Sobolev extensions and restrictions *

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Abstract

We give several characterizations for a domain $\Omega \subset \mathbb{R}^n$ to be a Sobolev extension domain. In particular, we show that, for $p > 1$, there is a bounded linear extension operator for $W^{1,p}(\Omega)$ if and only if every function in $W^{1,p}(\Omega)$ is the restriction of a function in $W^{1,p}(\mathbb{R}^n)$ to Ω. In the course of the proof, we show that extension domains, for all $1 \leq p < \infty$, satisfy a uniform measure density condition. We apply our results to study complemented subspaces in $W^{1,p}$. Our techniques also allow us to show that the extension property is invariant under bi-Lipschitz mappings.

1 Introduction

In this paper, we study various properties and characterizations of Sobolev extension domains. For a domain $\Omega \subset \mathbb{R}^n$ and $1 \leq p \leq \infty$, the Sobolev space $W^{1,p}(\Omega)$ consists of all functions in $L^p(\Omega)$ whose first order partial derivatives belong to $L^p(\Omega)$. It is a Banach space with respect to the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. We say that a domain $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$*-extension domain* if there is a bounded linear operator

$$\mathcal{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n) \quad \text{such that} \quad \mathcal{E}u(x) = u(x) \text{ for } x \in \Omega. \tag{1}$$

According to a theorem of Jones, [11], uniform domains are $W^{1,p}$-extension domains. Domains with Lipschitz boundary are uniform, but the boundary of a uniform domain can be very irregular, with Hausdorff dimension strictly larger than $n-1$ (it can be arbitrarily close to n, but always strictly less than n). The class of extension domains is, however, larger than the class of uniform domains and there is no geometric description of it. The following theorem is a part of one of the main results of the paper, Theorem 9.

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Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain and let $1 < p \leq \infty$. Then there is a bounded linear extension operator $\mathcal{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ if and only if every function $u \in W^{1,p}(\Omega)$ admits an extension to $W^{1,p}(\mathbb{R}^n)$.*

This result is a far reaching generalization of a result of Hajłasz and Martio [9, Theorem 10], who proved the same claim under the *corkscrew condition*: there is $0 < c < 1$ such that for every $x \in \Omega$ and every $0 < r < \operatorname{diam}\Omega$ the intersection $\Omega \cap B(x, r)$ contains a ball of radius cr. This geometric condition was crucial both for the construction and the estimates for the extension operator. It was a surprise for us that Theorem 1 can be proven without *any* additional conditions on the domain Ω.

Theorem 1 motivates the following open problem that we wish state for the interested readers.

Question 1. *Is Theorem 1 true for $p = 1$?*

Extension operators are closely related to the restriction operator, called also the *trace operator*

$$\mathcal{T} : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\Omega), \qquad \mathcal{T}u = u|_\Omega. \tag{2}$$

Namely, if $\mathcal{E}$ is an extension operator, then $\mathcal{T} \circ \mathcal{E}$ is the identity on $W^{1,p}(\Omega)$. Actually, the fact that every function $u \in W^{1,p}(\Omega)$ admits an extension to $W^{1,p}(\mathbb{R}^n)$ is obviously equivalent to the statement that the trace operator (2) is surjective. Therefore Theorem 1 can be reformulated as follows: *for an arbitrary domain in a Euclidean space and $1 < p \leq \infty$ there exists a bounded linear extension operator (1) if and only if the trace operator (2) is onto.*

There are two cases when Theorem 1 is very easy, $p = \infty$ and $p = 2$. For an arbitrary closed set $F \subset \mathbb{R}^n$, let $\operatorname{Lip}_\infty(F) = \operatorname{Lip}(F) \cap L^\infty(F)$ be the space of bounded Lipschitz functions on F. It is a Banach space with the norm

$$\|f\|_L = \|f\|_\infty + \operatorname{Lip}(f) = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Every bounded Lipschitz function on a domain $\Omega \subset \mathbb{R}^n$ uniquely extends to a bounded Lipschitz function on the closure, so we can consider $\operatorname{Lip}_\infty(\Omega)$ to be equal to $\operatorname{Lip}_\infty(\overline{\Omega})$. It is well known that $W^{1,\infty}(\mathbb{R}^n) = \operatorname{Lip}_\infty(\mathbb{R}^n)$ and for an arbitrary domain Ω, $\operatorname{Lip}_\infty(\Omega) \subset W^{1,\infty}(\Omega)$ is a linear subspace. As a restriction of a Lipschitz function to Ω is Lipschitz we conclude that

$$\mathcal{T} : W^{1,\infty}(\mathbb{R}^n) \to \operatorname{Lip}_\infty(\Omega) \subset W^{1,\infty}(\Omega).$$

Accordingly, if the operator (2) for $p = \infty$ is surjective onto $W^{1,\infty}(\Omega)$, we conclude that $\operatorname{Lip}_\infty(\Omega) = W^{1,\infty}(\Omega)$ as sets, and hence the norms are equivalent by the Banach open mapping theorem. Since it is well known that there is a bounded linear extension operator $\mathcal{E} : \operatorname{Lip}_\infty(F) \to \operatorname{Lip}_\infty(\mathbb{R}^n)$, see Lemma 20,[1] the case $p = \infty$ of Theorem 1 follows immediately.

[1]The familiar way to extend a Lipschitz function $f : \mathbb{R}^n \supset F \to \mathbb{R}$ to a Lipschitz function on $\mathbb{R}^n$ is by the way of McShane's formula $\tilde{f}(x) = \inf_{y \in E}\{f(y) + \operatorname{Lip}(f)|x - y|\}$. Note, however, that this

If $p = 2$, Theorem 1 is easy as well, but for a reason different than in the case $p = \infty$. In this case, Theorem 1 is a direct consequence of the Hilbert structure of the space $W^{1,2}$. Indeed, if we knew that the trace operator (2) were surjective, then $\mathcal{T}|_{(\ker \mathcal{T})^\perp} : (\ker \mathcal{T})^\perp \to W^{1,2}(\Omega)$ would be an isomorphism and hence we could define the extension $\mathcal{E}$ as

$$\mathcal{E} = \left(\mathcal{T}|_{(\ker \mathcal{T})^\perp}\right)^{-1} : W^{1,2}(\Omega) \to (\ker \mathcal{T})^\perp \subset W^{1,2}(\mathbb{R}^n).$$

This argument cannot be applied for $p \neq 2$ as not every subspace of $W^{1,p}$ for $p \neq 2$ is complemented. Recall that a closed subspace Y of a Banach space X is *complemented* if there is another closed subspace Z of X such that $X = Y \oplus Z$. That is, $Y \cap Z = \{0\}$ and every element $x \in X$ can be written as $x = y + z$, with $y \in Y$ and $z \in Z$.

Proposition 2 *Let $\Omega \subset \mathbb{R}^n$ be a domain such that, for some $1 \leq p \leq \infty$, every $u \in W^{1,p}(\Omega)$ admits an extension to $W^{1,p}(\mathbb{R}^n)$. Then there exists a bounded linear extension operator (1) if and only if the subspace* $\ker \mathcal{T}$ *is complemented in $W^{1,p}(\mathbb{R}^n)$.*

Proof. The first condition means that the trace operator is surjective. If $\ker \mathcal{T}$ is complemented in $W^{1,p}(\mathbb{R}^n)$, i.e. $W^{1,p}(\mathbb{R}^n) = \ker \mathcal{T} \oplus Y$ for some closed subspace $Y \subset W^{1,p}(\mathbb{R}^n)$, then the operator

$$\mathcal{E} = (\mathcal{T}|_Y)^{-1} : W^{1,p}(\Omega) \to Y \subset W^{1,p}(\mathbb{R}^n)$$

is a bounded extension operator. To prove the opposite implication, suppose that $\mathcal{E}$ is a bounded linear extension operator. Note that $\mathcal{E}(W^{1,p}(\Omega)) \subset W^{1,p}(\mathbb{R}^n)$ is a closed subspace. This easily follows from the obvious inequality $\|\mathcal{E}(u)\|_{1,p;\mathbb{R}^n} \geq \|u\|_{1,p;\Omega}$. Every element $u \in W^{1,p}(\mathbb{R}^n)$ can be written as $u = (u - \mathcal{E}(\mathcal{T}(u)) + \mathcal{E}(\mathcal{T}(u))$. Since $u - \mathcal{E}(\mathcal{T}(u)) \in \ker \mathcal{T}$, $\mathcal{E}(\mathcal{T}(u)) \in \mathcal{E}(W^{1,p}(\Omega))$, and $\ker \mathcal{T} \cap \mathcal{E}(W^{1,p}(\Omega)) = \{0\}$, we conclude that $W^{1,p}(\mathbb{R}^n) = \ker \mathcal{T} \oplus \mathcal{E}(W^{1,p}(\Omega))$ and hence the space $\ker \mathcal{T}$ is complemented. The proof is complete.

Using Proposition 2, Theorem 1 can be equivalently formulated as follows.

Theorem 3 *Let $\Omega \subset \mathbb{R}^n$ be a domain and $1 < p \leq \infty$. If the trace operator (2) is surjective, then the subspace* $\ker \mathcal{T}$ *is complemented in $W^{1,p}(\mathbb{R}^n)$.*

Note that, for $1 < p < \infty$, the space $W^{1,p}(\mathbb{R}^n)$ is isomorphic to $L^p(\mathbb{R}^n)$, [24, Chapter 5]. Accordingly, Theorem 3 is not obvious because not every subspace of $L^p(\mathbb{R}^n)$, $p \neq 2$, is complemented. Actually, the property of being complemented is rather rare. Here are some examples. Sobczyk, [23], seems to be the first to provide examples of subspaces of L^p that are not complemented. Lindenstrauss and Tzafriri, [16], proved that a real Banach space, in which every closed subspace is complemented, is isomorphic to a

does not give a linear extension. To obtain a linear one, we need to use e.g. the Whitney extension, see Lemma 20.

