Hence we can choose $t \in]r, cr[$ so that the inequality (2.2) holds together with

$$\int_{S^{n-1}(x,t)} |Df|^p \lesssim \frac{1}{r} \int_{B(x,cr)} |Df|^p.$$

Moreover, since f is a homeomorphism, we have that

$$\mathrm{diam}(f(B(x,r))) \leq \mathrm{diam}(f(S^{n-1}(x,t))).$$

Thus we arrive at

$$\begin{aligned} \mathrm{diam}(f(B(x,r))) &\lesssim r^{1-\frac{n-1}{p}} \Big(\frac{1}{r} \int_{B(x,cr)} |Df|^p \Big)^{1/p} \\ &= r^{\frac{p-n}{p}} \Big(\int_{B(x,cr)} |Df|^p \Big)^{1/p}. \end{aligned} \tag{2.3}$$

By applying (1.1) and Hölder's inequality, we deduce further that

$$\begin{aligned} \Big(\int_{B(x,cr)} |Df|^p \Big)^{1/p} &\leq \Big(\int_{B(x,cr)} J_f^{\frac{p}{n}} K^{\frac{p}{n}} \Big)^{1/p} \\ &\leq \Big(\int_{B(x,cr)} J_f \Big)^{1/n} \Big(\int_{B(x,cr)} K^{\frac{p}{n-p}} \Big)^{\frac{n-p}{np}} \\ &\lesssim r^{\frac{n-p}{p}} \Big(\int_{B(x,cr)} J_f \Big)^{1/n} \Big(\fint_{B(x,cr)} K^{\frac{p}{n-p}} \Big)^{\frac{n-p}{np}}. \end{aligned} \tag{2.4}$$

By combining (2.3) and (2.4) we conclude that

$$\mathrm{diam}(f(B(x,r))) \lesssim \Big(\int_{B(x,cr)} J_f \Big)^{1/n} \Big(\fint_{B(x,cr)} K^{\frac{p}{n-p}} \Big)^{\frac{n-p}{np}}. \tag{2.5}$$

It follows from the assumption $\exp(\lambda K) \in L^1$ by Jensen's inequality that

$$\fint_{B(x,r)} K^{\frac{p}{n-p}} \leq c_1 (\log(1/r))^{\frac{p}{n-p}}$$

for any ball $B(x,r) \subset B^n$ with $r \leq \frac{1}{2}$. Here and throughout the proof we denote by c_i positive constants depending only on $\lambda, n, \int_{B^n} \exp(\lambda K)$ and the doubling constant of h. Combining this with (2.5) we obtain

$$\operatorname{diam}(f(B)) \leq c_2\Big(\int_{cB} J_f\Big)^{1/n}\Big(\log(1/\operatorname{diam}(B))\Big)^{1/n} \tag{2.6}$$

for all $B \in \mathcal{B}$.

For a point $x \in \partial B^n$ let $\mathcal{B}(x)$ consist of all the balls $B \in \mathcal{B}$ such that the radius I_x intersects the ball B. Furthermore, denote by $\mathcal{B}_j$ the balls corresponding to the jth generation of Whitney cubes, i.e. $\mathcal{B}_j$ consists of all the balls $B \in \mathcal{B}$ of radius $\frac{\sqrt{n}}{2}2^{-j}$. By using (2.6), the definition of E_k and Hölder's inequality we obtain the following chain of inequalities:

