SEPARATION CONDITIONS ON CONTROLLED MORAN CONSTRUCTIONS

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ABSTRACT. It is well known that the open set condition and the positivity of the t-dimensional Hausdorff measure are equivalent on self-similar sets, where t is the zero of the topological pressure. We prove an analogous result for a class of Moran constructions and we study different kinds of Moran constructions with this respect.

1. INTRODUCTION AND NOTATION

The origin of fractal mathematics goes back to the early works of Cantor [4]. He showed that a nonempty perfect subset of the real line is uncountable. At that time, fractal type behavior were seen in many examples, which, however, were considered to be only pathological counterexamples for some property. For example, the Weierstrass function is an example of a continuous and nondifferentiable function. The later development of geometric measure theory gave necessary tools for studying these kinds of objects. A nice overview for the beginning of fractal mathematics can be found in the book of Edgar [6].

Mainly because of Mandelbrot's intuition [20], fractals started to be seen as models of real world phenomena instead of pathological examples. Although there is no generally accepted definition for the term "fractal", the fundamental idea behind this notion is self-similarity: small pieces of a set appear to be similar to the whole set. A mathematical class of self-similar sets was introduced by Hutchinson [12]. A mapping $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ is called a *similitude mapping* if there is $s > 0$ such that $|\varphi(x) - \varphi(y)| = s|x - y|$ whenever $x, y \in \mathbb{R}^d$.

If the similitude mappings $\varphi_1, \ldots, \varphi_k$ are *contractive*, that is, all the Lipschitz constants are strictly less than one, then a nonempty compact set $E \subset \mathbb{R}^d$ is called *self-similar* provided that it satisfies

$$E = \varphi_1(E) \cup \cdots \cup \varphi_k(E).$$

From this, one can easily see that the set E consists of smaller and smaller pieces which are geometrically similar to E. However, the self-similar structure is hard to recognize if these pieces overlap too much. Hutchinson [12] used a separation

Date: April 26, 2006.
2000 *Mathematics Subject Classification.* Primary 28A80; Secondary 37C45.
Key words and phrases. Moran construction, iterated function system, limit set, invariant set, separation condition, open set condition, Hausdorff measure, Hausdorff dimension.

condition which guarantees that we can distinguish the pieces. The idea goes back to Moran [25] who studied similar constructions but without mappings. In the *open set condition*, it is required that there exists an open set V such that all the images $\varphi_i(V)$ are pairwise disjoint and contained in V. Lalley [17] used a stronger version of the open set condition. In the *strong open set condition*, it is required that the open set V above can be chosen such that $V \cap E \neq \emptyset$.

Assuming the open set condition, Hutchinson [12, §5.3] proved that the t-dimensional Hausdorff measure $\mathcal{H}^t$ of E is positive, where t is the zero of the so-called topological pressure. See also Moran [25, Theorem III] for the corresponding theorem for the Moran constructions. Schief [28, Theorem 2.1] showed, extending the ideas of Bandt and Graf [2], that the open set condition is not only sufficient but also a necessary condition for the positivity of the Hausdorff measure. In fact, he proved that $\mathcal{H}^t(E) > 0$ implies the strong open set condition. Later, Peres, Rams, Simon, and Solomyak [27, Theorem 1.1] showed that this equivalence also holds for self-conformal sets. See also Fan and Lau [9] and Fan, Lau, and Ye [10].

The main theme in this article is to study the relationship between various separation conditions and the Hausdorff measure on Moran constructions. More precisely, we study what can be said about Schief's result in this setting. Since the open set condition requires the use of mappings, we introduce a representative form for it to be used on Moran constructions. We also study measure theoretical properties of limit sets of various classes of Moran constructions and invariant sets of various iterated function systems. We generalize many classical results into these settings.

Before going into more detailed preliminaries, let us fix some notation to be used throughout this article. As usual, let I be a finite set with at least two elements. Put $I^* = \bigcup_{n=1}^\infty I^n$ and $I^\infty = I^{\mathbb{N}}$. Now for each $\mathtt{i} \in I^*$, there is $n \in \mathbb{N}$ such that $\mathtt{i} = (i_1, \ldots, i_n) \in I^n$. We call this n the *length* of $\mathtt{i}$ and we denote $|\mathtt{i}| = n$. The length of elements in I^∞ is infinity. Moreover, if $\mathtt{i} \in I^*$ and $\mathtt{j} \in I^* \cup I^\infty$, then with the notation $\mathtt{ij}$, we mean the element obtained by juxtaposing the terms of $\mathtt{i}$ and $\mathtt{j}$. For $\mathtt{i} \in I^*$ and $A \subset I^\infty$, we define $[\mathtt{i}; A] = \{\mathtt{ij} : \mathtt{j} \in A\}$ and we call the set $[\mathtt{i}] = [\mathtt{i}; I^\infty]$ a *cylinder set* of level $|\mathtt{i}|$. If $\mathtt{j} \in I^* \cup I^\infty$ and $1 \le n < |\mathtt{j}|$, we define $\mathtt{j}|_n$ to be the unique element $\mathtt{i} \in I^n$ for which $\mathtt{j} \in [\mathtt{i}]$. We also denote $\mathtt{i}^- = \mathtt{i}|_{|\mathtt{i}|-1}$. With the notation $\mathtt{i} \bot \mathtt{j}$, we mean that the elements $\mathtt{i}, \mathtt{j} \in I^*$ are *incomparable*, that is, $[\mathtt{i}] \cap [\mathtt{j}] = \emptyset$. We call a set $A \subset I^*$ incomparable if all of its elements are mutually incomparable. Finally, with the notation $\mathtt{i} \wedge \mathtt{j}$, we mean the common beginning of $\mathtt{i} \in I^*$ and $\mathtt{j} \in I^*$, that is, $\mathtt{i} \wedge \mathtt{j} = \mathtt{i}|_n = \mathtt{j}|_n$, where $n = \min\{k - 1 : \mathtt{i}|_k \neq \mathtt{j}|_n\}$.

Defining

$$|\mathtt{i} - \mathtt{j}| = \begin{cases} 2^{-|\mathtt{i} \wedge \mathtt{j}|}, & \mathtt{i} \neq \mathtt{j} \\ 0, & \mathtt{i} = \mathtt{j} \end{cases}$$

for each $\mathtt{i}, \mathtt{j} \in I^\infty$, the couple $(I^\infty, |\cdot|)$ is a compact metric space. We call $(I^\infty, |\cdot|)$ a *symbol space* and an element $\mathtt{i} \in I^\infty$ a *symbol*. If there is no danger of misunderstanding, let us call also an element $\mathtt{i} \in I^*$ a symbol. Define the *left shift* $\sigma\colon I^\infty \to I^\infty$ by setting

$$\sigma(i_1, i_2, \ldots) = (i_2, i_3, \ldots).$$

With the notation $\sigma(i_1, \ldots, i_n)$, we mean the symbol $(i_2, \ldots, i_n) \in I^{n-1}$. Observe that to be precise in our definitions, we need to work with "empty symbols", that is, symbols with zero length. However, this is left to the reader.

In this article, we study the *controlled Moran construction (CMC)*, that is, the collection $\{X_\mathtt{i} \subset \mathbb{R}^d : \mathtt{i} \in I^*\}$ of compact sets with positive diameter satisfying

(M1) $X_{\mathtt{i}i} \subset X_\mathtt{i}$ as $\mathtt{i} \in I^*$ and $i \in I$,
(M2) there exists a constant $D \geq 1$ such that

$$D^{-1} \leq \frac{\operatorname{diam}(X_{\mathtt{ij}})}{\operatorname{diam}(X_\mathtt{i}) \operatorname{diam}(X_\mathtt{j})} \leq D$$

whenever $\mathtt{i}, \mathtt{j} \in I^*$,
(M3) there exists $n \in \mathbb{N}$ such that

$$\max_{\mathtt{i} \in I^n} \operatorname{diam}(X_\mathtt{i}) < D^{-1}.$$

Here with the notation $\operatorname{diam}(A)$, we mean the diameter of a given set A. The fact that we can define a continuous mapping $\pi\colon I^\infty \to \mathbb{R}^d$ by setting $\{\pi(\mathtt{i})\} = \bigcap_{n=1}^\infty X_{\mathtt{i}|_n}$ ties the CMC and the symbol space together. We shall focus on measure theoretical properties of the compact *limit set*

$$E = \bigcup_{\mathtt{i} \in I^\infty} \bigcap_{n=1}^\infty X_{\mathtt{i}|_n} = \pi(I^\infty).$$

Assuming a natural separation condition, the ball condition (see Definition 3.3) and some natural regularity assumptions, there is a constant $c \geq 1$ such that the Hausdorff measure restricted to the set E satisfies

$$c^{-1} \operatorname{diam}(X_\mathtt{i})^t \leq \mathcal{H}^t|_E(X_\mathtt{i}) \leq c \operatorname{diam}(X_\mathtt{i})^t$$

whenever $\mathtt{i} \in I^*$ and

$$\mathcal{H}^t|_E(X_\mathtt{i} \cap X_\mathtt{j}) = 0$$

whenever $\mathtt{i} \perp \mathtt{j}$. Here $t \geq 0$ is the zero of the topological pressure P defined in (3.1). See Proposition 3.2, Theorem 3.5, Lemma 3.6, and Remark 3.8. From this, it follows immediately that $\dim_\mathrm{H}(E) = \dim_\mathrm{M}(E) = t$, where $\dim_\mathrm{H}$ denotes the Hausdorff dimension and $\dim_\mathrm{M}$ the upper Minkowski dimension. For a suitably chosen subclass of CMC's, the so-called tractable CMC's, we show in Theorem 3.9 that the positivity of $\mathcal{H}^t(E)$ conversely implies the uniform ball condition.

In Chapter 4, we introduce the *congruent CMC*, that is, the collection $\{X_\mathtt{i} \subset \mathbb{R}^d : \mathtt{i} \in I^*\}$ of compact sets with positive diameter satisfying

(C1) $X_{\mathtt{i}i} \subset X_{\mathtt{i}}$ as $\mathtt{i} \in I^*$ and $i \in I$,
(C2) there exist $\mathtt{i}, \mathtt{j} \in I^*$ such that $X_{\mathtt{i}} \cap X_{\mathtt{j}} = \emptyset$,
(C3) $\lim_{n\to\infty} \operatorname{diam}(X_{\mathtt{i}|_n}) = 0$ for every $\mathtt{i} \in I^\infty$,
(C4) there exists a constant $C \geq 1$ such that

$$\frac{\operatorname{dist}(X_{\mathtt{hi}}, X_{\mathtt{hj}})}{\operatorname{diam}(X_{\mathtt{h}})} \leq C \frac{\operatorname{dist}(X_{\mathtt{ki}}, X_{\mathtt{kj}})}{\operatorname{diam}(X_{\mathtt{k}})}$$

whenever $\mathtt{h}, \mathtt{k}, \mathtt{i}, \mathtt{j} \in I^*$.

Here with the notation $\operatorname{dist}(A, B)$, we mean the distance between given sets A and B. The fact that this kind of a collection is a tractable CMC together with the assumption (C4) justify the use of the name "congruent CMC". Observe that the role of the assumption (C2) is just to ensure that $\operatorname{diam}(E) > 0$. We shall show that the congruence of a CMC is bi-Lipschitz invariant and hence, the class of all congruent CMC's is sufficiently large. Furthermore, we prove in Theorem 4.3 that the Hausdorff dimension and the upper Minkowski dimension coincide even without assuming the ball condition. This is a consequence of a fact that the limit set is "approximately self-similar" by the assumption (C4). Generalizing the argument of Schief [28, Theorem 2.1] into this setting, we notice in Corollary 4.8 that the ball condition has a self-improvement property. In Proposition 4.9, we prove that assuming the ball condition, the Hausdorff dimension of the intersection $\pi([\mathtt{i}]) \cap \pi([\mathtt{j}])$ is strictly smaller than $\dim_{\mathrm{H}}(E)$ as $\mathtt{i} \perp \mathtt{j}$. Because of these properties, congruent CMC's can be thought as a natural generalization of conformal iterated function systems into the setting of Moran constructions.

With the *congruent iterated function system (IFS)*, we mean a collection $\{\varphi_i : i \in I\}$ of mappings satisfying

$$\underline{s}_{\mathtt{i}}|x - y| \leq |\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \leq \overline{s}_{\mathtt{i}}|x - y|,$$

where $0 < D^{-1}\overline{s}_{\mathtt{i}} \leq \underline{s}_{\mathtt{i}} \leq \overline{s}_{\mathtt{i}} < 1$ for a constant $D \geq 1$ and $\varphi_{\mathtt{i}} = \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}$ as $\mathtt{i} \in I^n$ and $n \in \mathbb{N}$. We also assume there exists a closed and nonempty set X such that $\varphi_i(X) \subset X$ for each $i \in I$. We show that a congruent IFS defines a congruent CMC. In Example 6.8, we note that each conformal IFS is congruent and also that there exist natural examples of congruent IFS's which are not conformal. We stress that the use of differentiable mappings is not a necessity in order to obtain the classical results. Indeed, from Lemma 5.3, Proposition 5.4, and Theorems 3.9 and 5.5, we derive the following.

Theorem 1.1. *For a congruent IFS, the following conditions are equivalent:*
(1) *The ball condition.*
(2) *The open set condition.*
(3) *The strong open set condition.*
(4) $\mathcal{H}^t(E) > 0$, *where t is the zero of the topological pressure.*

We also show that E is the closure of its interior provided that the underlying congruent IFS satisfies the open set condition and $\dim_{\mathrm{H}}(E) = d$.

The last chapter is devoted to examples.

2. Semiconformal measure

In this chapter we work only in the symbol space. We present sufficient conditions for the existence of the so-called semiconformal measure. Suppose the collection $\{s_{\mathtt{i}} > 0 : \mathtt{i} \in I^*\}$ satisfies the following two assumptions:

(S1) There exists a constant $D \geq 1$ such that
$$D^{-1} s_{\mathtt{i}} s_{\mathtt{j}} \leq s_{\mathtt{ij}} \leq D s_{\mathtt{i}} s_{\mathtt{j}}$$
whenever $\mathtt{i}, \mathtt{j} \in I^*$.

(S2) $\max_{\mathtt{i} \in I^n} s_{\mathtt{i}} \to 0$ as $n \to \infty$.

Given $t \geq 0$, we define the *topological pressure* to be
$$P(t) = \lim_{n \to \infty} \tfrac{1}{n} \log \sum_{\mathtt{i} \in I^n} s_{\mathtt{i}}^t.$$
The limit above exists by the standard theory of subadditive sequences since
$$\sum_{\mathtt{i} \in I^{n+m}} s_{\mathtt{i}}^t \leq D^t \sum_{\mathtt{ij} \in I^{n+m}} s_{\mathtt{i}}^t s_{\mathtt{j}}^t = D^t \sum_{\mathtt{i} \in I^n} s_{\mathtt{i}}^t \sum_{\mathtt{j} \in I^m} s_{\mathtt{j}}^t$$
using (S1).

As a function, $P\colon [0,\infty) \to \mathbb{R}$ is convex: Let $0 \leq t_1 \leq t_2$ and $\lambda \in (0,1)$. Now Hölder's inequality implies
$$\sum_{\mathtt{i} \in I^n} s_{\mathtt{i}}^{\lambda t_1 + (1-\lambda) t_2} = \sum_{\mathtt{i} \in I^n} (s_{\mathtt{i}}^{t_1})^{\lambda} (s_{\mathtt{i}}^{t_2})^{1-\lambda} \leq \Bigl(\sum_{\mathtt{i} \in I^n} s_{\mathtt{i}}^{t_1} \Bigr)^{\lambda} \Bigl(\sum_{\mathtt{i} \in I^n} s_{\mathtt{i}}^{t_2} \Bigr)^{1-\lambda}$$
from which the claim follows. According to (S2), we choose $n \in \mathbb{N}$ such that $\max_{\mathtt{i} \in I^n} s_{\mathtt{i}} < D^{-1}$. Then, using (S1), we observe that
$$P(t) \leq \lim_{k \to \infty} \tfrac{1}{kn} \log \Bigl(D^t \sum_{\mathtt{i} \in I^n} s_{\mathtt{i}}^t \Bigr)^k \leq \tfrac{1}{n} \log \bigl(D \max_{\mathtt{i} \in I^n} s_{\mathtt{i}} \bigr)^t + \tfrac{1}{n} \log \# I^n \longrightarrow -\infty$$
as $t \to \infty$. Hence there exists a unique $t \geq 0$ for which $P(t) = 0$.

Lemma 2.1. *Suppose $P(t) = 0$. Then*
$$D^{-t} \leq \sum_{\mathtt{i} \in I^n} s_{\mathtt{i}}^t \leq D^t$$
whenever $n \in \mathbb{N}$.