Hilbert space. Kadec and Mitjagin, [12], extend the result of Lindenstrauss and Tzafriri. Their paper is also a concise survey of concrete examples of uncomplemented subspaces. Bennett, Dor, Goodman, Johnson, and Newman, [2], proved that for every $1 < p < 2$ there is a subspace of $L^p[0,1]$, linearly isomorphic to a Hilbert space, which is not complemented. This extends a result of Rosenthal [22] who proved the case $1 < p < 4/3$. If $p > 2$, then every subspace in $L^p[0,1]$ linearly isomorphic to Hilbert space is complemented by a theorem of Kadec and Pełczyński [13]. Randrianantoanina, [21] proved that for $1 \le p < \infty$, not an even integer, if X and Y are subspaces of L^p such that X is complemented and Y is isometric to X, then Y is also complemented. However if $p \ge 4$ is an even integer, Randrianantoanina constructs isometric subspaces X and Y of L_p such that X is complemented while Y is not.

Theorem 3 suggests the following question.

Question 2. *Suppose $\Omega \subset \mathbb{R}^n$ is a domain and $1 \le p \le \infty$. Is $\ker \mathcal{T}$ necessarily complemented in $W^{1,p}(\mathbb{R}^n)$?*

The answer is obviously in the positive if $p = 2$. This is also the case if $p = \infty$. Indeed, for an arbitrary domain Ω the image of the trace is $\mathrm{Lip}_\infty(\Omega)$. Since there is a bounded linear extension operator from $\mathrm{Lip}_\infty(\Omega)$, the argument from the proof of Proposition 2 applies. If $1 \le p < \infty$, $p \ne 2$ and Ω is arbitrary we do not know the answer. However we can prove the following.

Theorem 4 *If $\Omega \subset \mathbb{R}^n$ is a domain that satisfies the measure density condition (3) below, and $1 < p \le \infty$, then $\ker \mathcal{T}$ is complemented in $W^{1,p}(\mathbb{R}^n)$.*

We say that an open set $\Omega \subset \mathbb{R}^n$ satisfies the *measure density condition* if there exists a constant $C > 0$ such that for all $x \in \Omega$ and all $0 < r \le 1$

$$|B(x,r) \cap \Omega| \ge Cr^n. \tag{3}$$

Question 3. *Is Theorem 4 true for $p = 1$?*

This question is weaker than the case $p = 1$ of Question 2 as we now assume the measure density condition. However, the positive answer to Question 3 would imply the positive answer to Question 1, see the remark following Theorem 5.

Note that, in general, even if a domain satisfies the measure density condition (3), the image of the trace need not be a closed subspace of $W^{1,p}(\Omega)$. This makes the situation even more difficult: there is no obvious candidate for the space that would complement $\ker \mathcal{T}$ in $W^{1,p}(\mathbb{R}^n)$. There are many examples of domains satisfying (3) such that the image of the trace operator is not closed in $W^{1,p}(\Omega)$. Indeed, Lewis [15] proved that if $\Omega \subset \mathbb{R}^2$ is a bounded Jordan domain, then functions in $C^\infty(\mathbb{R}^2)$ are dense in $W^{1,p}(\Omega)$ for each $1 < p < \infty$. Hence the image of the trace (2) is dense in $W^{1,p}(\Omega)$. Now, if Ω is not an extension domain, then the image of the trace is a proper subset of $W^{1,p}(\Omega)$ and hence not closed. As a corollary we have: *If $\Omega \subset \mathbb{R}^2$ is a bounded*

Jordan domain, and $1 < p < \infty$ then it is a $W^{1,p}$-extension domain if and only if the image of the trace (2) is closed in $W^{1,p}(\Omega)$. Maz'ya [18, Theorem 1.5.2] has constructed an example of a bounded Jordan domain in $\mathbb{R}^2$ such that a bounded extension operator $\mathcal{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ exists if and only if $1 \le p < 2$. As a corollary we obtain that in this case the image of the trace (2) is closed for $1 \le p < 2$ and is not closed for $p > 2$.

The phenomenon that the existence of an individual extension for every function does not imply the existence of a bounded linear extension is quite typical in functional analysis. Abstractly, if $X \subset Y$ is a closed subspace of a Banach space Y, every functional on X can be extended to a functional on Y by the Hahn-Banach Theorem. However, there is a bounded linear extension of functionals on X to functionals on Y if and only if $\{y^* \in Y^* : y^*|_X = 0\}$ is complemented in Y^*. This is an abstract version of Proposition 2. Proof is the same with $\pi : Y^* \to X^*$, $\pi(y^*) = y^*|_X$ playing the role of the trace. There are also more concrete examples. One such, particularly relevant in our context, is due to Peetre [19]. According to a theorem of Gagliardo, [3], there is a bounded and surjective trace operator $\mathcal{T} : W^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^{n-1})$, and hence every $u \in L^1(\mathbb{R}^{n-1})$ admits an extension to $W^{1,1}(\mathbb{R}^n)$. However, as was proven by Peetre, [19] (cf. [20]), there is no bounded linear extension operator $\mathcal{E} : L^1(\mathbb{R}^{n-1}) \to W^{1,1}(\mathbb{R}^n)$.

Cusps are often used as examples of domains that are not $W^{1,p}$-extension domains. It turns out, however, that extension domains must satisfy the measure density condition (3) which is clearly not satisfied by cusps. This observation immediately gives a much bigger class of examples.

Theorem 5 *If $\Omega \subset \mathbb{R}^n$ is a domain such that the trace operator (2) is surjective for some $1 \le p < \infty$, then Ω satisfies the measure density condition (3).*[2]

In particular, $W^{1,p}$-extension domains for $1 \le p < \infty$ satisfy (3). This fact was previously known for $W^{1,p}$-extension domains for $p > n-1$ (cf. [14] and references therein). Notice that, the measure density condition along with the Lebesgue differentiation theorem imply that the boundary of a $W^{1,p}$-extension domain is necessarily of volume zero. This answers the separate inquiries by Markus Biegert, Dagmar Medkova and Bill Ziemer.

The boundary of an extension domain can nevertheless be of Hausdorff dimension n. To see this, it suffices to consider the complement of a compact set E of Hausdorff dimension n, constructed as the n-fold product of a suitable compact set of dimension one but with vanishing length. Indeed, it easily follows by integrating by parts that then each function $u \in W^{1,p}(\mathbb{R}^n \setminus E)$ belongs to $W^{1,p}(\mathbb{R}^n)$.

Notice that Theorem 3 (and hence also Theorem 1) is a direct corollary of Theorems 5 and 4 for $1 < p < \infty$, and the case $p = \infty$ was proved earlier. Also notice

[2]This theorem is not true for $p = \infty$, see Theorem 10 and the comment following it.

that, for the same reason, if the answer to Question 3 were positive, then the answer to Question 1 would be positive as well.

Actually, we can prove that not only the extension property implies (3) but that a suitable Sobolev embedding for each function $u \in W^{1,p}(\Omega)$ is sufficient for (3).

The classical Sobolev embedding theorem says that if $1 \le p < n$, then $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$, where $p^* = np/(n-p)$; if $p > n$, then $W^{1,p}(\mathbb{R}^n) \subset C^{0,1-n/p}(\mathbb{R}^n)$, and in each case the embedding is continuous. The embedding in the limiting case $p = n$ is more delicate as it is given in a local form of the Trudinger inequality.[3]

Lemma 6 (Trudinger inequality) *There exist positive constants $c_1(n)$ and $c_2(n)$ such that if $u \in W^{1,n}(B)$, where $B \subset \mathbb{R}^n$ is an arbitrary ball, then*

$$\int_B \exp\left(\frac{c_1(n)|u-u_B|}{\|\nabla u\|_{n;B}}\right)^{n/(n-1)} \le c_2(n)|B|.$$

For a proof see e.g. [25], [4], [1].

Suppose now that $\Omega \subset \mathbb{R}^n$ is an arbitrary domain and that $u \in W^{1,n}(\Omega)$ admits an extension to $W^{1,n}(\mathbb{R}^n)$, i.e. there is $v \in W^{1,n}(\mathbb{R}^n)$ such that $v|_\Omega = u$. Then for any ball B we have

$$\begin{aligned} c_2(n)|B| &\ge \int_B \exp\left(\frac{c_1(n)|v-v_B|}{\|\nabla v\|_{n;B}}\right)^{n/(n-1)} \\ &\ge \int_{B\cap\Omega} \exp\left(\frac{c_1(n)|u-v_B|}{\|\nabla v\|_{n;\mathbb{R}^n}}\right)^{n/(n-1)} \\ &\ge \inf_{\gamma\in\mathbb{R}} \int_{B\cap\Omega} \exp\left(\alpha|u-\gamma|\right)^{n/(n-1)}, \end{aligned}$$

where $\alpha = c_1(n)\|\nabla v\|^{-1}_{n;\mathbb{R}^n}$ depends on u, but does not depend on the ball B. We proved the following result.

Corollary 7 *There is a constant $c(n) > 0$ such that for an arbitrary domain $\Omega \subset \mathbb{R}^n$, and an arbitrary $u \in W^{1,n}(\Omega)$ that admits an extension to $W^{1,n}(\mathbb{R}^n)$, there exists a constant $\alpha > 0$ such that for every ball $B \subset \mathbb{R}^n$*

$$\inf_{\gamma\in\mathbb{R}} \int_{B\cap\Omega} \exp\left(\alpha|u-\gamma|\right)^{n/(n-1)} \le c(n)|B|.$$

[3] Although this is a well-known theorem, we name it a lemma as we keep the name theorem only for the new results proved in the paper. There are global versions of the Trudinger inequality, but they are more complicated [1].

One may, however, expect that for some domains a weaker Trudinger type inequality is satisfied with an exponent $n/(n-1)$ replaced by some other, smaller, exponent s.

If $\Omega \subset \mathbb{R}^n$ is a domain such that every function $u \in W^{1,p}(\Omega)$ admits an extension to $W^{1,p}(\mathbb{R}^n)$, then the assumptions formulated in cases (a), (b) and (c) of the theorem below are satisfied. Hence Theorem 5 is a direct consequence of the more general Theorem 8 which is one of the two main results in the paper (Theorem 9 is the other one).

Theorem 8 *Let $\Omega \subset \mathbb{R}^n$ be a domain.*

(a) If $1 \le p < n$ and every function $u \in W^{1,p}(\Omega)$ belongs to $L^{p^*}(\Omega)$, $p^* = np/(n-p)$, then Ω satisfies (3).

