Curvature integral and Lipschitz parametrization in 1-regular metric spaces

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Abstract. We show that for a bounded 1-regular metric measure space (E, μ) the finiteness of the Menger curvature integral

$$\int_E \int_E \int_E c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3$$

guarantees that E is a Lipschitz image of a subset of a bounded subinterval of $\mathbb{R}$.

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1 Introduction

Let z_1, z_2 and z_3 be three points in a metric space (E, d). The *Menger curvature* of the triple (z_1, z_2, z_3) is

$$c(z_1, z_2, z_3) = \frac{2 \sin \sphericalangle z_1 z_2 z_3}{d(z_1, z_3)},$$

where

$$\sphericalangle z_1 z_2 z_3 = \arccos \frac{d(z_1, z_2)^2 + d(z_2, z_3)^2 - d(z_1, z_3)^2}{2d(z_1, z_2)d(z_2, z_3)}.$$

Note that $c(z_1, z_2, z_3)$ is the reciprocal of the radius of the circle passing through x_1, x_2 and x_3 whenever $\{x_1, x_2, x_3\} \subset \mathbb{R}^2$ is an isometric triple for $\{z_1, z_2, z_3\}$. We set

$$c^2(E) = \int_E \int_E \int_E c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3.$$

Through the paper μ is the 1-dimensional Hausdorff measure on E.

We say that a metric space (E, d) is 1-*regular* if there exists $M_0 < \infty$ such that

$$M_0^{-1} r \leq \mu(B(x, r)) \leq M_0 r \tag{1}$$

whenever $x \in E$ and $r \in]0, d(E)]$. Here $d(E)$ is the diameter of E and $B(x, r)$ will denote the closed ball in E with center $x \in E$ and radius $r > 0$. The smallest constant M_0 such that (1) holds is called the *regularity constant of* E. We denote

$$\ell(E) = \inf\{ \operatorname{Lip}(f) \, : \, f : A \to E \text{ is a surjection and } A \subset [0, 1] \}, \tag{2}$$

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where $\mathrm{Lip}(f) \in [0,\infty]$ is the Lipschitz constant of f. Note that if E is a subset of a Hilbert space H, then by the classical Kirszbraun-Valentine extension theorem we can take in (2) the infimum over all functions $f : [0,1] \to H$ for which $E \subset f([0,1])$ without that $\ell(E)$ changes. Further, if E is a connected metric space, $\ell(E)$ is at most a constant multiple of $\mu(E)$ (see [11] and [3]). In this paper we shall prove the following theorem:

Theorem 1.1. *Let (E,d) be a 1-regular metric space. Then $\ell(E) \leq C(c^2(E) + d(E))$, where $C < \infty$ depends only on the regularity constant of E.*

In [4] P. W. Jones gave a sufficient and necessary condition for $E \subset \mathbb{C}$ to be contained in a rectifiable curve by showing that

(i) $\ell(E) \leq C_1 \left(d(E) + \sum_{Q\in\mathcal{D}} \beta_E(Q)^2 d(Q) \right)$,

(ii) $\sum_{Q\in\mathcal{D}} \beta_E(Q)^2 d(Q) \leq C_2 \ell(E)$,

where C_1 and C_2 are some absolute constants, $\mathcal{D} = \{3Q : Q \text{ is a dyadic cube}\}$ and

$$\beta_E(Q) = \inf_L d(Q)^{-1} \sup \{ d(y,L) \, : \, y \in E \cap Q \}$$

for $Q \in \mathcal{D}$, where the infimum is taken over all lines. Here $3Q$ is the cube with the same center as Q and sides parallel to the sides of Q, but whose diameter is $3d(Q)$. Jones' proof for (i) works also if $E \subset \mathbb{R}^n$. The latter part has been extended to sets in $\mathbb{R}^n$ by Okikiolu in [8]. Then, of course, the constant C_2 must depend on n. In [11] Schul extended this theorem to sets in a Hilbert space H using the family

$$\{ \{ y \in H \, : \, d(y,x) \leq A2^{-k} \} \, : \, x \in \Delta_k, \, k \in \mathbb{Z} \}$$

in the place of $\mathcal{D}$. Here A is some fixed constant and $(\Delta_k)_k$ is a *net* for E, that is, Δ_k is a maximal subset of E such that $d(x_1,x_2) > 2^{-k}$ for any distinct points $x_1, x_2 \in \Delta_k$ and $\Delta_k \subset \Delta_{k+1}$ for all $k \in \mathbb{Z}$. The easier part of Jones' theorem has an extension also for general metric spaces. In [3] we showed that there is an absolute constant C such that $\ell(E) \leq C(d(E) + \beta(E))$ for any metric space E, where

$$\beta(E) = \inf \left\{ \sum_{k\in\mathbb{Z}} \sum_{x\in\Delta_k\setminus\Delta_{k-1}} \beta(x,2^{-k})^2 (2^{-k})^3 \, : \, (\Delta_k)_k \text{ is a net for } E \right\}$$

and $\beta(x,t) = \sup \{ c(z_1,z_2,z_3) \, : \, z_1,z_2,z_3 \in B(x,At), \, d(z_i,z_j) \geq t \; \forall i \neq j \}$ for $x \in E$ and $t > 0$, where A is some sufficiently large constant. An example given by Schul shows that there is not any absolute constant C such that $\beta(E) \leq C\ell(E)$ for any metric space E. In fact, there exists a plane set E equipped with the ℓ^1 metric such that $\ell(E) < \infty$ and $\beta(E) = \infty$. The part (i) has extended also to the Heisenberg group in [2].

David and Semmes proved in [1] that a closed 1-regular set $E \subset \mathbb{R}^n$ is contained in a 1-regular curve if and only if there is $C < \infty$ such that

$$\int_0^R \int_{E\cap B(z,R)} \beta_q(x,t,E)^2 \, d\mu x \, \frac{dt}{t} \leq CR \tag{3}$$

for all $z \in E$ and $R > 0$. Here $q \in [1, \infty]$ is arbitrary,

$$\beta_q(x,t,E) = \inf_L \left(t^{-1-q} \int_{E \cap B(x,t)} d(y,L)^q \, d\mu y \right)^{1/q}$$

for $q \in [1, \infty[$ and

$$\beta_\infty(x,t,E) = \inf_L t^{-1} \sup \{ \, d(y,L) \, : \, y \in E \cap B(x,t) \, \} \, ,$$

where the infima are taken over all lines in $\mathbb{R}^n$. For $q = \infty$ this was already proved by Jones. In fact, David and Semmes gave in [1] a version of this theorem for m-dimensional sets in $\mathbb{R}^n$, where m is any integer. In [9] Pajot gave a more direct proof for that a closed 1-regular set $E \subset \mathbb{R}^n$ lies in a 1-regular curve if (3) is satisfied. His construction also yields

$$\ell(E) \le C \left(d(E) + \int_0^{d(E)} \int_E \beta_q(x,t,E)^2 \, d\mu x \, \frac{dt}{t} \right), \tag{4}$$

where $C < \infty$ depends only on the regularity constant of E. The basic idea of our proof for Theorem 1.1 is inspired by Pajot's algorithm, which is itself a kind of variant of Jones' one in [4].

Mattila, Melnikov and Verdera used Menger curvature in [7] for proving that the L^2 boundedness of the Cauchy integral operator associated to a closed 1-regular set $E \subset \mathbb{C}$ implies that E is contained in a 1-regular curve. The starting point of their work was the relation that for any three points $z_1, z_2, z_3 \in \mathbb{C}$

$$c(z_1, z_2, z_3)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}},$$

where σ runs through all six permutations of $\{1,2,3\}$. This implies that the Cauchy operator is bounded in $L^2(E)$ if and only if there is $C < \infty$ such that $c^2(E \cap B(z,R)) \le CR$ for all $z \in E$ and $R > 0$. They showed that for some constant λ depending only on the regularity constant of E

$$\int_0^R \int_{E \cap B(z,R)} \beta_2(x,t,E)^2 \, d\mu x \, \frac{dt}{t} \le \lambda c^2(E \cap B(z, \lambda R)) \tag{5}$$

for all $z \in E$ and $0 < R < d(E)/\lambda$. The claim now follows from the result of David and Semmes. Note that we get from (5) and (4) that a bounded 1-regular set $E \subset \mathbb{R}^n$ lies in a rectifiable curve if $c^2(E) < \infty$.

Jones has later proved that for a 1-regular set $E \subset \mathbb{C}$

$$\int_0^R \int_{E \cap B(z,R)} \beta_\infty(x,t,E)^2 \, d\mu x \, \frac{dt}{t} \le Cc^2(E \cap B(z, CR))$$

for all $z \in E$ and $R > 0$, where $C < \infty$ depends only on the regularity constant of E. For the proof see [10]. Using this we get also $\beta(E) \le Cc^2(E)$ for some $C < \infty$ depending only on the regularity constant of E whenever E is a 1-regular set in $\mathbb{C}$. We can easily construct an example which shows that this is not true for general

1-regular metric spaces. For example, let $\delta > 0$ and consider the plane set $E_\delta = ([0,1] \times \{0\}) \cup (\{0,1\} \times [0,\delta])$ equipped with the ℓ^1 metric. Then $c^2(E_\delta)/\beta(E_\delta) \to 0$ as $\delta \to 0$.

Any Borel set $E \subset \mathbb{R}^n$ with $\mu(E) < \infty$ and $c^2(E) < \infty$ is rectifiable in sense that there are rectifiable curves $\Gamma_1, \Gamma_2, \ldots$ such that

$$\mu\left(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right) = 0.$$

This was first proved by David. Léger gave in [5] a different proof which also gives a version for higher dimensional sets in $\mathbb{R}^n$.

For related results see also [6].

2 Preliminaries of the proof of Theorem 1.1

We assume that E is a bounded 1-regular metric space with regularity constant M_0 such that $c^2(E) < \infty$. Let C_1, C_2 and $\delta < 1$ be positive constants such that $C_1(1-\delta) > 4(2-\delta)$ and $C_2(1-\delta) > 8(1+2C_1)(2-\delta)$, and let r_2, r_4 and R_2 be small positive constants depending on C_2 and δ. Then, let $r_5 > 0$ be a small constant depending on C_2, δ, r_2, r_4 and R_2. We also let r_3 and R_3 be large positive constants depending on C_2, δ and M_0, and then we let $\varepsilon_0 < 1$ be a sufficiently large positive constant depending on C_1, C_2, δ, M_0 and r_5. Finally, let $r_0 > 0$ be a small constant depending on R_2 and ε_0, and let $r_1 > 0$ be a small number depending on most of the above constants. See more details later. For any $x \in E$ and $n \in \mathbb{Z}$ we choose a point $q_n(x) \in B(x, r_1\delta^n)$ such that

$$\mu(B(x,r_1\delta^n)) \int_{S_n(x)} c(z_1,z_2,q_n(x))^2 \, d\mu^2(z_1,z_2) \leq \int_{B(x,r_1\delta^n)} \int_{S_n(x)} c(z_1,z_2,z_3)^2 \, d\mu^2(z_1,z_2) \, d\mu z_3,$$

where $S_n(z) = \{(\zeta,\eta) \in (B(z,r_3\delta^n) \setminus B(z,r_2\delta^n))^2 \, : \, d(\zeta,\eta) > r_4\delta^n\}$ for $z \in E$. We also set

$$\vartheta(x,n) = \sup\{\, \varepsilon \in [0,1] \, : \, \{z_1,z_2,z_3\} \in \mathcal{O}(\varepsilon) \ \forall (z_1,z_2,z_3) \in W(x,n) \,\},$$

where

$$W(x,n) = \{\, (z_1,z_2,z_3) \in B(x,R_3\delta^n)^3 \, : \, d(z_i,z_j) > R_2\delta^n \ \forall i \neq j \,\}$$

and $\mathcal{O}(\varepsilon)$ is the set of the metric spaces E such that $d(x,z) \geq d(x,y) + \varepsilon d(y,z)$ whenever $x,y,z \in E$ such that $d(x,z) = d(\{x,y,z\})$. We say that $E' \subset E$ *has an order*, if there is an injection $o : E' \to \mathbb{R}$ such that for all $x,y,z \in E'$ the condition $o(x) < o(y) < o(z)$ implies $d(x,z) > \max\{d(x,y), d(y,z)\}$. In that case the function o is called an *order*. If there is an order o on $\{x_1,\ldots,x_n\} \subset E$, $n \in \mathbb{N}$, such that $o(x_i) < o(x_{i+1})$ for $i = 1,\ldots,n-1$, we write shortly $x_1x_2\ldots x_n$. The notation $x_1x_2x_3|\varepsilon$ means that $x_1x_2x_3$ and $\{x_1,x_2,x_3\} \in \mathcal{O}(\varepsilon)$.

