Bloch's theorem for mappings of bounded and finite distortion

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Abstract

We study the covering properties of mappings of bounded and exponentially integrable distortion on the unit ball. We extend the results of Eremenko [2] by proving Bloch-type theorems for mappings of exponentially integrable distortion. In the case of mappings of bounded distortion, we formulate and prove Bloch's theorem in its most natural form.

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1 Introduction

We call a mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ *a mapping of finite distortion* if it satisfies

$$|Df(x)|^n \leq K(x,f)J(x,f) \qquad \text{a.e.,} \tag{1.1}$$

where $1 \leq K(x,f) < \infty$, and if also $J(\cdot, f) \in L^1_{\mathrm{loc}}(\Omega)$. Here $|Df(x)|$ is the operator norm of the differential of f at x, and $J(x,f)$ the Jacobian determinant of $Df(x)$. When $K(x,f) \leq K < \infty$ a.e., f is called a K-quasiregular mapping, or a mapping of bounded distortion.

The systematic study of quasiregular mappings was started in the 1960s by Reshetnyak, who proved that non-constant quasiregular mappings are continuous, discrete and open. By now, a rich theory of quasiregular mappings has been developed, see the monographs [16] and [18]. This theory extends geometric function theory to higher dimensions.

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Recently, Reshetnyak's results mentioned above have been extended to include a more general class of mappings, see [4], [5], [6], [7], [8], [11]. In particular, continuity, discreteness and openness have been proved for non-constant mappings of finite distortion satisfying

$$\exp(\lambda K(\cdot, f)) \in L^1_{\mathrm{loc}}(\Omega) \tag{1.2}$$

for some $\lambda > 0$. In fact, a sharp Orlicz-condition for these properties is now known, see [8].

In this paper we study issues that are related to two classical results in function theory: the theorems of Picard and Bloch. Recall that Picard's theorem says that an entire, non-constant analytic function can omit at most one finite point, while Bloch's theorem says that an analytic function f on the unit disc with $|f'(0)| = 1$ univalently covers a ball of radius C, where C is a universal constant.

One of the main achievements in the theory of quasiregular mappings is the value distribution theory, mainly developed by Rickman (see [18], Chapters IV and V). This theory includes a version of Picard's theorem, as well as a deep example [17] showing that Picard's theorem is not true in dimension three in the same form as in the plane. Recently, by using a normal family method, Eremenko [2] showed that also Bloch's theorem admits a generalization to higher dimensions, so that there exists a version for K-quasiregular mappings on the unit ball, as well as for entire quasimeromorphic mappings. Eremenko's method of proof was indirect, and it did not give any quantitative bounds for the corresponding Bloch radii. Therefore, he asked for a direct quantitative proof. Such a proof was given in [14] in the case of entire quasimeromorphic mappings. Below we will give such a proof for mappings on the ball. Bloch's theorem in connection with quasiregular holomorphic mappings in several complex variables has also been studied, cf. [1] and the references therein.

Recently, Koskela and Onninen [9] established a path family inequality for mappings of exponentially integrable distortion, that allows one to generalize the techniques used in the value distribution theory of quasiregular mappings. However, the covering properties of mappings of exponentially integrable distortion are still not understood. In particular, it is not known if there exists a reasonable generalization of the Rickman-Picard theorem. Naturally, understanding the nature of Bloch's theorem also helps to understand the problem of omitted values better.

The main theorem of this paper reads as follows.

Theorem 1.1. *Let $f : B(0,1) \to \mathbb{R}^n$, $n \geq 2$, be a mapping of finite distortion. Moreover, assume that there exist constants $\lambda, K > 0$ so that*

$$\int_{B(0,1)} \exp(\lambda K(x,f))\, dx \leq K. \tag{1.3}$$

Then there exist $y \in \mathbb{R}^n$ and a constant $C > 0$, only depending on n, K and λ, so that

$$B(y, C \operatorname{diam} f(B(0,1/2))) \subset f(B(0,1)).$$

Here and below we are using normalization based on the diameter of the image of $B(0,1/2)$. This is natural since, unlike in the case of analytic functions, assuming $|Df(0)| = a$ does not give any information on the global behavior of a mapping of finite or bounded distortion f. Of course, the constant $C > 0$ in Theorem 1.1 depends on K; for every $\epsilon > 0$ there exists a $K(\epsilon)$-quasiconformal homeomorphism f on the unit ball, with $\operatorname{diam} f(B(0,1/2)) = 1$, so that $f(B(0,1))$ does not cover any balls of radius ϵ.

As a corollary we have a covering theorem for entire mappings of finite distortion.

Corollary 1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a non-constant mapping of finite distortion. Moreover, assume that there exist constants $\lambda, K > 0$ so that*

$$\liminf_{R \to \infty} |B(0,R)|^{-1} \int_{B(0,R)} \exp(\lambda K(x,f))\, dx \leq K. \tag{1.4}$$

Then $f(\mathbb{R}^n)$ contains balls of arbitrarily large radius.

Rickman's theorem [18], Chapter IV, Theorem 2.1 says that an entire non-constant K-quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ can omit at most $C(n,K)$ points. In [15] it is proved that this result remains valid when "K-quasiregular" is replaced by (1.4) and the assumption that the map should not grow too fast, i.e. the map should be of finite lower order. Earlier, Koskela and Onninen [9] proved that, under an assumption slightly weaker than (1.4), the set of omitted values is of zero conformal modulus. It is an important problem to find out if Rickman's theorem remains true under the assumptions of Corollary 1.2. In the plane this is the case, which can be seen by using factorization results, cf. [5], Theorem 11.9.2.