$$\begin{aligned}
\mu(E_k)k &\leq \int_{\partial B^n} \operatorname{length}_d(f(I_x))\, d_\mu x \\
&\leq \int_{\partial B^n} \sum_{B\in\mathcal{B}(x)} \operatorname{diam}(f(B))\, d_\mu x \\
&\leq \sum_{B\in\mathcal{B}} \mu(S(B)) \operatorname{diam}(f(B)) \\
&\leq c_2 \sum_{B\in\mathcal{B}} \mu(S(B))\Big(\int_{cB} J_f\Big)^{1/n}\Big(\log(1/\operatorname{diam}(B))\Big)^{1/n} \\
&\leq c_2\Big(\sum_{B\in\mathcal{B}}\int_{cB} J_f\Big)^{1/n}\Big(\sum_{B\in\mathcal{B}} \mu(S(Q))^{\frac{n}{n-1}}(\log(1/\operatorname{diam}(B)))^{\frac{1}{n-1}}\Big)^{\frac{n-1}{n}} \\
&\leq c_3\Big(\int_{B^n} J_f\Big)^{1/n}\Big(\sum_{j=1}^{\infty}\sum_{B\in\mathcal{B}_j} \mu(S(B))^{\frac{n}{n-1}} j^{\frac{1}{n-1}}\Big)^{\frac{n-1}{n}},
\end{aligned} \tag{2.7}$$

where $S(B) \subset \partial B^n$ denotes the "shadow" of a ball B, i.e. those points $x \in \partial B^n$ for which $I_x \cap B \neq \emptyset$. Note that the balls cB, $B \in \mathcal{B}$, have bounded overlap. Furthermore, since $\operatorname{diam}(S(B)) \leq C2^{-j}$ for all $B \in \mathcal{B}_j$ with some $C > 0$ depending only on n, we deduce that

$$\begin{aligned}
\sum_{B\in\mathcal{B}_j} \mu(S(B))^{\frac{n}{n-1}} &\leq \max_{B\in\mathcal{B}_j} \mu(S(B))^{\frac{1}{n-1}} \sum_{B\in\mathcal{B}_j} \mu(S(B)) \\
&\lesssim \max_{B\in\mathcal{B}_j} \mu(S(B))^{\frac{1}{n-1}} \mu(E_k) \\
&\lesssim h(C2^{-j})^{\frac{1}{n-1}} \mu(E_k) \\
&\leq c_4 h(2^{-j})^{\frac{1}{n-1}} \mu(E_k),
\end{aligned} \tag{2.8}$$

where the last inequality follows by the doubling property of h. Finally, by (2.7) and (2.8) we conclude that

$$\mu(E_k)^{1/n} k \leq c_5\Big(\int_{B^n} J_f\Big)^{1/n}\Big(\sum_{j=1}^{\infty} h(2^{-j})^{\frac{1}{n-1}} j^{\frac{1}{n-1}}\Big)^{\frac{n-1}{n}},$$

where the sum on the right-hand side of the inequality converges by the assumption (1.2). Notice that $\int_{B^n} J_f < \infty$ because $f(B^n)$ has finite volume, see [9, Theorem 6.2.1]. Hence $\mu(E_k) \to 0$ when $k \to \infty$, but this is a contradiction with (2.1). It follows that $H^h(E) = 0$.

Remark 2.1. *The above proof gives us a stronger statement than what is asserted in Theorem 1.1. Indeed, one has the estimate $H^h_\infty(E_k) \leq C/k^n$, where the constant C may depend on h.*

3 Sharpness of the result

This section is devoted to proving Theorem 1.2. The main idea of our proof is the following: We will first construct a set E on the boundary of the unit square $Q = [-1, 1]^2$ (it is easier to work in the square) with $H^h(E) > 0$ and a homeomorphism $f : Q \to Q$ of finite exponentially integrable distortion so that it maps the set E to a set of zero logarithmic capacity on the boundary of Q. We will then map Q conformally to a 'logarithmic spiral' around the origin so that $f(E) \subset \partial Q$ (of zero logarithmic capacity) gets mapped to the origin.

For the sake of simplicity, we will write the detailed proof in the case that the set E has positive h-measure with the concrete dimension gauge

$$h(t) = \frac{1}{(\log \frac{1}{t})^2}.$$

After dealing with this special case we will discuss the modifications needed in the construction of E in order to obtain the estimate $H^h(E) > 0$ for any given dimension gauge h satisfying (1.3).