Let $x_0 \in E$ and let n_0 be the biggest integer such that $E \subset B(x_0, \delta^{n_0})$. Set $D_0^{n_0} = \{q_{n_0}(x_0)\}$. Let now $n > n_0$ and assume by induction that we have constructed $D_0^{n-1} \subset E$ such that for any $x,y \in D_0^{n-1}$, $x \neq y$, $d(x,y) > \delta^n$. Let $A_n' \subset E$ such that

- for any $x, y \in A'_n$, $x \neq y$, $d(x,y) > \delta^n$,

- for any $x \in A'_n$, $y \in D_0^{n-1}$, $d(x,y) > \delta^n$,

- for any $x \in E$ there exists $y \in A'_n \cup D_0^{n-1}$ such that $d(x,y) \leq \delta^n$.

Now $\#A'_n \leq 2M_0\delta^{-n}\mu(E) \leq 2M_0^2\delta^{-n}d(E)$. We set $A_n = q_n(A'_n)$ and $D_0^n = A_n \cup q_n(D_0^{n-1})$. Let $A_n = \{x_1^n, \ldots, x_{\#A_n}^n\}$ such that

$$d\left(x_k^n, D_{k-1}^{n-1}\right) = \max\left\{ d\left(x, D_{k-1}^{n-1}\right) \,:\, x \in A_n \right\}$$

for $k = 1, \ldots, \#A_n$. Here and in the sequel we denote $D_k^{n-1} = D_0^{n-1} \cup \{x_1^n, \ldots, x_k^n\}$ for $k = 1, \ldots, \#A_n$. By choosing $\delta \leq 1 - 2r_1$ we have for all $n \geq n_0$

(i) for any $x, y \in D_0^n$, $x \neq y$, $d(x,y) > (1-2r_1)\delta^n$,

(ii) for any $x \in E$ there exists $y \in D_0^n$ such that $d(x,y) \leq (1+r_1)\delta^n$.

For $m \geq n > n_0$ and $x \in D_0^{n-1} \cup D_0^n$ we denote

$$q_{m,n}(x) = \begin{cases} q_m \circ q_{m-1} \circ \cdots \circ q_{n+1}(x) & \text{if } x \in D_0^n, \\ q_m \circ q_{m-1} \circ \cdots \circ q_n(x) & \text{if } x \in D_0^{n-1}. \end{cases}$$

Here we interpret $q_{n,n}(x) = x$ if $x \in D_0^n$. Note that $x = q_n(x)$ for $x \in D_0^{n-1} \cap D_0^n$. We also use the convention $q_{n-1,n}(x) = x$ for any x.

We are going to construct a sequence $(G_k^n)_{n > n_0, 0 \leq k \leq \# D_0^{n+1}}$ of connected weighted graphs with no cycles. We will denote by V_k^n and E_k^n the sets of the vertices and the edges of G_k^n. For each (n,k) we will have $D_k^n \subset V_k^n$. For all $x, y \in D_k^n$ such that $\{x,y\} \in E_k^n$ we will have $w_k^n(\{x,y\}) \geq d(x,y)$, where $w_k^n : E_k^n \to]0,\infty[$ is the weight function on the graph G_k^n. We denote $l(G_k^n) = \sum_{e \in E_k^n} w_k^n(e)$ and for $y \in D_k^n$ we will use the notations

$$\begin{aligned} V_k^n(y) &= \{ z \in V_k^n \,:\, \{y,z\} \in E_k^n \}, \\ D_k^n(y) &= V_k^n(y) \cap D_k^n. \end{aligned}$$

Each vertex in $V_k^n \setminus D_k^n$ will have only one neighbour. Thus the subgraph of G_k^n induced by D_k^n will also be connected. We will denote this graph and the set of its edges by T_k^n and F_k^n. For each (n,k) we will define a 1-Lipschitz surjection $f_k^n : I_k^n \to D_k^n$, where $I_k^n \subset [0, 2l(T_k^n)]$. Here $l(T_k^n) = \sum_{e \in F_k^n} w_k^n(e)$. If $e \in F_k^n$, we denote

$$J_k^n(e) = \{ (s_1, s_2) \in I_k^n \times I_k^n \,:\, s_1 < s_2,\ f_k^n(\{s_1, s_2\}) = e \text{ and } I_k^n \cap]s_1, s_2[= \emptyset \}.$$

Further we will define a function $P_k^n : D_k^n \to \{ V : V \subset \{\{x,y\} : x, y \in V_k^n, x \neq y\} \}$ such that the following properties will be satisfied:

- Let $y \in D_k^n$. If $e_1 \neq e_2$ and $e_1, e_2 \in P_k^n(y)$, then $e_1 \cap e_2 = \emptyset$. If $v \in V_k^n(y)$, then $v \in e$ for some $e \in P_k^n(y)$. If $\{v_1, v_2\} \in P_k^n(y)$, then $\{v_1, v_2\} \subset V_k^n(y)$ and $v_1 \neq v_2$.

- $\#\{e \in P_k^n(y) \,:\, e \subset D_k^n(y)\} \leq 1$ for all $y \in D_k^n$.

- Let $e \in F_k^n$. Then $1 \le \#J_k^n(e) \le 2$ and $s_2 - s_1 = w_k^n(e)$ for all $(s_1, s_2) \in J_k^n(e)$.

For $n > n_0$ and $k \in \{0, \dots, \#A_{n+1}\}$ also the following condition, called the (n,k)-*property*, will be satisfied:

> If $y \in D_k^n$, $\{z_1, z_2\} \in P_k^n(y)$, $\{z_1, z_2\} \subset D_k^n(y)$ and $\max\{d(y,z_1), d(y,z_2)\} < C_2(1+r_1)\delta^n$, then $q_{m_1,n}(z_1) q_{m,n}(y) q_{m_2,n}(z_2) | \varepsilon_0$ for any $m, m_1, m_2 \ge n-1$.

In Section 3 we define the graph G_k^{n-1} by deforming the graph G_{k-1}^{n-1}. The main point of the proof is to control $l(G_k^{n-1}) - l(G_{k-1}^{n-1})$ by some integral estimate. For this we need that the vertices are well chosen. Thus we at every stage n "update" the vertices by applying q_n to them. We do this in Section 4. In Section 5 we show that $l(T_0^m)$ is uniformly bounded by a constant multiple of $c^2(E) + d(E)$, from which we get the final conclusion.

We define a graph $G_1^{n_0}$ with 4 vertices and 3 edges as follows. Put $V_1^{n_0} = D_1^{n_0} \cup \{b_1, b_2\}$, where $\{b_1, b_2\} \cap E = \emptyset$, and set

$$E_1^{n_0} = \left\{ \{q_{n_0}(x_0), x_1^{n_0+1}\}, \{q_{n_0}(x_0), b_1\}, \{x_1^{n_0+1}, b_2\} \right\},$$

Further we define $w_1^{n_0}$ and $P_1^{n_0}$ by setting

$$\begin{aligned} &w_1^{n_0}(\{q_{n_0}(x_0), x_1^{n_0+1}\}) = d(q_{n_0}(x_0), x_1^{n_0+1}), \\ &w_1^{n_0}(\{q_{n_0}(x_0), b_1\}) = w_1^{n_0}(\{x_1^{n_0+1}, b_2\}) = C_1 d(q_{n_0}(x_0), x_1^{n_0+1}), \\ &P_1^{n_0}(q_{n_0}(x_0)) = \{\{x_1^{n_0+1}, b_1\}\}, \\ &P_1^{n_0}(x_1^{n_0+1}) = \{\{q_{n_0}(x_0), b_2\}\}. \end{aligned}$$

Now

$$l(G_1^{n_0}) \le (1 + 2C_1) d(E). \tag{6}$$

We set $I_1^{n_0} = \{0, d(q_{n_0}(x_0), x_1^{n_0+1})\}$ and define $f_1^{n_0} : I_1^{n_0} \to D_1^{n_0}$ by setting $f_1^{n_0}(0) = q_{n_0}(x_0)$ and $f_1^{n_0}(d(q_{n_0}(x_0), x_1^{n_0+1})) = x_1^{n_0+1}$. In the following two sections we assume that $n > n_0$.

3 Construction of $G_{\#A_n}^{n-1}$

Let now $k \in \{1, \dots, \#A_n\}$ and assume by induction that we have constructed a graph $G_{k-1}^{n-1} = (V_{k-1}^{n-1}, E_{k-1}^{n-1})$ with a weight function $w_{k-1}^{n-1} : E_{k-1}^{n-1} \to]0, \infty[$ and a 1-Lipschitz surjection $f_{k-1}^{n-1} : I_{k-1}^{n-1} \to D_{k-1}^{n-1}$, where $I_{k-1}^{n-1} \subset [0, 2l(T_{k-1}^{n-1})]$. We also assume that we have defined $P_{k-1}^{n-1} : D_{k-1}^{n-1} \to \left\{ V : V \subset \{\{x,y\} : x, y \in V_{k-1}^{n-1},\, x \ne y\} \right\}$ such that the $(n-1, k-1)$-property and the other conditions mentioned in the previous section are satisfied. We denote $x = x_k^n$. Let $y \in D_{k-1}^{n-1}$ such that $d(x,y) = d(x, D_{k-1}^{n-1})$.

Case 1. $\vartheta(x,n) < \varepsilon_0$.

We set $V_k^{n-1} = V_{k-1}^{n-1} \cup \{x, b_1, b_2\}$, where $b_1 \ne b_2$, $\{b_1, b_2\} \cap (V_{k-1}^{n-1} \cup E) = \emptyset$, and define

$$E_k^{n-1} = E_{k-1}^{n-1} \cup \{\{x,y\}, \{x, b_1\}, \{y, b_2\}\}.$$

Further we define w_k^{n-1} and P_k^{n-1} by setting

$$w_k^{n-1}(e) = \begin{cases} d(x,y) & \text{for } e = \{x,y\}, \\ C_1 d(x,y) & \text{for } e \in \{\{x,b_1\},\{y,b_2\}\}, \\ w_{k-1}^{n-1}(e) & \text{for } e \in E_{k-1}^{n-1} \end{cases}$$

and

$$P_k^{n-1}(v) = \begin{cases} \{\{y,b_1\}\} & \text{for } v = x, \\ P_{k-1}^{n-1}(v) \cup \{\{x,b_2\}\} & \text{for } v = y, \\ P_{k-1}^{n-1}(v) & \text{for } v \in D_{k-1}^{n-1} \setminus \{y\}. \end{cases}$$

Let $t \in I_{k-1}^{n-1}$ such that $f_{k-1}^{n-1}(t) = y$. We set

$$I_k^{n-1} = J_1 \cup \{t + d(x,y)\} \cup J_2,$$

where $J_1 = I_{k-1}^{n-1} \cap [0,t]$ and $J_2 = \big(I_{k-1}^{n-1} \cap [t,\infty[\big) + 2d(x,y)$, and define f_k^{n-1} by setting

$$f_k^{n-1}(s) = \begin{cases} f_{k-1}^{n-1}(s) & \text{for } s \in J_1, \\ x & \text{for } s = t + d(x,y), \\ f_{k-1}^{n-1}(s - 2d(x,y)) & \text{for } s \in J_2. \end{cases}$$

Now the $(n-1,k)$-property is satisfied, $I_k^{n-1} \subset [0, 2l(T_k^{n-1})]$ and f_k^{n-1} is surjective and 1-Lipschitz.

