AN EGG-YOLK PRINCIPLE AND EXPONENTIAL INTEGRABILITY FOR QUASIREGULAR MAPPINGS

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Abstract. Quasiregular mappings $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ are a natural generalization of analytic functions from complex analysis and provide a theory which is rich with new phenomena. In this paper we extend a well-known result of A. Chang and D. Marshall on exponential integrability of analytic functions in the disk, to the case of quasiregular mappings defined in the unit ball of $\mathbb{R}^n$. To this end, we first establish an “egg-yolk” principle for such maps, which extends a recent result of the first author. Our work leaves open an interesting problem regarding $n$-harmonic functions.

1. Introduction

We will denote an $n$-dimensional ball with center $a$ and radius $r$ by $B^n(a,r)$. The unit ball is $B^n$. Sometimes the notation $rB^n$ for $B^n(0,r)$ is used. Similarly, the notations $S^{n-1}(a,r)$ and $S^{n-1}$ for the corresponding $(n-1)$-spheres will be used, respectively. The $s$-dimensional Hausdorff measure will be denoted by $\mathcal{H}_s$. The volume of $B^n$ is denoted by $\alpha_n$, and the $(n-1)$-measure of $S^{n-1}$ by $\omega_{n-1}$.

A mapping $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is called quasiregular (qr) if it belongs to the Sobolev class $W^{1,n}_{loc}(\Omega,\mathbb{R}^n)$, and, for some $K \geq 1$, it satisfies the distortion inequality

$$\|Df(x)\|^n \leq K J(x,f)$$

for almost every $x \in \Omega$, where $\|Df(x)\|$ is the operator norm of the matrix derivative $Df(x) = \left( \frac{\partial f}{\partial x_i} \right)_{i,j=1}^n$, which is well-defined for almost every $x \in \mathbb{R}^n$, and $J(x,f)$ is the Jacobian determinant of $f$ at $x$, i.e., $J(x,f) = \det Df(x)$. It is well-known that quasiregular mappings are continuous and almost everywhere differentiable, and, when non-constant, they are open and discrete. Also when $n = 2$ and $K = 1$ they are analytic functions. They provide a fruitful generalization of classical function theory to higher (real) dimensional spaces. We refer to [Res89] and [Ric93] for the basic theory of quasiregular mappings. The theory of these mappings is often referred to, in colorful language, as the quasiworld.

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The purpose of this paper is twofold. We extend the exponential integrability result of [CM85] to the quasiworld. But, to do this, we also need to extend an “egg-yolk principle for the inverse map” conjectured by D. Marshall in [Mar89], which has been shown to hold in the classical case in [PC].

1.1. **Exponential integrability.** The following result is proved in [CM85].

**Theorem A (Chang-Marshall [CM85]).** There is a universal constant $C < \infty$ so that if $f$ is analytic in $\mathbb{D}$, $f(0) = 0$, and

\begin{equation}
\int_{\mathbb{D}} |f'(z)|^2 \frac{dA(z)}{\pi} \leq 1,
\end{equation}

then

\begin{equation}
\int_0^{2\pi} \exp \left( |f^*(e^{i\theta})|^2 \right) \, d\theta \leq C.
\end{equation}

where $f^*$ is the trace of $f$ on $\partial \mathbb{D}$, i.e., $f^*(\zeta) = \lim_{t \uparrow 1} f(t\zeta)$ for $\mathcal{H}_1$-a.e. $\zeta \in \partial \mathbb{D}$.

This result is moreover “sharp”. Indeed, even though for any given $\beta > 0$ and any analytic function $f$ on $\mathbb{D}$, satisfying $f(0) = 0$ and (1.1), the integral

\begin{equation}
\int_0^{2\pi} \exp \left( \beta |f^*(e^{i\theta})|^2 \right) \, d\theta
\end{equation}

is finite, there is a family of functions, the Beurling functions

$$B_a(z) = \left( \log \frac{1}{1 - az} \right) \left( \log \frac{1}{1 - a^2} \right)^{-\frac{1}{2}} \quad 0 < a < 1$$

that are analytic in $\mathbb{D}$, satisfy $B_a(0) = 0$ and (1.1), with the property that for any given $\alpha > 1$, one can choose $a$ so that the integral

\begin{equation}
\int_0^{2\pi} \exp \left( \alpha |B_a(e^{i\theta})|^2 \right) \, d\theta
\end{equation}

is as large as desired.

In this paper we extend the Chang-Marshall result to quasiregular mappings.

**Theorem 1.1.** There exists a constant $C = C(n, K) < \infty$ so that if $f : \mathbb{B}^n \to \mathbb{R}^n$, $n \geq 2$, is a $K$-quasiregular mapping with $f(0) = 0$ and

\begin{equation}
\int_{\mathbb{B}^n} J(x, f) \, dx \leq \alpha_n,
\end{equation}

then

\begin{equation}
\int_{\mathbb{S}^{n-1}} \exp \left( (n - 1) \left( \frac{n}{2K} \right)^{\frac{1}{n}} |f^*(\zeta)| \right) \, d\mathcal{H}_{n-1}(\zeta) \leq C,
\end{equation}

where $f^*$ is the trace of $f$ on $\mathbb{S}^{n-1}$, i.e., $f^*(\zeta) = \lim_{t \uparrow 1} f(t\zeta)$ for $\mathcal{H}_{n-1}$-a.e. $\zeta \in \mathbb{S}^{n-1}$.
The trace $f^*$ in Theorem 1.1 is well-defined, since a quasiregular mapping $f : \mathbb{B}^n \to \mathbb{R}^n$ satisfying (1.2) has radial limits at almost every $\theta \in \mathbb{S}^{n-1}$, see [Ric93], VII Theorem 2.7.

For a mapping satisfying the assumptions of Theorem 1.1,

$$\int_{\mathbb{S}^{n-1}} \exp \left( \beta |f^*(\zeta)|^{\frac{n}{n-1}} \right) \, d\mathcal{H}_{n-1}(\zeta) < \infty$$

for every $\beta > 0$. This is a consequence of Theorem 1.5 as will be shown at the end of Section 5.