Proof. Since
$$P(t) = \inf_{n \in \mathbb{N}} \tfrac{1}{n} \Bigl(\log \sum_{\mathtt{i} \in I^n} s_{\mathtt{i}}^t + \log D^t \Bigr),$$

we have

$$\sum_{\mathtt{i}\in I^n} s_{\mathtt{i}}^t \geq D^{-t}e^{nP(t)}$$

as $n \in \mathbb{N}$. On the other hand,

$$P(t) \geq \lim_{k\to\infty} \tfrac{1}{kn} \log\Big(D^{-t}\sum_{\mathtt{i}\in I^n} s_{\mathtt{i}}^t\Big)^k = \log\Big(\sum_{\mathtt{i}\in I^n} s_{\mathtt{i}}^t\Big)^{1/n} + \log D^{-t/n}$$

implies

$$\sum_{\mathtt{i}\in I^n} s_{\mathtt{i}}^t \leq D^{t}e^{nP(t)}$$

for each $n \in \mathbb{N}$. □

Let l^∞ be the linear space of all bounded sequences on the real line. Recalling [26, Theorem 7.2], we say that the *Banach limit* is the mapping $L\colon l^\infty \to \mathbb{R}$ for which

(L1) L is linear,
(L2) L is positive, that is, $L\big((x_n)_{n\in\mathbb{N}}\big) \geq 0$ if $x_n \geq 0$ for all $n \in \mathbb{N}$,
(L3) $L\big((x_n)_{n\in\mathbb{N}}\big) = L\big((x_{n+1})_{n\in\mathbb{N}}\big)$,
(L4) $\liminf_{n\to\infty} x_n \leq L\big((x_n)_{n\in\mathbb{N}}\big) \leq \limsup_{n\to\infty} x_n$.

To simplify the notation, we denote $\mathrm{Lim}_n\, x_n = L\big((x_n)_{n\in\mathbb{N}}\big)$.

A Borel probability measure μ on I^∞ is called *t-semiconformal* if there exists a constant $c \geq 1$ such that

$$c^{-1}s_{\mathtt{i}}^t \leq \mu([\mathtt{i}]) \leq cs_{\mathtt{i}}^t$$

whenever $\mathtt{i} \in I^*$. We call a Borel probability measure μ on I^∞ *invariant* if $\mu([\mathtt{i}]) = \mu\big(\sigma^{-1}([\mathtt{i}])\big)$ for each $\mathtt{i} \in I^*$ and *ergodic* if $\mu(A) = 0$ or $\mu(A) = 1$ for every Borel set $A \subset I^\infty$ for which $A = \sigma^{-1}(A)$. The use of the Banach limit is a rather standard tool in producing an invariant measure from a given measure, for example, see [31, Corollary 1] and [23, Theorem 3.8]. In the following theorem, we shall find the semiconformal measure by applying the Banach limit to a collection of set functions.

Theorem 2.2. *Assuming $P(t) = 0$, there exists a unique invariant t-semiconformal measure. Furthermore, it is ergodic.*

Proof. Take $t \geq 0$ such that $P(t) = 0$. Define for each $\mathtt{i} \in I^*$ and $n \in \mathbb{N}$

$$\nu_n(\mathtt{i}) = \frac{\sum_{\mathtt{j}\in I^n} s_{\mathtt{ij}}^t}{\sum_{\mathtt{j}\in I^{|\mathtt{i}|+n}} s_{\mathtt{j}}^t}. \tag{2.1}$$

Letting $\nu(\mathtt{i}) = \mathrm{Lim}_n\, \nu_n(\mathtt{i})$, we have $\nu(\mathtt{i}) > 0$ and, using (L1) and (L3),

$$\begin{aligned}\sum_{j\in I} \nu(\mathtt{i}j) &= \sum_{j\in I} \operatorname*{Lim}_n \nu_n(\mathtt{i}j) = \operatorname*{Lim}_n \frac{\sum_{j\in I}\sum_{\mathtt{j}\in I^n} s^t_{\mathtt{i}j\mathtt{j}}}{\sum_{\mathtt{j}\in I^{|\mathtt{i}|+1+n}} s^t_{\mathtt{j}}} \\ &= \operatorname*{Lim}_n \nu_{n+1}(\mathtt{i}) = \operatorname*{Lim}_n \nu_n(\mathtt{i}) = \nu(\mathtt{i})\end{aligned} \tag{2.2}$$

whenever $\mathtt{i} \in I^*$. Since, by Lemma 2.1 and (S1),

$$\nu_n(\mathtt{i}) \le D^t \sum_{\mathtt{j}\in I^n} s^t_{\mathtt{ij}} \le D^{2t} \sum_{\mathtt{j}\in I^n} s^t_{\mathtt{i}} s^t_{\mathtt{j}} \le D^{3t} s^t_{\mathtt{i}}$$

and similarly the other way around, we have, using (L4),

$$D^{-3t} s^t_{\mathtt{i}} \le \nu(\mathtt{i}) \le D^{3t} s^t_{\mathtt{i}}. \tag{2.3}$$

Now, identifying $\mathtt{i} \in I^*$ with the cylinder $[\mathtt{i}]$, we notice, using (2.2), that ν is a probability measure on the semi-algebra of all cylinder sets. Hence, using the Carathéodory-Hahn Theorem (see [32, Theorem 11.20]), ν extends to a Borel probability measure on I^∞.

Define for each $n \in \mathbb{N}$ a Borel probability measure

$$\mu_n = \tfrac{1}{n} \sum_{j=0}^{n-1} \nu \circ \sigma^{-j}$$

and take μ to be an accumulation point of the set $\{\mu_n\}_{n\in\mathbb{N}}$ in the weak topology. For any $\mathtt{i} \in I^*$, we have

$$\bigl|\mu_n([\mathtt{i}]) - \mu_n\bigl(\sigma^{-1}([\mathtt{i}])\bigr)\bigr| = \tfrac{1}{n}\bigl|\nu([\mathtt{i}]) - \nu\bigl(\sigma^{-n}([\mathtt{i}])\bigr)\bigr| \le \tfrac{1}{n} \longrightarrow 0$$

as $n \to \infty$. Thus μ is invariant. We also have, using (2.3), (S1), and Lemma 2.1, that for each $\mathtt{i} \in I^*$

$$\begin{aligned}\mu([\mathtt{i}]) &= \lim_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} \nu \circ \sigma^{-j}([\mathtt{i}]) = \lim_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} \sum_{\mathtt{j}\in I^j} \nu([\mathtt{ji}]) \\ &\le D^{3t} \lim_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} \sum_{\mathtt{j}\in I^j} s^t_{\mathtt{ji}} \le D^{4t} \lim_{n\to\infty} \tfrac{1}{n} \sum_{j=0}^{n-1} s^t_{\mathtt{i}} \sum_{\mathtt{j}\in I^j} s^t_{\mathtt{j}} \le D^{5t} s^t_{\mathtt{i}}\end{aligned}$$

since cylinder sets have empty boundary. Estimating similarly into the other direction, we have shown that μ is a semiconformal measure with

$$D^{-5t} s^t_{\mathtt{i}} \le \mu([\mathtt{i}]) \le D^{5t} s^t_{\mathtt{i}} \tag{2.4}$$

as $\mathtt{i} \in I^*$.

We shall next prove that μ is ergodic. We have learned the following argument from the proof of [23, Theorem 3.8]. Assume on the contrary that there exists a

μ-measurable set $A \subset I^\infty$ such that $\sigma^{-1}(A) = A$ and $0 < \mu(A) < 1$. Fix $\mathtt{i} \in I^*$ and take an incomparable set $R \subset I^*$ for which $I^\infty \setminus A \subset \bigcup_{\mathtt{j}\in R}[\mathtt{j}]$ and

$$\sum_{\mathtt{j}\in R} \mu([\mathtt{ij}]) \le 2\mu([\mathtt{i}; I^\infty \setminus A]).$$

Using (2.4) and (S1), we infer

$$\begin{aligned}\mu([\mathtt{i}; I^\infty \setminus A]) &\ge 2^{-1}D^{-6t}s_{\mathtt{i}}^t \sum_{\mathtt{j}\in R} s_{\mathtt{j}}^t \ge 2^{-1}D^{-16t}\mu([\mathtt{i}])\sum_{\mathtt{j}\in R}\mu([\mathtt{j}]) \\ &\ge 2^{-1}D^{-16t}\mu([\mathtt{i}])\mu(I^\infty \setminus A).\end{aligned}$$

Therefore

$$\begin{aligned}\mu\big(\sigma^{-n}(A) \cap [\mathtt{i}]\big) &= \mu([\mathtt{i}; A]) = \mu([\mathtt{i}]) - \mu([\mathtt{i}; I^\infty \setminus A]) \\ &\le \big(1 - 2^{-1}D^{-16t}\mu(I^\infty \setminus A)\big)\mu([\mathtt{i}])\end{aligned} \tag{2.5}$$

for each $\mathtt{i} \in I^*$. Denote $\gamma = \big(1 - 2^{-1}D^{-16t}\mu(I^\infty \setminus A)\big)$ and $\eta = (1+\gamma^{-1})/2$. Take an incomparable set $R \subset I^*$ for which $A \subset \bigcup_{\mathtt{i}\in R}[\mathtt{i}]$ and $\sum_{\mathtt{i}\in R}\mu([\mathtt{i}]) \le \eta\mu(A)$. Since now, using (2.5),

$$\begin{aligned}\mu(A) &= \sum_{\mathtt{i}\in R}\mu(A \cap [\mathtt{i}]) = \sum_{\mathtt{i}\in R}\mu\big(\sigma^{-n}(A)\cap[\mathtt{i}]\big) \\ &\le \sum_{\mathtt{i}\in R}\gamma\mu([\mathtt{i}]) \le \gamma\eta\mu(A) < \mu(A),\end{aligned}$$

we have finished the proof of the ergodicity.

To prove the uniqueness, assume that $\tilde\mu$ is another invariant t-semiconformal measure. Now there exists $c \ge 1$ such that $\tilde\mu([\mathtt{i}]) \le c\mu([\mathtt{i}])$ whenever $\mathtt{i} \in I^*$. According to the uniqueness of the Carathéodory-Hahn extension, this inequality implies that also $\tilde\mu \le c\mu$. Using the ergodicity of the measure μ, it follows that $\tilde\mu = \mu$, see [30, Theorem 6.10]. The proof is finished. □

Remark 2.3. Observe that the use of the Banach limit in the proof of Theorem 2.2 is not a necessity. Defining for each $n \in \mathbb{N}$ a measure

$$\tilde\nu_n = \frac{\sum_{\mathtt{j}\in I^n} s_{\mathtt{j}}^t \delta_{\mathtt{jh}}}{\sum_{\mathtt{j}\in I^n} s_{\mathtt{j}}^t},$$

where $\delta_{\mathtt{h}}$ is a probability measure with support $\{\mathtt{h}\}$, we have for each $\mathtt{i} \in I^*$ and $n > |\mathtt{i}|$

$$\tilde\nu_n([\mathtt{i}]) = \sum_{\mathtt{j}\in I^{n-|\mathtt{i}|}} \tilde\nu_n([\mathtt{ij}]) = \frac{\sum_{\mathtt{j}\in I^{n-|\mathtt{i}|}} s_{\mathtt{ij}}^t}{\sum_{\mathtt{k}\in I^n} s_{\mathtt{k}}^t} = \nu_{n-|\mathtt{i}|}(\mathtt{i}),$$

where $\nu_n(\mathtt{i})$ is as in (2.1). Therefore, instead of using $\nu_n(\mathtt{i})$ and its Banach limit, we would have worked with $\tilde\nu_n$ and its weak limit.

Let us next prove two lemmas for future reference. Define for $\mathtt{i} \in I^*$

$$\Omega_{\mathtt{i}} = \{\mathtt{j} \in I^\infty : \sigma^{n-1}(\mathtt{j}) \in [\mathtt{i}] \text{ with infinitely many } n \in \mathbb{N}\}$$

and

$$\Omega_{\mathtt{i}}^0 = \{\mathtt{j} \in I^\infty : \sigma^{n-1}(\mathtt{j}) \notin [\mathtt{i}] \text{ for every } n \in \mathbb{N}\}.$$

Lemma 2.4. *Suppose μ is an invariant ergodic Borel probability measure on I^∞. Then $\mu(\Omega_{\mathtt{i}}^0) = 0$ and $\mu(\Omega_{\mathtt{i}}) = 1$ for every $\mathtt{i} \in I^*$ provided that $\mu([\mathtt{i}]) > 0$.*

Proof. Take $\mathtt{i} \in I^*$ such that $\mu([\mathtt{i}]) > 0$. Notice that $\sigma^{-1}(I^\infty \setminus \Omega_{\mathtt{i}}^0) \subset I^\infty \setminus \Omega_{\mathtt{i}}^0$ and due to the invariance of μ, it holds that $\mu\big(\sigma^{-1}(I^\infty \setminus \Omega_{\mathtt{i}}^0)\big) = \mu(I^\infty \setminus \Omega_{\mathtt{i}}^0)$. Since $\Omega_{\mathtt{i}} = \bigcap_{n=0}^\infty \sigma^{-n}(I^\infty \setminus \Omega_{\mathtt{i}}^0)$, we have $\sigma^{-1}(\Omega_{\mathtt{i}}) = \Omega_{\mathtt{i}}$ and using the ergodicity of μ, we have either $\mu(\Omega_{\mathtt{i}}) = 0$ or $\mu(\Omega_{\mathtt{i}}) = 1$. Since

$$\mu(\Omega_{\mathtt{i}}) = \lim_{n\to\infty} \mu\big(\sigma^{-n}(I^\infty \setminus \Omega_{\mathtt{i}}^0)\big) = \mu(I^\infty \setminus \Omega_{\mathtt{i}}^0) \geq \mu([\mathtt{i}]) > 0,$$

it follows that $\mu(I^\infty \setminus \Omega_{\mathtt{i}}^0) = \mu(\Omega_{\mathtt{i}}) = 1$. The proof is finished. □

Assume that I has at least three elements. For a fixed $j \in I$, we denote $I_j = I \setminus \{j\}$ and define

$$P_j(t) = \lim_{n\to\infty} \tfrac{1}{n} \log \sum_{\mathtt{i} \in I_j^n} s_{\mathtt{i}}^t.$$

Lemma 2.5. *Suppose $P(t) = 0$. Then $P_j(t) < 0$ for every $j \in I$.*

Proof. Using Theorem 2.2, we denote with μ the invariant ergodic Borel probability measure on I^∞ for which

$$c^{-1} s_{\mathtt{i}}^t \leq \mu([\mathtt{i}]) \leq c s_{\mathtt{i}}^t$$

for a constant $c \geq 1$ whenever $\mathtt{i} \in I^*$. Assume now on the contrary that there is $j \in I$ such that $P_j(t) = 0$. Using Theorem 2.2, we denote with μ_j the unique invariant t-semiconformal measure on I_j^∞. Observe that there exists a constant $c_j \geq 1$ such that

$$c_j^{-1} s_{\mathtt{i}}^t \leq \mu_j([\mathtt{i}]) \leq c_j s_{\mathtt{i}}^t$$

whenever $\mathtt{i} \in I_j^*$. Notice also that $\mu_j(I^\infty \setminus I_j^\infty) = 0$ and $\mu(I_j^\infty) = 0$ by Lemma 2.4. Defining $\lambda_j = \frac{1}{2}(\mu + \mu_j)$, we have for each $\mathtt{i} \in I_j^*$

$$\begin{aligned} \lambda_j([\mathtt{i}]) &= \lambda_j([\mathtt{i}] \setminus I_j^\infty) + \lambda_j([\mathtt{i}] \cap I_j^\infty) \\ &= \tfrac{1}{2}\mu([\mathtt{i}]) + \tfrac{1}{2}\mu_j([\mathtt{i}]) \leq \tfrac{1}{2}(c + c_j) s_{\mathtt{i}}^t \end{aligned}$$

and similarly the other way around. Hence also λ_j is invariant and t-semiconformal on I_j^∞. From the uniqueness, we infer $\lambda_j = \mu_j$, and therefore

$$1 = \mu_j(I_j^\infty) = \lambda_j(I_j^\infty) = \tfrac{1}{2}(\mu + \mu_j)(I_j^\infty) = \tfrac{1}{2}.$$

This contradiction finishes the proof. □

3. Controlled Moran construction

The collection of compact sets with positive diameter $\{X_{\mathtt{i}} \subset \mathbb{R}^d : \mathtt{i} \in I^*\}$ is called a *controlled Moran construction (CMC)* if

(M1) $X_{\mathtt{i}i} \subset X_{\mathtt{i}}$ as $\mathtt{i} \in I^*$ and $i \in I$,

(M2) there exists a constant $D \geq 1$ such that
$$D^{-1} \leq \frac{\operatorname{diam}(X_{\mathtt{ij}})}{\operatorname{diam}(X_{\mathtt{i}})\operatorname{diam}(X_{\mathtt{j}})} \leq D$$
whenever $\mathtt{i}, \mathtt{j} \in I^*$,

(M3) there exists $n \in \mathbb{N}$ such that
$$\max_{\mathtt{i} \in I^n} \operatorname{diam}(X_{\mathtt{i}}) < D^{-1}.$$