Let $(w_1,w_2,w_3) \in W(x,n)$ such that $\{w_1,w_2,w_3\} \notin \mathcal{O}(\varepsilon_0)$ and let $z_i \in B(w_i, r_0\delta^n)$ for $i = 1,2,3$. Denote $d_{ij} = d(w_i,w_j)$ and $d'_{ij} = d(z_i,z_j)$ for $i = 1,2,3$. Suppose that $d(z_1,z_3) = d(\{z_1,z_2,z_3\})$ and $d_{12} \geq d_{23}$. Then, by choosing r_0 small enough,

$$\begin{aligned} \frac{d'_{13} - d'_{12}}{d'_{23}} &\leq \frac{(d_{13} + 2r_0\delta^n) - (d_{12} - 2r_0\delta^n)}{d_{23} - 2r_0\delta^n} \leq \frac{d_{13} - d_{12} + 4r_0\delta^n}{(1 - 2r_0R_2^{-1})d_{23}} \\ &\leq \frac{R_2}{R_2 - 2r_0}\left(\varepsilon_0 + \frac{4r_0}{R_2}\right) = \frac{\varepsilon_0 R_2 + 4r_0}{R_2 - 2r_0} < 1. \end{aligned}$$

Letting $\alpha = \sphericalangle z_1 z_2 z_3$ we have

$$c(z_1,z_2,z_3)^2 = \frac{(2\sin\alpha)^2}{d(z_1,z_3)^2} \geq \frac{4(1-\cos^2\alpha)}{(2(R_3+r_0)\delta^n)^2} \geq \frac{1 - \max\{\varepsilon_5^2, 1/4\}}{((R_3+r_0)\delta^n)^2},$$

where

$$\varepsilon_5 = \frac{\varepsilon_0 R_2 + 4r_0}{R_2 - 2r_0}.$$

Using this and the regularity we get

$$\begin{aligned} &l(G_k^{n-1}) - l(G_{k-1}^{n-1}) = (1+2C_1)d(x,y) \leq (1+2C_1)(1+r_1)\delta^{n-1} = \frac{C_3\delta^{3n}r_0^3c_1}{M_0^3\delta^{2n}} \\ &\leq C_3 \int_{B(x,(R_3+r_0)\delta^n)} \int_{T_n^1(z_3)} \int_{T_n^1(z_3)\cap T_n^1(z_2)} c(z_1,z_2,z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3, \end{aligned} \tag{7}$$

where

$$c_1 = \frac{1 - \max\{\varepsilon_5^2, 1/4\}}{(R_3 + r_0)^2},$$
$$C_3 = \frac{M_0^3(1 + 2C_1)(1 + r_1)}{c_1 \delta r_0^3}$$

and $T_n^1(z) = B(z, 2(R_3 + r_0)\delta^n) \backslash B(z, (R_2 - 2r_0)\delta^n)$ for $z \in E$.

For the rest of the cases we assume that $\vartheta(x, n) \geq \varepsilon_0$.

Case 2. There exists $z \in D_{k-1}^{n-1}(y)$, $n' \leq n$, $k' \in \{1, \ldots, \#A_{n'}\}$ such that $k' \leq k$ if $n' = n$, $\{y', z'\} \in F_{k'-1}^{n'-1}$, $y = q_{n-1,n'}(y')$, $z = q_{n-1,n'}(z')$ and $C_2 d\left(x_{k'}^{n'}, \{y', z'\}\right) \leq d(y', z')$.

We define G_k^{n-1}, P_k^{n-1} and f_k^{n-1} as in Case 1. Now

$$l(G_k^{n-1}) - l(G_{k-1}^{n-1}) = (1 + 2C_1) d(x, y). \tag{8}$$

The construction will show that $\{q_{m,n}(y), q_{m,n}(z)\} \in F_0^m$ for all $m \geq n$.

For the rest of the cases we assume that the condition of Case 2 does not hold.
Case 3. There exists $z \in D_{k-1}^{n-1}(y)$ such that $d(x, z) \leq d(y, z)$.

We set $V_k^{n-1} = V_{k-1}^{n-1} \cup \{x\}$ and define

$$E_k^{n-1} = \left(E_{k-1}^{n-1} \backslash \{\{y, z\}\}\right) \cup \{\{y, x\}, \{x, z\}\}.$$

Further we define w_k^{n-1} by setting

$$w_k^{n-1}(e) = \begin{cases} d(y, x) & \text{for } e = \{y, x\}, \\ \max\left\{d(x, z), w_{k-1}^{n-1}(\{y, z\}) - d(y, x)\right\} & \text{for } e = \{x, z\}, \\ w_{k-1}^{n-1}(e) & \text{for } e \in E_{k-1}^{n-1} \backslash \{\{y, z\}\}. \end{cases}$$

Let $z', y' \in V_{k-1}^{n-1}$ such that $\{z', z\} \in P_{k-1}^{n-1}(y)$ and $\{y, y'\} \in P_{k-1}^{n-1}(z)$. We set

$$P_k^{n-1}(v) = \begin{cases} \{\{y, z\}\} & \text{for } v = x, \\ \left(P_{k-1}^{n-1}(v) \backslash \{\{z', z\}\}\right) \cup \{\{z', x\}\} & \text{for } v = y, \\ \left(P_{k-1}^{n-1}(v) \backslash \{\{y, y'\}\}\right) \cup \{\{x, y'\}\} & \text{for } v = z, \\ P_{k-1}^{n-1}(v) & \text{for } v \in D_{k-1}^{n-1} \backslash \{y, z\}. \end{cases}$$

Let $(t_1, t_2) \in J_{k-1}^{n-1}(\{y, z\})$. We set

$$I_{k,0}^{n-1} = J_1 \cup \{t_1 + w_k^{n-1}(\{f_{k-1}^{n-1}(t_1), x\})\} \cup J_2,$$

where $J_1 = I_{k-1}^{n-1} \cap [0, t_1]$ and $J_2 = \left(I_{k-1}^{n-1} \cap [t_2, \infty[\right) + l(G_k^{n-1}) - l(G_{k-1}^{n-1})$, and define $f_{k,0}^{n-1} : I_{k,0}^{n-1} \to D_k^{n-1}$ by setting

$$f_{k,0}^{n-1}(s) = \begin{cases} f_{k-1}^{n-1}(s) & \text{for } s \in J_1, \\ x & \text{for } s = t_1 + w_k^{n-1}(\{f_{k-1}^{n-1}(t_1), x\}), \\ f_{k-1}^{n-1}(s - l(G_k^{n-1}) + l(G_{k-1}^{n-1})) & \text{for } s \in J_2. \end{cases}$$

If $\#J^{n-1}_{k-1}(\{y,z\}) = 1$, we put $I^{n-1}_k = I^{n-1}_{k,0}$ and $f^{n-1}_k = f^{n-1}_{k,0}$. Else let $u_1, u_2 \in I^{n-1}_{k,0}$ such that $u_2 - u_1 = w^{n-1}_{k-1}(\{y,z\})$, $f^{n-1}_{k,0}(\{u_1,u_2\}) = \{y,z\}$ and $I^{n-1}_{k,0} \cap]u_1,u_2[= \emptyset$. We set

$$I^{n-1}_k = J_1 \cup \{u_1 + w^{n-1}_k(f^{n-1}_{k,0}(u_1), x)\} \cup J_2,$$

where $J_1 = I^{n-1}_{k,0} \cap [0,u_1]$ and $J_2 = \left(I^{n-1}_{k,0} \cap [u_2,\infty[\right) + l(G^{n-1}_k) - l(G^{n-1}_{k-1})$, and define f^{n-1}_k by setting

$$f^{n-1}_k(s) = \begin{cases} f^{n-1}_{k,0}(s) & \text{for } s \in J_1, \\ x & \text{for } s = u_1 + w^{n-1}_k(\{f^{n-1}_{k,0}(u_1), x\}), \\ f^{n-1}_{k,0}(s - l(G^{n-1}_k) + l(G^{n-1}_{k-1})) & \text{for } s \in J_2. \end{cases}$$

Now $I^{n-1}_k \subset [0, 2l(T^{n-1}_k)]$ and f^{n-1}_k is surjective and 1-Lipschitz.

We next show that the $(n-1,k)$-property is satisfied at z. Suppose that $\{z_1,z_2\} \in P^{n-1}_k(z)$ such that $\{z_1,z_2\} \subset D^{n-1}_k(z)$ and $\max\{d(z,z_1), d(z,z_2)\} < C_2(1+r_1)\delta^{n-1}$. If $x \notin \{z_1,z_2\}$, then $\{z_1,z_2\} \in P^{n-1}_{k-1}(z)$ and the $(n-1,k)$-property is satisfied at z by the $(n-1,k-1)$-property. Thus we may assume that $z_1 = x$, which implies $\{y,z_2\} \in P^{n-1}_{k-1}(z)$. Since $d(y,z) < C_2(1+r_1)\delta^{n-1}$, we have yzz_2 by the $(n-1,k-1)$-property. By choosing

$$R_2 \le 1 - \frac{2r_1}{1-\delta},$$
$$R_3 \ge \frac{(2C_2 - \varepsilon_0)(1+r_1)}{\delta} + \frac{r_1}{1-\delta}$$

we have $\{y, q_{m_1,n}(x), q_{m,n}(z), q_{m_2,n}(z_2)\} \in \mathcal{O}(\varepsilon_0)$ for any $m, m_1, m_2 \ge n-1$. Now $d(v_1,v_2) < Kd(v_3,v_4)$ for all $v_1,v_2,v_3,v_4 \in \{y,x,z,z_2\}$, $v_3 \ne v_4$, where

$$K = C_2\left(1 + \frac{1+r_1}{(1-2r_1)\delta}\right).$$

We choose $\varepsilon_0 \ge K/(K+1)$. Therefore, since yxz and yzz_2, $\{y,x,z,z_2\}$ has an order by Lemma 2.2 of [3]. So we must have xzz_2. Choosing $r_1 < \varepsilon_0(1-\delta-2r_1)$ the following lemma gives that the $(n-1,k)$-property is satisfied at z. Similarly we see that $(n-1,k)$ is satisfied at y and x.

Lemma 3.1. *Let $\{\zeta,\eta,\xi,\xi_1\} \subset E$ such that $\{\zeta,\eta,\xi\}, \{\zeta,\eta,\xi_1\} \in \mathcal{O}(\varepsilon_0)$.*

(i) *If $\zeta\eta\xi$ and $d(\xi,\xi_1) < \varepsilon_0 \min\{d(\zeta,\eta), d(\eta,\xi) + d(\eta,\xi_1)\}$, then $\zeta\eta\xi_1$.*

(ii) *If $\zeta\xi\eta$ and $d(\xi,\xi_1) < \varepsilon_0 \min\{d(\xi,\zeta) + d(\xi_1,\zeta), d(\xi,\eta) + d(\xi_1,\eta))\}$, then $\zeta\xi_1\eta$.*

Proof. (i) By the assumptions we have

$$\begin{aligned} &d(\zeta,\xi_1) + \varepsilon_0 d(\eta,\xi_1) - d(\zeta,\eta) \ge d(\zeta,\xi) - d(\xi,\xi_1) + \varepsilon_0 d(\eta,\xi_1) - d(\zeta,\eta) \\ &\ge d(\zeta,\eta) + \varepsilon_0 d(\eta,\xi) - d(\xi,\xi_1) + \varepsilon_0 d(\eta,\xi_1) - d(\zeta,\eta) \\ &= \varepsilon_0(d(\eta,\xi) + d(\eta,\xi_1)) - d(\xi,\xi_1) > 0 \end{aligned}$$

and

$$\begin{aligned}
&d(\zeta,\xi_1)+\varepsilon_0 d(\zeta,\eta)-d(\eta,\xi_1)\geq d(\zeta,\xi)+\varepsilon_0 d(\zeta,\eta)-d(\eta,\xi)-2d(\xi,\xi_1)\\
&\geq \varepsilon_0 d(\zeta,\eta)+d(\eta,\xi)+\varepsilon_0 d(\zeta,\eta)-d(\eta,\xi)-2d(\xi,\xi_1)\\
&=2\varepsilon_0 d(\zeta,\eta)-2d(\xi,\xi_1)>0.
\end{aligned}$$

Therefore, since $\{\zeta,\eta,\xi_1\}\in\mathcal{O}(\varepsilon_0)$, we must have $\zeta\eta\xi_1$.