Theorem 1.1 is sharp for $n = 2$, in the sense that for any $K \geq 1$ the constant $K^{-1}$ cannot be improved on. To see this, first map the unit disk onto the upper half plane by a Möbius transformation, so that $(1,0)$ is mapped to the origin. Then apply the radial stretching $z \mapsto \frac{|z|^K}{1+|z|^K}$, which is a $K$-quasiconformal map, and map back to the disk. Finally, apply the Beurling functions $B_u$. The compositions of these maps, $B_{K,a}$, are $K$-quasiregular maps satisfying the assumptions of Theorem 1.1, and for each $\beta > K^{-1}$,

$$\sup_{0 < a < 1} \int_0^{2\pi} \exp \left( \beta |B_{K,a}^*(e^{i\theta})|^2 \right) \, d\theta = \infty.$$

In dimensions higher than two the situation is different. Indeed, by the Liouville theorem of Gehring and Reshetnyak, see [Res89], Theorem 5.10, 1-quasiregular mappings in dimensions three or higher are Möbius transformations. Moreover, the $L^\infty$-norm of a Möbius transformation satisfying the assumptions of Theorem 1.1 is bounded by two. We expect that the constant $(n-1) \left( \frac{n}{2K} \right)^{\frac{1}{n-1}}$ is not sharp for any $n \geq 3$ and any $K \geq 1$. In particular, it would be interesting to determine whether the sharp constant stays bounded as $n$ tends to infinity. Spatial maps that are similar to the Beurling functions can be constructed by using cylinder maps ($K$-quasiconformal maps mapping $\mathbb{B}^n$ onto an infinite cylinder). The best dilatation constant $K$ for cylinder maps is not known, see [GV65], Section 8.

1.2. Further remarks. The Chang-Marshall theorem has the following two corollaries for harmonic and Sobolev functions.

**Corollary D.** There is a universal constant $C < \infty$ so that if $u : \mathbb{D} \to \mathbb{R}$ is harmonic with $u(0) = 0$ and

$$\int_\mathbb{D} |\nabla u(z)|^2 \, dA(z) / \pi \leq 1,$$

then

$$\int_0^{2\pi} \exp \left( u^*(e^{i\theta})^2 \right) \, d\theta \leq C$$

where $u^*$ is the trace of $u$ on $\partial \mathbb{D}$, i.e., $u^*(\zeta) = \lim_{t \downarrow 1} u(t\zeta) = u^*(\zeta)$ for $\mathcal{H}_1$-a.e. $\zeta \in \partial \mathbb{D}$.
Proof. Let \( \tilde{u} \) be the harmonic conjugate of \( u \) such that \( \tilde{u}(0) = 0 \). Then \( f = u + i\tilde{u} \) satisfies the hypothesis of Theorem C, since \( |f'| = |\nabla u| \). So
\[
\int_0^{2\pi} \exp\left(u^*(e^{i\theta})^2\right) d\theta \leq \int_0^{2\pi} \exp\left(u^*(e^{i\theta})^2 + \tilde{u}^*(e^{i\theta})^2\right) d\theta \leq C.
\]
\[\square\]

**Corollary E.** There is a universal constant \( C < \infty \) so that if \( v \in W^{1,2}(\mathbb{D}) \) with
\[
\int_{\partial \mathbb{D}} v^*(e^{i\theta}) d\theta = 0
\]
and
\[
\int_\mathbb{D} |\nabla v(z)|^2 dA(z)/\pi \leq 1,
\]
then
\[
\int_0^{2\pi} \exp\left(v^*(e^{i\theta})^2\right) d\theta \leq C
\]
where \( v^* \) is the Sobolev trace of \( v \) on \( \partial \mathbb{D} \).

For the concept of Sobolev trace see [Zie89], pages 189–191.

**Proof.** Let \( v^* \) be the trace of \( v \) on the circle \( \partial \mathbb{D} \). Solve the Dirichlet problem with these boundary values, to get \( u \) harmonic in \( \mathbb{D} \) with
\[
\int_\mathbb{D} |\nabla u|^2 dA/\pi \leq \int_\mathbb{D} |\nabla v|^2 dA/\pi \leq 1.
\]
Then Corollary D implies \( \int_0^{2\pi} \exp\left(u^*(e^{i\theta})^2\right) d\theta \leq C \), but \( u^* = v^* \). So the same is true for \( v^* \). \[\square\]

**Remark 1.2.** In terms of statements we have:

Theorem A \( \implies \) Corollary D \( \iff \) Corollary E

Corollary E could possibly be proved by “Sobolev” methods, see for instance the similar Theorem 3.2.1 of [AH96]. When a seemingly stronger normalization
\[
\int_{\mathbb{R}^n} u(x) dx = 0
\]
is assumed, the techniques below can be used to prove results like Corollary E in all dimensions, see comments at the end of Section 4.

**Remark 1.3.** Condition (1.1) says that the Euclidean area of \( f(\mathbb{D}) \) counting multiplicity is less or equal to \( \pi \). In [Ess87] it is shown that (1.1) can be replaced by the condition that the area of the set \( f(\mathbb{D}) \) is less or equal to \( \pi \), without counting multiplicity.
1.3. **Open Questions.** In view of Corollary D we ask:

**Question 1.4.** What is the best constant $\beta$ for which there exists $C > 0$ so that if $u \in W^{1,n}(\mathbb{B}^n)$, $n \geq 2$, is $n$-harmonic on $\mathbb{B}^n$, $u(0) = 0$, and

$$\int_{\mathbb{B}^n} |\nabla u(x)|^n \, dx \leq \alpha_n,$$

then

$$\int_{\mathbb{S}^{n-1}} \exp\left(\beta |u^*(\zeta)|^{\frac{n}{n-1}}\right) \, d\mathcal{H}_{n-1}(\zeta) \leq C?$$

1.4. **Beurling’s estimate.** In [Mar89], Don Marshall deduces Theorem A from an estimate of Beurling, Theorem B below. We denote $E_t = \{ x \in \mathbb{B}^n : |f(x)| = t \}$, and $F^*_s = \{ \theta \in \mathbb{S}^{n-1} : |f(\theta)| > s \}$. The following is an unpublished estimate of A. Beurling which is stated and proved in [Mar89]. Here “Cap” denotes logarithmic capacity.