Proof of Theorem 1.2. Let $t_0 = 1$ and let $t_k = \exp(-2^{k/2})$ for all $k = 1, 2, ...$ We construct a "Cantor-like" set $E \subset [0, 1] \times \{-1\}$ on the boundary of Q in the following way. Starting with the unit interval $[0, 1]$ of length t_0, we select two intervals, each of length t_1, one from the middle of $[0, 1/2]$ and one from the middle of $[1/2, 1]$. This gives us our first generation of basic intervals. We now iterate the process: given a $(k-1)$st generation basic interval I, we select two intervals, each of length t_k, one from the middle of each of the two halves of I. This will give us 2^k kth generation intervals each of length t_k. We define the set E to be the intersection over all generations of the unions of all kth generation basic intervals. A standard calculation reveals that $H^h(E) > 0$ for $h(t) = 1/(\log \frac{1}{t})^2$.

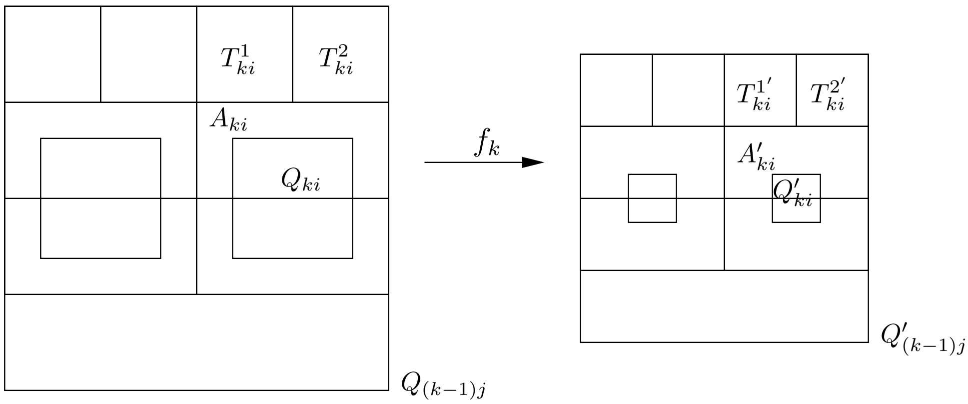
In the same way we also construct a set $E' \subset [0, 1] \times \{-1\}$ by choosing $t'_0 = 1$ and $t'_k = \exp(-2^k)$ for all $k = 1, 2, ...$ and carrying out the same construction outlined above. For the resulting Cantor set we have that $H^{h_0}(E') < \infty$ with the dimension gauge $h_0(t) = 1/\log(1/t)$. This implies that E' has zero logarithmic capacity.

Next we construct a homemorphism $f : Q \to Q$ such that $f(E) = E'$ and the restriction of f to the interior of Q has finite exponentially integrable distortion. We accomplish this by modifying the construction found in [6]. Given a kth generation basic interval I_{ki}, $i \in \{1, 2, ..., 2^k\}$, for E, let Q_{ki} denote the closed square of sidelength t_k centered at the center of I_{ki}. Then denote by P_{ki} a bigger square, concentric about Q_{ki} and of sidelength $t_{k-1}/2$. Set $A_{ki} = P_{ki} \setminus Q_{ki}$. Hence A_{ki} is a 'frame' or a 'spherical ring' when we work with the norm

$$\|x\| = \|x\|_\infty = \max\{|x_1|, |x_2|\}.$$

In the same way we define cubes Q'_{ki} and P'_{ki} with sidelengths t'_k and $t'_{k-1}/2$ respectively. Thus Q'_{ki} are formed using the kth generation basic intervals in the construction of E' while P'_{ki} are formed using the halves of the $(k-1)$st generation basic intervals. We then define $A'_{ki} = P'_{ki} \setminus Q'_{ki}$.