(ii) Now the assumption gives

$$\begin{aligned}
&d(\zeta,\eta)+\varepsilon_0 d(\xi_1,\eta)-d(\zeta,\xi_1)\\
&\geq d(\zeta,\xi)+\varepsilon_0 d(\xi,\eta)+\varepsilon_0 d(\xi_1,\eta)-d(\zeta,\xi)-d(\xi,\xi_1)\\
&=\varepsilon_0(d(\xi,\eta)+d(\xi_1,\eta))-d(\xi,\xi_1)>0
\end{aligned}$$

and similarly $d(\zeta,\eta)+\varepsilon_0 d(\zeta,\xi_1)>d(\xi_1,\eta)$. □

Since $R_2\leq 1-2r_1$ and $\delta R_3\geq C_2(1+r_1)$, we have

$$l(G_k^{n-1})-l(G_{k-1}^{n-1})\leq d(y,x)+d(x,z)-d(y,z)\leq(1-\varepsilon_0)d(y,x). \tag{9}$$

Let us now assume that there is $m\geq n$ such that $\{\{q_{m,n}(y),q_{m,n}(x)\},\{q_{m,n}(x),q_{m,n}(z)\}\}\cap F_0^m=\emptyset$. By the construction (see also Case 4 and Section 4) this implies that there exist $y_1,w_1,x_1,x_2,w_2,z_2\in E$ such that $y_1w_1x_1$, $x_2w_2z_2$,

$$\begin{aligned}
&\max\{d(y,y_1),d(z,z_2)\}\leq\frac{r_1\delta^n}{1-\delta},\\
&\max\{d(x,x_1),d(x,x_2)\}\leq\frac{r_1\delta^{n+1}}{1-\delta},\\
&d(y_1,x_1)<C_2\min\{d(y_1,w_1),d(w_1,x_1)\},\\
&d(x_2,z_2)<C_2\min\{d(x_2,w_2),d(w_2,z_2)\}
\end{aligned}$$

and

$$\begin{aligned}
&\min\{d(y_1,w_1),d(w_1,x_1),d(x_2,w_2),d(w_2,z_2)\}\\
&\leq\min\left\{d(w_1,w_2)+\frac{r_1\delta^{n+1}}{1-\delta},\ d(w_1,z)+\frac{r_1\delta^n}{1-\delta},\ d(y,w_2)+\frac{r_1\delta^n}{1-\delta}\right\}.
\end{aligned}$$

Denote

$$\begin{aligned}
r'&=\frac{1}{C_2}d(y,x)-d_0\delta^n,\\
C_1'&=M_0^2\left(\frac{1+r_1}{\delta}\left(C_2-\varepsilon_0+\frac{1}{C_2}\right)-d_0\right),
\end{aligned}$$

where

$$d_0=\left(1+\frac{1+\delta}{C_2}\right)\frac{r_1}{1-\delta}.$$

Below we will use

$$\max\left\{\frac{r_1}{1-\delta},\, r_5(1-2r_1)\right\} \le \varepsilon_0\left(\frac{1}{C_2}(1-2r_1)-d_0\right),$$
$$\max\left\{\frac{r_4}{\delta},\, R_2\right\} \le \left(\frac{1}{C_2}-2r_5\right)(1-2r_1)-d_0,$$
$$r_2 \le \delta\left(\left(\frac{1}{C_2}-r_5\right)(1-2r_1)-d_0\right)-r_1$$

and

$$r_3 \ge C_1' + \frac{(C_2-\varepsilon_0+2r_5)(1+r_1)}{\delta}+r_1,$$
$$R_3 \ge C_1' + \frac{2r_5(1+r_1)}{\delta}+r_1.$$

By the first part of Lemma 3.1 we have yw_1x and xw_2z. Let N_1 be the smallest integer such that $C_1'\delta^{N_1} < d(E)$ and assume that $n \ge N_1$. Denote $R' = M_0^2((C_2-\varepsilon_0)d(y,x)+r')$. By the regularity

$$\mu\left(B(x,R')\backslash B(x,d(x,z)+r')\right) \ge \mu(B(x,R')) - \mu(B(x,d(x,z)+r'))$$
$$\ge M_0^{-1}R' - M_0(d(x,z)+r') > 0$$

and so we find $w_3 \in B(x,R')\backslash B(x,d(x,z)+r')$. Now $d(z_1,z_2) > r'$ for any $z_1, z_2 \in \{y, w_1, x, w_2, z, w_3\}$, $z_1 \ne z_2$. We may assume that $d(w_3,x) \le d(w_3,z)$. The other case can be treated similarly.

Now $x = q_n(x')$ for some $x' \in A_n'$. Further by the construction there are $n_2, n_3 \in \{n-1, n\}$ such that $y = q_{n_2}(y')$ and $z = q_{n_3}(z')$ for some $y', z' \in E$. Denote $B_i = B(w_i, r_5d(y,x))$ for $i = 1,2,3$. Now

$$B_i \times B_j \subset S_n(x') \cap S_{n_2}(y') \cap S_{n_3}(z')$$

for $i, j \in \{1,2,3\}$, $i \ne j$. We also have

$$(B_y \times B_z) \cup ((B_y \cup B_z) \times (B_1 \cup B_2 \cup B_3)) \subset S_n(x'),$$
$$(B_x \times B_z) \cup ((B_x \cup B_z) \times (B_1 \cup B_2 \cup B_3)) \subset S_{n_2}(y'),$$
$$(B_x \times B_y) \cup ((B_x \cup B_y) \times (B_1 \cup B_2 \cup B_3)) \subset S_{n_3}(z'),$$

where $B_x = B(x, r_5d(y,x))$, $B_y = B(y, r_5d(y,x))$ and $B_z = B(z, r_5d(y,x))$. Thus

$$\min\left\{\, \mu^2(S_n(x')), \mu^2(S_{n_2}(y')), \mu^2(S_{n_3}(z'))\,\right\} \ge \frac{20r_5^2d(y,x)^2}{M_0^2}.$$

Denote

$$G = \frac{M_0^4r_3^2(2+\delta^2)\delta^{2n-2}}{r_5^2d(y,x)^2}$$

and let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 = \left\{ (\zeta,\eta) \in S_n(x') \,:\, \mu^2(S_n(x'))c(\zeta,\eta,x)^2 \geq G \int_{S_n(x')} c(z_1,z_2,x)^2 \, d\mu^2(z_1,z_2) \right\},$$
$$\Gamma_2 = \left\{ (\zeta,\eta) \in S_{n_2}(y') \,:\, \mu^2(S_{n_2}(y'))c(\zeta,\eta,y)^2 \geq G \int_{S_{n_2}(y')} c(z_1,z_2,y)^2 \, d\mu^2(z_1,z_2) \right\},$$
$$\Gamma_3 = \left\{ (\zeta,\eta) \in S_{n_3}(z') \,:\, \mu^2(S_{n_3}(z'))c(\zeta,\eta,z)^2 \geq G \int_{S_{n_3}(z')} c(z_1,z_2,z)^2 \, d\mu^2(z_1,z_2) \right\}.$$

If $c(x,y,z) = 0$, we have $l(G_k^{n-1}) - l(G_{k-1}^{n-1}) = 0$. Thus we may assume $c(x,y,z) > 0$. Then, since $(z_1,z_2) \mapsto c(z_1,z_2,z_3)$ is continuous on $\{(z_1,z_2) \in E^2 \,:\, z_1 \neq z_2 \neq z_3 \neq z_1\}$ in the product topology, we have by the regularity

$$\int_{S_n(x')} c(z_1,z_2,x)^2 \, d\mu^2(z_1,z_2) > 0,$$
$$\int_{S_{n_2}(y')} c(z_1,z_2,y)^2 \, d\mu^2(z_1,z_2) > 0,$$
$$\int_{S_{n_3}(z')} c(z_1,z_2,z)^2 \, d\mu^2(z_1,z_2) > 0.$$

Thus by the Tchebychev inequality

$$\begin{aligned}
\mu^2(\Gamma) &\leq \mu^2(\Gamma_1) + \mu^2(\Gamma_2) + \mu^2(\Gamma_3) \\
&\leq \frac{1}{G}\left(\mu^2(S_n(x')) + \mu^2(S_{n_2}(y')) + \mu^2(S_{n_3}(z'))\right) \\
&\leq \frac{1}{G}\left(M_0 r_3 - \frac{1}{M_0} r_2\right)^2 \left(\delta^{2n} + \delta^{2n_2} + \delta^{2n_3}\right) \qquad (10) \\
&\leq \frac{1}{G}\left(M_0 r_3 - \frac{1}{M_0} r_2\right)^2 \left(1 + \frac{2}{\delta^2}\right)\delta^{2n} < \frac{r_5^2 d(y,x)^2}{M_0^2}.
\end{aligned}$$

Denote $U_i = \{w \in B_1 \,:\, \{w\} \times B_i \subset \Gamma\}$ for $i = 2,3$. We next show that there exists $(u_1,u_2,u_3) \in B_1 \times B_2 \times B_3$ such that $(u_1,u_2) \notin \Gamma$ and $(u_1,u_3) \notin \Gamma$. Suppose this is false. Then $B_1 = U_2 \cup U_3$. Letting

$$p = \mu^2(S_n(x'))^{-1} G \int_{S_n(x')} c(z_1,z_2,x)^2 \, d\mu^2(z_1,z_2)$$

we have

$$\begin{aligned}
\{\, w \in B_1 \,:\, \{w\} \times B_2 \subset \Gamma_1 \,\} &= \{\, w \in B_1 \,:\, c(w,z_2,x)^2 \geq p \text{ for all } z_2 \in B_2 \,\} \\
&= \bigcap_{z_2 \in B_2} \{\, w \in B_1 \,:\, c(w,z_2,x)^2 \geq p \,\},
\end{aligned}$$

which is a closed set. Similarly $\{w \in B_1 \,:\, \{w\} \times B_i \subset \Gamma_j\}$ is closed for each $i \in \{2,3\}$

and $j \in \{1,2,3\}$. Thus U_1 and U_2 are closed and we get

$$\begin{aligned}\mu^2(\Gamma) &\geq \mu^2(U_2 \times B_2) + \mu^2(U_3 \times B_3)\\ &= \mu(U_2)\mu(B_2) + \mu(U_3)\mu(B_3)\\ &\geq (\mu(U_2) + \mu(U_3)) \min\{\mu(B_2), \mu(B_3)\}\\ &\geq \mu(B_1) \min\{\mu(B_2), \mu(B_3)\} \geq \frac{r_5^2 d(y,x)^2}{M_0^2},\end{aligned}$$

which contradicts (10).

For any $z_1, z_2 \in \{y, u_1, x, u_2, z, u_3\}$, $z_1 \neq z_2$, we have $d(z_1, z_2) \in]r, R]$, where $r = r' - 2r_5 d(y,x)$ and $R = R' + d(x,z) + 2r_5 d(y,x)$. Now $R \leq Kr$ for

$$K = \frac{((M_0^2+1)(C_2 - \varepsilon_0) + M_0^2 C_2^{-1} + 2r_5)(1-2r_1) - M_0^2 d_0}{(C_2^{-1} - 2r_5)(1-2r_1) - d_0}.$$

Therefore, choosing $\varepsilon_0^3 \geq (4K-1)/(4K+1)$, $\{y, u_1, x, u_2, z, u_3\}$ has an order by Lemma 2.3 of [3]. The latter part of Lemma 3.1 gives yu_1x and xu_2z. So we have yu_1xu_2z. Since

$$d(u_3,x) \geq d(w_3,x) - r_5 d(y,x) > d(x,z) + r' - r_5 d(y,x) > d(x,z) \geq d(y,x),$$

we must have $u_3yu_1xu_2z$ or $yu_1xu_2zu_3$. Using the assumption $d(w_3,x) \leq d(w_3,z)$ and Lemma 3.1 we get u_3xz. Thus we have $u_3yu_1xu_2z$.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, where $\varepsilon_1 = -\cos \sphericalangle u_1xu_2$, $\varepsilon_2 = -\cos \sphericalangle u_3yu_1$, $\varepsilon_3 = -\cos \sphericalangle u_1u_2z$ and $\varepsilon_4 = -\cos \sphericalangle u_3u_1z$. Then

$$\begin{aligned}d(y,z) &\geq d(u_3,z) - d(u_3,y)\\ &\geq \varepsilon_4 d(u_3,u_1) + d(u_1,z) - d(u_3,y)\\ &\geq \varepsilon_4(\varepsilon_2 d(u_3,y) + d(y,u_1)) + d(u_1,u_2) + \varepsilon_3 d(u_2,z) - d(u_3,y)\\ &\geq \varepsilon_4(\varepsilon_2 d(u_3,y) + d(y,u_1)) + d(u_1,x) + \varepsilon_1 d(x,u_2) + \varepsilon_3 d(u_2,z) - d(u_3,y)\\ &\geq \varepsilon(d(y,u_1) + d(u_1,x) + d(x,u_2) + d(u_2,z)) + (\varepsilon^2 - 1)d(u_3,y)\\ &\geq \varepsilon(d(y,x) + d(x,z)) + (\varepsilon^2 - 1)d(u_3,y).\end{aligned}$$