**Theorem B (Beurling).** Suppose $f$ is analytic in a neighborhood of $\mathbb{D}$ and suppose that $|f(z)| \leq M$ for $|z| \leq r < 1$, for some $0 < r < 1$. Then, for every $s > M$,

$$\text{Cap } F^*_s \leq r^{-\frac{1}{2}} \exp\left(-\pi \int_M^s \frac{dt}{|f(E_t)|}\right)$$

where $|f(E_t)|$ denotes the length of $f(E_t)$ counting multiplicity.

We establish a similar estimate in space. For a quasiregular map $f : \mathbb{B}^n \to \mathbb{R}^n$, $n \geq 2$, we denote the $(n-1)$-measure of $f(E_t)$ counting multiplicity by $A_{n-1}f(E_t)$:

$$A_{n-1}f(E_t) = \int_{\mathbb{S}^{n-1}(0,t)} \text{card } f^{-1}(y) \, d\mathcal{H}_{n-1}(y).$$

**Theorem 1.5.** Let $f$ be a $K$-quasiregular mapping defined in a neighborhood of $\mathbb{B}^n$, $n \geq 2$, and suppose that $|f(x)| \leq M$ for $|x| \leq r < 1$. Then, for every $s > M$,

$$\mathcal{H}_{n-1}(F^*_s) \leq C_1 \exp\left((1 - n)\left(\frac{\omega_{n-1}}{2K}\right)^{\frac{1}{n-1}} \int_M^s \frac{dt}{|A_{n-1}f(E_t)|^{\frac{1}{n-1}}}\right),$$

where $C_1$ depends only on $n$, $K$ and $r$.

1.5. **An egg-yolk principle for the inverse.** In [Mar89], Don Marshall conjectures an egg-yolk principle that would have simplified his argument for passing from Theorem B to Theorem A. This was proved in [PC] by the first author.

**Theorem C ([PC]).** There is a universal constant $0 < r_0 < 1$ such that whenever $f$ is analytic on $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ with $f(0) = 0$, and whenever $M > 0$ is such that

$$\int_{\{z \in \mathbb{D} : |f(z)| < M\}} |f'(z)|^2 \, dA(z) < \pi M^2,$$

then we have that $|z| < r_0$ implies $|f(z)| < M$. 
Here we prove that Theorem C extends to quasiregular maps, and this will allow us to deduce Theorem 1.1 from Theorem 1.5.

**Theorem 1.6.** Given \( n \geq 2 \) and \( K \geq 1 \), there exists a constant \( 0 < r_0(n,K) < 1 \), so that whenever \( f : B^n \to \mathbb{R}^n \) is a \( K \)-quasiregular mapping with \( f(0) = 0 \) and whenever \( M > 0 \) is such that
\[
\int_{\{x \in B^n : |f(x)| < M\}} J(x,f) \, dx < \alpha_n M^n,
\]
then we have that \( |x| < r_0 \) implies \( |f(x)| < M \).

Theorem 1.6 is equivalent to the following.

**Corollary 1.7.** For \( n \geq 2 \) and \( K \geq 1 \), there exists a constant \( 0 < r_0(n,K) < 1 \) so that if \( f : B^n \to \mathbb{R}^n \) is a \( K \)-quasiregular mapping with \( f(0) = 0 \), then \( 0 \leq M < \max_{|x| \leq r_0} |f(x)| \) implies
\[
\int_{\{x \in B^n : |f(x)| < M\}} J(x,f) \, dx \geq \alpha_n M^n.
\]

Theorem 1.6 no longer holds true if instead of (1.4) it is assumed that \( B^n \setminus f(B^n) \neq \emptyset \), see [PC], Remark 1.5.

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## 2. Proof of Theorem 1.6

We first recall the classical (conformal) modulus for path families in \( \mathbb{R}^n \). Let \( \Gamma \) be a family of paths \( \gamma \), i.e., continuous functions \( \gamma : I \to \mathbb{R}^n \), where \( I = [a,b] \) or \( [a,b). \) We say that a Borel measurable function \( \rho : \mathbb{R}^n \to [0, +\infty) \) is admissible for \( \Gamma \) if
\[
\int_{\gamma} \rho \, ds \geq 1 \quad \forall \gamma \in \Gamma.
\]

Then the modulus of \( \Gamma \) is
\[
\text{Mod} \, \Gamma := \inf \left\{ \int_{\mathbb{R}^n} \rho(x)^n \, dx : \rho \text{ admissible} \right\}.
\]

We recall two classical results concerning conformal modulus.

**Lemma 2.1** (Poletsky’s inequality, [Ric93], II Theorem 8.1). Let \( f : \Omega \to \mathbb{R}^n \) be a non-constant \( K \)-quasiregular mapping, and \( \Gamma \) a family of paths in \( \Omega \). Then
\[
\text{Mod} \, (f \Gamma) \leq K^{n-1} \text{Mod} \, \Gamma.
\]
Lemma 2.2 ([Väi71], Theorem 10.12). Suppose that $J$ is a measurable set of radii, and $p \in \mathbb{R}^n$. For each $r \in J$, consider distinct points $a_r, b_r$ in $\mathbb{S}^{n-1}(p, r)$. Set

$$\Gamma = \{ \gamma : [a, b) \to \mathbb{S}^{n-1}(p, r) | r \in J, \gamma \text{ connects } a_r \text{ and } b_r \}.$$ 

Then

$$\text{Mod} \Gamma \geq c_n \int_J \frac{dr}{r},$$

where $c_n > 0$ only depends on $n$.

Let $f$ satisfy the assumptions of Theorem 1.6. We lose no generality by assuming $M = 1$. Let $\delta$ denote the largest radius so that $f(B^n(0, \delta)) \subset B^n$.