In Theorem 1.1 the conclusion is that a large ball is covered by the image of f. The complete version of Bloch's theorem would replace "covered" by "univalently covered". We do not know if such a result is true under the assumptions of Theorem 1.1. Below we give assumptions that guarantee the complete version. Again, in the plane the complete version remains valid, which follows from the factorization results mentioned above, and some simple distortion estimates.

For completeness, we first note that Bloch's theorem is true under the assumption that the branch set should be empty. This is a rather straightforward consequence of the injectivity theorem in [10]. Now let $f : B(0,1) \to \mathbb{R}^n$ be a continuous, discrete and open mapping. We denote by $\mathcal{B}_f$ the supremum of all radii $r > 0$ so that there exists a domain $U \subset B(0,1)$ so that the restriction of f to U is one-to-one and maps onto a ball $B(y,r)$.

Theorem 1.3. *Let $f : B(0,1) \to \mathbb{R}^n$, $n \geq 2$, be a local homeomorphism satisfying the assumptions of Theorem 1.1. Then $\mathcal{B}_f \geq C \operatorname{diam} f(B(0,1/2))$, where $C > 0$ only depends on n, K and λ.*

Bloch's theorem also remains valid under a boundedness condition.

Theorem 1.4. *Let $f : B(0,1) \to \mathbb{R}^n$, $n \geq 2$, be a mapping satisfying the assumptions of Theorem 1.1. Moreover, assume that*

$$\operatorname{diam} f(B(0,1)) \leq \alpha \operatorname{diam} f(B(0,1/2)). \tag{1.5}$$

Then $\mathcal{B}_f \geq C \operatorname{diam} f(B(0,1/2))$, where $C > 0$ only depends on n, K, λ and α.

In the case of quasiregular mappings one can reduce the general situation to the situation where assumption (1.5) holds true. Therefore, a quantitative version of Eremenko's theorem mentioned above follows from Theorem 1.4

Theorem 1.5. *Let $f : B(0,1) \to \mathbb{R}^n$, $n \geq 2$, be a K-quasiregular mapping. Then $\mathcal{B}_f \geq C \operatorname{diam} f(B(0,1/2))$, where $C > 0$ only depends on n and K.*

We have not calculated any estimates for C here, since the expressions are quite complicated, compare [14]. Finally, we note that some of the results of this paper remain valid when the exponential integrability assumption is replaced by a slightly weaker subexponential integrability assumption, and that this assumption is the weakest possible in a certain sense, see [8]. We have chosen the exponential integrability assumption here in order to have simpler estimates.

2 Weighted conformal modulus

We will use methods that originate from the value distribution theory of quasiregular mappings, see [18], Chapter IV. In order to apply these methods in the case of unbounded distortion we use the modulus inequality established in [9], as well as estimates similar to ones given in [9] and [15]. In these estimates the familiar conformal modulus is replaced by a suitable weighted modulus, and the corresponding estimates are more complicated than the classical ones. In order to prove the existence of a univalent covering, we use a method established in [14].

Let Γ be a path family in a domain Ω. We call a Borel function $\rho : \Omega \to [0, \infty]$ admissible for Γ, if

$$\int_\gamma \rho \, ds \geq 1 \qquad \text{for all locally rectifiable } \gamma \in \Gamma.$$

Now let $\omega : \Omega \to [0, \infty]$ be a measurable function. The weighted p-modulus $M_{p,\omega}(\Gamma)$ of Γ is defined by

$$M_{p,\omega}(\Gamma) = \inf \Big\{ \int_{\mathbb{R}^n} \rho^p(x) \, \omega(x) \, dx : \; \rho : \Omega \to [0, \infty) \text{ is admissible for } \Gamma \Big\}.$$

When $\omega = 1$, we recover the usual p-modulus M_p. Also, $M_{n,\omega}$ is called the conformal modulus (with weight w), and we will denote it simply by M_ω.

When $A \subset \mathbb{R}^n$ is a Borel set and $f : A \to \mathbb{R}^n$ a mapping, we use the notation $N(y, f, A) = \text{card}\{x \in A : f(x) = y\}$. In what follows, we will denote by Γ_f the family of all locally rectifiable paths in A having a closed subpath on which f is not absolutely continuous. The so-called K_O-inequality for mappings of finite distortion will be used.

Theorem 2.1 ([15], Theorem 2.1). *Let $f : \Omega \to \mathbb{R}^n$ be a mapping of finite distortion satisfying* (1.2). *Let $A \subset \Omega$ be a Borel set with*

$$\sup_{y \in \mathbb{R}^n} N(y, f, A) < \infty.$$

Moreover, let Γ be a family of paths in A. If a function ρ is admissible for $f(\Gamma \setminus \Gamma_f)$, then

$$M_{K^{-1}(\cdot, f)}(\Gamma \setminus \Gamma_f) \leq \int_{\mathbb{R}^n} \rho^n(y) N(y, f, A) \, dy.$$

Moreover, $M_p(\Gamma_f) = 0$ for all $1 < p < n$.

We need a lower bound for the $K^{-1}(\cdot,f)$-modulus of certain path families. The following theorem extends a standard conformal modulus estimate (cf. [19], Theorem 10.12) to the current setting. A result like this was proved in [15], Theorem 2.5, and the next result is proved by using the same method. However, here we need estimates that are more general than the one given in [15]. Also see [3] and [12] for related estimates.