Denote

$$\begin{aligned}\lambda_1 &= c(x,u_1,u_2)^2 d(u_1,u_2)^2,\\ \lambda_2 &= c(y,u_1,u_3)^2 d(u_1,u_3)^2,\\ \lambda_3 &= c(z,u_1,u_2)^2 d(u_1,z)^2,\\ \lambda_4 &= c(z,u_1,u_3)^2 d(u_3,z)^2.\end{aligned}$$

Now

$$\begin{aligned}\lambda_1 &< \frac{Gd(u_1,u_2)^2}{\mu(B(x',r_1\delta^n))\mu^2(S_n(x'))} \int_{B(x',r_1\delta^n)} \int_{S_n(x')} c(z_1,z_2,z_3)^2 \, d\mu^2(z_1,z_2) \, d\mu z_3,\\ \lambda_2 &< \frac{Gd(u_1,u_3)^2}{\mu(B(y',r_1\delta^{n_2}))\mu^2(S_{n_2}(y'))} \int_{B(y',r_1\delta^{n_2})} \int_{S_{n_2}(y')} c(z_1,z_2,z_3)^2 \, d\mu^2(z_1,z_2) \, d\mu z_3,\\ \lambda_3 &< \frac{Gd(u_1,z)^2}{\mu(B(z',r_1\delta^{n_3}))\mu^2(S_{n_3}(z'))} \int_{B(z',r_1\delta^{n_3})} \int_{S_{n_3}(z')} c(z_1,z_2,z_3)^2 \, d\mu^2(z_1,z_2) \, d\mu z_3,\\ \lambda_4 &< \frac{Gd(u_3,z)^2}{\mu(B(z',r_1\delta^{n_3}))\mu^2(S_{n_3}(z'))} \int_{B(z',r_1\delta^{n_3})} \int_{S_{n_3}(z')} c(z_1,z_2,z_3)^2 \, d\mu^2(z_1,z_2) \, d\mu z_3.\end{aligned}$$

Using this we get

$$
\begin{aligned}
l(G_k^{n-1}) - l(G_{k-1}^{n-1}) &\le d(y,x) + d(x,z) - d(y,z) \\
&\le (1-\varepsilon)(d(y,x) + d(x,z)) + (1-\varepsilon^2) d(u_3,y) \\
&\le (1-\varepsilon^2)(d(y,x) + d(x,z) + d(u_3,y)) \\
&\le \frac{1}{4}\max\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}(d(y,x)+d(x,z)+d(u_3,y)) \qquad (11)\\
&\le C_4 \int\limits_{B(x,R_4\delta^n)} \int\limits_{T_n^2(z_3)} \int\limits_{T_n^2(z_3)\backslash B(z_2,r_4\delta^n)} c(z_1,z_2,z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3,
\end{aligned}
$$

where

$$
\begin{aligned}
R_4 &= \frac{(C_2 - \varepsilon_0)(1+r_1) + 2r_1}{\delta}, \\
T_n^2(z_3) &= B(z_3, (r_3 + r_1)\delta^{n-1}) \backslash B(z_3, (r_2 - r_1)\delta^n)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{M_0^3 G d(u_3,z)^2 (d(y,x) + d(x,z) + d(u_3,y))}{4 \cdot 20 r_5^2 d(y,x)^2 r_1 \delta^n} \\
&\le \frac{3M_0^7 r_3^2 (2+\delta^2)}{80(1-2r_1) r_5^4 r_1 \delta^2} \left(M_0^2 \left(C_2 - \varepsilon_0 + \frac{1}{C_2} - \frac{d_0}{1-2r_1} \right) + C_2 - \varepsilon_0 + 2r_5 \right)^3 = C_4.
\end{aligned}
$$

Case 4. $d(y,z) < d(x,z)$ for all $z \in D_{k-1}^{n-1}(y)$.

Assume that $\{z_1, z_2\} \in P_{k-1}^{n-1}(y)$ such that $\{z_1, z_2\} \subset D_{k-1}^{n-1}(y)$. Now $d(y,v) < C_2(1+r_1)\delta^{n-1}$ for all $v \in D_{k-1}^{n-1}(y)$. Thus by the $(n-1,k-1)$-property we have $z_1 y z_2$. Since $\delta R_3 \ge (1+C_2)(1+r_1)$ and $R_2 \le 1 - 2r_1$, we have $\{y, x, z_1, z_2\} \in \mathcal{O}(\varepsilon_0)$. Now $d(v_1,v_2) < K d(v_3,v_4)$ for all $v_1, v_2, v_3, v_4 \in \{z_1, x, y, z_2\}$, $v_3 \ne v_4$, and $\varepsilon_0 \ge K/(K+1)$ for $K = \max\{2C_2, (1+C_2)(1+r_1)(1-2r_1)^{-1}\}$. Since now xyz_1 and xyz_2, it follows from Lemma 2.2 of [3] that yz_1z_2 or yz_2z_1, which is a contradiction. Thus the assumption above is false and for fixed $z \in D_{k-1}^{n-1}(y)$ there exists $b \in V_{k-1}^{n-1} \backslash D_{k-1}^{n-1}$ such that $\{z,b\} \in P_{k-1}^{n-1}(y)$.

We set $V_k^{n-1} = V_{k-1}^{n-1} \cup \{x\}$ and define

$$
E_k^{n-1} = \left(E_{k-1}^{n-1} \backslash \{\{y,b\}\}\right) \cup \{\{x,y\},\{x,b\}\}.
$$

Further we define w_k^{n-1} and P_k^{n-1} by setting

$$
w_k^{n-1}(e) = \begin{cases} d(x,y) & \text{for } e = \{x,y\}, \\ w_{k-1}^{n-1}(\{y,b\}) & \text{for } e = \{x,b\}, \\ w_{k-1}^{n-1}(e) & \text{for } e \in E_{k-1}^{n-1} \backslash \{\{y,b\}\} \end{cases}
$$

and

$$
P_k^{n-1}(v) = \begin{cases} \{\{y,b\}\} & \text{for } v = x, \\ \left(P_{k-1}^{n-1}(v) \backslash \{\{z,b\}\}\right) \cup \{\{x,z\}\} & \text{for } v = y, \\ P_{k-1}^{n-1}(v) & \text{for } v \in D_{k-1}^{n-1} \backslash \{y\}. \end{cases}
$$

Now

$$l(G_k^{n-1}) - l(G_{k-1}^{n-1}) = d(x,y). \tag{12}$$

Since xyz, $\delta R_3 \geq (1+C_2)(1+r_1)\delta^{-1} + r_1(1-\delta)^{-1}$, $R_2 \leq 1 - 2r_1(1-\delta)^{-1}$ and $r_1 < \varepsilon_0(1-\delta-2r_1)$, we have the $(n-1,k)$-property at y by Lemma 3.1. The construction will show that for each $m \geq n$ there is $v \in D_0^m$ such that $\{v,b\} \in E_0^m$ and $w_0^m(\{v,b\}) = w_{k-1}^{n-1}(\{y,b\})$. We define I_k^{n-1} and f_k^{n-1} as in Case 1.

4 Construction of G_0^n

Denote $D_0^{n-1} = \{x_{\#A_n+1}^n, \ldots, x_{\#D_0^n}^n\}$. We define inductively $D_k^{n-1} = \big(D_{k-1}^{n-1}\backslash\{x_k^n\}\big) \cup \{q_n(x_k^n)\}$ for $k = \#A_n+1, \ldots, \#D_0^n$. Let $k \in \{\#A_n+1, \ldots, \#D_0^n\}$ and assume by induction that we have constructed a graph $G_{k-1}^{n-1} = (V_{k-1}^{n-1}, E_{k-1}^{n-1})$ with a weight function $w_{k-1}^{n-1} : E_{k-1}^{n-1} \to]0,\infty[$ and a 1-Lipschitz surjection $f_{k-1}^{n-1} : I_{k-1}^{n-1} \to D_{k-1}^{n-1}$, where $I_{k-1}^{n-1} \subset [0, 2l(T_{k-1}^{n-1})]$. We also assume that we have defined a function P_{k-1}^{n-1}.

We denote $x = x_k^n$. We set $V_k^{n-1} = \big(V_{k-1}^{n-1}\backslash\{x\}\big) \cup \{q_n(x)\}$ and define

$$E_k^{n-1} = \big(E_{k-1}^{n-1}\backslash\{\{x,v\} : v \in V_{k-1}^{n-1}(x)\}\big) \cup \{\{q_n(x),v\} : v \in V_{k-1}^{n-1}(x)\}.$$

Further we define $w_{k,0}^{n-1} : E_k^{n-1} \to]0,\infty[$ by setting

$$w_{k,0}^{n-1}(e) = \begin{cases} w_{k-1}^{n-1}(\{x,v\}) + r_1\delta^n & \text{for } e = \{q_n(x),v\}, \text{ where } v \in D_{k-1}^{n-1}(x), \\ w_{k-1}^{n-1}(\{x,v\}) & \text{for } e = \{q_n(x),v\}, \text{ where } v \in V_{k-1}^{n-1}(x)\backslash E, \\ w_{k-1}^{n-1}(e) & \text{for } e \in E_{k-1}^{n-1}\backslash\{\{x,v\} : v \in V_{k-1}^{n-1}(x)\}. \end{cases}$$

For any $v \in D_{k-1}^{n-1}(x)$ let $z(v) \in V_{k-1}^{n-1}(v)$ for which $\{x, z(v)\} \in P_{k-1}^{n-1}(v)$. We define P_k^{n-1} by setting

$$P_k^{n-1}(v) = \begin{cases} P_{k-1}^{n-1}(x) & \text{for } v = q_n(x), \\ \big(P_{k-1}^{n-1}(v)\backslash\{\{x,z(v)\}\}\big) \cup \{\{q_n(x),z(v)\}\} & \text{for } v \in D_{n-1}^{k-1}(x), \\ P_{k-1}^{n-1}(v) & \text{for } v \in D_{k-1}^{n-1}\backslash\big(D_{n-1}^{k-1}(x) \cup \{x\}\big). \end{cases}$$

Further we set $I_{k,0}^{n-1} = I_{k-1}^{n-1}$ and define $f_{k,0}^{n-1} : I_{k,0}^{n-1} \to D_k^{n-1}$ by setting

$$f_{k,0}^{n-1}(s) = \begin{cases} q_n(x) & \text{if } f_{k-1}^{n-1}(s) = x, \\ f_{k-1}^{n-1}(s) & \text{if } f_{k-1}^{n-1}(s) \neq x. \end{cases}$$

Let $\{y_1, \ldots, y_m\} = D_{k-1}^{n-1}(x)$ and $i \in \{1, \ldots, m\}$, where $m = \#D_{k-1}^{n-1}(x)$. Assume by induction that we have defined a function $f_{k,i-1}^{n-1} : I_{k,i-1}^{n-1} \to D_k^{n-1}$. Let $(t_1,t_2) \in J_{k-1}^{n-1}(\{x,y_i\})$. We set

$$I_{k,i,0}^{n-1} = \big(I_{k,i-1}^{n-1} \cap [0,t_1]\big) \cup \big(\big(I_{k,i-1}^{n-1} \cap [t_2,\infty[\big) + r_1\delta^n\big)$$

and define $f_{k,i,0}^{n-1} : I_{k,i,0}^{n-1} \to D_k^{n-1}$ by setting

$$f_{k,i,0}^{n-1}(s) = \begin{cases} f_{k,i-1}^{n-1}(s) & \text{for } s \in I_{k,i-1}^{n-1} \cap [0,t_1], \\ f_{k,i-1}^{n-1}(s - r_1\delta^n) & \text{for } s \in \big(I_{k,i-1}^{n-1} \cap [t_2,\infty[\big) + r_1\delta^n. \end{cases}$$