(b) If $p = n$ and there are constants $M > 0$ and $s > 0$ such that for every function $u \in W^{1,n}(\Omega)$ there is a constant $\alpha > 0$ such that for every $x \in \Omega$ and every $0 < r \le 1$

$$\inf_{\gamma \in \mathbb{R}} \int_{B(x,r)\cap\Omega} \exp\left(\alpha|u-\gamma|\right)^s \le Mr^n, \tag{4}$$

then Ω satisfies (3).

(c) If $n < p < \infty$ and every function $u \in W^{1,p}(\Omega)$ is uniformly locally Hölder continuous with the exponent $1 - n/p$, i.e.

$$|u(x) - u(y)| \le M|x-y|^{1-n/p}, \tag{5}$$

whenever $x, y \in \Omega$ satisfy $|x - y| \le r_0$, where M, r_0 are allowed to depend on u, then Ω satisfies (3).

Remark. Although we assume that every function u in $W^{1,p}(\Omega)$ belongs to the space that appears in the Sobolev embedding theorem, we *do not* require any estimates. Thus our conditions are weaker than the corresponding Sobolev embeddings.

If we combine the results discussed above with the results from Section 3, we arrive at the following theorem, one of the two main results of the paper.

Theorem 9 *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $1 < p < \infty$. Then the following conditions are equivalent:*

1. *For every $u \in W^{1,p}(\Omega)$ there exists $v \in W^{1,p}(\mathbb{R}^n)$, such that $v|_\Omega = u$.*
2. *The trace operator (2) is surjective.*
3. *There exists a bounded linear extension operator (1).*

4. The operator $\mathcal{E}^* : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ defined by (34) is bounded.

5. Ω satisfies the measure density condition (3) and $W^{1,p}(\Omega) = M^{1,p}(\Omega, |\cdot|, \mathcal{L}^n)$.

See Section 3 for the definition of the space $M^{1,p}$. Theorem 9 is a far reaching generalization of an analogous result in [9, Theorem 10], where the domain Ω in question was required to satisfy the corkscrew condition; here we prove it for *all* domains. See also a remark that follows Theorem 1.

There is also a counterpart of Theorem 9 for $p = \infty$, Theorem 10, but as in this case the space $W^{1,\infty}$ is closely related to the space of bounded Lipschitz functions, the result is easy and could be regarded as a mathematical folklore (cf. [28], [10], [30, Proposition 2]).

Theorem 10 *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain. Then the following conditions are equivalent:*

1. For every $u \in W^{1,\infty}(\Omega)$ there exists $v \in W^{1,\infty}(\mathbb{R}^n)$, such that $v|_\Omega = u$.

2. The trace operator (2) is surjective for $p = \infty$.

3. There exists a bounded linear extension operator (1) for $p = \infty$.

4. The operator $\mathcal{E}' : W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^n)$ defined by (33) is bounded.

5. $W^{1,\infty}(\Omega) = \mathrm{Lip}_\infty(\Omega)$.

6. Ω is uniformly locally quasiconvex.

We say that a domain $\Omega \subset \mathbb{R}^n$ is *uniformly locally quasiconvex* if there are constants $C > 0$ and $R > 0$ such that for every $x, y \in \Omega$ satisfying $|x-y| < R$ there is a rectifiable curve γ connecting x and y in Ω such that the length of γ is bounded from above by $C|x-y|$.

Note that the measure density condition does not appear in Theorem 10. In fact, there are obvious examples of quasiconvex domains that do not satisfy (3). Hence the existence of a bounded extension operator for $p = \infty$ does not imply (3), contrary to the case $1 \leq p < \infty$.

Since Theorem 10 is easy we will prove it now and postpone the proof of Theorem 9 until the end of Section 3.

Proof of Theorem 10. We have already proved Theorem 1 for $p = \infty$, which gives the equivalence between conditions (1), (2), and (3). The implication from (4) to (3) is obvious; the implication from (5) to (4) follows from Lemma 20 and the implication from (2) to (5) was established in the proof of Theorem 1 for $p = \infty$. This completes the proof of the equivalence of the conditions (1), (2), (3), (4) and (5). To prove the implication from (6) to (5) we need to show that $W^{1,\infty}(\Omega) \subset \mathrm{Lip}_{\infty}(\Omega)$. Let $f \in W^{1,\infty}(\Omega)$. If $|x-y| < R$ and γ is as in the definition of a uniformly locally quasiconvex domain, then $|f(x) - f(y)| \leq \int_\gamma \|\nabla f\|_\infty \leq C\|\nabla f\|_\infty |x-y|$. If $|x-y| > R$, then $|f(x) - f(y)| \leq 2\|f\|_\infty R^{-1}|x-y|$ and hence $f \in \mathrm{Lip}_{\infty}(\Omega)$. To complete the proof, it suffices to verify the implication from (3) to (6). For $x, y \in \Omega$, let $\varphi_x(y)$ be the infimum of lengths of curves that join x and y in Ω. Note that $\tilde{\varphi}_x = \min\{\varphi_x, 1\} \in W^{1,\infty}(\Omega)$ with $\|\nabla \tilde{\varphi}_x\|_\infty = 1$. Now (3) yields that $\{\mathcal{E}\tilde{\varphi}_x\}_{x\in\Omega}$ is a bounded family of functions in $\mathrm{Lip}_{\infty}(\mathbb{R}^n)$ and hence $\tilde{\varphi}_x(y) = |\tilde{\varphi}_x(x) - \tilde{\varphi}_x(y)| = |\mathcal{E}\tilde{\varphi}_x(x) - \mathcal{E}\tilde{\varphi}_x(y)| \leq C|x-y|$, whenever $x, y \in \Omega$. Now if $|x-y| \leq R = C^{-1}$ we have that $1 \geq C|x-y| \geq \tilde{\varphi}_x(y) = \varphi_x(y)$ and hence (6) follows. The proof is complete.

Theorem 11 *Let $\Omega, G \subset \mathbb{R}^n$ be two domains that are bi-Lipschitz homeomorphic. Then Ω is a $W^{1,p}$-extension domain for some $1 < p \leq \infty$ if and only if G is a $W^{1,p}$-extension domain.*

If $p = \infty$, the claim easily follows from Theorem 10, but if $1 < p < \infty$ this is far from being obvious. If we knew that there were a bi-Lipschitz homeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\Omega) = G$, the claim would easily follow even for $p = 1$. However, in general, a bi-Lipschitz homeomorphism $T : \Omega \to G$ cannot be extended beyond Ω (cf. [26],[27]).

Question 4. *Is Theorem 11 true for $p = 1$?*

Proof of Theorem 11. We may assume that $1 < p < \infty$. Let $T : \Omega \to G$ be a bi-Lipschitz homeomorphism. Suppose that one of the domains, say Ω, is a $W^{1,p}$-extension domain. By Theorem 9, Ω satisfies (3) and $W^{1,p}(\Omega) = M^{1,p}(\Omega, |\cdot|, \mathcal{L}^n)$. Now G satisfies (3) as bi-Lipschitz homeomorphisms preserve the measure density condition. Moreover, the transformation $\Phi(u) = u \circ T$ induces isomorphisms of spaces, $\Phi : W^{1,p}(G) \to W^{1,p}(\Omega)$, and $\Phi : M^{1,p}(G, |\cdot|, \mathcal{L}^n) \to M^{1,p}(\Omega, |\cdot|, \mathcal{L}^n)$. Therefore $W^{1,p}(G) = M^{1,p}(G, |\cdot|, \mathcal{L}^n)$ and again we can apply Theorem 9. The proof is complete.

The paper is organized as follows. In Section 2, we prove Theorem 8. This will also complete the proof of Theorem 5 as it is a direct corollary of Theorem 8. In Section 3 we prove Theorem 4. This will be a direct consequence of a stronger Theorem 18 (see the first remark following Theorem 18). Actually, Theorem 18 provides an exact description of traces of functions in $W^{1,p}(\mathbb{R}^n)$ on Ω when $1 < p < \infty$ and Ω satisfies the measure density condition. Now Theorems 5 and 4 immediately imply Theorem 3 and, as we know, this theorem is equivalent with Theorem 1. Finally, Theorem 9 will be proven at the end of Section 3.

Some of the results of the paper can be generalized to the setting of Sobolev spaces on metric spaces. This will be the subject of a forthcoming paper [8] (see also the third remark following Theorem 18).

Notation. Symbols C or c will be used to designate general constants; the same symbol can be used for different constants even within one string of estimates. Writing e.g., $c(n,p)$ we will emphasize that the constant depends on n and p only. The Lebesgue measure of a set $A \subset \mathbb{R}^n$ will be denoted either by $\mathcal{L}^n(A)$ of by $|A|$. If u is an integrable function defined on a measurable set of positive measure, then

$$u_A = \fint_A u\, dx = \frac{1}{|A|}\int_A u\, dx$$

will denote the average value of u over A. The L^p-norm and the Sobolev norm of u over a domain Ω will be denoted by $\|u\|_p$ and $\|u\|_{1,p}$, respectively. In case of need for emphasizing over which domain the norm is evaluated, we write $\|u\|_{p;\Omega}$ and $\|u\|_{1,p;\Omega}$, respectively. We say that a function $u : \Omega \to \mathbb{R}$ is uniformly locally Hölder continuous with exponent λ if there are $r_0 > 0$ and $C > 0$ such that $|u(x) - u(y)| \leq C|x-y|^\lambda$ for all $x, y \in \Omega$ with $|x-y| < r_0$. The oscillation of a function u over a set E is defined by $\operatorname{osc}_E u = \sup_{x,y\in E} |u(x) - u(y)|$. The volume of the unit ball in $\mathbb{R}^n$ is denoted by ω_n. If x is given and $0 < r < R$, then $A(R,r) = B(x,R) \setminus B(x,r)$ will denote the corresponding annulus.

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2 Proof of Theorem 8

First we treat the cases $1 \leq p < n$ and $p > n$. The case $p = n$ is treated as the last, most difficult case.