Theorem 2.2. *Let $f : B(0,1) \to \mathbb{R}^n$ be as in Theorem 1.1. Let E and F be two sets in $B(0,1)$. Assume that there exists a point $a \in B(0,1)$ so that $S^{n-1}(a,t)$ intersects both E and F for all $0 < r < t < R$. If $B(a,R) \subset B(0,1)$ and if Γ is the family of all paths joining E and F in $B(a,R) \setminus \overline{B}(a,r)$, then there exist constants $C_1, C_2 > 0$, only depending on n, so that*

$$M_{K^{-1}(\cdot,f)}(\Gamma \setminus \Gamma_f) \geq \mathcal{L}(R, R/r), \tag{2.1}$$

where

$$\mathcal{L}(R,R/r) = \begin{cases} \frac{C_1 \lambda \log\frac{R}{r}}{\log(C_2 K (R-r)^{-1} R^{1-n})}, & \log\frac{R}{r} \leq 1000\log(C_2 K R^{-n}), \\ C_1 \lambda \log\left(\frac{\log\frac{R}{r}}{\log(C_2 K R^{-n})}\right), & \log\frac{R}{r} > 1000\log(C_2 K R^{-n}). \end{cases}$$

In what follows, we will frequently use the notation $\mathcal{L}(R,R/r)$. This function also depends on n, K and λ, but we will consider them fixed.

Proof. In this proof $C > 0$ will be a constant that only depends on n but may vary from line to line. We first assume that $R < 10r$. Let ρ be an admissible function for $M_{K^{-1}(\cdot,f)}(\Gamma \setminus \Gamma_f)$, so that $\rho^n K^{-1}(\cdot,f)$ is integrable. Moreover, fix $p \in (n-1,n)$. As in the proof of [15], Theorem 2.5, we see that, for almost every $t \in (r,R)$,

$$1 \leq C t^{p+1-n} \int_{S^{n-1}(a,t)} \rho^p(x)\, dS(x) \tag{2.2}$$

(Lemma 2.3 of [12] gets used here). Here dS means integration against the surface measure. By integrating over t, we have

$$(R-r)\,\mathrm{essinf}_{t\in(r,R)} \int_{S^{n-1}(a,t)} \rho^p(x)\, dS(x) \leq \int_{B(a,R)\setminus B(a,r)} \rho^p(x)\, dx. \tag{2.3}$$

We denote $E = B(a,R) \setminus B(a,r)$. Then $|E|$ is bounded from above and below by dimensional constants times $(R-r)R^{n-1}$. Combining (2.2) and (2.3) gives

$$1 \leq Cr\Big(|E|^{-1} \int_E \rho^p(x)\, dx\Big)^{\frac{1}{p}}. \tag{2.4}$$

By writing $\rho^p(x) = \rho^p(x)K^{-p/n}(x, f)K^{p/n}(x, f)$ and using Hölder's inequality, we see that the right hand side of (2.4) is smaller than

$$Cr\Big(|E|^{-1}\int_E \rho^n(x)K^{-1}(x,f)\,dx\Big)^{\frac{1}{n}}\Big(|E|^{-1}\int_E K^{\frac{p}{n-p}}(x,f)\,dx\Big)^{\frac{n-p}{np}}. \tag{2.5}$$

In order to estimate the second integral in (2.5), we notice that the function $t \mapsto \exp(\lambda t^{(n-p)/p})$ is convex for large enough t. Since we may, if necessary, assume that $K(\cdot, f)$ is bounded from below by a fixed constant, we may use Jensen's inequality. Thus we have

$$\begin{aligned}\Big(|E|^{-1}\int_E K^{\frac{p}{n-p}}(x,f)\,dx\Big)^{\frac{n-p}{np}} &\leq \Big(\lambda^{-1}\log\Big(|E|^{-1}\int_E \exp(\lambda K(x,f))\,dx\Big)\Big)^{\frac{1}{n}}\\ &\leq \Big(\lambda^{-1}\log(|E|^{-1}K)\Big)^{\frac{1}{n}} \qquad (2.6)\end{aligned}$$

Combining (2.4), (2.5) and (2.6) gives

$$\begin{aligned}M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f) &\geq C\lambda\frac{R-r}{r}\log^{-1}(C^{-1}K(R-r)^{-1}R^{1-n})\\ &\geq \frac{C\lambda\log\frac{R}{r}}{\log(C^{-1}K(R-r)^{-1}R^{1-n})}. \qquad (2.7)\end{aligned}$$

Now we assume that $R > 10r$. Denote by k the largest integer smaller than $\log\frac{R}{r}$. Then

$$M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f) \geq \sum_{i=1}^{k} M_{K^{-1}(\cdot,f)}(\Gamma_i\setminus\Gamma_f), \tag{2.8}$$

where Γ_i is the family of all paths joining E and F in $B(a, e^{-i+1}R) \setminus B(a, e^{-i}R)$. Hence, by (2.7) and (2.8),

$$M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f) \geq C\lambda\sum_{i=1}^{k}\frac{1}{i+\log(C^{-1}KR^{-n})}. \tag{2.9}$$

When $k \leq \log\frac{R}{r} \leq 1000\log(C^{-1}KR^{-n})$, the right hand term of (2.9) is larger than

$$\frac{C\lambda\log\frac{R}{r}}{\log(C^{-1}KR^{-n})}.$$

On the other hand, if we assume that $\log\frac{R}{r} > 1000\log(C^{-1}KR^{-n})$, and denote by L the smallest integer larger than $\log(C^{-1}KR^{-n})$, we have

$$M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f) \geq C\lambda\sum_{L}^{k} i^{-1} \geq C\lambda\log\frac{k}{L} \geq C\lambda\log\Big(\frac{\log\frac{R}{r}}{\log(C^{-1}KR^{-n})}\Big).$$

The proof is complete.