In order to prove Theorem 1.6 we need to show that $\delta \geq C(n, K)$. Also, we let

$$F_0 = B^n \setminus f(B^n),$$

$$F_1 = \{ y \in B^n : \text{card } f^{-1}(y) = 1 \},$$

$$F_m = \{ y \in B^n : \text{card } f^{-1}(y) \geq 2 \} = B^n \setminus (F_0 \cup F_1).$$

By (1.4) and a change of variables, we have

$$\alpha_n > \int_{\{ x \in B^n : f(x) \in B^n \}} J(x, f) \, dx = \int_{B^n} \text{card } f^{-1}(y) \, dy.$$ 

Therefore $F_0 \neq \emptyset$.

We first prove Theorem 1.6 under the assumption

$$(2.1) \quad |F_0| \geq \alpha_n 100^{-n}.$$ 

We denote by $T$ the set of those radii $0 < r < 1$ for which

$$\mathbb{S}^{n-1}(0, r) \cap F_0 \neq \emptyset.$$ 

Lemma 2.3. Assume that (2.1) holds true. Then

$$\int_T \frac{dr}{r} \geq n^{-1} 100^{-n}.$$ 

Proof. Since $r < 1$, we have

$$\int_T \frac{dr}{r} = \omega_{n-1}^{-1} \int_T \int_{\mathbb{S}^{n-1}(0, r)} r^{-n} \, d\mathcal{H}_{n-1} \, dr \geq \omega_{n-1}^{-1} \int_{\mathbb{R}^n} \chi_{\{ y : y \in T \}}(x) \, dx$$

$$= \omega_{n-1}^{-1} |\{ y : y \in T \}| \geq \omega_{n-1}^{-1} |F_0| \geq \alpha_n \omega_{n-1}^{-1} 100^{-n} = n^{-1} 100^{-n}.\qed$$

Proposition 2.4. Theorem 1.6 holds true under assumption (2.1).
Proof. By definition of $T$, for each $r \in T$, we can choose points $q_r \in F_0 \cap S^{n-1}(0, r)$. Also, since $\overline{f(B^n(0, \delta))}$ is a connected set containing 0 and a point in $S^{n-1}$, for each $r \in T$, we can choose points $a_r \in B^n(0, \delta)$ such that $f(a_r) \in S^{n-1}(0, r)$. Then, for every path $\gamma$ starting at $f(a_r)$ and joining $f(a_r)$ to $q_r$ in $S^{n-1}$, every maximal lift $\gamma'$ of $\gamma$ starting at $a_r$ accumulates on $S^{n-1}$ (see [Ric93], II.3 for the definition of a maximal lift). Hence, if we denote the family of all such lifts, for any $r \in T$, by $\Gamma$, we have

\begin{equation}
\text{Mod } \Gamma \leq \omega_n \left( \log \delta^{-1} \right)^{1-n}.
\end{equation}

On the other hand, by Lemmas 2.2 and 2.3,

\begin{equation}
\text{Mod } f\Gamma \geq c_n \int_T \frac{dr}{r} \geq c_n n^{-1} 100^{-n}.
\end{equation}

By combining (2.2), (2.3) and Lemma 2.1, we have

\begin{equation*}
c_n n^{-1} 100^{-n} \leq K^{-1} \omega_n \left( \log \delta^{-1} \right)^{1-n},
\end{equation*}

Thus Theorem 1.6 holds in this case with

\begin{equation*}
r_0(n, K) = \exp \left( - \left( 100^n c_n^{-1} n K^{-1} \omega_n \right) \frac{1}{n-1} \right).
\end{equation*}

\[\Box\]

We now treat the case when (2.1) fails. First we establish a geometric lemma.

**Lemma 2.5.** Fix $q \in F_0$. Then there exists a point $w \in B^n$, and $1/4 \leq s < 1$, such that for all $r \in (s, \sqrt{3}s)$, we have $q \in B^n(w, r)$ and $S^{n-1}(w, r) \cap f(B^n(0, \delta)) \neq \emptyset$.

**Proof.** First assume $|q| \leq 1/2$. Then, since $\overline{f(B^n(0, \delta))}$ is a connected set containing 0 and a point in $S^{n-1}$,

\[S^{n-1}(0, r) \cap f(B^n(0, \delta)) \neq \emptyset \quad \forall r \in \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right).\]

Hence we may choose $w = 0$, $s = 1/2$.

Thus assume $|q| > 1/2$. Choose $p \in B^n(0, \delta)$ such that $|f(p)| = |q|$. Consider the triangle with vertices 0, $f(p)$ and $q/2$. Then, if the angle of the triangle at $q/2$ is less than $\pi/2$, we have, for each $r \in (|q|/2, \sqrt{3}|q|/2)$,

\[0, q \in B^n(q/2, r), \quad f(p) \notin B^n(q/2, r).\]

Since $0, f(p) \in f(B^n(0, \delta))$, there exists, for each such $r$, a point $\eta_r \in B^n(0, \delta)$ such that $f(\eta_r) \in S^{n-1}(q/2, r)$. Hence we may choose $w = q/2$ and $s = |q|/2 > 1/4$ in this case.
If the angle is greater than or equal to $\pi/2$, we have, for each $r \in (|q|/2, \sqrt{3}|q|/2)$,
\[
f(p), q \in B^n \left( \frac{f(p) + q}{2}, r \right), \quad 0 \notin B^n \left( \frac{f(p) + q}{2}, r \right).
\]
Hence we may in this case choose $w = (f(p) + q)/2$ and $s = |q|/2$. \hfill \square

Let $q$, $w$, and $s$ be as in Lemma 2.5. We denote by $G$ the set of all radii $r \in (s, \sqrt{3}s)$ for which

\[
F_1 \cap S^{n-1}(w, r) \neq \emptyset.
\]

**Lemma 2.6.** If (2.1) fails, then

\[
\int_G \frac{dr}{r} \geq n^{-1}100^{-n}.
\]

**Proof.** As in Lemma 2.3, we have

\[
\int_G \frac{dr}{r} \geq \omega_{n-1} \int_G \int_{S^{n-1}(w, r)} d\mathcal{H}_{n-1} r^{-n} dr \geq \omega_{n-1} |F_1 \cap (B^n(w, \sqrt{3}s) \setminus \overline{B^n(w, s)})|.
\]

By our assumption (1.4) and a change of variables,

\[
|F_1| + 2|F_m| \leq \int_{B^n} \text{card } f^{-1}(y) dy = \int_{\{x \in B^n : f(x) \in B^n\}} J(x, f) dx < \alpha_n = |F_0| + |F_1| + |F_m|.
\]

So

\[
|F_m| \leq |F_0| \leq \alpha_n 100^{-n},
\]

where the last inequality holds true since we assume the converse of (2.1).