If $\#J_{k-1}^{n-1}(\{x,y_i\}) = 1$, we put $I_{k,i}^{n-1} = I_{k,i,0}^{n-1}$ and $f_{k,i}^{n-1} = f_{k,i,0}^{n-1}$. Else let $u_1, u_2 \in I_{k,i,0}^{n-1}$ such that $u_2 - u_1 = w_{k-1}^{n-1}(\{x,y_i\})$, $f_{k,i,0}^{n-1}(\{u_1,u_2\}) = \{x,y_i\}$ and $I_{k,i,0}^{n-1} \cap]u_1,u_2[= \emptyset$. We set

$$I_{k,i}^{n-1} = \left(I_{k,i,0}^{n-1} \cap [0,u_1]\right) \cup \left(\left(I_{k,i,0}^{n-1} \cap [u_2,\infty[\right) + r_1\delta^n\right)$$

and define $f_{k,i}^{n-1}$ by setting

$$f_{k,i}^{n-1}(s) = \begin{cases} f_{k,i,0}^{n-1}(s) & \text{for } s \in I_{k,i,0}^{n-1} \cap [0,u_1], \\ f_{k,i,0}^{n-1}(s - r_1\delta^n) & \text{for } s \in \left(I_{k,i,0}^{n-1} \cap [u_2,\infty[\right) + r_1\delta^n. \end{cases}$$

Denote

$$P = \Big\{ \{v_1,v_2\} \in P_{k-1}^{n-1}(x) \;: \max\{d(q_n(x), q_{n,n}(v_1)), d(q_n(x), q_{n,n}(v_2))\} < C_2(1+r_1)\delta^n \text{ and } \{v_1,v_2\} \subset D_{k-1}^{n-1}(x) \Big\}.$$

If $P = \emptyset$, we set $w_k^{n-1} = w_{k,0}^{n-1}$, $I_k^{n-1} = I_{k,m}^{n-1}$ and $f_k^{n-1} = f_{k,m}^{n-1}$. From now on we assume that $\{y,z\} \in P$. Let us assume that $\min\{s : f_{k,0}^{n-1}(s) = y\} < \min\{s : f_{k,0}^{n-1}(s) = z\}$ and define w_k^{n-1} by setting

$$w_k^{n-1}(e) = \begin{cases} \rho & \text{for } e = \{y, q_n(x)\}, \\ \tau & \text{for } e = \{q_n(x), z\}, \\ w_{k,0}^{n-1}(e) & \text{for } e \in E_k^{n-1} \backslash \{\{y,q_n(x)\}, \{q_n(x),z\}\}, \end{cases}$$

where

$$\rho = \max\left\{w_{k-1}^{n-1}(\{y,x\}) - r_1\delta^n, d(y,q_n(x))\right\},$$
$$\tau = \max\left\{w_{k-1}^{n-1}(\{y,x\}) + w_{k-1}^{n-1}(\{x,z\}) - \rho, d(q_n(x),z)\right\}.$$

Let $\{e_1,e_2\} = \{\{y,q_n(x)\},\{q_n(x),z\}\}$ and $i \in \{1,2\}$ and assume by induction that we have defined a function $f_{k,m+i-1}^{n-1} : I_{k,m+i-1}^{n-1} \to D_k^{n-1}$. Let $t_1, t_2 \in I_{k,m+i-1}^{n-1}$ such that $t_2 - t_1 = w_{k,0}^{n-1}(e_i)$, $f_{k,m+i-1}^{n-1}(\{t_1,t_2\}) = e_i$ and $I_{k,m+i-1}^{n-1} \cap]t_1,t_2[= \emptyset$. We set

$$I_{k,m+i,0}^{n-1} = J_1 \cup J_2,$$

where $J_1 = I_{k,m+i-1}^{n-1} \cap [0,t_1]$ and $J_2 = \left(I_{k,m+i-1}^{n-1} \cap [t_2,\infty[\right) + w_k^{n-1}(e_i) + t_1 - t_2$, and define $f_{k,m+i,0}^{n-1} : I_{k,m+i,0}^{n-1} \to D_k^{n-1}$ by setting

$$f_{k,m+i,0}^{n-1}(s) = \begin{cases} f_{k,m+i-1}^{n-1}(s) & \text{for } s \in J_1, \\ f_{k,m+i-1}^{n-1}(s - w_k^{n-1}(e_i) - t_1 + t_2) & \text{for } s \in J_2. \end{cases}$$

If there exist $u_1, u_2 \in I_{k,m+i,0}^{n-1}$ such that $u_1 \neq t_1$, $u_2 - u_1 = w_{k,0}^{n-1}(e_i)$, $f_{k,m+i,0}^{n-1}(\{u_1,u_2\}) = e_i$ and $I_{k,m+i,0}^{n-1} \cap]u_1,u_2[= \emptyset$, we set

$$I_{k,m+i}^{n-1} = J_1 \cup J_2,$$

where $J_1 = I_{k,m+i,0}^{n-1} \cap [0,u_1]$ and $J_2 = \left(I_{k,m+i,0}^{n-1} \cap [u_2,\infty[\right) + w_k^{n-1}(e_i) + u_1 - u_2$, and define $f_{k,m+i}^{n-1} : I_{k,m+i}^{n-1} \to D_k^{n-1}$ by setting

$$f_{k,m+i}^{n-1}(s) = \begin{cases} f_{k,m+i,0}^{n-1}(s) & \text{for } s \in J_1, \\ f_{k,m+i,0}^{n-1}(s - w_k^{n-1}(e_i) - t_1 + t_2) & \text{for } s \in J_2. \end{cases}$$

Else we put $I^{n-1}_{k,m+i} = I^{n-1}_{k,m+i,0}$ and $f^{n-1}_{k,m+i} = f^{n-1}_{k,m+i,0}$.

We set $I^{n-1}_k = I^{n-1}_{k,m+2}$ and $f^{n-1}_k = f^{n-1}_{k,m+2}$. By the construction there exists $\{y', z'\} \in P^{n-1}_{\#A_n}(x)$ such that $\{y', z'\} \subset D^{n-1}_{\#A_n}$, $q_{n,n}(y') = q_{n,n}(y)$ and $q_{n,n}(z') = q_{n,n}(z)$. Since $\delta \le 1 - 2r_1$, we have $\max\{d(x, y'), d(x, z')\} < C_2(1 + r_1)\delta^n + 2r_1\delta^n \le C_2(1 + r_1)\delta^{n-1}$. Thus $yq_n(x)z|\varepsilon_0$ by the $(n-1, \#A_n)$-property and we have

$$\begin{aligned}
&w^{n-1}_k(\{y, q_n(x)\}) + w^{n-1}_k(\{q_n(x), z\}) - w^{n-1}_{k-1}(\{y, x\}) - w^{n-1}_{k-1}(\{x, z\}) \\
&\le \max\{d(y, q_n(x)) + d(q_n(x), z) - d(y, x) - d(x, z), 0\} \\
&\le d(y, q_n(x)) + d(q_n(x), z) - d(y, z) \\
&\le (1 - \varepsilon_0) \min\{d(y, q_n(x)), d(q_n(x), z)\}.
\end{aligned} \tag{13}$$

If $\vartheta(q_n(x), n) < \varepsilon_0$ we get as in Case 1

$$\begin{aligned}
&w^{n-1}_k(\{y, q_n(x)\}) + w^{n-1}_k(\{q_n(x), z\}) - w^{n-1}_{k-1}(\{y, x\}) - w^{n-1}_{k-1}(\{x, z\}) \le h\delta^n \\
&\le C_5 \int_{B(q_n(x),(R_3+r_0)\delta^n)} \int_{T^1_n(z_3)} \int_{T^1_n(z_3)\cap T^1_n(z_2)} c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3,
\end{aligned} \tag{14}$$

where $h = \min\{2r_1, (1-\varepsilon_0)(C_2(1+r_1)+r_1)\}$ and $C_5 = M_0^3 h c_1^{-1} r_0^{-3}$. We now assume that $\vartheta(q_n(x), n) \ge \varepsilon_0$ and there is $m \ge n$ such that $\{\{q_{m,n}(y), q_{m,n}(x)\}, \{q_{m,n}(x), q_{m,n}(z)\}\} \cap F^m_0 = \emptyset$. Denote

$$C'_2 = M_0^2 \left(C_2(1 + r_1) + r_1 + d_1\right),$$

where $d_1 = C_2^{-1}(1-r_1) - d_0$. Let N_2 be the smallest integer such that $C'_2\delta^{N_2} < d(E)$. By assuming $n \ge N_2$ and using $\max\{r_1(1 - \delta)^{-1}, r_5\} \le \varepsilon_0 d_1$, $\max\{r_4\delta^{-1}, R_2\} \le d_1 - 2r_5$, $r_2 \le \delta(d_1 - r_5) - r_1$, $r_3 \ge C'_2 + C_2(1 + r_1) + 2(r_1 + r_5)$, $R_3 \ge C'_2 + r_1 + 2r_5$ and $\varepsilon_0^3 \ge (4K - 1)/(4K + 1)$, where

$$K = \frac{C'_2 + C_2(1 + r_1) + r_1 + 2r_5}{d_1 - 2r_5},$$

we get as in Case 3

$$\begin{aligned}
&w^{n-1}_k(\{y, q_n(x)\}) + w^{n-1}_k(\{q_n(x), z\}) - w^{n-1}_{k-1}(\{y, x\}) - w^{n-1}_{k-1}(\{x, z\}) \\
&\le d(y, q_n(x)) + d(q_n(x), z) - d(y, z) \\
&\le C_6 \int_{B(q_n(x),R_5\delta^n)} \int_{T^2_n(z_3)} \int_{T^2_n(z_3)\setminus B(z_2,r_4\delta^n)} c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
R_5 &= C_2(1 + r_1) + \left(1 + \frac{2}{\delta}\right) r_1, \\
C_6 &= \frac{3M_0^7 r_3^2 (2 + \delta^2) \left(C'_2 + C_2(1 + r_1) + r_1 + 2r_5\right)^3}{80 r_5^4 r_1 \delta^2}.
\end{aligned}$$

If $k = \#D^n_0$, we now set $V^n_0 = V^{n-1}_k$, $E^n_0 = E^{n-1}_k$, $w^n_0 = w^{n-1}_k$, $P^n_0 = P^{n-1}_k$, $I^n_0 = I^{n-1}_k$ and $f^n_0 = f^{n-1}_k$. Since $(C_2(1 + r_1) + 2r_1)\delta \le C_2(1 + r_1)$, the $(n, 0)$-property is satisfied. Note also that $\{q_{m,n}(v_1), q_{m,n}(v_2)\} \in F^m_0$ for all $m \ge n$ if $\{v_1, v_2\} \in F^n_0$ such that $d(v_1, v_2) \ge C_2(1 + r_1)\delta^n$.

5 End of the proof

By iterating the above algorithm, we construct a sequence $(G_0^n)_{n>n_0}$ of graphs and a sequence $f_0^n : I_0^n \to D_0^n$ of 1-Lipschitz surjections such that $I_0^n \subset [0, 2l(T_0^n)]$ for all $n > n_0$.