$\square$

We will also use comparison results for the counting function. These results are proved by using the modulus estimates of [9], and the methods used in the case of quasiregular mappings, see [18], Chapter IV. In what follows, the following term will repeateadly appear:

$$C_3\lambda^{1-n}\log^{1-n}\Big(\frac{\log(C_4Kr^{-n})}{\log(C_4KR^{-n})}\Big) =: \mathcal{U}(R,R/r). \tag{2.10}$$

If we assume that $R \geq 2r$, then C_3, $C_4 > 0$ only depend on n. In what follows, we will frequently use the notation $\mathcal{U}(R,R/r)$. This term is an upper bound for the $K^{n-1}(\cdot,f)$-modulus of a spherical ring, see [9], Theorem 5.3.

For a mapping of finite distortion $f : \Omega \to \mathbb{R}^n$ and a Borel set $E \subset \Omega$, define the counting function $n(E,y)$ by

$$n(E,y) = \sum_{x\in f^{-1}(y)\cap E} i(x,f),$$

where $i(x,f)$ is the local index, see [18], Chapter I, Section 4. For an $(n-1)$-dimensional sphere $S^{n-1}(y,t) \subset \mathbb{R}^n$, define the average $\nu(E,y,t)$ of the counting function $n(E,\cdot)$ over the sphere $S^{n-1}(y,t)$ by

$$\nu(E,y,t) = \frac{1}{\omega_{n-1}}\int_{S^{n-1}} n(E,y+tx)\,dx,$$

where S^{n-1} is the unit sphere and ω_{n-1} its surface measure. We shall use the notation $\nu(a,r,y,t)$ if $E = B(a,r)$. The following comparison results for the counting function are proved in [15], Lemmas 2.6 and 2.7.

Lemma 2.3. *Let $f : B(0,1) \to \mathbb{R}^n$ be as in Theorem 1.1. Suppose that $B(a,\theta r) \subset\subset B(0,1)$, $\theta \geq 2$. Moreover, let $y \in \mathbb{R}^n$ and $s,t > 0$. Then*

$$\nu(a,\theta r,y,s) \geq \nu(a,r,y,t) - \omega_{n-1}^{-1}\mathcal{U}(\theta r,\theta)|\log(t/s)|^{n-1}.$$

Lemma 2.4. *Let $f : B(0,1) \to \mathbb{R}^n$ be as in Theorem 1.1. Moreover, let E and F be two disjoint continua in $\overline{B}(a,r)$ so that $f(E) \subset B(z,s)$ and $f(F) \subset \mathbb{R}^n \setminus B(z,t)$, $s < t$. Let $\theta \geq 2$, and suppose $B(a,\theta r) \subset\subset B(0,1)$. Then*

$$\nu(a,\theta r,z,t) \geq \omega_{n-1}^{-1}\Big(\log\frac{t}{s}\Big)^{n-1}\Big(M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f) - \mathcal{U}(\theta r,\theta)\Big),$$

where Γ is the family of all paths joining E and F in $B(a,r)$.

We will also use the following simple lemma.

Lemma 2.5. *Let $f : \overline{B}(a,r) \to \mathbb{R}^n$ be a continuous map. Then, for each $M \in \mathbb{N}$ there exists a ball $B(b, r/M) \subset B(a,r)$ so that*

$$\operatorname{diam} f(B(b,r/M)) \geq \operatorname{diam} f(B(a,r))/M.$$

Proof. Let $x_1, x_2 \in \overline{B}(a,r)$ be points so that

$$\operatorname{diam} f(B(a,r)) = |f(x_1) - f(x_2)|.$$

Moreover, let I be a line segment with endpoints x_1 and x_2. Since $\mathrm{l}(I)$, the length of I, does not exceed $2r$, I can be covered by M balls $B_j \subset B(a,r)$ of radius r/M. Then

$$\operatorname{diam} f(B(a,r)) \leq \mathrm{l}(f(I)) \leq \sum_j \operatorname{diam} f(B_j) \leq M \max_j \operatorname{diam} f(B_j).$$

□

3 Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. Without loss of generality, $\operatorname{diam} f(B(0,1/2)) = 1$. By considering, if necessary, $B(0, 1-\epsilon)$ with small $\epsilon > 0$ instead of $B(0,1)$, we may assume that f is bounded. We fix positive numbers

$$B = \exp((2\omega_{n-1}C_3)^{1/(n-1)}\lambda^{-1}), \quad B' = \exp((2C_3)^{1/(n-1)}\lambda^{-1}),$$

$$b = \exp((C_1\lambda)^{-1}) \quad \text{and} \quad k = 2(BB' - 1)(1 + nb) + 2nb.$$

Here and in what follows, the constants C_1, C_2, C_3 and C_4 are as in Theorem 2.2 and (2.10).