In the picture below there is an illustration of the squares $Q_{(k-1)j}$ and $Q'_{(k-1)j}$ for some k and j. Thus the horizontal line across the center of the left-hand side square illustrates some $(k-1)$th generation basic interval of length t_{k-1}, while the corresponding line in the right-hand side square has length t'_{k-1}. Note that the two squares T^1_{ki} and T^2_{ki} both have sidelength $t_{k-1}/4$, and the squares $T^{1'}_{ki}$ and $T^{2'}_{ki}$ both have sidelength $t'_{k-1}/4$.



Now define $f_1 : Q \to Q$,

$$f_1 = \begin{cases} \text{the identity outside } \;]0,1[\times]-1,-\tfrac{1}{2}[, \\ \text{a similarity of } Q_{1i} \cap Q \text{ onto } Q'_{1i} \cap Q, \\ \text{a similarity of } T^1_{1i} \text{ onto } T^{1'}_{1i}, \\ \text{a similarity of } T^2_{1i} \text{ onto } T^{2'}_{1i}, \\ \text{a 'radial stretching' of } A_{1i} \cap Q \text{ onto } A'_{1i} \cap Q. \end{cases}$$

Note that the only distortion for f_1 comes from the 'radial stretching' and this all 'lives' in the (upper halves of the) annuli A_{1i} We iterate this construction by defining for all $k \geq 2$ the function $f_k : Q \to Q$,

$$f_k = \begin{cases} f_{k-1} \text{ outside the union } \bigcup_i Q_{(k-1)i}, \\ \text{a similarity of } Q_{ki} \cap Q \text{ onto } Q'_{ki} \cap Q, \\ \text{a similarity of } T^1_{ki} \text{ onto } T^{1'}_{ki}, \\ \text{a similarity of } T^2_{ki} \text{ onto } T^{2'}_{ki}, \\ \text{a `radial stretching' of } A_{ki} \cap Q \text{ onto } A'_{ki} \cap Q. \end{cases}$$

Again, the only (new) distortion for f_k 'lives' in the annuli A_{ki}. In this way we obtain a homeomorphism $f = \lim_{k\to\infty} f_k$ which satisfies $f(E) = E'$.

Let us next define these radial stretchings more precisely. Consider the spherical rings (in the $\|\cdot\|_\infty$-metric)

$$A = \{x : \ r < \|x\| < R\} \quad \text{and} \quad A' = \{y : \ r' < \|y\| < R'\}$$

and the radial homeomorphism

$$y = \varphi(x) = \frac{x}{\|x\|}\rho(\|x\|), \quad \text{where } \rho(t) = at^b \quad \text{with some } a, b > 0.$$

Now $\varphi(A) = A'$ provided that

$$r' = ar^b \quad \text{and} \quad R' = aR^b,$$

which implies that

$$b = \frac{\log R' - \log r'}{\log R - \log r}.$$

Moreover, an elementary reasoning shows that

$$|D\varphi(x)| \approx \max\Big\{\frac{\rho(\|x\|)}{\|x\|},\ \rho'(\|x\|)\Big\} \quad \text{and} \quad J_\varphi(x) \approx \frac{\rho'(\|x\|)\rho(\|x\|)}{\|x\|},$$

see for example [10, Lemma 4.1]. It follows that

$$K_\varphi(x) \approx \begin{cases} b & \text{if } b \geq 1 \\ b^{-1} & \text{if } b < 1. \end{cases} \tag{3.1}$$

Now we examine the distortion of f in A_{ki}. Recall that f maps A_{ki} to A'_{ki}, where the inner, outer radii (in the $\|\cdot\|_\infty$-metric) are $r = t_k/2$, $R = t_{k-1}/4$ for A_{ki} and $r' = t'_k/2$, $R' = t'_{k-1}/4$ for A'_{ki}, respectively. A calculation reveals that

$$\begin{aligned} b_k = \frac{\log(t'_{k-1}/4) - \log(t'_k/2)}{\log(t_{k-1}/4) - \log(t_k/2)} &= \frac{-2^{k-1} - \log 4 + 2^k + \log 2}{-2^{(k-1)/2} - \log 4 + 2^{k/2} + \log 2} \\ &= \frac{2^{k-1} - \log 2}{(1 - \frac{1}{\sqrt{2}})2^{k/2} - \log 2} \\ &\approx 2^{k/2}, \end{aligned}$$

and thus we deduce by (3.1) that f has distortion

$$K_f \leq c2^{k/2} \quad \text{in} \;\; A_{ki} \cap Q \tag{3.2}$$

with some absolute constant $c > 0$.