Let $n > n_0$, $k \in \{1, \dots, \#A_n\}$ and $y \in D_{k-1}^{n-1}$. Denote

$$\mathcal{I} = \left\{ i \in \{k, \dots, \#A_n\} \,:\, \vartheta(x_i^n, n) \geq \varepsilon_0 \text{ and } d(x_i^n, y) = d(x_i^n, D_{i-1}^{n-1}) \right\}$$

and further for $j = 0, 1, 2, \dots$ set

$$\mathcal{I}_j = \left\{ i \in \mathcal{I} \,:\, (1+\varepsilon_0)^{-j-1} d < d(x_i^n, y) \leq (1+\varepsilon_0)^{-j} d \right\},$$

where $d = \max\{d(x_i^n, y) \,:\, i \in \mathcal{I}\} \leq (1+r_1)\delta^{n-1}$. Let $j \in \{0, 1, 2, \dots\}$. We show that $\#\mathcal{I}_j \leq 2$. Suppose this fails and there exist $i_1, i_2, i_3 \in \mathcal{I}_j$ with $i_1 < i_2 < i_3$. Since $R_2 \leq 1 - 2r_1$ and $\delta R_3 \geq 2(1+r_1)$, we have $\{y, x_{i_1}^n, x_{i_2}^n, x_{i_3}^n\} \in \mathcal{O}(\varepsilon_0)$. Denote $d_l = d(x_{i_l}^n, y)$ for $l = 1, 2, 3$. Since $\varepsilon_0 \geq 1/2$,

$$d_1 + \varepsilon_0 d_3 + \varepsilon_0(d_2 + \varepsilon_0 d_3) - (d_1 + d_2) > (2\varepsilon_0 - 1)(1+\varepsilon_0)^{-j} d \geq 0.$$

Thus we have $z_1 z_2 y$ for some $z_1, z_2 \in \{x_{i_1}^n, x_{i_2}^n, x_{i_3}^n\}$. This implies $d(z_1, z_2) \leq d(z_1, y) - \varepsilon_0 d(z_2, y) \leq (1+\varepsilon_0)^{-j-1} d$, which is a contradiction. So we have

$$\sum_{i \in \mathcal{I}} d(x_i^n, y) = \sum_{j=0}^{\infty} \sum_{i \in \mathcal{I}_j} d(x_i^n, y) \leq \sum_{j=0}^{\infty} 2(1+\varepsilon_0)^{-j} d = \frac{2(1+\varepsilon_0)d}{\varepsilon_0}.$$

Let $n_0 < n' \leq m$, $k' \in \{1, \dots, \#A_{n'}\}$, and assume that $\{y', z'\} \in F_{k'-1}^{n'-1}$. Then, since $\delta \leq 1 - 2r_1$,

$$\frac{d(y', z')}{d(q_{m,n'}(y'), q_{m,n'}(z'))} < \frac{(1-\delta)(1-r_1)}{1-\delta-2r_1}. \tag{16}$$

Suppose that $C_2 d(x_{k'}^{n'}, y') \leq d(y', z')$. If now $n' < n \leq m$ and $x \in A_n$, we have

$$d(x, D_0^{n-1}) \leq (1+r_1)\delta^{n-1} < \frac{(1+r_1)\delta^{n-n'-1} d(x_{k'}^{n'}, y')}{1-2r_1} \leq \frac{(1+r_1)\delta^{n-n'-1} d(y', z')}{(1-2r_1)C_2}.$$

Using these estimates and (8) we get

$$\begin{aligned} &\sum_{k \in \Lambda_{n'}(y') \cup \Lambda_{n'}(z'),\, k \geq k'} \left(l(G_k^{n'-1}) - l(G_{k-1}^{n'-1}) \right) + \sum_{n=n'+1}^{m} \sum_{k \in \Lambda_n(y') \cup \Lambda_n(z')} \left(l(G_k^{n-1}) - l(G_{k-1}^{n-1}) \right) \\ &\leq M_1 d(q_{m,n'}(y'), q_{m,n'}(z')) \leq M_1 w_0^m \left(\{q_{m,n'}(y'), q_{m,n'}(z')\} \right) \end{aligned}$$

for all $m \geq n'$, where

$$\Lambda_n(v) = \left\{ k \in \{1, \dots, \#A_n\} \,:\, \vartheta(x_k^n, n) \geq \varepsilon_0 \text{ and } d(x_k^n, q_{n-1,n'}(v)) = d(x_k^n, D_{k-1}^{n-1}) \right\}$$

for $v \in D_{\#A_{n'}}^{n'-1}$ and

$$M_1 = \frac{4(1+\varepsilon_0)(1+2C_1)(1-\delta)(1-r_1)}{C_2 \varepsilon_0 (1-\delta-2r_1)} \left(1 + \frac{1+r_1}{(1-2r_1)(1-\delta)} \right).$$

From this we get

$$\sum_{n=n_0+1}^{m} \sum_{k\in\Lambda_n^1} \big(l(G_k^{n-1}) - l(G_{k-1}^{n-1})\big) \le M_1 l(T_0^m) \tag{17}$$

for all $m > n_0$, where

$$\Lambda_n^1 = \{\, k \in \{1,\dots,\#A_n\} \,:\, \text{Case 2 applies to } x_k^n \text{ at stage } n \,\}.$$

Let $n > n_0$, $k \in \{1,\dots,\#A_n\}$ and $\{y,b\} \in E_{k-1}^{n-1}$, where $b \in V_{k-1}^{n-1} \backslash D_{k-1}^{n-1}$. Denote

$$\mathcal{I} = \big\{\, i \in \{k,\dots,\#A_n\} \,:\, \{x_i^n, b\} \in E_i^{n-1} \big\}$$

and further for $j = 0,1,2,\dots$ let

$$\mathcal{I}_j = \big\{\, i \in \mathcal{I} \,:\, (1+\varepsilon_0)^{-j-1} d < d(x_i^n, D_{i-1}^{n-1}) \le (1+\varepsilon_0)^{-j} d \,\big\},$$

where $d = \max\{d(x_i^n, D_{i-1}^{n-1}) : i \in \mathcal{I}\} \le (1+r_1)\delta^{n-1}$. We show that $\#\mathcal{I}_j \le 2$ for all j. Suppose that this fails and for some j there exist $i_1, i_2, i_3 \in \mathcal{I}_j$, $i_1 < i_2 < i_3$, such that $d(x_{i_l}^n, x_{i_{l-1}}^n) = d(x_{i_l}^n, D_{i_l-1}^{n-1})$ for $l = 2,3$. Denote $x_l = x_{i_l}^n$ for $l = 1,2,3$ and let $x_0 \in E$ such that $d(x_1, x_0) = d(x_1, D_{i_1-1}^{n-1})$. Now $x_l x_{l+1} x_{l+2}$ for $l = 0,1$. Namely, if this is not true for fixed l, there exists a nonempty set $\{y_1,\dots,y_p\} \subset D_{i_{l+2}-1}^{n-1}$ such that $y_p x_{l+1} x_{l+2}$, $x_l y_1 x_{l+1}$ and $y_q y_{q+1} x_{l+1}$ for $q = 1,\dots,p-1$. Since $(1+\varepsilon_0)^{-j-1} d < d(z_1,z_2) \le 3(1+\varepsilon_0)^{-j} d$ for each distinct points $z_1, z_2 \in \{x_0,x_1,x_2,x_3,y_1,\dots,y_p\} \subset B(x_1, 2(1+\varepsilon_0)^{-j} d)$, $\vartheta(x_1,n) \ge \varepsilon_0$ and we have chosen $\delta R_3 \ge 2(1+r_1)$, $R_2 \le 1 - 2r_1$ and

$$\varepsilon_0^3 \ge \frac{12(1+\varepsilon_0)-1}{12(1+\varepsilon_0)+1},$$

$\{x_0,x_1,x_2,x_3,y_1,\dots,y_p\}$ has an order by Lemma 2.3 of [3], from which we conclude $x_l x_{l+1} x_{l+2}$. Since $\max\{d(x, D_{i_1-1}^{n-1}) : x \in A_k\} = d(x_1,x_0) < d(x_2,x_0)$, there exists $z \in D_{i_1-1}^{n-1} \backslash \{x_0\}$ such that $d(x_2,z) \le d(x_1,x_0)$. As above, $\{x_0,x_1,x_2,x_3,z\}$ has an order. Since $d(x_l,x_{l-1}) = d(x_{i_l}^n, D_{i_l-1}^{n-1})$ for $l = 1,2,3$, we must have $x_0x_1x_2x_3z$. From this we get $d(x_2,z) \ge d(x_2,x_3) + \varepsilon_0 d(x_3,z) > (1+\varepsilon_0)^{-j} d \ge d(x_1,x_0)$, which is a contradiction. Thus we have

$$\sum_{i\in\mathcal{I}} d(x_i^n, D_{i-1}^{n-1}) = \sum_{j=0}^{\infty} \sum_{i\in\mathcal{I}_j} d(x_i^n, D_{i-1}^{n-1}) \le \sum_{j=0}^{\infty} 2(1+\varepsilon_0)^{-j} d = \frac{2(1+\varepsilon_0)d}{\varepsilon_0}.$$

Using this and (12) we get

$$\sum_{n=n_0+1}^{m} \sum_{k\in\Lambda_n^2} \big(l(G_k^{n-1}) - l(G_{k-1}^{n-1})\big) \le M_1' \big(l(G_0^m) - l(T_0^m)\big) \tag{18}$$

for all $m > n_0$, where

$$\Lambda_n^2 = \{\, k \in \{1,\dots,\#A_n\} \,:\, \text{Case 4 applies to } x_k^n \text{ at stage } n \,\},$$

$$M_1' = \frac{2(1+\varepsilon_0)}{C_1\varepsilon_0}\left(1 + \frac{1+r_1}{(1-2r_1)(1-\delta)}\right).$$

Since $\delta^{n_0+1} < d(E) \leq C_1' \delta^{N_1-1}$ (see pages 4 and 11), we have $N_1 - n_0 < 2 - \log C_1' / \log \delta$. Using this and $\# A_n \leq 2M_0^2 \delta^{-n} d(E)$ we get

$$\sum_{n=n_0+1}^{N_1-1} \# A_n \cdot (1-\varepsilon_0)(1+r_1)\delta^{n-1} < C_1'' d(E),$$

where

$$C_1'' = \left(1 - \frac{\log C_1'}{\log \delta}\right) \frac{2M_0^2(1-\varepsilon_0)(1+r_1)}{\delta}.$$

Thus by using (9), (16) and (11) we get

$$\begin{aligned} &\sum_{n=n_0+1}^{m} \sum_{k\in\Lambda_n^3} \left(l(G_k^{n-1}) - l(G_{k-1}^{n-1})\right) \leq C_1'' d(E) + M_2 l(T_0^m) \\ &+ C_4 \sum_{n=N_1}^{m} \sum_{k\in\Lambda_n^3} \int_{B(x_k^n, R_4\delta^n)} \int_{T_n^2(z_3)} \int_{T_n^2(z_3)\setminus B(z_2, r_4\delta^n)} c(z_1,z_2,z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3, \end{aligned} \tag{19}$$

for all $m > n_0$, where

$$\begin{aligned} \Lambda_n^3 &= \{\, k \in \{1,\ldots,\# A_n\} \,:\, \text{Case 3 applies to } x_k^n \text{ at stage } n \,\}, \\ M_2 &= \frac{(1-\varepsilon_0)(1-\delta)(1-r_1)}{1-\delta-2r_1}. \end{aligned}$$

Since $N_2 - n_0 < 2 - \log C_2' / \log \delta$ (see page 17) and $\# D_0^n \leq 2M_0^2 \delta^{-n} d(E)$ for $n > n_0$, we have

$$\min\{2r_1, 1-\varepsilon_0\} d(E) + \sum_{n=n_0+2}^{N_2-1} \# D_0^{n-1} \cdot h\delta^n < C_2'' d(E),$$

where

$$C_2'' = \min\{2r_1, 1-\varepsilon_0\} - \frac{2M_0^2 h\delta \log C_2'}{\log \delta}.$$

Let $n_0 < n' \leq m$ and assume that $b \in V_0^{n'} \setminus D_0^{n'}$. For any $n \geq n'$ let $k_n^1(b) \in \{1,\ldots,\# D_0^n\}$ be the unique index such that $b \in V_{\# A_n}^{n-1}(x_{k_n^1(b)}^n)$. Denote also by $y_n(b)$ the unique vertex in $D_{\max\{k_n^1(b), \# A_n\}}^{n-1}$ for which $\{q_{n,n}(y_n(b)), b\} \in P_0^n(q_{n,n}(x_{k_n^1(b)}^n))$. We have

$$\begin{aligned} &\sum_{n\geq n',\, k_n^1(b) > \# A_n} \left(w_{k_n^1(b)}^{n-1}(\{q_n(x_{k_n^1(b)}^n), y_n(b)\}) - w_{k_n^1(b)-1}^{n-1}(\{x_{k_n^1(b)}^n, y_n(b)\})\right) \\ &\leq \sum_{n=n'}^{\infty} r_1 \delta^n = \frac{r_1 \delta^{n'}}{1-\delta} \end{aligned}$$

and

$$w_0^m(q_{m,m}(x_{k_m^1(b)}^m), b) = w_0^{n'}(q_{n',n'}(x_{k_{n'}^1(b)}^{n'}), b) > C_1(1-2r_1)\delta^{n'}.$$