Next we define a continuous function

$$\rho : B(0,1) \to \mathbb{R}^n, \quad \rho(x) = (1 - |x|)^k \operatorname{diam} f(B(x, (1-|x|)/2)).$$

Then, since $\operatorname{diam} f(B(0,1/2)) = 1$ and since f is bounded,

$$1 \leq \sup_{x \in B(0,1)} \rho(x) \leq L < \infty.$$

We choose $x \in B(0,1)$ so that $\rho(x) \geq L/2$, and denote

$$(1 - |x|)/2 = R.$$

Then we have

$$(3.1)\qquad (2R)^{-k} \leq \operatorname{diam} f(B(x,3R/2)) \leq C(n,k) \operatorname{diam} f(B(x,R)).$$

Now we choose numbers $M_1 > M_2 > M_3 > 10$ as follows:

$$M_3 = (C_4 K R^{-n})^{\frac{B-1}{n}} 2^B, \quad M_2 = (C_4 K R^{-n})^{\frac{B'-1}{n}} M_3^{B'}$$

and

$$M_1 = (C_2 K R^{-n})^b M_2^{1+nb}.$$

Then

$$(3.2)\qquad M_1 = C(n,K,\lambda) R^{\frac{-k}{2}}.$$

Moreover, without loss of generality we may assume that M_1 is an integer. The choices of M_i:s are made in order to have the following:

$$(3.3)\qquad \begin{aligned} \mathcal{U}(M_3^{-1}R, M_2/M_3) &= \frac{1}{2} \\ \mathcal{U}(2^{-1}R, M_3/2) &= \frac{1}{2\omega_{n-1}}, \\ \mathcal{L}(M_2^{-1}R, M_1/M_2) &\geq 1. \end{aligned}$$

By Lemma 2.5, there exists a ball $B(y, M_1^{-1}R) \subset B(x,R)$ so that

$$(3.4)\qquad \operatorname{diam} f(B(y, M_1^{-1}R)) \geq \operatorname{diam} f(B(x,R))/M_1.$$

We denote by r the largest radius so that $U = U(y,f,r)$, the y-component of $f^{-1}(B(f(y),r))$, lies inside $B(y, M_2^{-1}R)$. Then, by [18], I Lemma 4.7,

$$f(U(y,f,r)) = B(f(y),r).$$

Our goal is to give a lower bound for r. By our choice of M_1, (3.1) and (3.4),

$$T =: \operatorname{diam} f(B(y, M_1^{-1}R))/2 \geq C(n,K,\lambda),$$

and thus we may assume that $r < T$. Now there exists a point

$$z \in S^{n-1}(y, M_1^{-1}R)$$

so that

$$|f(z) - f(y)| \geq T.$$

Hence there exists a ray γ connecting $f(z)$ and ∞ outside $B(f(y),T)$, and a lift γ' of γ connecting z and $S^{n-1}(0,1)$. Then, in particular, both U and γ' intersect $S^{n-1}(y,t)$ for all $t\in(M_1^{-1}R, M_2^{-1}R)$. Hence, if we denote by Γ the family of all paths joining U and γ' in $B(y,M_2^{-1}R)\setminus B(y,M_1^{-1}R)$, Theorem 2.2 and (3.3) yield

$$M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f)\geq \mathcal{L}(M_2^{-1}R, M_1/M_2)\geq 1. \tag{3.5}$$

On the other hand, by Lemma 2.4,

$$\begin{aligned} M_{K^{-1}(\cdot,f)}(\Gamma\setminus\Gamma_f) &\leq \omega_{n-1}\nu(y,M_3^{-1}R,f(y),T)\Big(\log\frac{T}{r}\Big)^{1-n}\\ &+\mathcal{U}(M_3^{-1}R, M_2/M_3). \end{aligned} \tag{3.6}$$

Since $\mathcal{U}(M_3^{-1}R, M_2/M_3)=1/2$ by (3.3), combining (3.5) and (3.6) gives

$$\Big(\log\frac{T}{r}\Big)^{n-1}\leq 2\omega_{n-1}\nu(y,M_3^{-1}R,f(y),T). \tag{3.7}$$

By applying Lemma 2.3, we have

$$\begin{aligned} \nu(y,M_3^{-1}R,f(y),T) &\leq \nu(y,R/2,f(y),\operatorname{diam} f(B(y,R/2)))\\ &+\mathcal{U}(R/2,M_3/2)\Big(\log\frac{\operatorname{diam} f(B(y,R/2))}{T}\Big)^{n-1}. \end{aligned} \tag{3.8}$$

We have

$$\nu(y,R/2,f(y),\operatorname{diam} f(B(y,R/2)))=0,$$

and by (3.1), (3.3) and (3.4),

$$\mathcal{U}(R/2,M_3/2)\Big(\log\frac{\operatorname{diam} f(B(y,R/2))}{T}\Big)^{n-1}\leq (2\omega_{n-1})^{-1}\log^{n-1}(M_1C(n,K,\lambda)).$$

Hence,

$$\Big(\log\frac{T}{r}\Big)^{n-1}\leq \log^{n-1}(M_1C(n,K,\lambda)),$$

which yields, when combined with (3.1), (3.2) and (3.4),

$$r\geq C(n,K,\lambda)M_1^{-2}R^{-k}\geq C(n,K,\lambda).$$

The proof is complete. □

Proof of Corollary 1.2. By (1.4), we find a sequence of radii R_i so that $R_i\to\infty$, and so that the map

$$f_i: B(0,1)\to\mathbb{R}^n,\quad f_i(x)=f(R_ix),$$

satisfies the assumptions of Theorem 1.1, with K and λ independent of i. By [9], Corollary 1.3, $\operatorname{diam} f(B(0,R_i/2))\to\infty$ as $R_i\to\infty$. Applying Theorem 1.1 to each f_i gives the claim. □

4 Proof of Theorem 1.3

As noted in the introduction, this result in the case $n = 2$ easily follows from known results. Hence it suffices to consider the case $n \geq 3$. Without loss of generality, $\operatorname{diam} f(B(0, 1/2)) = 1$. Now, by [10], there exists an integer $C_1^{-1} > 0$, only depending on n, K and λ, so that f restricted to $B(a, C_1)$ is one-to-one for all $a \in B(0, 1/2)$. Now, by Lemma 2.5, there exists $y \in B(0, 1/2)$ so that $\operatorname{diam} f(B(y, C_1/2)) \geq C_1$ and so that the restriction of f to $B(y, C_1)$ is one-to-one.