Case $1 \leq p < n$. For $x \in \Omega$ and $0 < r \leq 1$, there exists a unique $0 < \tilde{r} < r$ such that

$$|B(x,\tilde{r}) \cap \Omega| = |A(r,\tilde{r}) \cap \Omega| = \frac{1}{2}|B(x,r)\cap\Omega|, \tag{6}$$

where

$$A(r,\tilde{r}) = B(x,r) \setminus B(x,\tilde{r}).$$

Lemma 12 *There is a constant $c > 0$ such that*

$$r - \tilde{r} \leq c|B(x,r)\cap\Omega|^{1/n} \tag{7}$$

for all $x \in \Omega$ and all $0 < r \leq 1$.

Before proving the lemma, let us show that it implies (3). Let $x \in \Omega$ and $0 < r \leq 1$. Define a sequence $r_0 > r_1 > r_2 > \ldots > 0$ by induction:

$$r_0 = r, \qquad r_{j+1} = \tilde{r}_j.$$

Clearly $|B(x,r_j)\cap\Omega| = 2^{-j}|B(x,r)\cap\Omega|$. Hence $r_j \to 0$ and

$$r_j - r_{j+1} \le c2^{-j/n}|B(x,r)\cap\Omega|^{1/n}$$

by (7). This in turn yields

$$\begin{aligned} r = \sum_{j=0}^{\infty}(r_j - r_{j+1}) &\le C\left(\sum_{j=0}^{\infty} 2^{-j/n}\right)|B(x,r)\cap\Omega|^{1/n} \\ &\le C'|B(x,r)\cap\Omega|^{1/n}, \end{aligned}$$

which implies the measure density condition (3).

Proof of the lemma. By contradiction, suppose that there exist $x_i \in \Omega$ and $0 < r_i \le 1$ for $i = 1, 2, 3, \ldots$ such that[4]

$$r_i - \tilde{r}_i \ge 4^i |B(x, r_i)\cap\Omega|^{1/n}. \tag{8}$$

Let

$$u_i(y) = \begin{cases} 1 & \text{for } y \in B(x_i, \tilde{r}_i)\cap\Omega, \\ \frac{r_i - |x_i - y|}{r_i - \tilde{r}_i} & \text{for } y \in A(r_i, \tilde{r}_i)\cap\Omega, \\ 0 & \text{for } y \in \Omega\setminus B(x_i, r_i). \end{cases}$$

Clearly, u_i is a Lipschitz function bounded by 1 and

$$|\nabla u_i| = \frac{1}{r_i - \tilde{r}_i}\chi_{A(r_i,\tilde{r}_i)\cap\Omega}.$$

Let $k_i > 0$ be defined by the equality

$$\|u_i\|_{p^*} = k_i(\|u_i\|_p + \|\nabla u_i\|_p).$$

Then

$$\begin{aligned} |B(x_i,\tilde{r}_i)\cap\Omega|^{1/p^*} &\le \|u_i\|_{p^*} = k_i(\|u_i\|_p + \|\nabla u_i\|_p) \\ &\le k_i\left(|B(x_i,r_i)\cap\Omega|^{1/p} + \frac{|A(r_i,\tilde{r}_i)\cap\Omega|^{1/p}}{r_i - \tilde{r}_i}\right) \\ &= k_i\left(2^{1/p} + \frac{1}{r_i - \tilde{r}_i}\right)|A(r_i,\tilde{r}_i)\cap\Omega|^{1/p} \\ &\le k_i(2^{1/p}+1)\frac{1}{r_i - \tilde{r}_i}|B(x_i,\tilde{r}_i)\cap\Omega|^{1/p}. \end{aligned}$$

[4]Here the sequence $\{r_i\}$ is *not* the same as the sequence $\{r_j\}$ constructed earlier. Just a coincidence of the notation.

We employed (6) and the obvious inequality $2^{1/p} \le 2^{1/p}/(r_i - \tilde{r}_i)$. Since $1/p - 1/p^* = 1/n$, we conclude

$$r_i - \tilde{r}_i \le k_i c(p) |B(x_i, \tilde{r}_i) \cap \Omega|^{1/n},$$

which together with (8) and (6) yields $k_i \ge c4^i$. Let

$$a_i = 2^{-i} \left(\|u_i\|_p + \|\nabla u_i\|_p \right)^{-1}.$$

We have

$$\|a_i u_i\|_{p^*} = 2^{-i} k_i \ge c2^i \qquad \text{and} \qquad \|a_i u_i\|_{1,p} = 2^{-i}.$$

Now we define

$$u = \sum_{i=1}^{\infty} a_i u_i. \tag{9}$$

Since $\sum_{i=1}^{\infty} \|a_i u_i\|_{1,p} < \infty$, the series (9) converges to some $u \in W^{1,p}(\Omega)$. On the other hand, $a_i u_i \ge 0$, and hence

$$\|u\|_{p^*} \ge \|a_i u_i\|_{p^*} \ge c2^i, \qquad \text{for } i = 1, 2, 3, \ldots$$

and thus $\|u\|_{p^*} = \infty$. This contradicts the assumption on the L^{p^*}-integrability of functions in $W^{1,p}(\Omega)$. The proof of the lemma and hence that for the case $1 \le p < n$ is complete.

Case $n < p < \infty$. Suppose that the measure density condition (3) is not satisfied. Given $\varepsilon > 0$ and a positive integer k, we can find $x \in \Omega$ and $0 < r \le 1$ satisfying $|B(x,r) \cap \Omega| < \varepsilon 2^{-kn} r^n$. Hence $|B(x, \hat{r}) \cap \Omega| < \varepsilon \hat{r}^n$, for $\hat{r} = r/2^k$. This implies that, when violating (3), we may require that the radius of the ball be arbitrarily small. This is to say that there are sequences $x_i \in \Omega$ and $0 < r_i \to 0$ such that

$$|B(x_i, r_i) \cap \Omega| < 4^{-ip} r_i^n.$$

We may assume that $\Omega \setminus B(x_i, r_i) \ne \emptyset$, since $r_i \to 0$ and we will only be interested in sufficiently large i. Given i we define

$$u_i(y) = \begin{cases} 1 - \frac{|x_i - y|}{r_i} & \text{for } y \in B(x_i, r_i) \cap \Omega, \\ 0 & \text{for } y \in \Omega \setminus B(x_i, r_i). \end{cases}$$

Clearly, u_i is a Lipschitz function bounded by 1 and

$$|\nabla u_i| = \frac{1}{r_i} \chi_{B(x_i, r_i) \cap \Omega}.$$

Since $\Omega \setminus B(x_i, r_i) \ne \emptyset$, we have $\operatorname{osc}_{B(x_i, r_i) \cap \Omega} u_i = 1$. Moreover,

$$\begin{aligned} \|2^i r_i^{1-n/p} u_i\|_{1,p} &= 2^i r_i^{1-n/p} \left(\|u_i\|_p + \|\nabla u_i\|_p \right) \\ &\le 2^i r_i^{1-n/p} \left(|B(x_i, r_i) \cap \Omega|^{1/p} + \frac{|B(x_i, r_i) \cap \Omega|^{1/p}}{r_i} \right) \\ &\le 2^i r_i^{1-n/p} \cdot 2r_i^{-1} \cdot (4^{-ip} r_i^n)^{1/p} = 2 \cdot 2^{-i}. \end{aligned} \tag{10}$$

Each of the functions $v_i = 2^i r_i^{1-n/p} u_i$ is Lipschitz-continuous and hence Hölder-continuous with the exponent $1 - n/p$. However, it is easily seen that the constant in the Hölder-continuity estimate for v_i has to blow up as $i \to \infty$. Indeed

$$|v_i(x) - v_i(y)| \leq 2^{i+n/p}|x-y|^{1-n/p} \qquad \text{for } x, y \in \Omega,$$

but, on the other hand, for $x \in B(x_i, r_i/4) \cap \Omega$ and $y \in A(r_i, 3r_i/4) \cap \Omega$, we have

$$|v_i(x) - v_i(y)| \geq 2^{i-2+n/p}|x-y|^{1-n/p}.$$

Estimate (10) implies that the series $\sum_{i=1}^{\infty} v_i$ converges to some function $\tilde{u} \in W^{1,p}(\Omega)$ and one might expect that the lack of the uniform Hölder estimate for the functions v_i would imply that the function $\tilde{u}$ not be in the class $C^{0,1-n/p}$, not even locally. There is, however, one technical difficulty: the supports of the functions v_i need not be disjoint and perhaps this can cause some cancellation phenomenon. To overcome this problem, we choose a suitable subsequence $\{i_j\}_j$ and define our function by the series (11). Correct choice of $\{i_j\}_j$ guarantees that $v_{i_{j+1}}$ has so bad a Hölder-continuity constant that it cannot be overtaken by the Hölder-continuity estimate of the function $\sum_{k=1}^{j} v_{i_k}$. On the other hand, the total measure of the supports of the functions v_{i_k} for $k \geq j+2$ is so tiny that the function $\sum_{k=j+2}^{\infty} v_{i_k}$ changes $\sum_{k=1}^{j+1} v_{i_k}$ on a very small set and hence it cannot destroy the bad Hölder-continuity estimate of $\sum_{k=1}^{j+1} v_{i_k}$. Although for many of the readers this idea would be sufficient, we prefer to provide a detailed construction with rigorous arguments. Since the construction is quite technical, the reader should keep in mind the idea to see what is really being done.