On the other hand, since $w \in B^n$ and $s \geq 1/4$, we have

\[
|(B^n(w, \sqrt{3}s) \setminus \overline{B^n(w, s)}) \cap B^n| \geq \alpha_n 10^{-n},
\]

and combining (2.5) and (2.6) yields

\[
|F_1 \cap (B^n(w, \sqrt{3}s) \setminus \overline{B^n(w, s)})| = \ |(B^n(w, \sqrt{3}s) \setminus \overline{B^n(w, s)}) \cap B^n| - \ |(B^n(w, \sqrt{3}s) \setminus \overline{B^n(w, s)}) \cap (F_0 \cup F_m)| \geq \alpha_n 100^{-n}.
\]

The Lemma follows by combining (2.4) and (2.7). \hfill \square

For each $r \in G$, choose points $p_r \in f^{-1}(F_1)$, $a_r \in B^n(0, \delta)$ such that

\[
f(p_r), f(a_r) \in S^{n-1}(w, r).
\]

Denote

\[
G_1 = \{r \in G : |p_r| \geq \delta^{1/2}\},
\]

\[
G_2 = \{r \in G : |p_r| < \delta^{1/2}\} = G \setminus G_2.
\]
Then, by Lemma 2.6, either (2.1) holds, or else we have one of
\[
\int_{G_1} \frac{dr}{r} \geq 2^{-1}n^{-1}100^{-n},
\]
(2.8)
or
\[
\int_{G_2} \frac{dr}{r} \geq 2^{-1}n^{-1}100^{-n}.
\]
(2.9)

**Proposition 2.7.** Theorem 1.6 holds true under assumption (2.8).

**Proof.** For each \( r \in G_1 \) and each \( \gamma \) starting at \( f(a_r) \) and joining \( f(a_r) \) to \( f(p_r) \) in \( S^{n-1}(w, r) \), consider a maximal lift \( \gamma' \) of \( \gamma \) starting at \( a_r \). Then, since \( \text{card } f^{-1}(f(p_r)) = 1 \), either \( \gamma' \) accumulates to \( S^{n-1} \), or \( \gamma' \) ends at \( p_r \); in any case, \( \gamma' \) starts at \( B_n(0, \delta) \) and leaves \( B_n(0, \delta_1^{1/2}) \). Denote the family of all such \( \gamma' \) by \( \Gamma \). Then we have
\[
\text{Mod } \Gamma \leq \omega_{n-1} \left( \log \frac{\delta_1^{1/2}}{\delta} \right)^{1-n} \omega_{n-1} \left( \log \delta \right)^{1-n}.
\]
(2.10)

On the other hand, combining Lemma 2.2 and (2.8) yields
\[
\text{Mod } f \Gamma \geq c_n 2^{-1}n^{-1}100^{-n}.
\]
(2.11)

Furthermore, combining (2.10), (2.11) and Lemma 2.1 gives
\[
c_n 2^{-1}n^{-1}100^{-n} \leq K^{n-1}\omega_{n-1} \left( \log \delta \right)^{1-n},
\]
Thus Theorem 1.6 holds in this case with
\[
r_0(n, K) = \exp \left( -2 \left( 100^n 2c_n^{-1}nK^{n-1}\omega_{n-1} \right)^{1/n} \right).
\]
\[
\square
\]

In order to finish the proof of Theorem 1.6, we need the following auxiliary result.

**Lemma 2.8.** For each \( r \in G_2 \) there exists \( \tau_r \in S^{n-1}(0, \delta_1^{1/2}) \) such that \( f(\tau_r) \in S^{n-1}(w, r) \).

**Proof.** Let \( U_r \) be any component of \( f^{-1}(B^n(w, r)) \) intersecting \( B^n(0, \delta) \). Such a component exists by Lemma 2.5. Also, by Lemma 2.5, \( B^n(w, r) \setminus f(B^n) \neq \emptyset \), and hence \( f|_{U_r} : U_r \rightarrow B^n(w, r) \) is not onto. Thus, by [Ric93], I Lemma 4.7,
\[
S^{n-1}(0, t) \cap U_r \neq \emptyset \quad \forall t \in (\delta, 1).
\]

Choose \( k_r \in U_r \cap S^{n-1}(0, \delta_1^{1/2}) \), and consider all paths joining \( k_r \) to \( -k_r \) in \( S^{n-1}(0, \delta_1^{1/2}) \). If none of the images of these paths intersects \( S^{n-1}(w, r) \), we have
\[
f(S^{n-1}(0, \delta_1^{1/2})) \subset B^n(w, r).
\]
(2.12)
Since \( f \) is open,
\[
\partial f(B^n(0, \delta_1^{1/2})) \subset f(S^{n-1}(0, \delta_1^{1/2})),
\]
and since \( f(\mathbb{B}^n(0, \delta^{\frac{1}{4}})) \) is bounded, (2.12) further implies

\[
(2.13) \quad f(\mathbb{B}^n(0, \delta^{\frac{1}{4}})) \subset \mathbb{B}^n(w, r).
\]

By Lemma 2.5 there are, however, points \( x \in \mathbb{B}^n(0, \delta) \) such that \( f(x) \notin \mathbb{B}^n(w, r) \) which contradicts (2.13). The proof is complete. \( \square \)

**Proposition 2.9.** Theorem 1.6 holds true under assumption (2.9).

**Proof.** For each \( r \in \mathbb{R}^2 \), and each \( \gamma \) starting at \( f(\tau_r) \) (where \( \tau_r \) is as in Lemma 2.8) and joining \( f(\tau_r) \) to \( f(p_r) \) in \( S_{n-1}(w, r) \), consider a maximal lift \( \gamma' \) of \( \gamma \) starting at \( \tau_r \).