Denote by Γ the family of all paths joining

$$(f_{|B(y,C_1)})^{-1}(\mathbb{R}^n \setminus B(f(y), R))$$

and

$$(f_{|B(y,C_1)})^{-1}(B(f(y), r))$$

in $B(y, C_1)$, where R and r are the smallest and largest radii, respectively, so that

$$B(f(y), r) \subset f(B(y, C_1/2)) \subset B(f(y), R).$$

By Theorem 2.2 and [10], Lemma 3.1, we have

$$M_{K^{-1}(\cdot,f)}(\Gamma \setminus \Gamma_f) \geq \mathcal{L}(\sqrt{3}C_1/4, \sqrt{3}).$$

On the other hand, since $R \geq \operatorname{diam} f(B(y, C_1/2))/2$, we have

$$Mf(\Gamma) \leq \omega_{n-1} \log^{1-n} \frac{\operatorname{diam} f(B(y, C_1/2))}{2r} \leq \omega_{n-1} \log^{1-n} \frac{C_1}{2r}.$$

Since $\mathcal{B}_f \geq r$ and $M_{K^{-1}(\cdot,f)}(\Gamma \setminus \Gamma_f) \leq Mf(\Gamma)$ by Theorem 2.1, we have

$$\mathcal{B}_f \geq \frac{C_1}{2} \exp(-(\mathcal{L}(\sqrt{3}C_1/4, \sqrt{3})/\omega_{n-1})^{1/(1-n)}).$$

The proof is complete.

5 Proofs of Theorem 1.4 and Theorem 1.5

Proof of Theorem 1.4. The first part of the proof is similar to the proof of Theorem 1.1. However, since the parameters have to be chosen in a different way here, we will give full details.

Without loss of generality, $\operatorname{diam} f(B(0, 1/2)) = 1$. We fix an integer $M > 2$, to be chosen later. By Lemma 2.5, there exists a ball $B(y, M^{-1}) \subset B(0, 1/2)$ so that $\operatorname{diam} f(B(y, M^{-1})) \geq 2M^{-1}$.

Next, we denote by r the largest radius so that the y-component $U = U(y,f,r)$ of $f^{-1}(B(f(y),r))$ lies inside $B(y,2^{-4})$. We will give a lower bound for r. First, we may assume that $r < 1/M$. Since $\operatorname{diam} f(B(y,M^{-1})) \geq 2M^{-1}$, there exists a point $z \in S^{n-1}(y,M^{-1})$ so that $|f(z)-f(y)| \geq M^{-1}$. Then, for a ray γ connecting $f(z)$ and ∞ outside $B(f(y),M^{-1})$, there exists a lift γ' connecting z and $S^{n-1}(0,1)$. Then, in particular, both U and γ' intersect $S^{n-1}(y,t)$ for all $t \in (M^{-1},2^{-4})$. Hence, if we denote by Γ the family of all paths joining U and γ' in $B(y,2^{-4}) \setminus B(y,M^{-1})$, Theorem 2.2 yields

$$M_{K^{-1}(\cdot,f)}(\Gamma \setminus \Gamma_f) \geq \mathcal{L}(2^{-4},2^{-4}M). \tag{5.1}$$

On the other hand, by Lemma 2.4,

$$M_{K^{-1}(\cdot,f)}(\Gamma \setminus \Gamma_f) \leq \omega_{n-1}\nu(y,2^{-3},f(y),M^{-1})\Big(\log\frac{1}{Mr}\Big)^{1-n} + \mathcal{U}(2^{-3},2). \tag{5.2}$$

Now we choose M to be large enough, so that

$$\mathcal{U}(2^{-3},2) < \frac{1}{2}\mathcal{L}(2^{-4},2^{-4}M).$$

Then, combining (5.1) and (5.2) yields

$$\Big(\log\frac{1}{Mr}\Big)^{n-1} \leq 2\omega_{n-1}\mathcal{L}(2^{-4},2^{-4}M)^{-1}\nu(y,2^{-3},f(y),M^{-1}). \tag{5.3}$$

Since

$$\sup_{x\in B(y,2^{-3})}|f(x)-f(y)| \leq \sup_{x\in B(0,1)}|f(x)-f(y)| \leq \alpha,$$

Lemma 2.3 yields

$$\nu(y,2^{-3},f(y),M^{-1}) \leq \nu(y,2^{-2},f(y),\alpha) + \omega_{n-1}^{-1}\mathcal{U}(2^{-2},2)\log^{n-1}(\alpha M). \tag{5.4}$$

Since $\nu(y,2^{-2},f(y),\alpha)=0$, (5.3) and (5.4) give

$$\Big(\log\frac{1}{Mr}\Big)^{n-1} \leq 2\mathcal{L}(2^{-4},2^{-4}M)^{-1}\mathcal{U}(2^{-2},2)\log^{n-1}(\alpha M) =: A^{n-1},$$

i.e.

$$r \geq M^{-1}\exp(-A) =: r_0. \tag{5.5}$$

We conclude that $U(y,f,r_0)$ is a normal domain. Now, when we apply Lemma 2.3 again for $r_0/2$ and α, we have

$$\nu(y,2^{-4},f(y),r_0/2) \leq \nu(y,2^{-3},f(y),\alpha) + \omega_{n-1}^{-1}\mathcal{U}(2^{-3},2)\Big(\log\frac{2\alpha}{r_0}\Big)^{n-1}.$$