Since $|B(x_i, r_i) \cap \Omega| \to 0$ as $i \to \infty$, we can choose a subsequence $\{i_j\}_{j=1}^{\infty}$ such that

$$E_j = \Big(B(x_{i_j}, \frac{r_{i_j}}{4}) \cap \Omega\Big) \setminus \bigcup_{k=j+1}^{\infty} (B(x_{i_k}, r_{i_k}) \cap \Omega) \neq \emptyset$$

and also

$$F_j = \Big(A(r_{i_j}, \frac{3}{4} r_{i_j}) \cap \Omega\Big) \setminus \bigcup_{k=j+1}^{\infty} (B(x_{i_k}, r_{i_k}) \cap \Omega) \neq \emptyset$$

for $j = 1, 2, 3, \ldots$ We may also require that

$$2^{n/p} \left(2^{i_{j+1}-2} - \sum_{k=1}^{j} 2^{i_k} \right) \geq j.$$

Define

$$u = \sum_{j=1}^{\infty} v_{i_j} = \sum_{j=1}^{\infty} 2^{i_j} r_{i_j}^{1-n/p} u_{i_j}. \tag{11}$$

It follows from (10) that the series converges to some $u \in W^{1,p}(\Omega)$. Now it remains to prove that u is *not* uniformly locally Hölder continuous with the exponent $1 - n/p$. By contradiction, suppose that there is $R > 0$ and $M > 0$ such that

$$|u(x) - u(y)| \leq M|x-y|^{1-n/p} \qquad \text{for all } x, y \in \Omega \text{ with } |x-y| \leq R. \tag{12}$$

Let $x \in E_{j+1}$ and $y \in F_{j+1}$. Then

$$u(x) = \sum_{k=1}^{j+1} v_{i_k}(x) \quad \text{and} \quad u(y) = \sum_{k=1}^{j+1} v_{i_k}(y).$$

This is because the definitions of E_{j+1} and F_{j+1} give $v_{i_k}(x) = 0$ for $k \geq j+2$ and $v_{i_k}(y) = 0$ for $k \geq j+2$. Thus

$$\begin{aligned}
|u(x) - u(y)| &\geq |v_{i_{j+1}}(x) - v_{i_{j+1}}(y)| - \sum_{k=1}^{j} |v_{i_k}(x) - v_{i_k}(y)| \\
&\geq 2^{i_{j+1}-2+n/p}|x-y|^{1-n/p} - \left(\sum_{k=1}^{j} 2^{i_k+n/p}\right)|x-y|^{1-n/p} \\
&= 2^{n/p}\left(2^{i_{j+1}-2} - \sum_{k=1}^{j} 2^{i_k}\right)|x-y|^{1-n/p} \\
&\geq j|x-y|^{1-n/p}.
\end{aligned}$$

Since $x, y \in B(x_{i_{j+1}}, r_{i_{j+1}})$, taking sufficiently large j, we can guarantee that $|x-y| < 2r_{i_{j+1}} < R$ and $j > M$. This is a contradiction with (12). The proof for the case $n < p < \infty$ is complete.

Case $p = n$. For $x \in \Omega$ and $0 < r \leq 1$, we choose $0 < \tilde{\tilde{r}} < \tilde{r} < r$ such that

$$|B(x, \tilde{\tilde{r}}) \cap \Omega| = \frac{1}{2}|B(x, \tilde{r}) \cap \Omega| = \frac{1}{4}|B(x, r) \cap \Omega|. \tag{13}$$

Note that

$$|A(\tilde{r}, \tilde{\tilde{r}}) \cap \Omega| = |B(x, \tilde{\tilde{r}}) \cap \Omega| \quad \text{and} \quad |A(r, \tilde{r}) \cap \Omega| = |B(x, \tilde{r}) \cap \Omega|. \tag{14}$$

We define

$$u(y) = \begin{cases} (\tilde{r} - \tilde{\tilde{r}})|A(\tilde{r}, \tilde{\tilde{r}}) \cap \Omega|^{-1/n}/4 & \text{for } y \in B(x, \tilde{\tilde{r}}) \cap \Omega, \\ (\tilde{r} - |x-y|)|A(\tilde{r}, \tilde{\tilde{r}}) \cap \Omega|^{-1/n}/4 & \text{for } y \in A(\tilde{r}, \tilde{\tilde{r}}) \cap \Omega, \\ 0 & \text{for } y \in \Omega \setminus B(x, \tilde{r}). \end{cases} \tag{15}$$

We have

$$|\nabla u| = \frac{\chi_{A(\tilde{r}, \tilde{\tilde{r}}) \cap \Omega}}{4|A(\tilde{r}, \tilde{\tilde{r}}) \cap \Omega|^{1/n}} \qquad \text{and} \qquad |u| \leq \frac{\tilde{r} - \tilde{\tilde{r}}}{4|A(\tilde{r}, \tilde{\tilde{r}}) \cap \Omega|^{1/n}} \chi_{B(x, \tilde{r}) \cap \Omega}.$$

Hence

$$\left(\int_\Omega |\nabla u|^n\right)^{1/n} \leq 1/4$$

and

$$\left(\int_\Omega |u|^n\right)^{1/n} \le \frac{\tilde r - \tilde{\tilde r}}{4|A(\tilde r, \tilde{\tilde r}) \cap \Omega|^{1/n}} |B(x, \tilde r) \cap \Omega|^{1/n} = (\tilde r - \tilde{\tilde r}) 2^{1/n}/4 < 1/2,$$

because $\tilde r - \tilde{\tilde r} < 1$. Combining these two inequalities yields $\|u\|_{1,n} < 1$.

Lemma 13 *There is a constant $\beta > 0$ depending on Ω, M and s, such that for every $x \in \Omega$ and every $0 < r \le 1$ the function u defined by (15) satisfies*

$$\inf_{\gamma \in \mathbb{R}} \int_{B(x,r)\cap\Omega} \exp(\beta |u - \gamma|)^s \le (M + \omega_n) r^n. \tag{16}$$

Remark. The assumption of the theorem yields that for each x and all $0 < r < 1$, the function u defined by (15) satisfies (16) with a constant $\beta = \alpha$ depending on u. The lemma says that (16) holds for all functions defined by (15) with the same constant independent of u.

Proof of the lemma. Suppose the claim of the lemma is not true. Then there are sequences $x_i \in \Omega$ and $0 < r_i \le 1$, such that

$$\inf_{\gamma \in \mathbb{R}} \int_{B(x_i,r_i)\cap\Omega} \exp(4^{-i} |u_i - \gamma|)^s > (M + \omega_n) r_i^n,$$

where u_i is defined by (15) with x and r replaced by x_i and r_i.

Since $\sum_{i=1}^\infty \|2^{-i} u_i\|_{1,n} < \sum_{i=1}^\infty 2^{-i} < \infty$, for every subsequence $\{u_{i_j}\}_j$, we have

$$u = \sum_{j=1}^\infty 2^{-i_j} u_{i_j} \in W^{1,n}(\Omega). \tag{17}$$

Our aim is to show that the function u defined by (17) for a subsequence $\{u_{i_j}\}_j$ that will be specified later, cannot satisfy (4) with any choice of α.

For $\gamma = 0$ we have

$$\begin{aligned} (M + \omega_n) r_i^n &< |B(x_i, r_i) \cap \Omega| \exp\left(4^{-i} \frac{\tilde r_i - \tilde{\tilde r}_i}{4|A(\tilde r_i, \tilde{\tilde r}_i) \cap \Omega|^{1/n}}\right)^s \\ &\le \omega_n r_i^n \exp\left(4^{-i} \frac{\tilde r_i - \tilde{\tilde r}_i}{4|A(\tilde r_i, \tilde{\tilde r}_i) \cap \Omega|^{1/n}}\right)^s . \end{aligned} \tag{18}$$

Hence

$$2^i \left(\ln\left(1 + \frac{M}{\omega_n}\right)\right)^{1/s} \le 2^{-i} \frac{\tilde r_i - \tilde{\tilde r}_i}{4|A(\tilde r_i, \tilde{\tilde r}_i) \cap \Omega|^{1/n}}.$$

This, in turn, yields

$$2^{-i}\frac{\tilde{r}_i-\tilde{\tilde{r}}_i}{4|A(\tilde{r}_i,\tilde{\tilde{r}}_i)\cap\Omega|^{1/n}}\to\infty \qquad \text{as } i\to\infty. \tag{19}$$

Since

$$|B(x_i,\tilde{\tilde{r}}_i)\cap\Omega|^{1/n}\left(2^{-i}\frac{\tilde{r}_i-\tilde{\tilde{r}}_i}{4|A(\tilde{r}_i,\tilde{\tilde{r}}_i)\cap\Omega|^{1/n}}\right)\le\|2^{-i}u_i\|_n<2^{-i}\to 0\quad \text{as } i\to\infty,$$

(19) and (13) imply that

$$|B(x_i,r_i)\cap\Omega|\to 0 \qquad \text{as } i\to\infty. \tag{20}$$

Because each function u_i is bounded, (19) implies that we can choose a subsequence u_{i_j} such that

$$\frac{1}{2}2^{-i_j}\frac{\tilde{r}_{i_j}-\tilde{\tilde{r}}_{i_j}}{8|A(\tilde{r}_{i_j},\tilde{\tilde{r}}_{i_j})\cap\Omega|^{1/n}}>\sum_{k=1}^{j-1}2^{-i_k}u_{i_k} \tag{21}$$

for all $j\ge 2$. Condition (20) implies that, in addition to (21), we may assume that

$$\left|\left\{x:\ \sum_{k=j+1}^{\infty}2^{-i_k}u_{i_k}(x)\neq 0\right\}\right|<\sum_{k=j+1}^{\infty}|B(x_{i_k},r_{i_k})\cap\Omega|<\frac{1}{8}|B(x_{i_j},r_{i_j})\cap\Omega| \tag{22}$$

for all $j\ge 1$.

For a subsequence u_{i_j}, satisfying (21) and (22), we define u by formula (17). By the assumption of the theorem, there is $\alpha>0$ such that, for every $x\in\Omega$ and $0<r\le 1$, inequality (4) is satisfied.

Now (19) implies that there is i_0 (depending on α) such that

$$\frac{1}{8}\exp\left(\alpha 2^{-i}\frac{\tilde{r}_i-\tilde{\tilde{r}}_i}{16|A(\tilde{r}_i,\tilde{\tilde{r}}_i)\cap\Omega|^{1/n}}\right)^s\ge\exp\left(4^{-i}\frac{\tilde{r}_i-\tilde{\tilde{r}}_i}{4|A(\tilde{r}_i,\tilde{\tilde{r}}_i)\cap\Omega|^{1/n}}\right)^s \tag{23}$$

for all $i\ge i_0$.