Then, since \( \text{card} f^{-1}(f(p_r)) = 1 \), either \( \gamma' \) accumulates to \( S_{n-1}(0, \delta^{\frac{1}{4}}) \), or \( \gamma' \) ends at \( p_r \). We denote the family of all such \( \gamma' \) for which the first case occurs by \( \Gamma_1 \), the family of all \( \gamma' \) for which the second case occurs by \( \Gamma_2 \), and \( \Gamma = \Gamma_1 \cup \Gamma_2 \).

Then, since each \( \gamma' \in \Gamma_1 \) connects \( S_{n-1}(0, \delta^{\frac{1}{4}}) \) to \( S_{n-1} \),

\[
(2.14) \quad \text{Mod } \Gamma_1 \leq \omega_{n-1} \left( \log \delta^{\frac{1}{4}} \right)^{1-n}.
\]

Similarly, since \( p_r \in \mathbb{B}^n(0, \delta^{\frac{1}{4}}) \) for all \( r \in \mathbb{R}^2 \),

\[
(2.15) \quad \text{Mod } \Gamma_2 \leq \omega_{n-1} \left( \log \frac{\delta^{\frac{1}{4}}}{\delta^{\frac{1}{2}}} \right)^{1-n} = \omega_{n-1} \left( \log \delta^{\frac{1}{4}} \right)^{1-n}.
\]

By Lemma 2.2 and (2.9),

\[
(2.16) \quad \text{Mod } f \Gamma \geq c_n 2^{-1} n^{-1} 100^{-n}.
\]

Hence, combining (2.14), (2.15), (2.16) and Lemma 2.1 yields

\[
c_n 2^{-1} n^{-1} 100^{-n} \leq \text{Mod } f \Gamma \leq K^{n-1} \text{Mod } \Gamma \leq K^{n-1}(\text{Mod } \Gamma_1 + \text{Mod } \Gamma_2) \leq 2 K^{n-1} \omega_{n-1} \left( \log \delta^{\frac{1}{4}} \right)^{1-n},
\]

Thus Theorem 1.6 holds in this case with

\[
r_0(n, K) = \exp \left( -4 \left( 100^n 4 c_n^{-1} n K^{n-1} \omega_{n-1} \right)^{\frac{1}{n-1}} \right).
\]

\( \square \)

The proof of Theorem 1.6 follows by combining Propositions 2.4, 2.7 and 2.9.

3. **Beurling’s modulus estimate**

Suppose \( f \) is \( K \)-quasiregular in a neighborhood of the closed unit ball \( \bar{\mathbb{B}}^n \), and for some fixed \( 0 < r < 1 \) let \( M := \max_{z \in \mathbb{B}^n} |f| \). Recall that for \( s > M \) we define \( F_s = \{ \zeta \in S^{n-1} : |f(\zeta)| \geq s \} \) and for \( M < t < s \) we have \( E_t = \{ x \in \mathbb{B}^n : |f(x)| = t \} \).
Consider the family $\Gamma_s$ consisting of the paths in $\mathbb{R}^n$ starting at $r\mathbb{B}^n$ and ending at $F_s^\star$. We claim that

$$\text{Mod } \Gamma_s \leq K \left( \int_M \left( A_{n-1} f (E_t) \right)^{-\frac{1}{n-1}} \right)^{1-n}.$$ 

Recall that $A_{n-1} f (E_t) = \int_{S^{n-1}(0,t)} \text{card } f^{-1}(y) \, d\mathcal{H}_{n-1}(y)$.

**Proof.** Set $\rho : \mathbb{R}^n \rightarrow [0, \infty),$

$$\rho(x) = \left( \int_M \left( A_{n-1} f (E_u) \right)^{-\frac{1}{n-1}} \right)^{-1} \| Df(x) \| \left( A_{n-1} f (E_t) \right)^{-\frac{1}{n-1}} \text{ when } |f(x)| = t \in (M, s),$$

and $\rho(x) = 0$ otherwise. Then, for each $\gamma \in \Gamma_s$,

$$\int_{\gamma} \rho \, ds \geq \left( \int_M \left( A_{n-1} f (E_u) \right)^{-\frac{1}{n-1}} \right)^{-1} \int_{f(\gamma)} \left( A_{n-1} f (E_{|x|}) \right)^{-\frac{1}{n-1}} \, ds \geq 1.$$

Moreover, if we denote

$$I(M, s) = \int_M \left( A_{n-1} f (E_u) \right)^{-\frac{1}{n-1}}$$

and

$$A(M, s) = f^{-1}(\mathbb{B}^n(0, s) \setminus \mathbb{B}^n(0, M)),$$

we have

$$\text{Mod } \Gamma_s \leq \int_{\mathbb{R}^n} \rho(x)^n \, dx = I(M, s)^{-n} \int_{A(M, s)} \frac{\| Df(x) \|^n}{\left( A_{n-1} f (E_{|x|}) \right)^{-\frac{1}{n-1}}} \, dx$$

$$\leq K I(M, s)^{-n} \int_{A(M, s)} \frac{J(x, f)}{\left( A_{n-1} f (E_{|x|}) \right)^{-\frac{1}{n-1}}} \, dx$$

$$= K I(M, s)^{-n} \int_{f(A(M, s))} \frac{\text{card } f^{-1}(y)}{\left( A_{n-1} f (E_{|y|}) \right)^{-\frac{1}{n-1}}} \, dy$$

$$= K I(M, s)^{-n} \int_M \left( A_{n-1} f (E_t) \right)^{-\frac{1}{n-1}} \int_{S^{n-1}(0,t)} \text{card } f^{-1}(\varphi) \, d\mathcal{H}_{n-1}(\varphi) \, dt$$

$$= K I(M, S)^{1-n}. \quad \square$$

## 4. Capacity and Symmetrization

We recall that a condenser is a pair $(\Omega, K)$ with $\Omega \subset \mathbb{R}^n$, $\Omega$ open and $K$ compact with $\emptyset \neq K \subset \Omega$. Also, the conformal capacity of $(\Omega, K)$ is

$$\text{Cap}(\Omega, K) := \inf \{ \| \nabla u \|^n_{L^n(\Omega)} : u \in W^{1,p}_0(\Omega), u|_{V} \geq 1, \text{ for some } V \text{ open }, V \supset K \}$$
where \( W^{1,p}_0(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) (the smooth functions compactly supported in \( \Omega \)) in the norm

\[
\|u\|_{W^{1,p}_0(\Omega)} = \left( \int_{\Omega} |u(x)|^p + |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}.
\]