Again, since $\nu(y, 2^{-3}, f(y), \alpha) = 0$, we conclude that there exists a point $p \in B(f(y), r_0)$ so that

$$\mu(p, f, U(y, f, r_0)) \leq \omega_{n-1}^{-1} \mathcal{U}(2^{-3}, 2) \Big(\log \frac{2\alpha}{r_0} \Big)^{n-1} =: m_0.$$

Here μ denotes the topological degree, see [18], Chapter I, Section 4. Since the topological degree is constant in the normal domain $U(y, f, r_0)$, we have

$$\mu(f, U(y, f, r_0)) \leq m_0. \tag{5.6}$$

In order to establish a lower bound for $\mathcal{B}_f$ by using (5.5) and (5.6), we proceed inductively. The method we are going to use allows us to calculate an explicit expression for the lower bound. However, since the calculations that are needed are rather complicated, we will not worry about explicit constants. The essential point is that they only depend on n, K, λ and α. We will need the following topological result from [13]. A mapping is proper if the preimage of an arbitrary compact set is compact.

Lemma 5.1 ([13], Theorem 2). *Let G be an open, connected, relatively compact subset of $\mathbb{R}^n$ and Y an n-dimensional manifold (possibly with boundary). If $f : \overline{G} \mapsto Y$ is a continuous, proper, finite-to-one open mapping which is not a homeomorphism,*

$$f^{-1}(\partial Y) \subseteq (\overline{G} \setminus G), \qquad D = \max\{d(x, \overline{G} \setminus G) : x \in G\},$$

and

$$C = \max\{\operatorname{diam}(f^{-1}(f(x))) : x \in \overline{G} \setminus G\},$$

then $D \leq C$.

We assume that there exist $y_k \in B(y, 2^{-4})$ and r_k, m_k, only depending on n, K, λ, α and k, so that $1 \leq m_k < m_{k-1} < \ldots < m_0$, $U(y_k, f, r_k) \subset B(y_0, 2^{-3})$ and

$$\mu(f, U(y_k, f, r_k)) \leq m_k. \tag{5.7}$$

The assumption for $k = 0$ holds true by (5.5) and (5.6). In order to finish the proof of Theorem 1.4, it suffices to show the existence of r_{k+1} and m_{k+1}; since $m_{k+1} < m_k < \ldots < m_0$, after $L \leq m_0$ steps we have $\mu(f, U(y_L, f, r_L)) = 1$. This means that $f_{|U(y_L, f, r_L)}$ is one-to-one and onto $B(f(y_L), r_L)$.

By Lemma 5.1, we find a point $p \in S^{n-1}(f(y_k), r_k/2)$, so that

$$\operatorname{diam} f^{-1}(p) \cap \partial U(y_k, f, r_k/2) \geq D = \max_{x \in U(y_k, f, r_k/2)} \mathrm{d}(x, \partial U(y_k, f, r_k/2)).$$

By [9], Corollary 5.5, we have the continuity estimate

$$r_k/2 = |f(z) - f(y_k)| \le \varphi(n, K, \lambda, \alpha, |z - y_k|)$$

for all $z \in \partial U(y_k, f, r_k/2)$, where $\varphi(n, K, \lambda, \alpha, t) \to 0$ as $t \to 0$. Hence we have

$$D \ge C_{k+1}, \tag{5.8}$$

where $C_{k+1} > 0$ only depends on n, K, λ, α and r_k. Now fix a radius $r < r_k/2$, and consider

$$V_{k+1} = f^{-1}(B(p, r)) \cap U(y_k, f, r_k).$$

Then, by (5.7), we know that if V_{k+1} consists of more than one component, then we can choose one of them, U_{k+1}, so that $\mu(f, U_{k+1}) \le m_{k+1} < m_k$. Hence, in order to complete the proof it suffices to show that V_{k+1} consists of more than one component when $r < C$ and $C > 0$ only depends on n, K, λ, α and r_k.

Assume that V_{k+1} consists of one component. Then, by (5.8), $\operatorname{diam} V_{k+1} \ge C_{k+1}$. Consider the family Γ of all paths joining V_{k+1} and $\partial U(y_k, f, r_k)$. Then $f\Gamma \subset \Gamma'$, where Γ' is the family of all paths joining $B(p, r)$ and $B(y_k, r_k)$. By Theorem 2.1 and (5.7),

$$M_{K^{-1}(\cdot, f)}(\Gamma \setminus \Gamma_f) \le m_k M(\Gamma'). \tag{5.9}$$

Moreover, by Theorem 2.2,

$$M_{K^{-1}(\cdot, f)}(\Gamma \setminus \Gamma_f) \ge \mathcal{L}(2^{-3}, 2^{-3} C_{k+1}^{-1}), \tag{5.10}$$

while

$$M(\Gamma') \le \omega_{n-1} \Big(\log \frac{r_k}{2r} \Big)^{1-n}. \tag{5.11}$$

By combining (5.9), (5.10) and (5.11), we have

$$r \ge C(n, K, \lambda, \alpha, r_k),$$

and we may choose $r_{k+1} = C(n, K, \lambda, \alpha, r_k)/2$, and any $y_{k+1} \in f^{-1}(p) \cap U(y_k, f, r_k)$. The proof is complete. $\square$

Proof of Theorem 1.5. Without loss of generality, $\operatorname{diam} f(B(0,1/2)) = 1$. By considering, if necessary, $B(0,1-\epsilon)$ for small $\epsilon > 0$ instead of $B(0,1)$, we may assume that f is bounded. We define

$$\rho : B(0,1) \to [0,\infty), \quad \rho(x) = \operatorname{diam} f(B(x,(1-|x|)/2)).$$