Assume now that $\{y,z\} \in F_{\# A_{n'}}^{n'-1}$ such that $\{q_{n,n'}(y), q_{n,n'}(z)\} \in F_0^n$ for all $n \geq n'$. For $x \in D_{\# A_{n'}}^{n'-1}$ and $n \geq n'$ let $k_n^2(x) \in \{1,\ldots,\# D_0^n\}$ such that $q_{n-1,n'}(x) = x_{k_n^2(x)}^n$. Denote also

$$n(x_1,x_2) = \inf\{\, n \geq n' \,:\, v_n(x_1,x_2) \in E \text{ and } q_{n-1,n'}(x_1) \notin A_n \,\}$$

for $\{x_1, x_2\} \in F^{n'-1}_{\#A_{n'}}$, where $v_n(x_1, x_2)$ is the unique vertex in $V^{n-1}_{\max\{k^2_n(x_1), \#A_n\}}$ such that $\{q_{n,n'}(x_2), q_{n,n'}(v_n(x_1, x_2))\} \in P^n_0(q_{n,n'}(x_1))$. Now

$$
\begin{aligned}
&\sum_{n=n(y,z)}^{m} \big(w^{n-1}_{k^2_n(y)}(\{q_{n,n'}(y), v_n(y,z)\}) + w^{n-1}_{k^2_n(y)}(\{q_{n,n'}(y), p_n(z,y)\}) \\
&\qquad - w^{n-1}_{k^2_n(y)-1}(\{q_{n-1,n'}(y), v_n(y,z)\}) - w^{n-1}_{k^2_n(y)-1}(\{q_{n-1,n'}(y), p_n(z,y)\})\big) \\
&+ \sum_{n=n(z,y)}^{m} \big(w^{n-1}_{k^2_n(z)}(\{q_{n,n'}(z), v_n(z,y)\}) + w^{n-1}_{k^2_n(z)}(\{q_{n,n'}(z), p_n(y,z)\}) \\
&\qquad - w^{n-1}_{k^2_n(z)-1}(\{q_{n-1,n'}(z), v_n(z,y)\}) - w^{n-1}_{k^2_n(z)-1}(\{q_{n-1,n'}(z), p_n(y,z)\})\big) \\
&\leq M_3 w^m_0(\{q_{m,n'}(y), q_{m,n'}(z)\}),
\end{aligned}
$$

where $p_n(x_1, x_2) \in D^{n-1}_{k^2_n(x_2)}$ such that $q_{n,n}(p_n(x_1, x_2)) = q_{n,n'}(x_1)$ for $\{x_1, x_2\} \in F^{n'-1}_{\#A_{n'}}$ and

$$M_3 = \frac{4r_1}{1 - \delta + 2r_1}.$$

Using these estimates, (13), (14) and (15) we get

$$
\begin{aligned}
&\sum_{n=n_0+1}^{m} \sum_{k=\#A_n+1}^{\#D^n_0} \big(l(G^{n-1}_k) - l(G^{n-1}_{k-1})\big) \\
&\leq C''_2 d(E) + M_3 l(T^m_0) + M'_2\big(l(G^m_0) - l(T^m_0)\big) \\
&+ C_5 \sum_{n=n_0+1}^{m} \sum_{x \in H^1_n} \int_{B(x,(R_3+r_0)\delta^n)} \int_{T^1_n(z_3)} \int_{T^1_n(z_3) \cap T^1_n(z_2)} c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3 \\
&+ C_6 \sum_{n=N_2}^{m} \sum_{x \in H^2_n} \int_{B(x,R_5\delta^n)} \int_{T^3_n(z_3)} \int_{T^3_n(z_3) \setminus B(z_2, r_4\delta^n)} c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3
\end{aligned} \tag{20}
$$

for all $m > n_0$, where

$$
\begin{aligned}
M'_2 &= \frac{r_1}{C_1(1 - 2r_1)(1 - \delta)}, \\
H^1_n &= \{\, x \in q_n(D^{n-1}_0) \;:\; \vartheta(x, n) < \varepsilon_0 \,\}, \\
H^2_n &= \{\, x \in q_n(D^{n-1}_0) \;:\; \vartheta(x, n) \geq \varepsilon_0 \,\}.
\end{aligned}
$$

Combining the estimates (6), (7), (17), (18), (19), and (20) we get for all $m > n_0$

$$
\begin{aligned}
&l(T^m_0) \leq (1 + 2C_1 + C''_1 + C''_2) d(E) + (M_1 + M_2 + M_3) l(T^m_0) \\
&+ C_0 \sum_{n=n_0+1}^{m} \sum_{x \in D^n_0} \int_{B(x,R_0\delta^n)} \int_{T_n(z_3)} \int_{T_n(z_3) \setminus B(z_2, \rho_0\delta^n)} c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3 \\
&+ (M'_1 + M'_2 - 1)\big(l(G^m_0) - l(T^m_0)\big),
\end{aligned}
$$

where

$$C_0 = \max\{C_3, C_4, C_5, C_6\},$$
$$R_0 = \max\{R_3 + r_0, R_4, R_5\},$$
$$\rho_0 = \min\{R_2 - 2r_0, r_4\},$$
$$T_n(z) = B(z, R_1\delta^n) \backslash B(z, \rho_1\delta^n)$$

for $z \in E$, where

$$R_1 = \max\{2(R_3 + r_0), (r_3 + r_1)\delta^{-1}\},$$
$$\rho_1 = \min\{R_2 - 2r_0, r_2 - r_1\}.$$

Let $n > n_0$, $y \in E$ and $D = B(y, (R_0 + r_1)\delta^n) \cap \left(A'_n \cup D_0^{n-1}\right)$. Then

$$M_0\left((R_0 + r_1)\delta^n + \delta^n/2\right) \geq \mu\left(B(y, (R_0 + r_1)\delta^n + \delta^n/2)\right)$$
$$\geq \sum_{x \in D} \mu\left(B(x, \delta^n/2)\right) \geq \frac{\#D \cdot \delta^n}{2M_0},$$

from which we get

$$\#\left(B(y, R_0\delta^n) \cap D_0^n\right) = \#D \leq M_0^2\left(2(R_0 + r_1) + 1\right).$$

Suppose now that $k_1 < k_2$ and $T_{k_1}(y) \cap T_{k_2}(y) \neq \emptyset$. Then $\rho_1\delta^{k_1} < R_1\delta^{k_2}$, which gives

$$k_2 - k_1 < \frac{\log R_1 - \log \rho_1}{-\log \delta}.$$

Thus we have for all $m > n_0$

$$\sum_{n=n_0+1}^{m} \sum_{x \in D_0^n} \int_{B(x, R_0\delta^n)} \int_{T_n(z_3)} \int_{T_n(z_3)\backslash B(z_2, \rho_0\delta^n)} c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3$$
$$\leq C'_0 \int_E \sum_{n=n_0+1}^{m} \int_{T_n(z_3)} \int_{\mathcal{T}(z_2, z_3)} c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3$$
$$\leq C'_0 C''_0 \int_E \int_E \int_{\mathcal{T}(z_2, z_3)} c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3$$
$$= C'_0 C''_0 \int_{\mathcal{T}} c(z_1, z_2, z_3)^2 \, d\mu^3(z_1, z_2\, z_3),$$

where $C'_0 = M_0^2(2(R_0 + r_1) + 1)$, $C''_0 = (\log \rho_1 - \log R_1)/\log \delta$ and

$$\mathcal{T} = \left\{ (z_1, z_2, z_3) \in E^3 \,:\, d(z_i, z_j) < K_0 d(z_k, z_l) \text{ for all } i, j, k, l \in \{1, 2, 3\},\ k \neq l \right\},$$
$$\mathcal{T}(z_2, z_3) = \{ z \in E \,:\, (z, z_2, z_3) \in \mathcal{T} \},$$

where

$$K_0 = \frac{R_1 \max\{2\rho_0, \rho_1\}}{\rho_0 \rho_1}.$$

By choosing the constants suitably we have $M_1 + M_2 + M_3 < 1$ and $M_1' + M_2' \leq 1$. Thus there exists a constant C (depending on M_0) such that

$$2l(T_0^m) \leq C(c^2(E) + d(E))$$

for all $m > n_0$. We denote $I_n = I_0^n$ and $f_n = f_0^n$ for $n > n_0$. Since now $I_n \subset [0, C(c^2(E) + d(E))]$ for all $n > n_0$, there exists a compact set $I \subset [0, C(c^2(E) + d(E))]$ such that $I_n \to I$ in the Kuratowski sense:

(i) If $a = \lim_{k\to\infty} a_{n_k}$ for some subsequence (a_{n_k}) of a sequence (a_n) such that $a_n \in I_n$ for any n, then $a \in I$.

(ii) If $a \in I$, then there exists a sequence (a_n) such that $a_n \in I_n$ for any n and $a = \lim_{n\to\infty} a_n$.

Let $a \in I$ and let $(a_n)_n$ be a sequence such that $a_n \in I_n$ for any n and $a_n \to a$ as $n \to \infty$. Let $m \geq n > n_0$. By the construction there is $b \in I_m$ such that

$$-\frac{4r_1\delta^{n+1}}{1-\delta} < -\sum_{k=n+1}^{m} 4r_1\delta^k \leq b - a_n \leq 2\left(l(T_0^m) - l(T_0^n)\right)$$

and $f_n(a_n) = f_m(b)$. Using this we get

$$\begin{aligned} d(f_m(a_m), f_n(a_n)) &= d(f_m(a_m), f_m(b)) \leq |a_m - b| \leq |a_m - a_n| + |a_n - b| \\ &\leq |a_m - a_n| + \max\left\{4r_1\delta^{n+1}(1-\delta)^{-1}, 2\left(l(T_0^m) - l(T_0^n)\right)\right\}. \end{aligned}$$

From this we see that $(f_n(a_n))$ is a Cauchy sequence in E. Thus we can define $f : I \to \overline{E}$, where $\overline{E}$ is the completion of E, by setting for $a \in I$

$$f(a) = \lim_{n\to\infty} f_n(a_n),$$

where (a_n) is a sequence such that $a_n \in I_n$ for any n and $a_n \to a$ as $n \to \infty$. Clearly $f(a)$ does not depend on the choice of the sequence (a_n). Let $a, b \in I$ and let $a_n \to a$ and $b_n \to b$ such that $a_n, b_n \in I_n$ for any n. Now, since f_n is 1-Lipschitz for each n,

$$\begin{aligned} d(f(a), f(b)) &\leq d(f(a), f_n(a_n)) + d(f_n(a_n), f_n(b_n)) + d(f_n(b_n), f(b)) \\ &\leq d(f(a), f_n(a_n)) + |a_n - b_n| + d(f_n(b_n), f(b)) \to |a - b| \end{aligned}$$

as $n \to \infty$. So f is 1-Lipschitz. It is also surjective. To check this let $x \in D_0^k$ for some $k \geq n_0$. Then there is $c_k \in I_k$ such that $x = f_k(c_k)$. By the construction we have a sequence $(c_n)_{n\geq k}$ such that $c_n \in I_n$, $f_n(c_n) = x$ and $|c_{n+1} - c_n| \leq \max\{4r_1\delta^{n+1}, 2(l(T_0^{n+1}) - l(T_0^n))\}$ for any $n \geq k$. From this we see that (c_n) is a Cauchy sequence and thus there is $c \in [0, C(c^2(E) + d(E))]$ such that $c_n \to c$. Now $c \in I$ by (i) and $x = \lim_{n\to\infty} f_n(c_n) = f(c)$. Thus $\bigcup_{n=n_0}^{\infty} D_0^n \subset f(I)$. Since $\bigcup_{n=n_0}^{\infty} D_0^n \subset \overline{E}$ is dense and $f(I)$ is compact, we have $E \subset f(I) = \overline{E}$. Finally, we restrict f to $f^{-1}(E)$. The proof of Theorem 1.1 is now complete.

We actually showed that

$$\ell(E) \leq C\left(\int_{\mathcal{T}} c(z_1, z_2, z_3)^2\, d\mu^3(z_1, z_2\, z_3) + d(E)\right). \tag{21}$$

A slight modification of the proof gives that we can take K_0 in the definition of $\mathcal{T}$ as a universal constant such that (21) holds for some C depending only on the regularity constant of E.

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