Lemma 3.1. *Given CMC, there are constants $c > 0$ and $0 < \varrho < 1$ such that $\max_{\mathtt{i} \in I^n} \operatorname{diam}(X_{\mathtt{i}}) \leq c\varrho^n$ for all $n \in \mathbb{N}$.*

Proof. Using (M3), we find $k \in \mathbb{N}$ and $0 < a < 1$ such that $\operatorname{diam}(X_{\mathtt{i}}) < a/D$ for every $\mathtt{i} \in I^k$. Fix $n > k$, take $\mathtt{i} \in I^n$ and denote $\mathtt{i} = \mathtt{i}_1\mathtt{i}_2 \cdots \mathtt{i}_l$, where $l-1$ is the integer part of n/k, $\mathtt{i}_j \in I^k$ for $j \in \{1, \ldots, l-1\}$, and $0 < |\mathtt{i}_l| \leq k$. Since now, by (M2),
$$\begin{aligned}\operatorname{diam}(X_{\mathtt{i}}) &\leq D^{l-1}\operatorname{diam}(X_{\mathtt{i}_1})\operatorname{diam}(X_{\mathtt{i}_2})\cdots\operatorname{diam}(X_{\mathtt{i}_{l-1}})\operatorname{diam}(X_{\mathtt{i}_l}) \\ &\leq D^{l-1}\big(a/D\big)^{l-1}\max_{0<|\mathtt{i}|\leq k}\operatorname{diam}(X_{\mathtt{i}}) \leq a^{-1}\max_{0<|\mathtt{i}|\leq k}\operatorname{diam}(X_{\mathtt{i}})\big(a^{1/k}\big)^n,\end{aligned}$$
the proof is finished. ∎

Using the assumption (M1) and Lemma 3.1, we define a *projection mapping* $\pi\colon I^\infty \to X$ such that
$$\{\pi(\mathtt{i})\} = \bigcap_{n=1}^{\infty} X_{\mathtt{i}|_n}$$
as $\mathtt{i} \in I^\infty$. It is clear that π is continuous. The compact set $E = \pi(I^\infty)$ is called a *limit set* (of the CMC). We call a Borel probability measure m on E *t-semiconformal* if there exists a constant $c \geq 1$ such that
$$c^{-1}\operatorname{diam}(X_{\mathtt{i}})^t \leq m(X_{\mathtt{i}}) \leq c\operatorname{diam}(X_{\mathtt{i}})^t$$
whenever $\mathtt{i} \in I^*$ and
$$m(X_{\mathtt{i}} \cap X_{\mathtt{j}}) = 0$$
whenever $\mathtt{i}\perp\mathtt{j}$. Observe that in Chapter 2 we defined a semiconformal measure on I^∞. The overlapping terminology should not be confusing as it is clear from the content which definition we use. Furthermore, for each $t \geq 0$, we set
$$P(t) = \lim_{n\to\infty}\tfrac{1}{n}\log\sum_{\mathtt{i}\in I^n}\operatorname{diam}(X_{\mathtt{i}})^t \tag{3.1}$$

provided that the limit exists. It follows straight from the definition that if there exists a t-semiconformal measure on E then $P(t) = 0$. Recalling Lemma 2.1, the equation $P(t) = 0$ gives a natural upper bound for the Hausdorff dimension of E, $\dim_{\mathrm{H}}(E) \le t$. Finally, we say that a CMC satisfies a *bounded overlapping property* if $\sup_{x\in E} \sup\bigl\{\#R : R \subset \{\mathtt{i} \in I^* : x \in X_{\mathtt{i}}\} \text{ is incomparable}\bigr\} < \infty$.

Proposition 3.2. *Given CMC, the limit in* (3.1) *exists and there is a unique* $t \ge 0$ *such that* $P(t) = 0$*. Assuming* $P(t) = 0$*, there exists an invariant ergodic Borel probability measure* μ *on* I^∞ *and constants* $c, c' > 0$ *such that*

$$c^{-1}\operatorname{diam}(X_{\mathtt{i}})^t \le \mu([\mathtt{i}]) \le c\operatorname{diam}(X_{\mathtt{i}})^t$$

whenever $\mathtt{i} \in I^*$ *and denoting* $m = \mu \circ \pi^{-1}$*, we have* $\mathcal{H}^t(A) \le c' m(A)$ *for every* m*-measurable* $A \subset E$*. Furthermore, if in addition the CMC satisfies the bounded overlapping property and for each* $\mathtt{i}, \mathtt{j} \in I^*$ *and* $\mathtt{h} \in I^\infty$ *it holds that* $\pi(\mathtt{ih}) \in X_{\mathtt{ij}}$ *whenever* $\pi(\mathtt{h}) \in X_{\mathtt{j}}$ *then* m *is a* t*-semiconformal measure on* E*.*

Proof. According to (M2) and Lemma 3.1, the collection $\{\operatorname{diam}(X_{\mathtt{i}}) : \mathtt{i} \in I^*\}$ satisfies (S1) and (S2). The proof of the first claim is now trivial. Suppose $P(t) = 0$ and denote with μ the t-semiconformal measure on I^∞ associated to this collection, see Theorem 2.2. For fixed $x \in E$ and $r > 0$ take $\mathtt{i} = (i_1, i_2, \ldots) \in I^\infty$ such that $\pi(\mathtt{i}) = x$ and choose n to be the smallest integer for which $X_{\mathtt{i}|_n} \subset B(x, r)$. Denoting $m = \mu \circ \pi^{-1}$ and using (M2), we obtain

$$\begin{aligned} m\bigl(B(x,r)\bigr) &\ge m(X_{\mathtt{i}|_n}) \ge \mu([\mathtt{i}|_n]) \ge c^{-1}\operatorname{diam}(X_{\mathtt{i}|_n})^t \\ &\ge c^{-1}D^{-1}\operatorname{diam}(X_{\mathtt{i}|_{n-1}})^t \operatorname{diam}(X_{i_n})^t \\ &\ge c^{-1}D^{-1}\min_{i\in I}\operatorname{diam}(X_i)^t r^t, \end{aligned}$$

which, according to [8, Proposition 2.2(b)], gives the second claim. Here with the notation $B(x, r)$, we mean the open ball centered at x with radius r. Furthermore, if the bounded overlapping property is satisfied then the proof of [14, Theorem 3.7] shows that

$$m(X_{\mathtt{i}} \cap X_{\mathtt{j}}) = 0$$

whenever $\mathtt{i}\perp\mathtt{j}$ provided that for each $\mathtt{i}, \mathtt{h}, \mathtt{k} \in I^*$ it holds $\mu\bigl([\mathtt{i}; \pi^{-1}(X_{\mathtt{h}} \cap X_{\mathtt{k}})]\bigr) \le m(X_{\mathtt{ih}} \cap X_{\mathtt{ik}})$. This is guaranteed by our extra assumption. Hence

$$\begin{aligned} m(X_{\mathtt{i}}) &= m\Bigl(X_{\mathtt{i}} \setminus \bigcup_{\mathtt{i}\perp\mathtt{j}} X_{\mathtt{j}} \cap X_{\mathtt{i}}\Bigr) \\ &= \mu\Bigl(\pi^{-1}(X_{\mathtt{i}}) \setminus \bigcup_{\mathtt{i}\perp\mathtt{j}} \pi^{-1}(X_{\mathtt{j}} \cap X_{\mathtt{i}})\Bigr) = \mu([\mathtt{i}]), \end{aligned}$$

which finishes the proof of the last claim. □

In the definition that follows, we introduce a natural separation condition to be used on Moran constructions. Given CMC, define for $r > 0$

$$Z(r) = \{\mathtt{i} \in I^* : \operatorname{diam}(X_{\mathtt{i}}) \le r < \operatorname{diam}(X_{\mathtt{i}^-})\}$$

and if in addition $x \in E$, we set

$$Z(x,r) = \{\mathtt{i} \in Z(r) : X_{\mathtt{i}} \cap B(x,r) \neq \emptyset\}.$$

It is often useful to know the cardinality of the set $Z(x,r)$. We say that a CMC satisfies a *finite clustering property* if $\sup_{x\in E} \limsup_{r\downarrow 0} \#Z(x,r) < \infty$. Furthermore, if $\sup_{x\in E}\sup_{r>0} \#Z(x,r) < \infty$ then the CMC is said to satisfy a *uniform finite clustering property.*

Definition 3.3. We say that a CMC satisfies a *ball condition* if there exists a constant $0 < \delta < 1$ such that for each $x \in E$ there is $r_0 > 0$ such that for every $0 < r < r_0$ there exists a set $\{x_{\mathtt{i}} \in \operatorname{conv}(X_{\mathtt{i}}) : \mathtt{i} \in Z(x,r)\}$ such that the collection $\{B(x_{\mathtt{i}}, \delta r) : \mathtt{i} \in Z(x,r)\}$ is disjoint. If $r_0 > 0$ above can be chosen to be infinity for every $x \in E$ then the CMC is said to satisfy a *uniform ball condition.* Here with the notation $\operatorname{conv}(A)$, we mean the convex hull of a given set A.

We shall next prove that the (uniform) ball condition and the (uniform) finite clustering property are equivalent.

Lemma 3.4. *Given compact and connected set $A \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, there exists points $x_1, \dots, x_k \in A$ such that the collection of balls $\big\{B\big(x_i, (2k)^{-1}\operatorname{diam}(A)\big) : i \in \{1,\dots,k\}\big\}$ is disjoint and $\#\big\{i \in \{1,\dots,k\} : B\big(x_i,(2k)^{-1}\operatorname{diam}(A)\big) \cap B\big(x,(2k)^{-1}\operatorname{diam}(A)\big) \neq \emptyset\big\} \le 2$ for every $x \in \mathbb{R}^n$.*

Proof. Choose $y_1, y_k \in A$ such that $|y_1 - y_k| = \operatorname{diam}(A)$. Denote the line going through y_1 and y_k with L and define for each $i \in \{2,\dots,k-1\}$ a point $y_i = \big(1-\frac{i}{k}\big)y_1 + \frac{i}{k}y_k \in L$. Using the connectedness of A, we find for each $i \in \{1,\dots,k\}$ a point $x_i \in A$ for which the inner product $(x_i - y_i)\cdot(y_k - y_1) = 0$. The proof is finished. □

Theorem 3.5. *A CMC satisfies the (uniform) ball condition exactly when it satisfies the (uniform) finite clustering property.*

Proof. We shall prove the non-uniform case. The uniform case follows similarly. Assuming the ball condition, take $x \in E$ and $0 < r < r_0$. Choose for each $\mathtt{i} \in Z(x,r)$ a point $x_{\mathtt{i}} \in \operatorname{conv}(X_{\mathtt{i}})$ such that the balls $B(x_{\mathtt{i}}, \delta r)$ are disjoint as $\mathtt{i} \in Z(x,r)$. Now clearly

$$B(x_{\mathtt{i}}, \delta r) \subset B\big(x, (2+\delta)r\big)$$

for every $\mathtt{i} \in Z(x,r)$. Hence

$$\begin{aligned}\#Z(x,r)\delta^d r^d \alpha(d) &= \sum_{\mathtt{i}\in Z(x,r)} \mathcal{H}^d\big(B(x_\mathtt{i},\delta r)\big) = \mathcal{H}^d\Bigg(\bigcup_{\mathtt{i}\in Z(x,r)} B(x_\mathtt{i},\delta r)\Bigg)\\ &\le \mathcal{H}^d\big(B\big(x,(2+\delta)r\big)\big) = (2+\delta)^d r^d \alpha(d),\end{aligned}$$

where $\alpha(d)$ denotes the d-dimensional Hausdorff measure of the unit ball. This shows that the CMC satisfies the finite clustering property.

Conversely, by the finite clustering property, there exists $M > 0$ such that for each $x \in E$ there is $r_0 > 0$ such that $\#Z(x,r) < M$ whenever $0 < r < r_0$. Choose $\delta = (4MD)^{-1}\min_{i\in I}\operatorname{diam}(X_i)$ and for fixed $x \in E$ and $0 < r < r_0$ denote the symbols of $Z(x,r)$ with $\mathtt{i}_1,\dots,\mathtt{i}_n$, where $n = \#Z(x,r)$. We shall define the points $x_{\mathtt{i}_1},\dots,x_{\mathtt{i}_n}$ needed in the ball condition inductively. Choose $x_{\mathtt{i}_1}$ to be any point of $\operatorname{conv}(X_{\mathtt{i}_1})$ and assume the points $x_{\mathtt{i}_1},\dots,x_{\mathtt{i}_k}$, where $k \in \{1,\dots,n-1\}$, have already been chosen such that the collection of balls $\big\{B(x_{\mathtt{i}_i},\delta r) : i \in \{1,\dots,k\}\big\}$ is disjoint. Using Lemma 3.4, we find points $y_1,\dots,y_{2n} \in \operatorname{conv}(X_{\mathtt{i}_{k+1}})$ such that the collection $\big\{B\big(y_j,(4n)^{-1}\operatorname{diam}(X_{\mathtt{i}_{k+1}})\big) : j \in \{1,\dots,2n\}\big\}$ is disjoint. Since, using (M2),

$$\delta r \le (4MD)^{-1}\min_{i\in I}\operatorname{diam}(X_i)\operatorname{diam}(X_{\mathtt{i}^-}) \le (4n)^{-1}\operatorname{diam}(X_\mathtt{i})$$

for every $\mathtt{i} \in Z(x,r)$, Lemma 3.4 says also that the balls $B(x_{\mathtt{i}_i},\delta r)$, $i \in \{1,\dots,k\}$, can intersect at most $2k$ of balls $B\big(y_j,(4n)^{-1}\operatorname{diam}(X_{\mathtt{i}_{k+1}})\big)$, $j \in \{1,\dots,2n\}$. Hence, choosing $x_{\mathtt{i}_{k+1}} \in \{y_1,\dots,y_{2n}\}$ such that $B\big(x_{\mathtt{i}_{k+1}},(4n)^{-1}\operatorname{diam}(X_{\mathtt{i}_{k+1}})\big) \cap B(x_{\mathtt{i}_i},\delta r) = \emptyset$ for every $i \in \{1,\dots,k\}$, we have finished the proof. □

It is evident that the bounded overlapping property does not imply the finite clustering property and in Example 6.1, we show that the converse does not hold either. The natural condition according to which $\sup_{x\in E, r>0}\sup\big\{\#R : R \subset \{\mathtt{i} \in I^* : X_\mathtt{i} \cap B(x,r) \ne \emptyset \text{ and } \operatorname{diam}(X_{\mathtt{i}^-}) > r\} \text{ is incomparable}\big\} < \infty$ clearly implies both the bounded overlapping property and the uniform finite clustering property. See also [23, Lemma 2.7]. However, we do not need this condition as under a minor technical assumption, the finite clustering property implies the bounded overlapping property.

Lemma 3.6. *If a CMC satisfies the finite clustering property then it satisfies the bounded overlapping property provided that*

$$X_\mathtt{i} \cap E = \pi([\mathtt{i}])$$

for each $\mathtt{i} \in I^*$.

Proof. Set $M = \sup_{x\in E}\limsup_{r\downarrow 0}\#Z(x,r)$. Fix $x \in E$ and assume that $R \subset I^*$ is a finite and incomparable set such that $x \in X_\mathtt{i}$ for each $\mathtt{i} \in R$. Choose $r > 0$

small enough so that $\#Z(x,r) \le M$ and

$$\min_{\mathtt{j}\in Z(x,r)} |\mathtt{j}| > \max_{\mathtt{i}\in R} |\mathtt{i}|.$$

According to the assumption, $x \in \bigcap_{\mathtt{i}\in R} \pi([\mathtt{i}])$, and hence, for each $\mathtt{i} \in R$ there exists a unique $\mathtt{i}^* \in Z(x,r)$ such that $\mathtt{i}^*|_n = \mathtt{i}$ for some $n \in \mathbb{N}$. The incomparability of R now implies that $\mathtt{i}^* \ne \mathtt{j}^*$ for distinct $\mathtt{i},\mathtt{j} \in R$. Consequently, $\#R \le \#Z(x,r) \le M$. $\square$

Let us examine how the Hausdorff measure is related to the ball condition. Bear in mind that the finite clustering property and the ball condition are equivalent.

Theorem 3.7. *If a CMC satisfies the uniform finite clustering property, $P(t) = 0$, and m is the measure of Proposition 3.2 then there exist constants $r_0 > 0$ and $K \ge 1$ such that*

$$K^{-1} r^t \le m\bigl(B(x,r)\bigr) \le K r^t$$

whenever $x \in E$ and $0 < r < r_0$. Consequently, $\dim_{\mathrm{H}}(E) = \dim_{\mathrm{M}}(E) = t$.