Then, since $\operatorname{diam} f(B(0,1/2)) = 1$, and since f is bounded,

$$1 \leq \sup_{x \in B(0,1)} \rho(x) =: L < \infty.$$

We choose $x_0 \in B(0,1)$ so that $\rho(x_0) \geq L/2$. Now $B(x_0, 3(1-|x_0|)/4) \subset B(0,1)$, and

$$\operatorname{diam} f(B(x_0, 3(1-|x_0|)/4)) \leq CL,$$

where $C > 0$ only depends on n. By Lemma 2.5, we find a ball $B(y,(1-|x_0|)/8) \subset B(x_0,(1-|x_0|)/2)$ so that

$$A_1 \leq \operatorname{diam} f(B(y,(1-|x_0|)/4)) \leq A_2 \operatorname{diam} f(B(y,(1-|x_0|)/8)),$$

where $A_1, A_2 > 0$ only depend on n. Now

$$g : B(0,1) \to \mathbb{R}^n, \quad g(x) = f(y + (1-|x_0|)x/4))$$

is a K-quasiregular map, $\operatorname{diam} g(B(0,1/2)) \geq A_1/A_2 \operatorname{diam} f(B(0,1/2)) = A_1/A_2$,

$$\operatorname{diam} g(B(0,1)) / \operatorname{diam} g(B(0,1/2)) \leq A_2$$

and $\mathcal{B}_g \leq \mathcal{B}_f$. Hence the claim follows from Theorem 1.4. ☐

Remark 5.2.

1. The proofs of Theorems 1.4 and 1.5 show that the K-quasiregularity assumption in Theorem 1.5 can be relaxed by requiring that
$$|B|^{-1} \int_B \exp(\lambda K(x,f))\, dx \leq K$$
for all balls $B \subset B(0,1)$ whose radius equals the distance of B to $S^{n-1}(0,1)$.

2. The method used to prove univalency from the information that one has a normal domain with bounded topological degree does not give all that sharp estimates. In particular, in the case of unbounded distortion one would like to have a simpler criterion for univalency, especially when trying to prove Theorem 1.5 in the setting of Theorem 1.1.

References

[1] Chen, H., Gauthier, P. M.: *Bloch constants in several variables*, Trans. Amer. Math. Soc., 353 (2001), no. 4, 1371–1386.

[2] Eremenko, A.: *Bloch radius, normal families and quasiregular mappings*, Proc. Amer. Math. Soc., 128 (2000), no. 2, 557–560.

[3] Herron, D., Koskela, P.: *Mappings of finite distortion: Gauge dimension of generalized quasicircles*, Illinois J. Math., 47 (2003), no. 4, 1243–1259.

[4] Iwaniec, T., Koskela, P., Onninen, J.: *Mappings of finite distortion: Monotonicity and continuity*, Invent. Math., 44 (2001), no. 3, 507–531.

[5] Iwaniec, T., Martin, G.: Geometric Function Theory and Nonlinear Analysis, Oxford Mathematical Monographs, 2001.

[6] Kauhanen, J., Koskela, P., Malý, J.: *Mappings of finite distortion: Discreteness and openness*, Arch. Rational Mech. Anal., 160 (2001), 135–151.

[7] Kauhanen, J., Koskela, P., Malý, J.: *Mappings of finite distortion: Condition N*, Michigan Math. J., 9 (2001), no. 1, 169–181.

[8] Kauhanen, J., Koskela, P., Malý, J., Onninen, J., Zhong, X.: *Mappings of finite distortion: Sharp Orlicz-conditions*, Rev. Mat. Iberoamericana, 19 (2003), no. 3, 857–872.

[9] Koskela, P., Onninen, J.: *Mappings of finite distortion: Capacity and modulus inequalities*, J. Reine Angew. Math., to appear.

[10] Koskela, P., Onninen, J., Rajala, K.: *Mappings of finite distortion: injectivity radius of a local homeomorphism*, Future trends in Geometric Function Theory, 169–174, Rep. Univ. Jyväskylä Dep. Math. Stat., 92, Univ. of Jyväskylä, 2003.

[11] Manfredi, J., Villamor, E.: *An extension of Reshetnyak's theorem*, Indiana Univ. Math. J., 47 (1998), no. 3, 1131–1145.

[12] Manfredi, J., Villamor, E.: *Traces of monotone functions in weighted Sobolev spaces*, Illinois J. Math., 45 (2001), no. 2, 403–422.

[13] McAuley, L. F., Robinson, E. E.: *On Newman's theorem for finite-to-one open mappings on manifolds*, Proc. Amer. Math. Soc., 87 (1983), no. 3, 561–566.

[14] Rajala, K.: *A lower bound for the Bloch radius of K-quasiregular mappings*, Proc. Amer. Math. Soc., 132 (2004), no. 9, 2593–2601.

[15] Rajala, K.: *Mappings of finite distortion: the Rickman-Picard theorem for mappings of finite lower order*, J. Anal. Math., 94 (2004), 235–248.

[16] Reshetnyak, Yu. G.: Space Mappings with Bounded Distortion, Trans. of Mathematical Monographs, Amer. Math. Soc., vol. 73, 1989.

[17] Rickman, S.: *The analogue of Picard's theorem for quasiregular mappings in dimension three*, Acta Math., 154 (1985), no. 3–4, 195–242.

[18] Rickman, S.: Quasiregular mappings, Springer-Verlag, 1993.

[19] Väisälä, J.: Lectures on n-dimensional quasiconformal mappings, Springer-Verlag, 1971.

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