Inequality (4) yields in particular that, for every j with $i_j\ge i_0$, there is $\gamma\in\mathbb{R}$ such that

$$\begin{aligned}\left(M+\frac{\omega_n}{2}\right)r_{i_j}^n &\ge \int_{B(x_{i_j},r_{i_j})\cap\Omega}\exp(\alpha|u-\gamma|)^s\\ &\ge \int_{B(x_{i_j},r_{i_j})\cap\Omega}\exp\left|\alpha|2^{-i_j}u_{i_j}-\gamma|-\alpha\sum_{k\neq j}2^{-i_k}u_{i_k}\right|^s=\heartsuit.\end{aligned}$$

We employed here the elementary inequality:

$$\left|a+\sum_{n=1}^{\infty} b_n\right| \geq \left||a|-\sum_{n=1}^{\infty} b_n\right| \qquad \text{for } a \in \mathbb{R} \text{ and } b_n \geq 0.$$

Obviously

$$|2^{-i_j} u_{i_j} - \gamma| \geq 2^{-i_j} \frac{\tilde{r}_{i_j} - \tilde{\tilde{r}}_{i_j}}{8|A(\tilde{r}_{i_j}, \tilde{\tilde{r}}_{i_j}) \cap \Omega|^{1/n}}$$

on at least one of the sets

$$B(x_{i_j}, \tilde{\tilde{r}}_{i_j}) \cap \Omega \qquad \text{and} \qquad A(r_{i_j}, \tilde{r}_{i_j}) \cap \Omega.$$

Since each of the sets has measure greater than or equal to $|B(x_{i_j}, r_{i_j}) \cap \Omega|/4$, (21) and (22) yield

$$\alpha |2^{-i_j} u_{i_j} - \gamma| - \alpha \sum_{k \neq j} 2^{-i_k} u_{i_k} \geq \alpha 2^{-i_j} \frac{\tilde{r}_{i_j} - \tilde{\tilde{r}}_{i_j}}{16|A(\tilde{r}_{i_j}, \tilde{\tilde{r}}_{i_j}) \cap \Omega|^{1/n}}$$

on a subset of $B(x_{i_j}, r_{i_j}) \cap \Omega$ that has measure no less than $|B(x_{i_j}, r_{i_j}) \cap \Omega|/8$. Thus (23) and (18) imply

$$\begin{aligned}
\heartsuit \quad &\geq \quad \frac{1}{8}|B(x_{i_j}, r_{i_j}) \cap \Omega| \exp\left(\alpha 2^{-i_j} \frac{\tilde{r}_{i_j} - \tilde{\tilde{r}}_{i_j}}{16|A(\tilde{r}_{i_j}, \tilde{\tilde{r}}_{i_j}) \cap \Omega|^{1/n}}\right)^s \\
&\geq \quad |B(x_{i_j}, r_{i_j}) \cap \Omega| \exp\left(4^{-i_j} \frac{\tilde{r}_{i_j} - \tilde{\tilde{r}}_{i_j}}{4|A(\tilde{r}_{i_j}, \tilde{\tilde{r}}_{i_j}) \cap \Omega|^{1/n}}\right)^s > (M+\omega_n) r_{i_j}^n,
\end{aligned}$$

provided $i_j \geq i_0$, which is a contradiction with the left-hand side on the above sequence of inequalities. The proof of the lemma is complete.

Lemma 14 *There exist constants $c_1 > 0$ and $c_2 > \omega_n$ depending on Ω, M and s such that for every $x \in \Omega$ and $0 < r \leq 1$ we have*

$$\tilde{r} - \tilde{\tilde{r}} \leq c_1 |B(x, \tilde{r}) \cap \Omega|^{1/n} \left(\ln\left(\frac{c_2 r^n}{|B(x, \tilde{r}) \cap \Omega|}\right)\right)^{1/s}. \tag{24}$$

Remark. Condition $c_2 > \omega_n$ is used to guarantee that the logarithm in (24) is positive.

Proof of the lemma. According to Lemma 13

$$\inf_{\gamma \in \mathbb{R}} \int_{B(x,r) \cap \Omega} \exp(\beta |u - \gamma|)^s \leq (M + \omega_n) r^n, \tag{25}$$

where u is defined by (15). For every $\gamma \in \mathbb{R}$

$$|u - \gamma| \geq \frac{\tilde{r} - \tilde{\tilde{r}}}{8|A(\tilde{r}, \tilde{\tilde{r}}) \cap \Omega|^{1/n}}$$

for all points in the set $B(x,\tilde{\tilde{r}}) \cap \Omega$ or for all points in the set $A(r,\tilde{r}) \cap \Omega$. Since each of these sets has measure greater than or equal to $|B(x,\tilde{r}) \cap \Omega|/2$, inequality (25) yields

$$\frac{1}{2}|B(x,\tilde{r}) \cap \Omega| \exp\left(\frac{\beta}{8}\frac{\tilde{r}-\tilde{\tilde{r}}}{|A(\tilde{r},\tilde{\tilde{r}}) \cap \Omega|^{1/n}}\right)^s \leq (M+\omega_n)r^n.$$

Hence

$$\tilde{r}-\tilde{\tilde{r}} \leq \frac{8}{\beta}|A(\tilde{r},\tilde{\tilde{r}}) \cap \Omega|^{1/n}\left(\ln\left(\frac{2(M+\omega_n)r^n}{|B(x,\tilde{r}) \cap \Omega|}\right)\right)^{1/s}$$

and now (14) and (13) yield the desired estimate. The proof of the lemma is complete.

Lemma 15 *If the measure density condition (3) holds for all $x \in \Omega$ and all $r \leq 1$ such that $r \leq 10\tilde{r}$, where $\tilde{r}$ is defined by (13), then (3) holds for all $x \in \Omega$ and all $r \leq 1$.*[5]

Proof. Let $r \leq 1$. If $\Omega \subset B(x,r)$, then

$$|B(x,r) \cap \Omega| \geq |\Omega| \geq |\Omega| r^n$$

and hence (3) is satisfied. If $r \leq 10\tilde{r}$, then (3) is also satisfied. Thus we may assume that $\Omega \setminus B(x,r) \neq \emptyset$ and that $r > 10\tilde{r}$. Take $x' \in B(x,r) \cap \Omega$ such that $|x-x'| = \tilde{r}+r/5$. Such an x' exists because $\Omega \setminus B(x,r) \neq \emptyset$ and Ω is connected. Let $R = 2\tilde{r}+r/5$. We have

$$B(x,\tilde{r}) \subset B(x',R) \subset B(x,r)$$

and

$$B(x',R/2) \subset B(x',r/5) \subset A(r,\tilde{r}).$$

Hence $B(x,\tilde{r})$ and $B(x',R/2)$ are disjoint subsets of $B(x',R)$ and thus

$$\begin{aligned}
|B(x',R/2) \cap \Omega| &\leq \frac{1}{2}(|A(r,\tilde{r}) \cap \Omega| + |B(x',R/2) \cap \Omega|) \\
&= \frac{1}{2}(|B(x,\tilde{r}) \cap \Omega| + |B(x',R/2) \cap \Omega|) \\
&\leq \frac{1}{2}|B(x',R) \cap \Omega|.
\end{aligned}$$

This, in turn, implies that $\tilde{R} \geq R/2$, and so the measure density condition is satisfied by the ball $B(x',R)$ and hence

$$|B(x,r) \cap \Omega| \geq |B(x',R) \cap \Omega| \geq CR^n \geq 5^{-n}Cr^n.$$

The proof of the lemma is complete.

[5]Perhaps with a different constant C.

Now we are ready to complete the proof of the theorem. We need to prove (3) for all $x \in \Omega$ and all $0 < r \leq 1$. According to Lemma 15 we may assume that $r \leq 10\tilde{r}$. Define a sequence by setting

$$r_0 = r, \qquad r_{j+1} = \tilde{r}_j.$$

Lemma 14 yields

$$r_{j+1} - r_{j+2} \leq c_1 |B(x, r_{j+1}) \cap \Omega|^{1/n} \left(\ln \left(\frac{c_2 r_j^n}{|B(x, r_{j+1}) \cap \Omega|} \right) \right)^{1/s}.$$

Since

$$|B(x, r_{j+1}) \cap \Omega| = 2^{-j} |B(x, \tilde{r}) \cap \Omega|, \tag{26}$$

we conclude

$$r_{j+1} - r_{j+2} \leq c_1 2^{-j/n} |B(x, \tilde{r}) \cap \Omega|^{1/n} \left(\ln \left(\frac{c_2 2^j r_j^n}{|B(x, \tilde{r}) \cap \Omega|} \right) \right)^{1/s}.$$

It follows from (26) that $r_j \to 0$ as $j \to \infty$, and hence

$$\tilde{r} = \sum_{j=0}^{\infty} (r_{j+1} - r_{j+2}) \leq c_1 |B(x, \tilde{r}) \cap \Omega|^{1/n} \sum_{j=0}^{\infty} 2^{-j/n} \left(\ln \left(\frac{c_2 2^j r^n}{|B(x, \tilde{r}) \cap \Omega|} \right) \right)^{1/s}.$$

The sum on the right-hand side is bounded (up to a constant factor depending on s only) by

$$\sum_{j=0}^{\infty} 2^{-j/n} j^{1/s} (\ln 2)^{1/s} + \left(\sum_{j=0}^{\infty} 2^{-j/n} \right) \left(\ln \left(\frac{c_2 r^n}{|B(x, \tilde{r}) \cap \Omega|} \right) \right)^{1/s}.$$

These sums converge to some constants depending on n and s only, and hence we obtain

$$\tilde{r} \leq c |B(x, \tilde{r}) \cap \Omega|^{1/n} \left(1 + \left(\ln \left(\frac{c_2 r^n}{|B(x, \tilde{r}) \cap \Omega|} \right) \right)^{1/s} \right). \tag{27}$$

Denote

$$|B(x, \tilde{r}) \cap \Omega| = \varepsilon \tilde{r}^n.$$

Since

$$|B(x, r) \cap \Omega| = 2 |B(x, \tilde{r}) \cap \Omega| = 2\varepsilon \tilde{r}^n \geq 2 \cdot 10^{-n} \varepsilon r^n,$$

it suffices to show that ε is bounded from below by some positive constant depending on Ω, M and s only. Inequality (27) gives

$$c\varepsilon^{1/n} \left(1 + \left(\ln(c_2 10^n \varepsilon^{-1}) \right)^{1/s} \right) \geq 1.$$

Now it suffices to observe that the expression on the left hand side converges to 0 if $\varepsilon \to 0$, and since it is bounded from below by a positive constant, ε must also be bounded from below by a positive constant. This ends the proof of the theorem.