Proof. Suppose $P(t) = 0$ and $m = \mu \circ \pi^{-1}$ is the measure of Proposition 3.2. Seeing that $\pi^{-1}\bigl(B(x,r)\bigr) \subset \bigcup_{\mathtt{i}\in Z(x,r)} [\mathtt{i}]$, we get for fixed $x \in E$ and $r > 0$

$$\begin{aligned} m\bigl(B(x,r)\bigr) &\le \mu\biggl(\bigcup_{\mathtt{i}\in Z(x,r)} [\mathtt{i}]\biggr) \le \sum_{\mathtt{i}\in Z(x,r)} \mu([\mathtt{i}]) \\ &\le c \sum_{\mathtt{i}\in Z(x,r)} \mathrm{diam}(X_{\mathtt{i}})^t \le \#Z(x,r) c r^t, \end{aligned}$$

which, together with the uniform finite clustering property and the proof of Proposition 3.2, gives the first claim.

The second claim follows immediately from [22, Theorem 5.7]. $\square$

Remark 3.8. We remark that in Theorem 3.7, the measure m can be replaced with the Hausdorff measure $\mathcal{H}^t|_E$ by recalling [8, Proposition 2.2]. In fact, it is sufficient to assume the finite clustering property instead of the uniform finite clustering property to see that $\mathcal{H}^t|_E$ is proportional to m. Especially, under this assumption, it holds that $0 < \mathcal{H}^t(E) < \infty$.

One could easily prove that if $\mathcal{H}^t|_E$ is t-semiconformal for some $t \ge 0$ then there exists a set $A \subset E$ with $\mathcal{H}^t(E \setminus A) = 0$ such that $\sup_{x\in A} \limsup_{r\downarrow 0} \#Z(x,r) < \infty$. Since this hardly generalizes to the whole set E without any additional assumption, we propose the following definition. We say that a CMC is *tractable* if there exists a constant $C \ge 1$ such that for each $r > 0$ we have

$$\mathrm{dist}(X_{\mathtt{hi}}, X_{\mathtt{hj}}) \le C \,\mathrm{diam}(X_{\mathtt{h}}) r \tag{3.2}$$

whenever $\mathtt{h} \in I^*$, $\mathtt{i}, \mathtt{j} \in Z(r)$, and $\operatorname{dist}(X_{\mathtt{i}}, X_{\mathtt{j}}) \le r$. See Example 6.2 for an example of a nontractable CMC. Compare the following theorem to [28, Theorem 2.1] and [27, Theorem 1.1].

Theorem 3.9. *A tractable CMC satisfies the uniform finite clustering property provided that $P(t) = 0$ and $\mathcal{H}^t(E) > 0$.*

Proof. Assume on the contrary that for each $N \in \mathbb{N}$ there are $x'_N \in E$ and $r'_N > 0$ such that $\#Z(x'_N, r'_N) \ge N$. For fixed $N \in \mathbb{N}$ choose $\mathtt{i} \in Z(x'_N, r'_N)$ so that $x'_N = \pi(\mathtt{i}\mathtt{k}_0)$ for some $\mathtt{k}_0 \in I^\infty$. We define

$$\Omega_{\mathtt{i}} = \{\mathtt{k} \in I^\infty : \sigma^{n-1}(\mathtt{k}) \in [\mathtt{i}] \text{ with infinitely many } n \in \mathbb{N}\}$$

and taking arbitrary $\mathtt{k} \in \Omega_{\mathtt{i}}$ and $n \in \mathbb{N}$ for which $\sigma^n(\mathtt{k}) \in [\mathtt{i}]$, we denote $x = \pi(\mathtt{k})$ and $\mathtt{h} = \mathtt{k}|_n$. Finally, pick $\mathtt{j}_1, \ldots, \mathtt{j}_N \in Z(x'_N, r'_N)$ such that $\mathtt{j}_i \ne \mathtt{j}_j$ as $i \ne j$. Since now $\operatorname{dist}(X_{\mathtt{i}}, X_{\mathtt{j}_i}) \le r'_N$ for every $i \in \{1, \ldots, N\}$, we have, according to the assumption, that $\operatorname{dist}(X_{\mathtt{hi}}, X_{\mathtt{hj}_i}) \le C \operatorname{diam}(X_{\mathtt{h}}) r'_N$. Hence

$$\begin{aligned}\pi([\mathtt{hj}_i]) \subset X_{\mathtt{hj}_i} &\subset B\bigl(x, \operatorname{diam}(X_{\mathtt{hi}}) + \operatorname{dist}(X_{\mathtt{hi}}, X_{\mathtt{hj}_i}) + \operatorname{diam}(X_{\mathtt{hj}_i})\bigr) \\ &\subset B\bigl(x, (2D + C) \operatorname{diam}(X_{\mathtt{h}}) r'_N\bigr)\end{aligned}$$

for each $i \in \{1, \ldots, N\}$ recalling that $x \in X_{\mathtt{hi}}$. Therefore

$$\pi\Bigl(\bigcup_{i=1}^N [\mathtt{hj}_i]\Bigr) \subset B(x, r_n),$$

where $r_n = (2D + C) \operatorname{diam}(X_{\mathtt{k}|_n}) r'_N$, and

$$\begin{aligned}\frac{m\bigl(B(x, r_n)\bigr)}{r_n^t} &\ge \frac{\sum_{i=1}^N \mu([\mathtt{hj}_i])}{r_n^t} \ge \frac{c^{-1} \sum_{i=1}^N \operatorname{diam}(X_{\mathtt{hj}_i})^t}{r_n^t} \\ &\ge \frac{c^{-1} D^{-t} \operatorname{diam}(X_{\mathtt{h}})^t \sum_{i=1}^N \operatorname{diam}(X_{\mathtt{j}_i})^t}{(2D + C)^t \operatorname{diam}(X_{\mathtt{h}})^t r'_N} \ge C_0 N,\end{aligned}$$

where μ is the measure of Proposition 3.2, $m = \mu \circ \pi^{-1}$, and the constant $C_0 > 0$ does not depend on n or N. Since $r_n \downarrow 0$ as $n \to \infty$, we obtain

$$\limsup_{r \downarrow 0} \frac{m\bigl(B(x, r)\bigr)}{r^t} \ge C_0 N$$

for all $x \in \pi(\Omega_{\mathtt{i}})$, which, according to [8, Proposition 2.2(b)], gives

$$\mathcal{H}^t\bigl(\pi(\Omega_{\mathtt{i}})\bigr) \le 2^t C_0^{-1} N^{-1} m\bigl(\pi(\Omega_{\mathtt{i}})\bigr). \tag{3.3}$$

Since $1 = \mu(\Omega_{\mathtt{i}}) \le m\bigl(\pi(\Omega_{\mathtt{i}})\bigr) \le 1$ by Lemma 2.4, we have, using (3.3) and Proposition 3.2,

$$\begin{aligned}\mathcal{H}^t(E) &\le \mathcal{H}^t\bigl(\pi(\Omega_{\mathtt{i}})\bigr) + \mathcal{H}^t\bigl(E \setminus \pi(\Omega_{\mathtt{i}})\bigr) \\ &\le 2^t C_0^{-1} N^{-1} m\bigl(\pi(\Omega_{\mathtt{i}})\bigr) + c' m\bigl(E \setminus \pi(\Omega_{\mathtt{i}})\bigr) \le 2^t C_0^{-1} N^{-1},\end{aligned}$$

which leads to a contradiction as $N \to \infty$. $\square$

To summarize the implications of the previous theorem, we finish this chapter with the following corollary.

Corollary 3.10. *For a tractable CMC, the following are equivalent:*
1. *The ball condition.*
2. *The uniform ball condition.*
3. $\mathcal{H}^t(E) > 0$, *where* $P(t) = 0$.
4. *There exist constants* $r_0 > 0$ *and* $K \geq 1$ *such that*
$$K^{-1} r^t \leq \mathcal{H}^t|_E\big(B(x,r)\big) \leq K r^t$$
whenever $x \in E$, $0 < r < r_0$, *and* $P(t) = 0$.

4. Congruent Moran construction

In a tractable CMC, we require that the relative positions of the sets $X_{\mathtt{i}}$, $\mathtt{i} \in I^*$, follow the rule given in (3.2). The only restriction for the shapes of these sets comes from (M2) and (M3). Assuming more on the shape, we are able to prove that the Hausdorff dimension and the upper Minkowski dimension of the limit set coincide and if the uniform ball condition is satisfied then the dimension of the intersection of incomparable cylinder sets is small. We say that a CMC is *congruent* if there is a constant $C^* \geq 1$ such that

$$\frac{\operatorname{dist}(X_{\mathtt{hi}}, X_{\mathtt{hj}})}{\operatorname{diam}(X_{\mathtt{h}})} \leq C^* \frac{\operatorname{dist}(X_{\mathtt{ki}}, X_{\mathtt{kj}})}{\operatorname{diam}(X_{\mathtt{k}})}$$

whenever $\mathtt{h}, \mathtt{k}, \mathtt{i}, \mathtt{j} \in I^*$. Observe that this is equivalent to the existence of a constant $C \geq 1$ for which

$$C^{-1} \operatorname{diam}(X_{\mathtt{h}}) \operatorname{dist}(X_{\mathtt{i}}, X_{\mathtt{j}}) \leq \operatorname{dist}(X_{\mathtt{hi}}, X_{\mathtt{hj}}) \leq C \operatorname{diam}(X_{\mathtt{h}}) \operatorname{dist}(X_{\mathtt{i}}, X_{\mathtt{j}}) \tag{4.1}$$

whenever $\mathtt{h}, \mathtt{i}, \mathtt{j} \in I^*$. We notice immediately that a congruent CMC is tractable which indicates, for example, that the finite clustering property and the uniform finite clustering property are equivalent.

Let us first introduce natural mappings for a congruent CMC.

Lemma 4.1. *If a CMC is congruent then for each* $\mathtt{i} \in I^*$ *there exists a mapping* $\varphi_{\mathtt{i}} \colon E \to E$ *such that* $\varphi_{\mathtt{i}}\big(\pi(\mathtt{h})\big) = \pi(\mathtt{ih})$ *as* $\mathtt{h} \in I^\infty$ *and*

$$C^{-1} \operatorname{diam}(X_{\mathtt{i}})|x-y| \leq |\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \leq C \operatorname{diam}(X_{\mathtt{i}})|x-y|$$

whenever $x, y \in E$.

Proof. Fix $\mathtt{i} \in I^*$ and $\mathtt{h}, \mathtt{k} \in I^\infty$. Take $\varepsilon > 0$ and using Lemma 3.1, choose $n \in \mathbb{N}$ such that $\operatorname{diam}(X_{\mathtt{i}(\mathtt{h}|_n)}) + \operatorname{diam}(X_{\mathtt{i}(\mathtt{k}|_n)}) < \varepsilon$. Now, using (4.1), we have

$$\begin{aligned}
|\pi(\mathtt{ih}) - \pi(\mathtt{ik})| &\leq \operatorname{diam}(X_{\mathtt{i}(\mathtt{h}|_n)}) + \operatorname{dist}(X_{\mathtt{i}(\mathtt{h}|_n)}, X_{\mathtt{i}(\mathtt{k}|_n)}) + \operatorname{diam}(X_{\mathtt{i}(\mathtt{k}|_n)}) \\
&\leq C \operatorname{diam}(X_{\mathtt{i}}) \operatorname{dist}(X_{\mathtt{h}|_n}, X_{\mathtt{k}|_n}) + \varepsilon \\
&\leq C \operatorname{diam}(X_{\mathtt{i}}) |\pi(\mathtt{h}) - \pi(\mathtt{k})| + \varepsilon.
\end{aligned} \tag{4.2}$$

On the other hand, choosing $n \in \mathbb{N}$ such that $\operatorname{diam}(X_{\mathtt{h}|_n}) + \operatorname{diam}(X_{\mathtt{k}|_n}) < \varepsilon$, we get similarly

$$\begin{aligned}|\pi(\mathtt{ih}) - \pi(\mathtt{ik})| &\geq \operatorname{dist}(X_{\mathtt{i}(\mathtt{h}|_n)}, X_{\mathtt{i}(\mathtt{k}|_n)}) \\ &\geq C^{-1} \operatorname{diam}(X_\mathtt{i}) \operatorname{dist}(X_{\mathtt{h}|_n}, X_{\mathtt{k}|_n}) \\ &\geq C^{-1} \operatorname{diam}(X_\mathtt{i})\big(|\pi(\mathtt{h}) - \pi(\mathtt{k})| - \operatorname{diam}(X_{\mathtt{h}|_n}) - \operatorname{diam}(X_{\mathtt{k}|_n})\big) \\ &\geq C^{-1} \operatorname{diam}(X_\mathtt{i})|\pi(\mathtt{h}) - \pi(\mathtt{k})| - C^{-1} \operatorname{diam}(X_\mathtt{i})\varepsilon.\end{aligned}$$

The claim follows now by letting $\varepsilon \downarrow 0$ since according to (4.2), we may define a mapping $\varphi_\mathtt{i} \colon E \to E$ by setting $\varphi_\mathtt{i}\big(\pi(\mathtt{h})\big) = \pi(\mathtt{ih})$ as $\mathtt{h} \in I^\infty$. □

It follows that the measure of Proposition 3.2 is semiconformal on a congruent CMC satisfying the finite clustering property in the following sense.

Lemma 4.2. *If a congruent CMC satisfies the finite clustering property, $P(t) = 0$, and m is the measure of Proposition 3.2 then*

$$m\big(\varphi_\mathtt{i}(E) \cap \varphi_\mathtt{j}(E)\big) = 0$$

whenever $\mathtt{i} \perp \mathtt{j}$. Here $\varphi_\mathtt{i}$, $\mathtt{i} \in I^*$, are the mappings of Lemma 4.1.

Proof. Since Lemma 4.1 clearly implies that $\operatorname{diam}\big(\varphi_\mathtt{i}(E)\big)$ is proportional to $\operatorname{diam}(X_\mathtt{i})$, the CMC formed by the sets $\varphi_\mathtt{i}(E)$, $\mathtt{i} \in I^*$, has the same topological pressure as the original CMC. Notice that $\operatorname{diam}(E) > 0$ by the finite clustering property. By the uniqueness of the invariant semiconformal measure on I^∞, also the semiconformal measures determined by these CMC's on I^∞ are the same. Noting that the finite clustering property remains satisfied in the new setting and trivially $\varphi_\mathtt{i}(E) \cap E = \pi([\mathtt{i}])$ for each $\mathtt{i} \in I^*$, Lemma 3.6 implies the bounded overlapping property. By the congruence, it is evident that for each $\mathtt{i}, \mathtt{j} \in I^*$ and $\mathtt{h} \in I^\infty$ it holds that $\pi(\mathtt{ih}) \in \varphi_{\mathtt{ij}}(E)$ whenever $\pi(\mathtt{h}) \in \varphi_\mathtt{j}(E)$ and hence Proposition 3.2 completes the proof. □

Using the mappings of Lemma 4.1, we are able to prove that the Hausdorff dimension and the upper Minkowski dimension of the limit set of a congruent CMC coincide even without assuming the ball condition.

Theorem 4.3. *If a CMC is congruent and $t = \dim_{\mathrm{H}}(E)$ then $\dim_{\mathrm{M}}(E) = t$ and $\mathcal{H}^t(E) < \infty$.*

Proof. We may assume that $\operatorname{diam}(E) > 0$. Let $\varphi_\mathtt{i}$, $\mathtt{i} \in I^*$, be the mappings of Lemma 4.1. Notice that, using (M2), there exists a constant $\delta > 0$ such that

$$\operatorname{diam}(X_{\mathtt{i}i}) \geq \delta \operatorname{diam}(X_\mathtt{i}) \tag{4.3}$$

whenever $\mathtt{i} \in I^*$ and $i \in I$. Take $x_0 \in E$, $\mathtt{h} \in I^\infty$ such that $x_0 = \pi(\mathtt{h})$, and $0 < r < C \operatorname{diam}(E)^2$. Then choose $n \in \mathbb{N}$ such that $\mathtt{h}|_n \in Z\big(C^{-1} \operatorname{diam}(E)^{-1} r\big)$.

Since $x_0 = \varphi_{\mathtt{h}|_n}\big(\pi(\sigma^n(\mathtt{h}))\big)$, we have

$$\begin{aligned}|x_0 - \varphi_{\mathtt{h}|_n}(y)| &\le C \operatorname{diam}(X_{\mathtt{h}|_n})\big|\pi\big(\sigma^n(\mathtt{h})\big) - y\big| \\ &\le C \operatorname{diam}(X_{\mathtt{h}|_n}) \operatorname{diam}(E) < r\end{aligned}$$

for every $y \in E$. Hence

$$\varphi_{\mathtt{h}|_n}(E) \subset E \cap B(x_0, r).$$

On the other hand, using (4.3),

$$\begin{aligned}|\varphi_{\mathtt{h}|_n}(x) - \varphi_{\mathtt{h}|_n}(y)| &\ge C^{-1} \operatorname{diam}(X_{\mathtt{h}|_n})|x-y| \\ &\ge C^{-2} \operatorname{diam}(E)^{-1}\delta r|x-y|\end{aligned}$$

whenever $x, y \in E$. Therefore for each $x_0 \in E$ and $0 < r < C \operatorname{diam}(E)^2$ there is a mapping $g\colon E \to E \cap B(x_0, r)$ and a constant $a = C^{-2} \operatorname{diam}(E)^{-1}\delta$ such that

$$|g(x) - g(y)| \ge ar|x-y|$$

whenever $x, y \in E$. The claim follows now from [8, Theorem 3.2]. □

The following simple proposition shows the bi-Lipschitz invariance of a congruent CMC. Therefore the collection of all congruent CMC's is sufficiently large. Observe that despite of this property the geometry of the limit set may change a lot under a bi-Lipschitz map, see [21, Lemma 3.2].