Next we estimate the integral of $\exp(\lambda K_f)$ over the interior of Q by applying the estimate (3.2) to all spherical rings A_{ki}, $k = 1, 2, ..., i = 1, ..., 2^k$. Since each of these rings has measure no larger than the area of P_{ki} or $(t_{k-1}/2)^2 \leq \exp(-2^{(k+1)/2})$, we conclude that

$$\begin{aligned}\int_{\operatorname{int} Q} \exp(\lambda K_f) &\lesssim \sum_{k=1}^{\infty} 2^k \; \exp\Big(- 2^{(k+1)/2}\Big) \exp\Big(\lambda c 2^{k/2}\Big) \\ &\lesssim \sum_{k=1}^{\infty} \exp\Big(k - 2^{(k+1)/2} + \lambda c 2^{k/2}\Big) \\ &< \infty,\end{aligned}$$

when we choose $\lambda \leq \frac{1}{c}$. Hence the restriction of f to the interior of Q has finite exponentially integrable distortion.

To establish the theorem, we take a conformal mapping $g_1 : \operatorname{int} Q \to \Omega \subset H$ such that it maps the set E' of zero logarithmic capacity to $\{0\} \subset \partial\Omega$ in terms of a limit, i.e. $g_1(x) \to 0$ when $x \to E'$. Here H denotes the upper half plane. See a recent result of Bishop for the existence of such a mapping [3]. Then we take another conformal mapping g_2, which maps the domain Ω to a 'logarithmic spiral' S so that $g_2(x) \to 0$ when $x \to 0$. By a logarithmic spiral S we mean the image of the set

$$S_0 = \{x + iy \in \mathbb{C} : \; 2\pi(e^x - 2) < y < 2\pi(e^x - 1) \text{ and } x > 0 \text{ and } y > 0\}$$

under the analytic mapping $\exp(-z)$. Note that the area of S is finite and that the composition $g_2 \circ g_1 \circ f$ maps E to origin (in terms of a limit).

The claim now follows by considering the composition $g_2 \circ g_1 \circ \tilde{f} \circ g_0 : B^2 \to S$, where

$$g_0 : B^2 \to \operatorname{int} Q \quad , \quad g_0(x) = \frac{|x|}{\|x\|} x$$

is the "natural" stretching that maps the unit disk to the unit square and $\tilde{f}$ is the restriction of f to the interior of Q.

Let us close the proof by indicating the important modifications needed in the above construction in order to obtain a set E for which $H^h(E) > 0$ with a given gauge function h satisfying (1.3). Observe that by choosing $t_k = \exp(-2^{k/2}/k^{1/2})$ and $t'_k = \exp(-2^k/k)$ in the construction of E and E', we would have $H^{\tilde{h}}(E) > 0$ with the dimension gauge

$$\tilde{h}(t) = \frac{1}{(\log \frac{1}{t})^2 \log \log \frac{1}{t}}, \tag{3.3}$$

while E' would still have zero logarithmic capacity. Carrying out the construction of f and choosing g_0, g_1 and g_2 exactly as above gives us the desired conclusion with the improved dimension gauge of (3.3). In the same way one obtains the conclusion of Theorem 1.2 for arbitrary h satisfying (1.3).

Remark 3.1. *The construction of the homeomorphism $f : Q \to Q$ above can be easily extended to higher dimensions. It appears to be unknown if the resulting set E' of conformal capacity zero can be mapped to the origin "through" a logarithmic spiral. We expect this to be the case.*

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