3 $M^{1,p}$ spaces and the proof of Theorems 1 and 9

The proof of Theorem 4 employs Sobolev spaces on metric spaces introduced by Hajłasz [7] (cf. [9], [5]). If X is a metric space equipped with a Borel measure μ, then, for a measurable function u on X, we define $D(u)$ as the collection of all non-negative measurable functions g on X such that

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)), \text{ a.e.} \tag{28}$$

Then for $1 \leq p < \infty$ we define

$$M^{1,p}(X,d,\mu) = \{u \in L^p(\mu) : \ D(u) \cap L^p(\mu) \neq \emptyset\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{M^{1,p}} = \|u\|_p + \inf_{g \in D(u)} \|g\|_p.$$

If $X = \mathbb{R}^n$, $|\cdot|$ denotes the Euclidean metric, and $\mathcal{L}^n$ is the Lebesgue measure, then we have the following result.

Lemma 16 ([7]) *If $1 < p < \infty$, then $M^{1,p}(\mathbb{R}^n, |\cdot|, \mathcal{L}^n) = W^{1,p}(\mathbb{R}^n)$ in the sense that the spaces are equal as sets and the norms are equivalent.*

Lemma 17 ([9], Lemma 6) *If $\Omega \subset \mathbb{R}^n$ is an arbitrary domain and $1 \leq p < \infty$, then for $u \in M^{1,p}(\Omega, |\cdot|, \mathcal{L}^n)$, $\|u\|_{1,p;\Omega} \leq c(n)\|u\|_{M^{1,p}(\Omega)}$. In particular, $M^{1,p}(\Omega, |\cdot|, \mathcal{L}^n) \subset W^{1,p}(\Omega)$ is a linear subspace.*

Note that Lemma 17 holds for $p \geq 1$, while Lemma 16 is not true when $p = 1$, see [6, Example 3].

The two lemmas imply that if $1 < p < \infty$ and $u \in W^{1,p}(\mathbb{R}^n) = M^{1,p}(\mathbb{R}^n, |\cdot|, \mathcal{L}^n)$, then the restriction of u to Ω belongs to $M^{1,p}(\Omega, |\cdot|, \mathcal{L}^n)$. Thus, for an arbitrary domain Ω and $1 < p < n$, we have

$$\mathcal{T} : W^{1,p}(\mathbb{R}^n) = M^{1,p}(\mathbb{R}^n, |\cdot|, \mathcal{L}^n) \to M^{1,p}(\Omega, |\cdot|, \mathcal{L}^n) \subset W^{1,p}(\Omega). \tag{29}$$

Hence it is natural to consider bounded linear extension operators

$$\mathcal{E} : M^{1,p}(\Omega, |\cdot|, \mathcal{L}^n) \to M^{1,p}(\mathbb{R}^n, |\cdot|, \mathcal{L}^n). \tag{30}$$

We will prove the following result.

Theorem 18 *If $\Omega \subset \mathbb{R}^n$ is a domain that satisfies the measure density condition (3), then, for every $1 < p < \infty$, there is a bounded linear extension operator (30).*

Remarks. (1) Let us note that the theorem easily implies Theorem 4 by an obvious modification to the proof of Proposition 2.

(2) Theorem 18 shows that the space $M^{1,p}(\Omega, |\cdot|, \mathcal{L}^n)$ gives the exact description of the space of traces of functions in $W^{1,p}(\mathbb{R}^n)$, whenever $1 < p < \infty$ and the domain Ω satisfies the measure density condition.

(3) The proof of Theorem 18 relies heavily on the boundedness of the maximal operator in L^p, and hence this argument cannot be applied to the case $p = 1$. It turns out, however, that the theorem is also true for $p = 1$, but the proof is substantially more difficult, see the forthcoming paper [8]. Actually, Theorem 18 holds for all $1 \le p < \infty$ in the general setting of metric spaces, see [8]. We would like to emphasize that the space $M^{1,1}(\Omega, |\cdot|, \mathcal{L}^n)$ does not describe traces of $W^{1,1}(\mathbb{R}^n)$ functions because not every function in $W^{1,1}$ belongs to $M^{1,1}$. The case $p = 1$ of the theorem is still not enough for the positive answer to Question 1, but at least gives some hope that the answer could be in the positive.

In the proof of Theorem 18, we will not only prove the existence of an extension operator, but we will construct such an operator explicitly. Even more than that: we construct an extension operator for $M^{1,p}(F, |\cdot|, \mathcal{L}^n)$, where $F \subset \mathbb{R}^n$ is an arbitrary closed set that satisfies the measure density condition similar to (3)

$$|F \cap B(x,r)| \ge Cr^n \qquad \text{for all } x \in F \text{ and } 0 < r \le 1. \tag{31}$$

Note that, since the boundary of Ω in Theorem 18 has measure zero (by the Lebesgue differentiation theorem), the space $M^{1,p}(\Omega, |\cdot|, \mathcal{L}^n)$ is the same as the space $M^{1,p}(F, |\cdot|, \mathcal{L}^n)$, where $F = \overline{\Omega}$.

Actually it would be easier to prove the estimates on the extension operator if we could use (31) for all $r > 0$. To get such a condition we need a trick. Let

$$F_2 = \{x \in \mathbb{R}^n : \operatorname{dist}(x, F) \ge 2\} \qquad \text{and} \qquad F_1 = \{x \in \mathbb{R}^n : \operatorname{dist}(x, F_2) \le 1\}.$$

Lemma 19 *If a closed set F satisfies (31), then there is a constant $C > 0$ such that*

$$|(F \cup F_1) \cap B(x,r)| \ge Cr^n \tag{32}$$

for all $x \in F \cup F_1$ and all $r > 0$.

Proof. If $x \in F$ and $r \le 3$, then (32) is obvious. If $x \in F_1$ and $r \le 3$ then there is $y \in F_2$ with $|x - y| \le 1$. Since $B(y,1) \subset F_1$, it easily follows that $B(x,r) \cap F_1$ contains a ball of radius $r/6$ and hence (32) is satisfied. Now suppose that $x \in F \cup F_1$ and $r \ge 3$. The ball $B(x,r)$ contains a family of pairwise disjoint balls of radius 3, consisting of at least $C(n)r^n$ balls. Take one such a ball $B(a,3) \subset B(x,r)$ from the family. If $B(a,2) \cap F = \emptyset$, then $B(a,1) \subset F_1$ and hence

$$|(F \cup F_1) \cap B(a,3)| \ge |B(a,1)| = \omega_n.$$

If $y \in B(a,2) \cap F \neq \emptyset$, then $B(y,1) \subset B(a,3)$, and hence

$$|(F \cup F_1) \cap B(a,3)| \geq |B(y,1) \cap F| \geq C$$

by (31). Adding up the estimates over all balls from the disjointed family in $B(x,r)$, we arrive at (32). The proof is complete.

If u is a measurable function defined in F, then we extend it first to F_1 by 0 and then extend this new function from $F \cup F_1$ to a function in $\mathbb{R}^n$ using (32) for all $r > 0$. To construct the extension operator from $F \cup F_1$ we will need the Whitney decomposition of an open set into cubes and an associated partition of unity.

For a closed set $E \subset \mathbb{R}^n$, the open set $\mathbb{R}^n \setminus E$ has the *Whitney decomposition* into cubes $\mathbb{R}^n \setminus E = \bigcup_{i \in I} Q_i$, where all the cubes Q_i are dyadic and have pairwise disjoint interiors. There is also an associated Lipschitz partition of unity $\{\varphi_i\}_{i \in I}$, $0 \leq \varphi_i \leq 1$, so that the following properties are satisfied:

1. $\operatorname{dist}(2Q_i, E) \leq \operatorname{diam} 2Q_i \leq 4 \operatorname{dist}(2Q_i, E)$;
2. Every point of $\mathbb{R}^n \setminus E$ is covered by at most 4^n different cubes $2Q_i$;
3. For each $i \in I$, $\operatorname{supp} \varphi_i \subset 2Q_i \subset \mathbb{R}^n \setminus E$;
4. $\sum_{i \in I} \varphi_i(x) \equiv 1$ on $\mathbb{R}^n \setminus E$, and, for every $i \in I$, the Lipschitz constant of φ_i is bounded by $C(n)(\operatorname{diam} Q_i)^{-1}$.

Here $2Q$ denotes the cube with the same center as Q, with parallel sides, and with the diameter twice that of Q.

Actually, Whitney needed the above construction for the proof of the celebrated Whitney extension theorem, [29], [17], that we describe next in the simplest possible setting.

Let $F \subset \mathbb{R}^n$ be a closed set. Let $\{Q_i\}_{i \in I}$ and $\{\varphi_i\}_{i \in I}$ be the Whitney decomposition and the associated Lipschitz partition of unity constructed for $E = F \cup F_1$.

For each $i \in I$, let $a_i \in E$ be a closest point to Q_i, that is $r_i := \operatorname{dist}(a_i, Q_i) = \operatorname{dist}(Q_i, E)$. For a Lipschitz function u on F, we first define $\tilde{u}(x)$ to be equal to $u(x)$ for $x \in F$ and 0 for $x \in F_1$ and then we set

$$\mathcal{E}' u(x) = \begin{cases} \tilde{u}(x) & \text{for } x \in F \cup F_1, \\ \sum_{i \in I} \varphi_i(x) \tilde{u}(a_i) & \text{for } x \in \mathbb{R}^n \setminus (F \cup F_1). \end{cases} \tag{33}$$

Note that $\mathcal{E}' u(x) = 0$ for $x \in \mathbb{R}^n$ with $\operatorname{dist}(x, F) \geq 2$.

Lemma 20 *For an arbitrary closed set $F \subset \mathbb{R}^n$, $\mathcal{E}' : \operatorname{Lip}_\infty(F) \to \operatorname{Lip}_\infty(\mathbb{R}^n)$ is a bounded linear extension operator.*

Let us emphasize that in Lemma 20 we *do not* assume (31). The lemma is a special case of a well known theorem of Whitney. One can easily prove Lemma 20 using the proof of the more difficult Theorem 21 as a hint. The steps will be similar but the proof easier. We leave details to the reader as an exercise.