Proposition 4.4. *If $\{X_{\mathtt{i}} : \mathtt{i} \in I^*\}$ is a congruent CMC with E as a limit set and $h\colon \mathbb{R}^d \to \mathbb{R}^d$ is a bi-Lipschitz mapping then $\{h(X_{\mathtt{i}}) : \mathtt{i} \in I^*\}$ is a congruent CMC with $h(E)$ as a limit set.*

Proof. Fix constants $a, b > 0$ such that

$$a|x-y| \le |h(x) - h(y)| \le b|x-y|$$

for every $x, y \in X$. The condition (M1) is clearly satisfied and since $a \operatorname{diam}(X_{\mathtt{i}}) \le \operatorname{diam}\big(h(X_{\mathtt{i}})\big) \le b \operatorname{diam}(X_{\mathtt{i}})$ as $\mathtt{i} \in I^*$ and $a \operatorname{dist}(X_{\mathtt{i}}, X_{\mathtt{j}}) \le \operatorname{dist}\big(h(X_{\mathtt{i}}), h(X_{\mathtt{j}})\big) \le b \operatorname{dist}(X_{\mathtt{i}}, X_{\mathtt{j}})$ as $\mathtt{i}, \mathtt{j} \in I^*$, also the conditions (M2), (M3), and (4.1) are satisfied. The proof is finished. □

Examining the method used in [28, Theorem 2.1], one is easily convinced by the usefulness of the set of symbols W defined by

$$\begin{aligned}W(\mathtt{i}) = \big\{\mathtt{j} \in I^* :\ &\mathtt{j}' \in Z\big(\operatorname{diam}(X_{\mathtt{i}'})\big) \text{ and} \\ &\operatorname{dist}(X_{\mathtt{i}'}, X_{\mathtt{j}'}) \le 3 \operatorname{diam}(X_{\mathtt{i}'}),\ \text{where} \\ &\mathtt{i}' = \sigma^{|\mathtt{i}\wedge\mathtt{j}|}(\mathtt{i}) \text{ and } \mathtt{j}' = \sigma^{|\mathtt{i}\wedge\mathtt{j}|}(\mathtt{j})\big\}\end{aligned} \tag{4.4}$$

as $\mathtt{i} \in I^*$. See also [10, §2] and [27, §3]. Notice that $\mathtt{i} \in W(\mathtt{i})$. The constant 3 in (4.4) is somewhat arbitrary. The reader can easily see that any constant strictly larger than 2 would suffice. Let us next prove two technical lemmas.

Lemma 4.5. *Given CMC, the set $W(\mathtt{i})$ is incomparable for every $\mathtt{i} \in I^*$. Furthermore, if $\mathtt{j} \in W(\mathtt{i})$ then*

$$D^{-3} \min_{i \in I} \operatorname{diam}(X_i) \operatorname{diam}(X_{\mathtt{i}}) \le \operatorname{diam}(X_{\mathtt{j}}) \le D^2 \operatorname{diam}(X_{\mathtt{i}}).$$

Proof. Fix $\mathtt{i} \in I^*$. Observe that if $\mathtt{i} \ne \mathtt{j} \in W(\mathtt{i})$ then clearly $\mathtt{i} \perp \mathtt{j}$. Take $\mathtt{j}, \mathtt{h} \in W(\mathtt{i})$. If now $|\mathtt{j} \wedge \mathtt{i}| < |\mathtt{h} \wedge \mathtt{i}|$, it must be $\mathtt{j} \perp \mathtt{h}$ since otherwise $\mathtt{j} = \mathtt{i} \wedge \mathtt{j}$, which contradicts with the first observation. If $|\mathtt{j} \wedge \mathtt{i}| = |\mathtt{h} \wedge \mathtt{i}| =: k$ then $\sigma^k(\mathtt{j}), \sigma^k(\mathtt{h}) \in Z\big(\operatorname{diam}(X_{\sigma^k(\mathtt{i})})\big)$ and hence $\mathtt{j} \perp \mathtt{h}$.

To prove the second claim, fix $\mathtt{i} \in I^*$, take $\mathtt{j} \in W(\mathtt{i})$, and denote $\mathtt{i}' = \sigma^{|\mathtt{i} \wedge \mathtt{j}|}(\mathtt{i})$ and $\mathtt{j}' = \sigma^{|\mathtt{i} \wedge \mathtt{j}|}(\mathtt{j})$. Since $\mathtt{j}' \in Z\big(\operatorname{diam}(X_{\mathtt{i}'})\big)$, we have, using (M2),

$$\operatorname{diam}(X_{\mathtt{i}'}) \ge \operatorname{diam}(X_{\mathtt{j}'}) \ge D^{-1} \min_{i \in I} \operatorname{diam}(X_i) \operatorname{diam}(X_{\mathtt{i}'}).$$

Therefore, according to (M2),

$$\begin{aligned}
\operatorname{diam}(X_{\mathtt{j}}) &\ge D^{-1} \operatorname{diam}(X_{\mathtt{i} \wedge \mathtt{j}}) \operatorname{diam}(X_{\mathtt{j}'}) \\
&\ge D^{-2} \min_{i \in I} \operatorname{diam}(X_i) \operatorname{diam}(X_{\mathtt{i} \wedge \mathtt{j}}) \operatorname{diam}(X_{\mathtt{i}'}) \\
&\ge D^{-3} \min_{i \in I} \operatorname{diam}(X_i) \operatorname{diam}(X_{\mathtt{i}})
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{diam}(X_{\mathtt{j}}) &\le D \operatorname{diam}(X_{\mathtt{i} \wedge \mathtt{j}}) \operatorname{diam}(X_{\mathtt{j}'}) \\
&\le D \operatorname{diam}(X_{\mathtt{i} \wedge \mathtt{j}}) \operatorname{diam}(X_{\mathtt{i}'}) \le D^2 \operatorname{diam}(X_{\mathtt{i}}).
\end{aligned}$$

The proof is finished. □

Lemma 4.6. *If a congruent CMC satisfies the finite clustering property then*

$$\sup_{\mathtt{i} \in I^*} W(\mathtt{i}) < \infty.$$

Proof. Suppose $\varphi_{\mathtt{i}}$, $\mathtt{i} \in I^*$, are the mappings of Lemma 4.1, $P(t) = 0$, and $m = \mu \circ \pi^{-1}$ is the measure of Proposition 3.2. According to Corollary 3.10 and Theorems 3.5 and 3.7, there exists a constant $K \ge 1$ such that for every $x \in E$ and $r > 0$

$$m\big(B(x,r)\big) \le K r^t.$$

Fix $\mathtt{i} \in I^*$, take $\mathtt{j} \in W(\mathtt{i})$, and denote $\mathtt{i}' = \sigma^{|\mathtt{i} \wedge \mathtt{j}|}(\mathtt{i})$ and $\mathtt{j}' = \sigma^{|\mathtt{i} \wedge \mathtt{j}|}(\mathtt{j})$. Since $\mathtt{j} \in W(\mathtt{i})$ and $\mathtt{j}' \in Z\big(\operatorname{diam}(X_{\mathtt{i}'})\big)$, we have $\operatorname{dist}(X_{\mathtt{i}'}, X_{\mathtt{j}'}) \le \operatorname{diam}(X_{\mathtt{i}'})$ and

$$\begin{aligned}
\operatorname{dist}(X_{\mathtt{i}}, X_{\mathtt{j}}) &\le C \operatorname{diam}(X_{\mathtt{i} \wedge \mathtt{j}}) \operatorname{dist}(X_{\mathtt{i}'}, X_{\mathtt{j}'}) \\
&\le 3C \operatorname{diam}(X_{\mathtt{i} \wedge \mathtt{j}}) \operatorname{diam}(X_{\mathtt{i}'}) \le 3CD \operatorname{diam}(X_{\mathtt{i}}).
\end{aligned}$$

Using Lemma 4.5, we obtain

$$\begin{aligned}
X_{\mathtt{j}} &\subset B\big(x, \operatorname{diam}(X_{\mathtt{i}}) + 3CD \operatorname{diam}(X_{\mathtt{i}}) + \operatorname{diam}(X_{\mathtt{j}})\big) \\
&\subset B\big(x, (1 + 3CD + D^2) \operatorname{diam}(X_{\mathtt{i}})\big)
\end{aligned}$$

for a point $x \in \pi([\mathtt{i}])$ whenever $\mathtt{j} \in W(\mathtt{i})$. Hence

$$m\Big(\bigcup_{\mathtt{j}\in W(\mathtt{i})} X_{\mathtt{j}}\Big) \le m\big(B\big(x,(1+3CD+D^2)\operatorname{diam}(X_{\mathtt{i}})\big)\big)$$
$$\le K(1+3CD+D^2)^t \operatorname{diam}(X_{\mathtt{i}})^t.$$

Since, on the other hand, we have a constant $c \ge 1$ such that

$$m\Big(\bigcup_{\mathtt{j}\in W(\mathtt{i})} X_{\mathtt{j}}\Big) \ge m\Big(\bigcup_{\mathtt{j}\in W(\mathtt{i})} \varphi_{\mathtt{j}}(E)\Big) = \sum_{\mathtt{j}\in W(\mathtt{i})} m\big(\varphi_{\mathtt{j}}(E)\big)$$
$$\ge \sum_{\mathtt{j}\in W(\mathtt{i})} \mu([\mathtt{j}]) \ge c^{-1} \sum_{\mathtt{j}\in W(\mathtt{i})} \operatorname{diam}(X_{\mathtt{j}})^t$$
$$\ge \#W(\mathtt{i})c^{-1}D^{-3t} \min_{i\in I} \operatorname{diam}(X_i)^t \operatorname{diam}(X_{\mathtt{i}})^t$$

using Lemmas 4.2 and 4.5, we conclude

$$\#W(\mathtt{i}) \le \frac{cKD^{3t}(1+3CD+D^2)^t}{\min_{i\in I}\operatorname{diam}(X_i)^t}$$

whenever $\mathtt{i} \in I^*$. □

The following theorem generalizes a crucial point of [28, Theorem 2.1] into the setting of CMC's. See also [10, Theorem 3.3] and [27, §3].

Theorem 4.7. *If a congruent CMC satisfies the finite clustering property then there are a constant $\delta > 0$ and a symbol $\mathtt{h} \in I^*$ such that*

$$\operatorname{dist}(X_{\mathtt{ih}}, X_{\mathtt{jh}}) > \delta\big(\operatorname{diam}(X_{\mathtt{i}}) + \operatorname{diam}(X_{\mathtt{j}})\big)$$

whenever $\mathtt{i}\perp\mathtt{j}$.

Proof. Using Lemma 4.6, we choose $\mathtt{h} \in I^*$ such that $\#W(\mathtt{h}) = \sup_{\mathtt{i}\in I^*} \#W(\mathtt{i})$. Therefore clearly

$$\#\{\mathtt{ij} : \mathtt{j} \in W(\mathtt{h})\} = \#W(\mathtt{h}) \ge \#W(\mathtt{ih})$$

for every $\mathtt{i} \in I^*$. Since it follows immediately from the definition (4.4) that

$$\{\mathtt{ij} : \mathtt{j} \in W(\mathtt{h})\} \subset W(\mathtt{ih}),$$

we infer

$$W(\mathtt{ih}) = \{\mathtt{ij} : \mathtt{j} \in W(\mathtt{h})\} \tag{4.5}$$

whenever $\mathtt{i} \in I^*$.

Take next $\mathtt{i}, \mathtt{j} \in I^*$ such that $\mathtt{i}\perp\mathtt{j}$ and denote $\mathtt{i}' = \sigma^{|\mathtt{i}\wedge\mathtt{j}|}(\mathtt{i})$ and $\mathtt{j}' = \sigma^{|\mathtt{i}\wedge\mathtt{j}|}(\mathtt{j})$. Let $y_{\mathtt{j}'} = \pi(\mathtt{k}) \in X_{\mathtt{j}'\mathtt{h}}$, where $\mathtt{k} \in [\mathtt{j}'\mathtt{h}]$, and choose $k \in \mathbb{N}$ such that $\mathtt{k}|_k \in Z\big(\operatorname{diam}(X_{\mathtt{i}'\mathtt{h}})\big)$. Since $\mathtt{k}|_1 = \mathtt{j}'|_1 \ne \mathtt{i}'|_1$, we have, using (4.5),

$$\mathtt{k}|_k \notin W(\mathtt{i}'\mathtt{h}).$$

Hence the definition (4.4) yields $\operatorname{dist}(X_{\mathtt{k}|_k}, X_{\mathtt{i}'\mathtt{h}}) > 3\operatorname{diam}(X_{\mathtt{i}'\mathtt{h}})$. Since $y_{\mathtt{j}'} \in X_{\mathtt{k}|_k}$, we also have $\operatorname{dist}(y_{\mathtt{j}'}, X_{\mathtt{i}'\mathtt{h}}) > 3\operatorname{diam}(X_{\mathtt{i}'\mathtt{h}})$. Similarly, changing the roles of $\mathtt{i}$ and $\mathtt{j}$ above, we find $y_{\mathtt{i}'} \in X_{\mathtt{i}'\mathtt{h}}$ such that $\operatorname{dist}(y_{\mathtt{i}'}, X_{\mathtt{j}'\mathtt{h}}) > 3\operatorname{diam}(X_{\mathtt{j}'\mathtt{h}})$. This implies that

$$\begin{aligned}|y_{\mathtt{i}'} - y_{\mathtt{j}'}| &\geq 3\max\{\operatorname{diam}(X_{\mathtt{i}'\mathtt{h}}), \operatorname{diam}(X_{\mathtt{j}'\mathtt{h}})\} \\ &\geq \tfrac{3}{2}\big(\operatorname{diam}(X_{\mathtt{i}'\mathtt{h}}) + \operatorname{diam}(X_{\mathtt{j}'\mathtt{h}})\big).\end{aligned}$$

Since, on the other hand,

$$|y_{\mathtt{i}'} - y_{\mathtt{j}'}| \leq \operatorname{diam}(X_{\mathtt{i}'\mathtt{h}}) + \operatorname{dist}(X_{\mathtt{i}'\mathtt{h}}, X_{\mathtt{j}'\mathtt{h}}) + \operatorname{diam}(X_{\mathtt{j}'\mathtt{h}}),$$

we infer

$$\operatorname{dist}(X_{\mathtt{i}'\mathtt{h}}, X_{\mathtt{j}'\mathtt{h}}) \geq \tfrac{1}{2}\big(\operatorname{diam}(X_{\mathtt{i}'\mathtt{h}}) + \operatorname{diam}(X_{\mathtt{j}'\mathtt{h}})\big).$$

Thus, using (4.1) and (M2),

$$\begin{aligned}\operatorname{dist}\big(X_{\mathtt{ih}}, X_{\mathtt{jh}}\big) &\geq C^{-1}\operatorname{diam}(X_{\mathtt{i}\wedge\mathtt{j}})\operatorname{dist}\big(X_{\mathtt{i}'\mathtt{h}}, X_{\mathtt{j}'\mathtt{h}}\big) \\ &\geq (2C)^{-1}\operatorname{diam}(X_{\mathtt{i}\wedge\mathtt{j}})\big(\operatorname{diam}(X_{\mathtt{i}'\mathtt{h}}) + \operatorname{diam}(X_{\mathtt{j}'\mathtt{h}})\big) \\ &\geq (2CD)^{-1}\big(\operatorname{diam}(X_{\mathtt{ih}}) + \operatorname{diam}(X_{\mathtt{jh}})\big) \\ &\geq (2CD^2)^{-1}\operatorname{diam}(X_{\mathtt{h}})\big(\operatorname{diam}(X_{\mathtt{i}}) + \operatorname{diam}(X_{\mathtt{j}})\big)\end{aligned}$$

whenever $\mathtt{i}\perp\mathtt{j}$. Therefore, choosing $\delta = (3CD^2)^{-1}\operatorname{diam}(X_{\mathtt{h}})$, we have finished the proof. $\square$

As a corollary, we notice that for a congruent Moran construction, we may choose the balls in the ball condition to be centered at E and placed in such manner that also larger collections (than required in the definition) of them are disjoint.