By Proposition II.10.2 of [Ric93], if \( \Gamma(\Omega,K) \) is the family of all paths \( \gamma : [a,b) \to \Omega \) such that \( \gamma(a) \in K \) and \( \lim_{t \to b} \gamma(t) \in \partial \Omega \), then

\[
(4.1) \quad \text{Cap}(\Omega,K) = \text{Mod} \Gamma(\Omega,K).
\]

We are mainly interested in measuring the sets \( F^\star_s \) defined in Section 3, which are compact subsets of \( S^{n-1} \). Therefore, we will fix \( 0 < r < 1 \) to be determined later, consider the spherical ring \( A(r) = \{ x \in \mathbb{R}^n : r < |x| < 1/r \}, \) \( 0 < r < 1, \) and compute \( \text{Cap}(A(r), F^\star_s) \).

By the symmetry rule, cf. [GM05] IV(3.4), if \( F \subset S^{n-1} \), we have:

\[
(4.2) \quad \text{Mod} \Gamma_s = \frac{1}{2} \text{Mod} \Gamma(A(r), F) = \frac{1}{2} \text{Cap}(A(r), F).
\]

Also, if \( F \subset S^{n-1} \), let \( C(F) \) be the spherical cap centered at \( e_1 = (1,0,\ldots,0) \) with \( \mathcal{H}_{n-1}(C(F)) = \mathcal{H}_{n-1}(F) \). By spherical symmetrization, see [Geh61],

\[
(4.3) \quad \text{Cap}(A(r), C(F)) \leq \text{Cap}(A(r), F).
\]

By [Geh61], Theorem 4, we see that, when \( \mathcal{H}_{n-1}(F) \leq \epsilon(r,n) \),

\[
(4.4) \quad \text{Cap}(A(r), C(F)) \geq \omega_{n-1} \log^{1-n} \frac{C_2}{\mathcal{H}_{n-1}(F)^{\frac{1}{n-1}}},
\]

where \( C_2 > 0 \) depends only on \( r \) and \( n \) (the results in [Geh61] are stated for \( n = 3 \) only, but they extend to all dimensions).

Putting (3.1), (4.1), (4.2), (4.3), and (4.4) together, we obtain (1.3) and thus we have proved Theorem 1.5 for \( \mathcal{H}_{n-1}(F^\star_s) \leq \epsilon(r,n) \). If \( \mathcal{H}_{n-1}(F^\star_s) > \epsilon(r,n) \), then the arguments above show that

\[
(4.5) \quad \text{Mod} \Gamma_s \geq C(r,n).
\]

Combining (4.5) with (3.1) yields

\[
\int_M^s \frac{dt}{\mathcal{A}_{n-1}(f(E_t))^{\frac{1}{n-1}}} \leq C(r,n,K).
\]

Hence increasing \( C_1 \) if necessary gives Theorem 1.5 for all \( s > M \).

We finish this section by briefly commenting on the real-valued case mentioned in the introduction. Suppose that \( u : \mathbb{B}^n \to \mathbb{R} \) belongs to \( W^{1,n}(\mathbb{B}^n) \) and satisfies

\[
(4.6) \quad \int_{\mathbb{B}^n} u(x) \, dx = 0.
\]
Then, by the Poincaré inequality and (4.6),

$$|A_T| = |\{ x \in \frac{1}{2} B^n : |u| \leq T \}| \geq C(n)$$

for large enough $T$ depending only on $n$ and the Sobolev norm of $u$. Hence, by applying arguments similar to the ones above to the $n$-capacity related to the sets $A_T$ and $U^*_s = \{ y \in S^{n-1} : |u^*(y)| \geq s \}$, we have an estimate for the $\mathcal{H}_{n-1}$-measure of $U^*_s$ in terms of $s$, $T$ and $\int_{\{ x \in B^n : |u| \leq s \}} |\nabla u(x)|^n \, dx$.

5. **Exponential integrability**

In this section we prove Theorem 1.1 by using the results established in previous sections and arguments similar to those used in [Mar89]. Let $f$ be a $K$-quasiregular mapping defined in a neighborhood of $B^n$ and satisfying $f(0) = 0$ and (1.2). We denote

$$\beta = (n - 1) \left( \frac{n}{2K} \right)^{\frac{1}{n-1}}.$$

Then

$$\alpha_n^{\frac{1}{n-1}} \beta = (n - 1) \left( \frac{\omega_{n-1}}{2K} \right)^{\frac{1}{n-1}}.$$

Notice that we lose no generality by assuming that $f$ is defined in a neighborhood of $B^n$: if we consider a sequence $(r_j)$ increasing to one, and functions $f_j, f_j(x) = f(r_j x)$, then the existence of radial limits at almost every $\varphi \in S^{n-1}$ and Fatou’s lemma yield

$$\int_{S^{n-1}} \exp \left( \beta |f^*(\varphi)|^{\frac{n}{n-1}} \right) \, d\mathcal{H}_{n-1}(\varphi) \leq \liminf_j \int_{S^{n-1}} \exp \left( \beta |f_j^*(\varphi)|^{\frac{n}{n-1}} \right) \, d\mathcal{H}_{n-1}(\varphi).$$

By the Cavalieri principle,

(5.1) $$\int_{S^{n-1}} \exp \left( \beta |f^*(\varphi)|^{\frac{n}{n-1}} \right) \, d\mathcal{H}_{n-1}(\varphi) = \omega_{n-1} + \frac{\beta n}{n-1} \int_0^\infty s^{\frac{1}{n-1}} \mathcal{H}_{n-1}(F_s^*) \exp(\beta s^{\frac{n}{n-1}}) \, ds.$$