For a measurable function u defined on F, we cannot use (33) as u might not be defined at every point of F. A natural suggestion is to take averages of u instead of values of u at single points. This leads to the formula

$$\mathcal{E}^* u(x) = \begin{cases} \tilde{u}(x) & \text{for } x \in F \cup F_1, \\ \sum_{i\in I} \varphi_i(x)\tilde{u}_{B_i \cap (F\cup F_1)} & \text{for } x \in \mathbb{R}^n \setminus (F\cup F_1), \end{cases} \tag{34}$$

where $B_i = B(a_i, r_i)$. To be more precise, we need to assume that u is locally integrable with respect to the Lebesgue measure restricted to F and that the intersection of F with an arbitrary ball centered at F has positive measure, as otherwise $\tilde{u}_{B_i\cap(F\cup F_1)}$ would not make sense.

Now, if $\Omega \subset \mathbb{R}^n$ is an open set whose boundary has measure zero, $\mathcal{L}^n(\partial\Omega) = 0$, then measurable functions on Ω are equal a.e. to measurable functions on $F = \overline{\Omega}$, and hence (34) can be applied to measurable functions on Ω as well. In particular, it can be applied in domains satisfying the measure density condition.

For a measurable function g, defined on a closed set $F \subset \mathbb{R}^n$, we define the *Hardy–Littlewood maximal function* by

$$\mathcal{M}_F g(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap F} |g(z)|\, dz, \qquad \text{for } x \in \mathbb{R}^n.$$

If $F = \mathbb{R}^n$, then we simply write $\mathcal{M}g$ without using $\mathbb{R}^n$ as a subscript. By the classical theorem of Hardy–Littlewood, the maximal function is a bounded operator in L^p for $1 < p < \infty$, that is, $\|\mathcal{M}_F g\|_{p;\mathbb{R}^n} \leq C(n,p)\|g\|_{p;F}$, see [24].

We will prove the following result which is a slight improvement on Theorem 18.

Theorem 21 *If $F \subset \mathbb{R}^n$ is a closed set that satisfies the measure density condition (31), then, for $1 < p < \infty$, the operator $\mathcal{E}^* : M^{1,p}(F, |\cdot|, \mathcal{L}^n) \to M^{1,p}(\mathbb{R}^n, |\cdot|, \mathcal{L}^n)$ defined by (34) is bounded.*

Proof. First note that the function $\tilde{u}$ belongs to $M^{1,p}(F\cup F_1, |\cdot|, \mathcal{L}^n)$ and that the $M^{1,p}$ norm of $\tilde{u}$ on $F \cup F_1$ is bounded (up to a constant factor) by the $M^{1,p}$ norm of u on F. Indeed, it is easy to see that if $g \in D(u) \cap L^p(F)$, then the function h defined as $g(x) + |u(x)|$ for $x \in F$ and 0 for $x \in F_1$, belongs to $D(\tilde{u}) \cap L^p(F \cup F_1)$.

Therefore, replacing F by $F \cup F_1$, u by $\tilde{u}$ and applying Lemma 19, we may assume that the set F satisfies the condition

$$|F \cap B(x,r)| \geq Cr^n \qquad \text{for all } x \in F \text{ and all } r > 0. \tag{35}$$

The operator $\mathcal{E}^*$ is given by

$$\mathcal{E}^*u(x) = \begin{cases} u(x) & \text{for } x \in F, \\ \sum_{i\in I} \varphi_i(x) u_{B_i\cap F} & \text{for } x \in \mathbb{R}^n \setminus F, \end{cases}$$

where now the Whitney decomposition is applied to $\mathbb{R}^n \setminus F$.

It suffices to prove two facts. The first one is that $\mathcal{E}^*u \in L^p(\mathbb{R}^n)$ with

$$\|\mathcal{E}^*u\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{L^p(F)}. \tag{36}$$

The second one is that, for every $g \in D(u) \cap L^p(F)$, the inequality

$$|\mathcal{E}^*u(x) - \mathcal{E}^*u(y)| \le C|x-y|(h(x)+h(y)), \tag{37}$$

where

$$h(x) = \begin{cases} |u(x)| + \mathcal{M}_F g(x) & \text{for } x \in F, \\ \mathcal{M}_F g(x) & \text{for } x \in \mathbb{R}^n \setminus F, \end{cases}$$

holds for a.e. $x, y \in \mathbb{R}^n$. Then the claim will easily follow from the Hardy–Littlewood theorem. Before we proceed to prove the above two claims, let us introduce some notation and make some auxiliary observations. For $x \in \mathbb{R}^n \setminus F$ let $\overline{x} \in F$ be such that $|x - \overline{x}| = \operatorname{dist}(x, F)$. Let also $I_x = \{i \in I : x \in 2Q_i\}$. There is a constant C depending on n only such that

$$B_i \subset B(x, C|x-\overline{x}|) \qquad \text{for all } i \in I_x.$$

Write $B_x = B(x, C|x - \overline{x}|)$. Note that

$$⨍_{B_i\cap F} |u| \le C\mathcal{M}_F u(x) \qquad \text{for every } i \in I_x.$$

This follows from the estimate

$$⨍_{B_i\cap F} |u| \le \frac{|B_x|}{|B_i \cap F|}\left(\frac{1}{|B_x|}\int_{B_x\cap F} |u|\right) \le C\mathcal{M}_F u(x) \tag{38}$$

along with the estimate (35). Using the same argument, we can prove the inequality

$$⨍_{B_i\cap F} g(w)\, dw + ⨍_{B_x\cap F} g(z)\, dz \le C\mathcal{M}_F g(x) \tag{39}$$

for all $i \in I_x$.

Note also that if $i \notin I_x$, then $\varphi_i(x) = 0$, and hence

$$\mathcal{E}^*u(x) = \sum_{i\in I} \varphi_i(x) u_{B_i\cap F} = \sum_{i\in I_x} \varphi_i(x) u_{B_i\cap F}.$$

Now we can prove the above two claims. The first one is easy. Indeed, for $x \in \mathbb{R}^n \setminus F$, (38) yields

$$|\mathcal{E}^* u(x)| = \left| \sum_{i \in I_x} \varphi_i(x) u_{B_i \cap F} \right| \le \sum_{i \in I_x} \fint_{B_i \cap F} |u| \le C \mathcal{M}_F u(x), \tag{40}$$

because the number of elements in I_x is bounded by 4^n. Now (36) follows from the Hardy–Littlewood maximal theorem.

We split the proof of (37) into four cases.

Case 1: $x, y \in F$. We have

$$|\mathcal{E}^* u(x) - \mathcal{E}^* u(y)| = |u(x) - u(y)| \le |x - y| (\mathcal{M}_F g(x) + \mathcal{M}_F g(y)),$$

because $g(x) \le \mathcal{M}_F g(x)$ whenever x is a Lebesgue point of g.

Case 2: $x \in \mathbb{R}^n \setminus F$ and $y \in F$. Since $\sum_{i \in I_x} \varphi_i(x) = 1$, we have

$$\begin{aligned} |\mathcal{E}^* u(x) - u_{B_x \cap F}| &= \left| \sum_{i \in I_x} \varphi_i(x) (u_{B_i \cap F} - u_{B_x \cap F}) \right| \\ &\le \sum_{i \in I_x} \fint_{B_i \cap F} \fint_{B_x \cap F} |u(w) - u(z)| \, dw dz \\ &\le C |x - \overline{x}| \sum_{i \in I_x} \fint_{B_i \cap F} \fint_{B_x \cap F} (g(w) + g(z)) \, dw dz \\ &\le C |x - \overline{x}| \mathcal{M}_F g(x). \end{aligned} \tag{41}$$

The last inequality follows from (39). Hence

$$|\mathcal{E}^* u(x) - \mathcal{E}^* u(y)| = |\mathcal{E}^* u(x) - u(y)| \le |\mathcal{E}^* u(x) - u_{B_x \cap F}| + |u_{B_x \cap F} - u(y)| = A + B.$$

The first term A is estimated by (41) and, for the second term, we have

$$B \le \fint_{B_x \cap F} |u(z) - u(y)| \, dz \le \fint_{B_x \cap F} |z - y| (g(z) + g(y)) \, dz \le C |x - y| (g(y) + \mathcal{M}_F g(x)).$$

Here we used the observation that $|z - y| \le C |x - y|$ whenever $z \in B_x \cap F$ and again inequality (39).

Case 3: $x, y \in \mathbb{R}^n \setminus F$ and $|x - y| \ge \min\{\operatorname{dist}(x, F), \operatorname{dist}(y, F)\}$. The argument is very similar to that in the Case 2 and is left to the reader (we need to use both $\overline{x}$ and $\overline{y}$).

CASE 4: $x, y \in \mathbb{R}^n \setminus F$ and $|x-y| < \min\{\operatorname{dist}(x,F), \operatorname{dist}(y,F)\}$. Since $\sum_{i\in I_x\cup I_y}(\varphi_i(x)-\varphi_i(y)) = 0$, we have

$$\begin{aligned}
|\mathcal{E}^*u(x) - \mathcal{E}^*u(y)| &= \left|\sum_{i\in I_x\cup I_y}(\varphi_i(x)-\varphi_i(y))(u_{B_i\cap F} - u_{B_x\cap F})\right| \\
&\le C\sum_{i\in I_x\cup I_y}\frac{|x-y|}{|x-\overline{x}|}\fint_{B_i\cap F}\fint_{B_x\cap F}|u(w)-u(z)|\,dwdz \\
&\le C|x-y|\mathcal{M}_F g(x).
\end{aligned}$$

In the last but one inequality we employed the fact that all the functions φ_i, for $i \in I_x \cup I_y$, are Lipschitz continuous with the Lipschitz constant bounded by $C|x-\overline{x}|^{-1}$, and the proof of the last inequality follows from estimates very similar to those in (41). This completes the proof of the theorem.

Proof of Theorem 9. Conditions (1) and (2) are obviously equivalent. The equivalence between (2) and (3) follows from Theorem 1. The implication from (4) to (3) is obvious. To prove the implication from (2) to (5), note that (29) gives (5) as equality of sets and the equivalence of norms follows from the Banach open mapping theorem. The measure density condition follows from Theorem 5. Finally, the implication from (5) to (4) follows from Theorem 21 applied to $F = \overline{\Omega}$ (cf. the remark following (31)) along with Lemma 16. The proof is complete.

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