Corollary 4.8. *If a congruent CMC satisfies the ball condition then there are a constant $\delta > 0$ and a point $x \in E$ such that*

$$B\big(\varphi_{\mathtt{i}}(x), \delta\operatorname{diam}(X_{\mathtt{i}})\big) \cap B\big(\varphi_{\mathtt{j}}(x), \delta\operatorname{diam}(X_{\mathtt{j}})\big) = \emptyset$$

whenever $\mathtt{i}\perp\mathtt{j}$. Here $\varphi_{\mathtt{i}}$, $\mathtt{i} \in I^*$, are the mappings of Lemma 4.1.

Proof. Assuming that $\delta > 0$ and $\mathtt{h} \in I^*$ are as in Theorem 4.7, the claim follows immediately from Theorems 3.5 and 4.7 by choosing $x \in \pi([\mathtt{h}])$. $\square$

Compare the following improvement of Lemma 4.2 to [24, Theorem 3.3] and [18, Theorem 1.6].

Proposition 4.9. *If a congruent CMC satisfies the ball condition then*

$$\dim_{\mathrm{H}}\big(\varphi_{\mathtt{i}}(E) \cap \varphi_{\mathtt{j}}(E)\big) < \dim_{\mathrm{H}}(E)$$

whenever $\mathtt{i}\perp\mathtt{j}$. Here $\varphi_{\mathtt{i}}$, $\mathtt{i} \in I^*$, are the mappings of Lemma 4.1.

Proof. Let $\delta > 0$ and $\mathtt{h} \in I^*$ be as in Theorem 4.7 and define

$$A = \bigcup_{\mathtt{k} \in I^*} \varphi_{\mathtt{k}}\big(\pi([\mathtt{h}])\big).$$

According to Theorem 4.7, we have $\varphi_{\mathtt{i}}\big(\pi([\mathtt{h}])\big) \cap \varphi_{\mathtt{j}}\big(\pi([\mathtt{h}])\big) = \emptyset$ whenever $\mathtt{i} \perp \mathtt{j}$, and hence also

$$\varphi_{\mathtt{i}}(A) \cap \varphi_{\mathtt{j}}(A) = \emptyset$$

as $\mathtt{i} \perp \mathtt{j}$. Thus we get

$$\begin{aligned} \varphi_{\mathtt{i}}(E) \cap \varphi_{\mathtt{j}}(E) &= \big(\varphi_{\mathtt{i}}(E \setminus A) \cap \varphi_{\mathtt{j}}(A)\big) \cup \big(\varphi_{\mathtt{i}}(E) \cap \varphi_{\mathtt{j}}(E \setminus A)\big) \\ &\subset \varphi_{\mathtt{i}}(E \setminus A) \cup \varphi_{\mathtt{j}}(E \setminus A) \end{aligned}$$

whenever $\mathtt{i} \perp \mathtt{j}$ from which the Lipschitz continuity implies

$$\begin{aligned} \dim_{\mathrm{H}}\big(\varphi_{\mathtt{i}}(E) \cap \varphi_{\mathtt{j}}(E)\big) &\le \dim_{\mathrm{H}}\big(\varphi_{\mathtt{i}}(E \setminus A) \cup \varphi_{\mathtt{j}}(E \setminus A)\big) \\ &\le \dim_{\mathrm{H}}(E \setminus A). \end{aligned}$$

Obviously, $\{X_{\mathtt{i}} : \mathtt{i} \in (I^{|\mathtt{h}|})^*\}$ is a CMC having E as a limit set, whereas $E \setminus A$ is contained in the limit set F of the subconstruction $\{X_{\mathtt{i}} : \mathtt{i} \in (I^{|\mathtt{h}|} \setminus \{\mathtt{h}\})^*\}$. Since it is evident that both of these CMC's satisfy the uniform finite clustering property, Lemma 2.5 and Theorem 3.7 imply that $\dim_{\mathrm{H}}(F) < \dim_{\mathrm{H}}(E)$. Consequently, $\dim_{\mathrm{H}}(E \setminus A) < \dim_{\mathrm{H}}(E)$ and the proof is finished. □

We shall finish this chapter with the following observation.

Proposition 4.10. *Suppose a collection of compact sets with positive diameter* $\{X_{\mathtt{i}} \subset \mathbb{R}^d : \mathtt{i} \in I^*\}$ *satisfies the following four conditions:*

(C1) $X_{\mathtt{i}i} \subset X_{\mathtt{i}}$ *as* $\mathtt{i} \in I^*$ *and* $i \in I$,
(C2) *there exist* $\mathtt{i}, \mathtt{j} \in I^*$ *such that* $X_{\mathtt{i}} \cap X_{\mathtt{j}} = \emptyset$,
(C3) $\lim_{n\to\infty} \operatorname{diam}(X_{\mathtt{i}|_n}) = 0$ *for every* $\mathtt{i} \in I^\infty$,
(C4) *there exists a constant* $C \ge 1$ *such that*

$$C^{-1} \operatorname{diam}(X_{\mathtt{h}}) \operatorname{dist}(X_{\mathtt{i}}, X_{\mathtt{j}}) \le \operatorname{dist}(X_{\mathtt{hi}}, X_{\mathtt{hj}}) \le C \operatorname{diam}(X_{\mathtt{h}}) \operatorname{dist}(X_{\mathtt{i}}, X_{\mathtt{j}})$$

whenever $\mathtt{h}, \mathtt{i}, \mathtt{j} \in I^*$.

Then the collection is a congruent CMC.

Proof. It suffices to prove (M2) and (M3). To show (M2), observe first that the assumptions (C1) and (C3) guarantee the existence of the limit set E and the claim in Lemma 4.1 follows from the assumptions (C1), (C3), and (C4). Let $\varphi_{\mathtt{i}}$, $\mathtt{i} \in I^*$, be the mappings of Lemma 4.1. Then

$$\operatorname{diam}\big(\varphi_{\mathtt{i}}(E)\big) \ge |\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \ge C^{-1} \operatorname{diam}(X_{\mathtt{i}})|x - y|$$

whenever $x, y \in E$. This implies, using (C2), that

$$\operatorname{diam}(X_{\mathtt{i}}) \le C \operatorname{diam}(E)^{-1} \operatorname{diam}\big(\varphi_{\mathtt{i}}(E)\big) \tag{4.6}$$

for every $\mathtt{i} \in I^*$. Since

$$\begin{aligned}\operatorname{diam}\bigl(\varphi_{\mathtt{i}\mathtt{j}}(E)\bigr) &= \sup_{x,y\in E}\bigl|\varphi_{\mathtt{i}}\bigl(\varphi_{\mathtt{j}}(x)\bigr) - \varphi_{\mathtt{i}}\bigl(\varphi_{\mathtt{j}}(y)\bigr)\bigr| \\ &\le C^2 \operatorname{diam}(X_{\mathtt{i}}) \operatorname{diam}(X_{\mathtt{j}}) \sup_{x,y\in E}|x-y|\end{aligned}$$

whenever $\mathtt{i}, \mathtt{j} \in I^*$, we get, by (4.6), that

$$\begin{aligned}\operatorname{diam}(X_{\mathtt{i}\mathtt{j}}) &\le C \operatorname{diam}(E)^{-1} \operatorname{diam}\bigl(\varphi_{\mathtt{i}\mathtt{j}}(E)\bigr) \\ &\le C^3 \operatorname{diam}(X_{\mathtt{i}}) \operatorname{diam}(X_{\mathtt{j}})\end{aligned}$$

whenever $\mathtt{i}, \mathtt{j} \in I^*$. On the other hand,

$$\operatorname{diam}\bigl(\varphi_{\mathtt{i}}(E)\bigr) = \sup_{x,y\in E}|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le C \operatorname{diam}(X_{\mathtt{i}}) \sup_{x,y\in E}|x-y|$$

implies, using (C2), that

$$\operatorname{diam}(X_{\mathtt{i}}) \ge C^{-1} \operatorname{diam}(E)^{-1} \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(E)\bigr) \tag{4.7}$$

for every $\mathtt{i} \in I^*$. Since

$$\begin{aligned}\operatorname{diam}\bigl(\varphi_{\mathtt{i}\mathtt{j}}(E)\bigr) &\ge \bigl|\varphi_{\mathtt{i}}\bigl(\varphi_{\mathtt{j}}(x)\bigr) - \varphi_{\mathtt{i}}\bigl(\varphi_{\mathtt{j}}(y)\bigr)\bigr| \\ &\ge C^{-2} \operatorname{diam}(X_{\mathtt{i}}) \operatorname{diam}(X_{\mathtt{j}})|x-y|\end{aligned}$$

whenever $x, y \in E$ and $\mathtt{i}, \mathtt{j} \in I^*$, we get, by (4.7), that

$$\begin{aligned}\operatorname{diam}(X_{\mathtt{i}\mathtt{j}}) &\ge C^{-1} \operatorname{diam}(E)^{-1} \operatorname{diam}\bigl(\varphi_{\mathtt{i}\mathtt{j}}(E)\bigr) \\ &\ge C^{-3} \operatorname{diam}(X_{\mathtt{i}}) \operatorname{diam}(X_{\mathtt{j}})\end{aligned}$$

whenever $\mathtt{i}, \mathtt{j} \in I^*$.

Let us then show (M3). Denote $M_n = \max_{\mathtt{i}\in I^n} \operatorname{diam}(X_{\mathtt{i}})$ as $n \in \mathbb{N}$ and choose $\mathtt{i}_1, \mathtt{i}_2, \ldots \in I^\infty$ such that

$$M_n = \operatorname{diam}(X_{\mathtt{i}_n|_n})$$

for every $n \in \mathbb{N}$. By the compactness of I^∞, the sequence $\{\mathtt{i}_n\}_{n\in\mathbb{N}}$ has a converging subsequence. Let $\mathtt{i} \in I^\infty$ be the limit point of such a subsequence. Now for each $j \in \mathbb{N}$ there is $n(j) \in \mathbb{N}$ such that $n(j) \ge j$ and $\mathtt{i}_{n(j)} \in [\mathtt{i}|_j]$. Since $\mathtt{i}_{n(j)}|_j = \mathtt{i}|_j$ for all $j \in \mathbb{N}$, we have, using (C1) and (C3),

$$\begin{aligned}M_{n(j)} &= \operatorname{diam}(X_{\mathtt{i}_{n(j)}|_{n(j)}}) \\ &\le \operatorname{diam}(X_{\mathtt{i}_{n(j)}|_j}) = \operatorname{diam}(X_{\mathtt{i}|_j}) \to 0\end{aligned}$$

as $j \to \infty$. The proof is finished by choosing $j \in \mathbb{N}$ such that $M_{n(j)} < C^{-3}$. $\square$

5. Congruent iterated function system

We assume that for each $i \in I$ there is a contractive injection $\varphi_i \colon \Omega \to \Omega$ defined on an open subset Ω of $\mathbb{R}^d$ and that there also exists a closed and nonempty $X \subset \Omega$ satisfying

$$\bigcup_{i \in I} \varphi_i(X) \subset X. \tag{5.1}$$

Here the *contractivity* of φ_i means that there is a constant $0 < s_i < 1$ such that

$$|\varphi_i(x) - \varphi_i(y)| \le s_i |x - y| \tag{5.2}$$

whenever $x, y \in \Omega$. The collection $\{\varphi_i : i \in I\}$ is then called an *iterated function system (IFS)*. As shown in [12, §3], an elegant application of the Banach fixed point theorem implies the existence of a unique compact and nonempty set $E \subset X$ for which

$$E = \bigcup_{i \in I} \varphi_i(E).$$

Such a set E is called an *invariant set* (for the corresponding IFS). As a side note, it is not necessary to require the mappings φ_i to be injective in order to ensure the existence of the invariant set. However, under this additional assumption, it follows from Brouwer's domain invariance theorem [5, Theorem IV.7.4] that $\varphi_i(U)$ is open whenever U is.

Observe that we may replace the set X by the closed neighborhood of the invariant set E. Indeed, we fix $0 < \varepsilon < \operatorname{dist}(E, \mathbb{R}^d \setminus \Omega)$ (if $\Omega = \mathbb{R}^d$, any positive ε will do) and take

$$X = \{x \in \Omega : |x - a| \le \varepsilon \text{ for some } a \in E\}.$$

The validity of (5.1) is then a consequence of the easily proven fact that

$$\operatorname{dist}\bigl(\varphi_i(A), E\bigr) \le s_i \operatorname{dist}(A, E) \tag{5.3}$$

whenever $A \subset \Omega$ and $i \in I$.

We say that an IFS is *controlled* if it defines a CMC, that is, there exists a compact set $A \subset \Omega$ such that the collection $\{\varphi_{\mathtt{i}}(A) : \mathtt{i} \in I^*\}$ is a CMC. The limit set of such a CMC is clearly E. Here $\varphi_{\mathtt{i}} = \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}$ as $\mathtt{i} = (i_1, \ldots, i_n) \in I^n$ and $n \in \mathbb{N}$.

Lemma 5.1. *A controlled IFS defines a tractable CMC.*

Proof. Choose a compact set $A \subset X$ such that the collection $\{\varphi_{\mathtt{i}}(A) : \mathtt{i} \in I^*\}$ is a CMC. Define for $\mathtt{i} \in I^*$

$$s'_{\mathtt{i}} = \inf\{s > 0 : |\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le s|x - y| \text{ for all } x, y \in X\}$$

and using the compactness, take $x, y \in A$ such that $|x - y| = \operatorname{diam}(A)$. Since now

$$|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(A)\bigr) = \operatorname{diam}(A)^{-1} \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(A)\bigr)|x - y|,$$

we have $s'_{\mathtt{i}} \le \operatorname{diam}(A)^{-1} \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(A)\bigr)$. Hence it follows that

$$\begin{aligned}\operatorname{dist}\bigl(\varphi_{\mathtt{ih}}(A), \varphi_{\mathtt{ik}}(A)\bigr) &\le s'_{\mathtt{i}} \operatorname{dist}\bigl(\varphi_{\mathtt{h}}(A), \varphi_{\mathtt{k}}(A)\bigr) \\ &\le \operatorname{diam}(A)^{-1} \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(A)\bigr) \operatorname{dist}\bigl(\varphi_{\mathtt{h}}(A), \varphi_{\mathtt{k}}(A)\bigr).\end{aligned}$$

This implies (3.2) and finishes the proof. □

Finally, we say that an IFS is *congruent* if the invariant set E has positive diameter and there are constants $0 < \underline{s}_{\mathtt{i}} \le \overline{s}_{\mathtt{i}} < 1$, $\mathtt{i} \in I^*$, and $D \ge 1$ for which $\overline{s}_{\mathtt{i}} \le D\underline{s}_{\mathtt{i}}$ as $\mathtt{i} \in I^*$ and

$$\underline{s}_{\mathtt{i}}|x-y| \le |\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le \overline{s}_{\mathtt{i}}|x-y| \tag{5.4}$$

whenever $x, y \in \Omega$. The following lemma shows that a congruent IFS defines a congruent CMC. The natural question whether the converse holds raises from Lemma 4.1. Sufficient geometric conditions on the limit set for the positive answer are provided in [1]. See also [29, Example 6.2].

Lemma 5.2. *If $\{\varphi_i : i \in I\}$ is a congruent IFS and a compact set A with positive diameter satisfies $\varphi_i(A) \subset A$ for every $i \in I$ then $\{\varphi_{\mathtt{i}}(A) : \mathtt{i} \in I^*\}$ is a congruent CMC. Furthermore, the mappings $\varphi_{\mathtt{i}}|_E$, $\mathtt{i} \in I^*$, are the mappings of Lemma 4.1.*

Proof. To be able to use Proposition 4.10, we have to verify the required assumptions (C1)–(C4). Observe first that (C1) is clearly satisfied and the positivity of $\operatorname{diam}(E)$ implies (C2). Notice also that the sets $\varphi_{\mathtt{i}}(A)$, $\mathtt{i} \in I^*$, are compact with positive diameter. Since for fixed $\mathtt{i} \in I^*$, we have $\underline{s}_{\mathtt{i}} \operatorname{diam}(A) \le \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(A)\bigr) \le \overline{s}_{\mathtt{i}} \operatorname{diam}(A)$ by (5.4), it follows that

$$C^{-1} \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(A)\bigr)|x-y| \le |\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le C \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(A)\bigr)|x-y|, \tag{5.5}$$

where $C = D\max\{\operatorname{diam}(A), \operatorname{diam}(A)^{-1}\}$ and $x, y \in A$. Hence, applying (5.2) to (5.5) several times, we get $\operatorname{diam}\bigl(\varphi_{\mathtt{i}}(A)\bigr) \le Cs_{i_1}\cdots s_{i_{|\mathtt{i}|}}$, which implies the assumption (C3). Since (5.5) gives also the assumption (C4), that is

$$\begin{aligned}C^{-1} \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(A)\bigr) \operatorname{dist}\bigl(\varphi_{\mathtt{h}}(A), \varphi_{\mathtt{k}}(A)\bigr) &\le \operatorname{dist}\bigl(\varphi_{\mathtt{ih}}(A), \varphi_{\mathtt{ik}}(A)\bigr) \\ &\le C \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(A)\bigr) \operatorname{dist}\bigl(\varphi_{\mathtt{h}}(A), \varphi_{\mathtt{k}}(A)\bigr)\end{aligned}$$

as $\mathtt{h}, \mathtt{k} \in I^*$, we have finished the proof of the first claim.