We choose $r_0 = r_0(n, K)$ as in Theorem 1.6, and let $M = \max_{|x| \leq r_0} |f(x)|$. Note that by Corollary 1.7 and (1.2), we have $M < 1$ and

(5.2) $$\int_{\{ x \in B^n : f(x) \in B^n(0,M) \}} J(x, f) \, dx = \int_0^M A_{n-1} f(E_t) \, dt \geq \alpha_n M^n.$$

Using (1.3) and (5.1), we are reduced to estimate

(5.3) $$\int_0^{\|f\|_\infty} s^{\frac{1}{n-1}} \exp(\beta s^{\frac{n}{n-1}} - \psi(s)) \, ds,$$

where $\psi(s) = 0$ for $0 < s \leq M$ and

$$\psi(s) = \alpha_n^{\frac{1}{n-1}} \beta \int_0^s \frac{dt}{A_{n-1} f(E_t)^{\frac{1}{n-1}}}.$$
for \( s \geq M \). We modify \( \psi \) as follows: for \( 0 < s \leq M \), set
\[
\tilde{\psi}(s) := \mu s,
\]
and for \( s \geq M \),
\[
\tilde{\psi}(s) := \psi(s) + \mu M,
\]
where
\[
\mu = \left( \frac{M \beta^{n-1} \alpha_n}{\int_0^M A_{n-1} f(E_t) \, dt} \right)^{\frac{1}{n-1}}.
\]
Note that \( \tilde{\psi} \) is strictly increasing for \( 0 < s \leq \|f\|_\infty \) and constant, equal to \( \|\tilde{\psi}\|_\infty \), for \( s > \|f\|_\infty \). Also \( \tilde{\psi}(0) = 0 \). Finally, \( \tilde{\psi} \leq \psi + \mu M \). So, by (5.2), and since \( M < 1 \), it is enough to estimate (5.3) with \( \psi \) replaced by \( \tilde{\psi} \).

Let \( \phi(y) := \tilde{\psi}^{-1}(y) \) for \( 0 < y \leq \|\tilde{\psi}\|_\infty \) and \( \phi(y) := \|f\|_\infty \) for \( y > \|\tilde{\psi}\|_\infty \), so that \( \phi \) is strictly increasing for \( 0 < y \leq \|\tilde{\psi}\|_\infty \) and \( \phi(0) = 0 \).

Changing variables \( y = \tilde{\psi}(s) \) the integral (5.3) becomes
\[
\int_0^{\|\tilde{\psi}\|_\infty} \exp\left( \beta \phi(y) \frac{n}{n-1} - y \right) \phi'(y) \phi(y)^{\frac{1}{n-1}} \, dy
\]
which, since \( \phi' \geq 0 \), is less than or equal to the same integral but from 0 to \( \infty \).

Integrating by parts we then need to estimate
\[
(5.4) \quad \int_0^\infty \exp(\beta \phi(y) \frac{n}{n-1} - y) \, dy = \int_0^\infty \exp((\beta \frac{n}{n-1} \phi(y)) \frac{n}{n-1} - y) \, dy.
\]

We have
\[
\beta \frac{n}{n-1} \phi'(y) = \begin{cases} 
\beta \frac{n}{n-1} \mu^{-1}, & 0 < y < \mu M, \\
\alpha_n^{-1} \beta \frac{n}{n-1} (A_{n-1} f(E_{\phi(y)})) \frac{1}{n-1}, & \mu M < y < \|\tilde{\psi}\|_\infty.
\end{cases}
\]
Thus, by changing variables with \( s = \phi(y) \), and by our choice of \( \mu \),
\[
\int_0^\infty (\beta \frac{n}{n-1} \phi'(y))^n \, dy = \int_0^{\mu M} \beta^{n-1} \mu^{-n} \, dy + \alpha_n^{-1} \beta^{-1} \int_{\mu M}^{\|\tilde{\psi}\|_\infty} (A_{n-1} f(E_{\phi(y)})) \frac{n}{n-1} \, dy
\]
\[= \beta^{n-1} M \mu^{1-n} + \alpha_n^{-1} \int_{\mu M}^{\|f\|_\infty} A_{n-1} f(E_t) \, dt
\]
\[\leq \alpha_n^{-1} \int_0^\infty A_{n-1} f(E_t) \, dt \leq 1.
\]

By applying equation (6), page 1080 of [Mos71] to \( \beta \frac{n}{n-1} \phi \), we conclude that (5.4) is bounded from above by a constant depending only on \( n \). The proof of Theorem 1.1 is complete.

We finally note that, under the assumptions of Theorem 1.1, the left hand side of (5.1) is finite for every \( \beta > 0 \). We fix \( M > 0 \), to be chosen later. After applying
Theorem 1.5 to the right hand term in (5.1), we need to show that
\[ \int_M^\infty s^{-\frac{1}{n-1}} \exp \left( \beta s^{\frac{n}{n-1}} - C \int_M^s \frac{dt}{(A_{n-1}f(E_t))^{\frac{1}{n-1}}} \right) ds \]
is finite, where $C > 0$.

By Hölder’s inequality,
\[ s - M = \int_M^s \left( \frac{(A_{n-1}f(E_t))^{\frac{1}{2}}}{(A_{n-1}f(E_t))^{\frac{1}{n-1}}} \right) \frac{dt}{(A_{n-1}f(E_t))^{\frac{1}{n-1}}} \leq \left( \int_M^s \frac{dt}{(A_{n-1}f(E_t))^{\frac{1}{n-1}}} \right)^{\frac{n-1}{n}} \left( \int_M^s A_{n-1}f(E_t) dt \right)^{\frac{1}{n}}. \]

By our assumption $\int_0^\infty A_{n-1}f(E_t) dt$ is finite. Thus, by choosing $M$ large enough so that
\[ \left( \int_M^\infty A_{n-1}f(E_t) dt \right)^{\frac{n-1}{n}} > \frac{2\beta}{C}, \]
and combining this with (5.6), we can estimate (5.5) from above by
\[ \int_M^\infty s^{-\frac{1}{n-1}} \exp \left( \beta s^{\frac{n}{n-1}} - 2(s - M)^{\frac{n}{n-1}} \right) ds, \]
which is clearly finite.

References


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