The second claim follows immediately since the collection $\{\varphi_{\mathtt{i}}(E) : \mathtt{i} \in I^*\}$ is a congruent CMC. □

We say that an IFS satisfies an *open set condition (OSC)*, if there exists a nonempty open set $U \subset \Omega$ such that

$$\varphi_{\mathtt{i}}(U) \cap \varphi_{\mathtt{j}}(U) = \emptyset$$

whenever $\mathtt{i} \perp \mathtt{j}$. See [25, Theorem III] and [12, §5.2] for the motivation of the definition. Adapting terminology from [3], we call any such nonempty open set U a *feasible set* for the OSC. As an immediate consequence of the definition, we notice that each nonempty open subset and each image $\varphi_{\mathtt{i}}(U)$ of a feasible set U

is feasible as well. Thus, using the observation (5.3) repeatedly, we see that the OSC is equivalent to the existence of a feasible set $U \subset X$. Recall that X is the fixed compact ε-neighborhood of the invariant set.

Lemma 5.3. *An IFS satisfies the OSC exactly when there exists a nonempty open set $V \subset X$ such that*

$$\varphi_i(V) \subset V$$

as $i \in I$ and

$$\varphi_i(V) \cap \varphi_j(V) = \emptyset$$

as $i \neq j$. Furthermore, there exists a feasible set intersecting E if and only if there exists a set V as above such that $V \cap E \neq \emptyset$.

Proof. Defining $V = \bigcup_{\mathtt{h} \in I^*} \varphi_{\mathtt{h}}(U)$, where $U \subset X$ is a feasible set for the OSC, we clearly have $\varphi_i(V) \subset V \subset X$ as $i \in I$. If $i \neq j$, it holds that

$$\varphi_{i\mathtt{h}}(U) \cap \varphi_{j\mathtt{h}}(U) = \emptyset$$

for every $\mathtt{h} \in I^*$ and hence

$$\Big(\bigcup_{\mathtt{h} \in I^*} \varphi_{i\mathtt{h}}(U) \Big) \cap \Big(\bigcup_{\mathtt{h} \in I^*} \varphi_{j\mathtt{h}}(U) \Big) = \emptyset.$$

Noting that the other direction is trivial we have finished the proof. □

Given IFS, we say that $A \subset \Omega$ is *forwards invariant* if $\varphi_i(A) \subset A$ as $i \in I$ and *backwards invariant* if $\varphi_i^{-1}(A) \subset A$ as $i \in I$. For $A \subset \Omega$ we define

$$F_A = \bigcup_{\mathtt{i} \perp \mathtt{j}} \varphi_{\mathtt{i}}^{-1}\big(\varphi_{\mathtt{j}}(A)\big)$$

and for a congruent IFS we set

$$O_A = \big\{ x \in \Omega : D \operatorname{dist}(x, A) < \operatorname{dist}\big(x, F_A \cup (\mathbb{R}^d \setminus \Omega)\big) \big\}.$$

Here the constant $D \geq 1$ is the same as in the definition of the congruent IFS. Observe that $F_A \subset \Omega$ is backwards invariant.

Proposition 5.4. *Suppose a given IFS is congruent. If $U \subset \Omega$ is a feasible set for the OSC then $O_U \neq \emptyset$. Furthermore, if there exists a set $A \subset \Omega$ such that $O_A \neq \emptyset$ then O_A is feasible.*

Proof. Let $U \subset \Omega$ be a nonempty open set for which $\varphi_{\mathtt{i}}(U) \cap \varphi_{\mathtt{j}}(U) = \emptyset$ whenever $\mathtt{i} \perp \mathtt{j}$. It follows that $U \cap F_U = \emptyset$ and since U is open, we get $U \subset O_U$.

Conversely, it suffices to show that $\varphi_{\mathtt{i}}(O_A) \cap \varphi_{\mathtt{j}}(O_A) = \emptyset$ as $\mathtt{i} \perp \mathtt{j}$. Suppose contrarily that there are $\mathtt{i}, \mathtt{j} \in I^*$ and $x, y \in O_A$ such that $\mathtt{i} \perp \mathtt{j}$ and $\varphi_{\mathtt{i}}(x) = \varphi_{\mathtt{j}}(y) =: z$. Observe that the inverse mapping $\varphi_{\mathtt{i}}^{-1} \colon \varphi_{\mathtt{i}}(\Omega) \to \Omega$ has a Lipschitz constant $\underline{s}_{\mathtt{i}}^{-1}$ for each $\mathtt{i} \in I^*$. According to Kirszbraun's theorem [11, §2.10.43], there exists a Lipschitz extension $\overline{\varphi}_{\mathtt{i}} \colon \Omega \to \mathbb{R}^d$ for the mapping $\varphi_{\mathtt{i}}^{-1}$ having the same Lipschitz constant. Since $\overline{\varphi}_{\mathtt{i}}\bigl(\varphi_{\mathtt{j}}(A)\bigr) \subset F_A \cup (\mathbb{R}^d \setminus \Omega)$ and $x \in O_A$, we have

$$\begin{aligned}\operatorname{dist}\bigl(z, \varphi_{\mathtt{j}}(A)\bigr) &= \operatorname{dist}\bigl(\varphi_{\mathtt{i}}(x), \varphi_{\mathtt{j}}(A)\bigr) \ge \underline{s}_{\mathtt{i}} \operatorname{dist}\bigl(x, \overline{\varphi}_{\mathtt{i}}\bigl(\varphi_{\mathtt{j}}(A)\bigr)\bigr) \\ &\ge \underline{s}_{\mathtt{i}} \operatorname{dist}\bigl(x, F_A \cup (\mathbb{R}^d \setminus \Omega)\bigr) > \underline{s}_{\mathtt{i}} D \operatorname{dist}(x, A) \\ &\ge \underline{s}_{\mathtt{i}} D \overline{s}_{\mathtt{i}}^{-1} \operatorname{dist}\bigl(\varphi_{\mathtt{i}}(x), \varphi_{\mathtt{i}}(A)\bigr) \ge \operatorname{dist}\bigl(z, \varphi_{\mathtt{i}}(A)\bigr)\end{aligned}$$

using (5.4). Changing the roles of $\mathtt{i}$ and $\mathtt{j}$ above, we end up with a contradiction. The proof is finished. □

We say that a congruent IFS $\{\varphi_i : i \in I\}$ satisfies the (uniform) ball condition if the congruent CMC $\{\varphi_{\mathtt{i}}(E) : \mathtt{i} \in I^*\}$ satisfies the (uniform) ball condition. Lemma 5.2 guarantees that this is well defined. Recalling the proof of Lemma 4.2, it is clear that in this definition the set E can be replaced with any forwards invariant compact set A with positive diameter. Observe also that if a congruent IFS satisfies the OSC then it satisfies the uniform ball condition. See also [14, Proposition 3.6].

Theorem 5.5. *A congruent IFS satisfies the ball condition exactly when $O_E \cap E \neq \emptyset$.*

Proof. Let us first prove that the uniform ball condition implies $O_E \cap E \neq \emptyset$. Recall that X is the closed ε-neighborhood of E. We may further assume that

$$F := \bigcup_{\mathtt{i} \perp \mathtt{j}} \varphi_{\mathtt{i}}^{-1}\bigl(\varphi_{\mathtt{j}}(E)\bigr) \cap X \neq \emptyset$$

seeing that $F = \emptyset$ implies $\operatorname{dist}(E, F_E) \ge \varepsilon$, which gives $E \subset O_E$. It is now sufficient to find a point $x \in E$ with $\operatorname{dist}(x, F) > 0$.

According to Theorem 3.5 and Corollary 4.8, there exist a point $x \in E$ and a constant $\delta > 0$ such that

$$|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{jh}}(x)| > \delta \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(X)\bigr)$$

whenever $\mathtt{i} \perp \mathtt{j}$ and $\mathtt{h} \in I^*$. It is easy to see that the set $\{\varphi_{\mathtt{jh}}(x) : \mathtt{h} \in I^*\}$ is dense in $\varphi_{\mathtt{j}}(E)$. So, in fact, we have

$$\operatorname{dist}\Bigl(\varphi_{\mathtt{i}}(x), \bigcup_{\mathtt{i} \perp \mathtt{j}} \varphi_{\mathtt{j}}(E)\Bigr) \ge \delta \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(X)\bigr)$$

for each $\mathtt{i} \in I^*$, which in turn implies that

$$|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \ge \delta \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(X)\bigr)$$

for each $y \in \varphi_{\mathtt{i}}^{-1}\bigl(\varphi_{\mathtt{j}}(E)\bigr)$ when $\mathtt{i} \perp \mathtt{j}$. On the other hand, the proof of Lemma 5.1 shows that there is a constant $C > 0$ such that

$$|\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le C \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(X)\bigr)|x - y|$$

whenever $x, y \in X$ and $\mathtt{i} \in I^*$. Combining the inequalities above gives

$$|x-y| \geq C^{-1}\delta$$

for each $y \in F$ and consequently $\operatorname{dist}(x, F) > 0$ as desired.

Since the other direction follows immediately from Proposition 5.4, the proof is finished. □

The following proposition generalizes [28, Corollary 2.3] and [27, Corollary 1.2] into the setting of congruent IFS's. Although the argument used here is similar to the proof of [27, Corollary 1.2], we give the details for the convenience of the reader.

Proposition 5.6. *If a congruent IFS satisfies the OSC and* $\dim_{\mathrm{H}}(E) = d$ *then the invariant set* E *is the closure of its interior.*

Proof. As the OSC implies the uniform finite clustering property, we have $P(d) = 0$. Hence there exists a constant $c > 0$ such that

$$\sum_{\mathtt{i} \in I^n} \underline{s}_{\mathtt{i}}^d \geq D^{-d} \operatorname{diam}(X)^{-d} \sum_{\mathtt{i} \in I^n} \operatorname{diam}\bigl(\varphi_{\mathtt{i}}(X)\bigr)^d \geq c, \tag{5.6}$$

see the defining equation (3.1) and Lemma 2.1. Choose the forwards invariant feasible set $V \subset X$ as in Lemma 5.3 and consider the set

$$T = V \setminus \bigcup_{i \in I} \varphi_i(V).$$

The facts that $\varphi_{\mathtt{i}}(T) \subset \varphi_{\mathtt{i}}(V)$ and $\varphi_{\mathtt{i}}(T) \cap \varphi_{\mathtt{ij}}(V) = \emptyset$ for every $\mathtt{i} \in I^*$ and $\mathtt{j} \in I^*$ easily lead to the conclusion that $\varphi_{\mathtt{i}}(T) \cap \varphi_{\mathtt{j}}(T) = \emptyset$ whenever $\mathtt{i} \neq \mathtt{j}$. Furthermore, since $\varphi_{\mathtt{i}}(T) \subset X$ for each $\mathtt{i} \in I^*$, we have

$$\begin{aligned} \infty > \mathcal{H}^d(X) \geq \mathcal{H}^d\Bigl(\bigcup_{\mathtt{i} \in I^*} \varphi_{\mathtt{i}}(T)\Bigr) &= \sum_{n \in \mathbb{N}} \sum_{\mathtt{i} \in I^n} \mathcal{H}^d\bigl(\varphi_{\mathtt{i}}(T)\bigr) \\ &\geq \mathcal{H}^d(T) \sum_{n \in \mathbb{N}} \sum_{\mathtt{i} \in I^n} \underline{s}_{\mathtt{i}}^d. \end{aligned} \tag{5.7}$$

Now (5.6) and (5.7) together imply that $\mathcal{H}^d(T) = 0$. This in turn shows that the set

$$V \setminus \overline{\bigcup_{i \in I} \varphi_i(V)} = V \setminus \bigcup_{i \in I} \varphi_i(\overline{V})$$

is empty, being an open set with zero measure. Here with the notation $\overline{A}$, we mean the closure of a given set A. This means that $\overline{V} = \bigcup_{i \in I} \varphi_i(\overline{V})$, giving $E = \overline{V}$ by the uniqueness of the invariant set. The proof is complete. □

A similitude IFS, introduced in [12], is the most obvious example of a congruent IFS. Suppose that for each $i \in I$ there is a mapping $\varphi_i \colon \mathbb{R}^d \to \mathbb{R}^d$ and a constant $0 < s_i < 1$ such that

$$|\varphi_i(x) - \varphi_i(y)| = s_i |x-y|$$

whenever $x, y \in \mathbb{R}^d$. Now for a closed ball B centered at the origin, we have $\varphi_i(B) \subset B$ whenever $i \in I$ provided that the radius of B is chosen large enough. The collection $\{\varphi_i : i \in I\}$ is then an IFS and we call it a *similitude IFS*.

The following proposition is a slightly more general result than [3, Theorem 1].

Proposition 5.7. *Given similitude IFS, the set O_A is forwards invariant and feasible for the OSC provided that $O_A \neq \emptyset$ and $A \subset X$ is forwards invariant.*

Proof. According to Proposition 5.4, it suffices to show that $\varphi_i(O_A) \subset O_A$ as $i \in I$. Assume on the contrary that there exist $i \in I$ and $x \in O_A$ such that $\varphi_i(x) \notin O_A$, that is,

$$D \operatorname{dist}\bigl(\varphi_i(x), A\bigr) \geq \operatorname{dist}\bigl(\varphi_i(x), F_A\bigr).$$

Notice that here D can be chosen to be one. Therefore, since $A \subset \varphi_i^{-1}(A)$ and $\varphi_i^{-1}(F_A) \subset F_A$ for every $i \in I$, we obtain

$$\begin{aligned}\operatorname{dist}(x, F_A) &> D \operatorname{dist}(x, A) \geq D \operatorname{dist}\bigl(x, \varphi_i^{-1}(A)\bigr) \\ &= s_i^{-1} D \operatorname{dist}\bigl(\varphi_i(x), A\bigr) \geq s_i^{-1} \operatorname{dist}\bigl(\varphi_i(x), F_A\bigr) \\ &= \operatorname{dist}\bigl(x, \varphi_i^{-1}(F_A)\bigr) \geq \operatorname{dist}(x, F_A).\end{aligned}$$

This contradiction finishes the proof. □

6. Examples

In the last chapter, we illustrate the preceding theory by providing the reader with several examples. We begin by showing that the uniform finite clustering property does not imply the bounded overlapping property.

Example 6.1. The standard Cantor $\frac{1}{3}$-set E can be defined as the invariant set of the similitude IFS formed by the mappings

$$\begin{aligned}\varphi_0(x) &= \tfrac{1}{3}x, \\ \varphi_1(x) &= \tfrac{1}{3}x + \tfrac{2}{3}\end{aligned}$$

on $\mathbb{R}$. We have $P(t) = 0$ for $t = \log 2/\log 3$ and it is well known that $\mathcal{H}^t(E) = 1$, see [7, Theorem 1.14]. Consider now the CMC $\{\varphi_{\mathtt{i}}(X) : \mathtt{i} \in I^*\}$, where $X = [0, 3]$ and $I = \{0, 1\}$. The positivity of $\mathcal{H}^t(E)$ implies the uniform finite clustering property for this tractable CMC by Theorem 3.9. However, using the facts $1 \in \varphi_0(X)$ and $\varphi_1(1) = 1$, we infer by induction that $1 \in \varphi_{1^k 0}(X)$ for every $k \in \mathbb{N}$, where $1^k = (1, \ldots, 1) \in I^k$ for each k. Since the infinite set $\{1^k 0 : k \in \mathbb{N}\}$ is incomparable, we conclude that the bounded overlapping property is not satisfied.

Example 6.2. In this example, we give a CMC which shows that the assumption concerning the relative positions of the sets $X_{\mathtt{i}}$ in the last claim of Proposition

3.2 is indispensable. Besides this, it is also an example of a nontractable CMC. Using the mappings $\varphi_{\mathtt{i}}$, $\mathtt{i} \in I^*$, from the previous example, set

$$X_0 = [0,1] \times [0,1],$$
$$X_1 = [0,1] \times [-1,0]$$

and for $j \in I$ and $\mathtt{i} \in I^*$

$$X_{j\mathtt{i}} = \begin{cases} \varphi_{\mathtt{i}}([0,1]) \times [0, 3^{-|\mathtt{i}|}], & \text{if } j = 0 \\ \varphi_{\mathtt{i}}([0,1]) \times [-3^{-|\mathtt{i}|}, 0], & \text{if } j = 1. \end{cases}$$

The CMC determined by these squares obviously has the limit $E = E_x \times \{0\} \subset \mathbb{R}^2$, where $E_x \subset \mathbb{R}$ is the standard Cantor $\frac{1}{3}$-set. It is equally obvious that the uniform ball condition is satisfied, which, according to Theorems 3.7 and 3.5 and Remark 3.8, implies that the measure m of Proposition 3.2 is proportional to $\mathcal{H}^t|_E$, where $t = \log 2/\log 3$ as in the previous example. Consequently, $m(J) > 0$ whenever J is one of the line segments $\varphi_{\mathtt{i}}([0,1]) \times \{0\}$, $\mathtt{i} \in I^*$. Especially,

$$m(X_{\mathtt{i}} \cap X_{\mathtt{j}}) > 0$$

for incomparable symbols $\mathtt{i}$ and $\mathtt{j}$ satisfying $\mathtt{i}|_1 \neq \mathtt{j}|_1$ and $\sigma(\mathtt{i}) = \sigma(\mathtt{j})$. We have hereby shown that the measure m is not t-semiconformal. On the other hand, Lemma 3.6 implies that the bounded overlapping property is satisfied, noting that clearly $X_{\mathtt{i}} \cap E = \pi([\mathtt{i}])$ for each $\mathtt{i} \in I^*$. Therefore, an extra assumption in Proposition 3.2 is really needed.

Furthermore, this CMC is not tractable. This can be deduced from the fact that

$$\operatorname{dist}(X_{0\mathtt{i}}, X_{1\mathtt{i}}) = 0$$

but

$$\operatorname{dist}(X_{00\mathtt{i}}, X_{01\mathtt{i}}) \geq \operatorname{dist}(X_{00}, X_{01}) = \tfrac{1}{3}$$

for every $\mathtt{i} \in I^*$.

Example 6.3. Suppose I is a finite set and for each $i \in I$ there is a mapping $\varphi_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\varphi_i(x,y) = (a_i x + c_i, b_i y + d_i),$$

where $0 < b_i \leq a_i < 1$ and $c_i, d_i \geq 0$. Denoting $L = 1 + \max_{i\in I}\{c_i, d_i\}/(1 - \max_{i \in I} a_i)$ and $X = [0,L]^2$, we have $\varphi_i(X) \subset X$ for every $i \in I$. Since $a_{\mathtt{i}} L \leq \operatorname{diam}\big(\varphi_{\mathtt{i}}(X)\big) \leq \sqrt{2} a_{\mathtt{i}} L$, where $\mathtt{i} = (i_1, \ldots, i_n) \in I^n$, $a_{\mathtt{i}} = a_{i_1} \cdots a_{i_n}$, and $n \in \mathbb{N}$, the collection $\{\varphi_{\mathtt{i}}(X) : \mathtt{i} \in I^*\}$ is a CMC and hence tractable by Lemma 5.1.

According to Theorem 3.5, Remark 3.8, and 3.9, this CMC satisfies the uniform ball condition if and only if $0 < \mathcal{H}^t(E) < \infty$, where E is the limit set and $\sum_{i \in I} a_i^t = 1$.

Example 6.4. Suppose $I = \{0,1\}$ and there are mappings $\varphi_0, \varphi_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\varphi_0(x,y) = (ax, by + d),$$
$$\varphi_1(x,y) = (ax + c, by),$$

where $0 < b \le a \le \frac{1}{2}$, $c > 0$, and $d \ge 0$. Let $L = 1 + \max\{c,d\}/(1-a)$ and $X = [0,L]^2$. As in Example 6.3, we notice that the collection $\{\varphi_{\mathtt{i}}(X) : \mathtt{i} \in I^*\}$ is a tractable CMC.

Let us examine the distances between the points $\mathrm{proj}_x\bigl(\varphi_{\mathtt{i}}(0,0)\bigr)$ as $|\mathtt{i}| = k$. Here proj_x denotes the orthogonal projection onto the x-axis. If $k = 1$, there is just one distance, c. If $k = 2$ then there are six possible distances, but it suffices to notice that from the two first level sets it can be found the first level distance d multiplied by a and that $\mathrm{proj}_x\bigl(\varphi_{2,1}(0,0)\bigr) - \mathrm{proj}_x\bigl(\varphi_{1,2}(0,0)\bigr) = c - ca > 0$. Hence, the six possible distances are bounded below by $\lambda_2 = \min\{ca, c - ca\}$. Similarly, if $k = 3$, the possible distances are bounded below by $\lambda_3 = \min\{a\lambda_2, c - ca - ca^2\} > 0$ and if $k = 4$, they are bounded by $\lambda_4 = \min\{a\lambda_3, c - ca - ca^2 - ca^3\} > 0$. Continuing in this manner, we find that for $k \in \mathbb{N}$ the possible distances are bounded below by $\lambda_k = \min\{ca^k, ca^{k-1} - ca^k, \ldots, c - ca - \cdots - ca^{k-1} - ca^k\}$. Noting that $1 - a - a^2 - \cdots - a^k \ge a^k$ for every $k \in \mathbb{N}$ by $0 < a \le \frac{1}{2}$, we get $\lambda_k = ca^k$. Hence the collection of balls $\bigl\{B\bigl(\varphi_{\mathtt{i}}(0,0), ca^k/3\bigr) : \mathtt{i} \in I^k\bigr\}$ is disjoint for each $k \in \mathbb{N}$. This implies the uniform ball condition.

We conclude that $0 < \mathcal{H}^t(E) < \infty$, where $t = -\log 2/\log a$.

Example 6.5. Suppose $I = \{0,1\}$ and there are mappings $\varphi_0, \varphi_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\varphi_0(x,y) = (ax, by + d),$$
$$\varphi_1(x,y) = (ax, by),$$

where $0 < b < a < 1$ and $d \ge 0$. Let $L = 1 + d/(1-a)$ and $X = [0,L]^2$. As in Example 6.3, we notice that the collection $\{\varphi_{\mathtt{i}}(X) : \mathtt{i} \in I^*\}$ is a tractable CMC.

We show that the uniform ball condition does not hold in this setting. Then Corollary 3.10 guarantees that the ball condition does not hold either. Recalling Theorem 3.5, it suffices to state that for each $N \in \mathbb{N}$ there is $x \in E$ and $r > 0$ such that $\#Z(x,r) \ge N$. Let $N \in \mathbb{N}$, $x \in E$, and $\mathtt{j} \in I^\infty$ such that $\pi(\mathtt{j}) = x$. Take $n \ge (\log b/\log a - 1)^{-1} \log 2N/\log 2$, $n \log b/\log a - 1 \le m < n \log b/\log a$, and choose $a^m L \le r < a^{m-1}L$. Notice that if $\mathtt{i} \in Z(r)$, we have $a^{|\mathtt{i}|}L \le \mathrm{diam}\bigl(\varphi_{\mathtt{i}}(X)\bigr) < r < a^{m-1}L$, giving $|\mathtt{i}| \ge m$. Since $r \ge a^m L > b^n L$, it holds, recalling the choice of the mappings, that $B(x,r) \cap \varphi_{\mathtt{i}}(X) \ne \emptyset$ whenever $\mathtt{i} \in I^m \cap [\mathtt{j}|_n]$. Now

$$\#Z(x,r) \ge \#\{\mathtt{i} \in I^m : \mathtt{i} \in [\mathtt{j}|_n]\} = 2^{m-n} \ge N.$$

We conclude that in this case $P(t) = 0$ implies $\mathcal{H}^t(E) = 0$. It is also worthwhile to notice that the IFS $\{\varphi_i : i \in I\}$ satisfies the OSC provided that $0 < b \le \frac{1}{2}$ and

$d > 0$. To see this, recall the calculation in Example 6.4 and consult Theorem 5.5.

Example 6.6. In this example, we identify $\mathbb{R}^2$ and $\mathbb{C}$ for notational reasons and set $\eta = \frac{1}{2} + \frac{i}{2}$. Let $I = \{0, 1\}$ and φ_0, φ_1 be the similitudes given by the equations

$$\begin{aligned}\varphi_0(z) &= \eta z,\\ \varphi_1(z) &= \overline{\eta} z + \eta,\end{aligned}$$

where $z \in \mathbb{C}$ and $\overline{\eta} = \frac{1}{2} - \frac{i}{2}$ is the complex conjugate of η. Notice that the Lipschitz constant of both mappings is $\frac{1}{\sqrt{2}}$. Hence, choosing any compact set X with positive diameter satisfying $\varphi_i(X) \subset X$ for each $i \in I$, we have $P(2) = 0$ by Lemma 5.2, the defining equation (3.1), and noting that $\operatorname{diam}\big(\varphi_{\mathtt{i}}(X)\big)$ is proportional to $(\frac{1}{\sqrt{2}})^{|\mathtt{i}|}$ for every $\mathtt{i} \in I^*$. The invariant set E of the IFS $\{\varphi_0, \varphi_1\}$ is known as *Lévy's dragon*, see [19]. We shall show that this IFS satisfies the uniform ball condition and hence, Theorems 3.5, 3.7, 5.5 and Propositions 5.4, 5.6 lead to the conclusion that E is the closure of its interior. Observe that in this example, the feasible set is virtually impossible to find.

We begin by setting $H = \mathbb{Z} + i\mathbb{Z}$, $N = [0, 1]^2$, $\triangle = \operatorname{conv}\{0, 1, \eta\}$, and $\triangle' = \operatorname{conv}\{0, 1, \overline{\eta}\}$. The triangles $\triangle$ and $\triangle'$ have now Lebesgue measure $\frac{1}{4}$. A straightforward calculation shows that

$$\varphi_{\mathtt{i}}(H) = \{\eta^{|\mathtt{i}|} h : h \in H\}$$

for each $\mathtt{i} \in I^*$, implying that whenever $\mathtt{i} \in I^*$ and $\mathtt{j} \in I^*$ have the same length, we have either $\varphi_{\mathtt{i}}(N) = \varphi_{\mathtt{j}}(N)$ or $\operatorname{int}\big(\varphi_{\mathtt{i}}(N)\big) \cap \operatorname{int}\big(\varphi_{\mathtt{j}}(N)\big) = \emptyset$. Here with the notation $\operatorname{int}(A)$, we mean the interior of a given set A. Since $\varphi_{\mathtt{i}}(\triangle)$ has one side common with $\varphi_{\mathtt{i}}(N)$ while the other two sides lie on the diagonals of $\varphi_{\mathtt{i}}(N)$ intersecting at $\varphi_{\mathtt{i}}(\eta)$, we conclude that if $|\mathtt{i}| = |\mathtt{j}|$ and $\varphi_{\mathtt{i}}\big(\operatorname{int}(\triangle)\big) \cap \varphi_{\mathtt{j}}\big(\operatorname{int}(\triangle)\big) \neq \emptyset$ then $\varphi_{\mathtt{i}}(\triangle) = \varphi_{\mathtt{j}}(\triangle)$.

We shall now show that if $|\mathtt{i}| = |\mathtt{j}|$ and $\mathtt{i} \neq \mathtt{j}$ then

$$\varphi_{\mathtt{i}}\big(\operatorname{int}(\triangle)\big) \cap \varphi_{\mathtt{j}}\big(\operatorname{int}(\triangle)\big) = \emptyset, \tag{6.1}$$

which, in turn, implies the uniform ball condition. See [6, p. 222] for an illustration. Observe that for each $\mathtt{i} \in I^*$, $\varphi_{\mathtt{i}}(\triangle')$ is the only triangle such that $\varphi_{\mathtt{i}}(\triangle) \cup \varphi_{\mathtt{i}}(\triangle')$ is a square and hence, if $\varphi_{\mathtt{i}}(\triangle) = \varphi_{\mathtt{j}}(\triangle)$ then also $\varphi_{\mathtt{i}}(\triangle') = \varphi_{\mathtt{j}}(\triangle')$. On the other hand,

$$\varphi_{\mathtt{i}}\big(\operatorname{int}(\triangle')\big) \subset \varphi_{\mathtt{i}^-}\big(\operatorname{int}(\triangle)\big) \tag{6.2}$$

for each $\mathtt{i} \in I^*$. Suppose contrarily that there exist $k \in \mathbb{N}$ and $\mathtt{i}, \mathtt{j} \in I^k$ such that $\mathtt{i} \neq \mathtt{j}$ and $\varphi_{\mathtt{i}}\big(\operatorname{int}(\triangle)\big) \cap \varphi_{\mathtt{j}}\big(\operatorname{int}(\triangle)\big) \neq \emptyset$. Letting $k \geq 2$ be minimal with this respect, we get

$$\varphi_{\mathtt{i}^-}\big(\operatorname{int}(\triangle)\big) \cap \varphi_{\mathtt{j}^-}\big(\operatorname{int}(\triangle)\big) = \emptyset.$$

As we now have $\varphi_{\mathtt{i}}(\triangle) = \varphi_{\mathtt{j}}(\triangle)$ and consequently $\varphi_{\mathtt{i}}(\triangle') = \varphi_{\mathtt{j}}(\triangle')$, we in fact have, using (6.2),

$$\begin{aligned}\varphi_{\mathtt{i}}\bigl(\mathrm{int}(\triangle')\bigr) &= \varphi_{\mathtt{i}}\bigl(\mathrm{int}(\triangle')\bigr) \cap \varphi_{\mathtt{j}}\bigl(\mathrm{int}(\triangle')\bigr) \\ &\subset \varphi_{\mathtt{i}^-}\bigl(\mathrm{int}(\triangle)\bigr) \cap \varphi_{\mathtt{j}^-}\bigl(\mathrm{int}(\triangle)\bigr) = \emptyset.\end{aligned}$$

This contradiction finishes the proof of (6.1).

Observe that our method gives also a lower bound for the two dimensional Lebesgue measure $\mathcal{L}^2$ of E. Using (6.1), we see that

$$\mathcal{L}^2\Bigl(\bigcup_{\mathtt{i}\in I^m} \varphi_{\mathtt{i}}(\triangle)\Bigr) = \tfrac{1}{4}$$

whenever $m \in \mathbb{N}$, giving $\mathcal{L}^2(E) \geq \frac{1}{4}$ since

$$E = \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{m\geq k} \bigcup_{\mathtt{i}\in I^m} \varphi_{\mathtt{i}}(\triangle)}.$$

Here with the notation $\overline{A}$, we mean the closure of a given set A.

Example 6.7. We set $D' \subset [0,1]^2$ to be the graph of a nondecreasing continuous function $F\colon [0,1] \to [0,1]$ satisfying $F(0) = 0$ and $F(1) = 1$. A well known nondifferentiable example of this kind of function is $x \mapsto \mathcal{H}^t|_E([0,x])$, where E is the standard $\frac{1}{3}$-Cantor set and $t = \log 2/\log 3$. In this case, the set D' is known as *Devil's stairs*. We set $D = D' \cup \{(x,x) : |x| > 1\}$, $L = \{(x,x) : x \in \mathbb{R}\}$, and proj_L to be the orthogonal projection onto L. Now for the mapping $f = (\mathrm{proj}_L|_D)^{-1}\colon L \to D$, we clearly have

$$|x-y| \leq |f(x) - f(y)| \leq \sqrt{2}|x-y|$$

whenever $x, y \in L$ and defining a mapping $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ by setting $g(x) = f\bigl(\mathrm{proj}_L(x)\bigr) + x - \mathrm{proj}_L(x)$ for each $x \in \mathbb{R}^2$, the reader can easily see that $g|_L = f$ and

$$\sqrt{8}^{-1}|x-y| \leq |g(x) - g(y)| \leq (\sqrt{2}+2)|x-y|$$

whenever $x, y \in \mathbb{R}^2$.

Since the set $L \cap [0,1]^2$ is clearly an invariant set of a similitude IFS satisfying the uniform ball condition, the set D' is an invariant set of a congruent IFS satisfying the uniform ball condition.

Example 6.8. Suppose I is a finite set and for each $i \in I$ there is a contractive conformal mapping $\varphi_i\colon \Omega \to \Omega$ defined on an open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$. Assuming there exists a closed and nonempty $X \subset \Omega$ satisfying

$$\bigcup_{i\in I} \varphi_i(X) \subset X,$$

the collection $\{\varphi_i : i \in I\}$ is an IFS and we call it a *conformal IFS*. Since conformal mappings are C^∞, we deduce from [23, Remark 2.3] that each conformal IFS is congruent. Observe that the converse does not necessarily hold. In [15, Example 2.1], it is constructed a congruent IFS, which is not conformal. Also Devil's stairs in Example 6.7 provides the reader with such an IFS, see [13, Theorem 2.1].

Example 6.9. Defining for $A \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$, and $r > 0$

$$\begin{aligned}\operatorname{por}(A,x,r) = \sup\{\varrho \geq 0 : \text{there is } z \in \mathbb{R}^d \text{ such that}\\ B(z,\varrho r) \subset B(x,r) \setminus A\},\end{aligned}$$

we say that a bounded set $A \subset \mathbb{R}^d$ is *uniformly porous* if there are $\varrho > 0$ and $r_0 > 0$ such that $\operatorname{por}(A,x,r) \geq \varrho$ for all $x \in A$ and $0 < r < r_0$. The notation of porosity has arisen from the study of dimensional estimates related to the boundary behavior of various mappings.

Following the proof of [16, Theorem 4.1], we notice that a uniformly porous set is contained in a limit set of a CMC satisfying the uniform ball condition such that $\dim_{\mathrm{M}}(E) \leq d - c\varrho^d$, see Theorem 3.7.

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