ON DISCRETE TIME HEDGING IN D-DIMENSIONAL OPTION PRICING MODELS

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Abstract. We study the approximation of certain stochastic integrals with respect to a d-dimensional diffusion by corresponding stochastic integrals with piecewise constant integrands i.e. an approximation of the form $\sum_{k=1}^{d}\int_0^T N_t^k dX_s^k \approx \sum_{k=1}^{d}\sum_{i=1}^{n} N_{t_{i-1}}^k (X_{t_i}^k - X_{t_{i-1}}^k)$. The approximation error is measured with respect to L^2 and it is shown that under certain assumptions the approximation rate is $n^{-1/2}$ when one optimizes over deterministic but not necessarily equidistant time-nets $0 = t_0 \le t_1 \le \cdots \le t_n = T$.

1. Introduction

Assume a Borel-function $f : \mathbb{R}^d \to \mathbb{R}$, $T > 0$ and a stochastic process $(X_t)_{t\in[0,T]}$ defined as a solution of

$$X_t^i = x_0^i + \int_0^t b_i(X_u)du + \sum_{j=1}^{d}\int_0^t \sigma_{ij}(X_u)dW_u^j,\ i \in \{1,\dots,d\}\,, \tag{1.1}$$

where $(W_t)_{t\in[0,T]}$ is a d-dimensional Brownian motion and the functions b and σ satisfy certain assumptions (cf. Chapter 2).

Consider the problem that a trader has to hedge, by a self-financing strategy, a European type option with maturity $T > 0$, where the pay-off of the option is described by a random variable $f(X_T)$. The perfect hedging strategy is determined by the process $(N_u)_{u\in[0,T]}$ in a stochastic integral representation of $f(X_T)$,

$$f(X_T) = V_0 + \sum_{k=1}^{d}\int_0^T N_u^k dX_u^k,$$

Date: September 8, 2005.

2000 *Mathematics Subject Classification.* 41A25; 60H05.

Key words and phrases. approximation, discrete time hedging, rate of convergence, stochastic integral.

where V_0 is the initial capital. In practice the continuous strategy has to be replaced by a discretely adjusted one. This leads to an approximation

$$\sum_{k=1}^{d}\int_0^T N_u^k dX_u^k \approx \sum_{k=1}^{d}\sum_{i=1}^{n} N_{t_{i-1}}^k (X_{t_i}^k - X_{t_{i-1}}^k),$$

where $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = T$ is a deterministic but not necessarily equidistant time-net.

We will measure and (to some extend) optimize the error of this approximation in L^2. Our interest lies in the rate of convergence of the approximation, when the approximation error is minimized over all time-nets with at most $n+1$ time-knots. This means that we are interested in the quantity

$$\inf_{\tau\in\mathcal{T}_n}\left\|\sum_{k=1}^{d}\int_0^T N_t^k dX_s^k - \sum_{k=1}^{d}\sum_{i=1}^{m} N_{t_{i-1}}^k (X_{t_i}^k - X_{t_{i-1}}^k)\right\|_{L^2} \tag{1.2}$$

as n tends to infinity, where

$$\mathcal{T}_n := \{(t_i)_{i=0}^m : 0 = t_0 < t_1 < \cdots < t_m = T, m \leq n\}.$$

Let us recall some results from the literature. Among others, the 1-dimensional case has been considered by Zhang [13], Gobet-Temam [8], and Geiss [5]. Geiss considered the approximation problem for general deterministic nets, which are not necessarily equidistant, and a closed form formula for the L_2-error was obtained. Based on this, in [7] several classes of examples were given, where the optimal rate of convergence $n^{-1/2}$ is attained by general deterministic nets (but, in general, not by equidistant ones). The result from [5] and [7] cannot be straightforward extended to the multi-dimensional case because part of the arguments from the 1-dimensional case do not seem to apply in the multi-dimensional situation.

The multi-dimensional case was, for example, studied by Zhang [13] and Temam [12] for equidistant nets. For C^1-functions with derivatives of polynomial growth (cf. [13, Proposition 3.1.6, Corollary 3.3.3]) Zhang established the rate $n^{-1/2}$. On the other side, Temam proved the rate $n^{-1/4}$ for the European digital option.

The aim of this paper is to improve the approximation rate of the European digital option in the multi-dimensional case from $n^{-1/4}$ to $n^{-1/2}$ by replacing the equidistant nets by general deterministic nets.

The paper is organized as follows: In Theorem 1 we show, for a certain class of functions f, that one gets the L^2-approximation rate of $n^{-1/2}$ by optimizing over all deterministic nets of cardinality $n+1$. Here we also allow a drift term in the underlying diffusion process (which is sometimes remarked, but not carried out, in the literature). In Section 4 we finish by some examples illustrating Theorem 1.

2. Preliminaries

In this chapter we introduce the setting we are working with and recall some known facts that are needed in order to prove our results.

We shall use the standard assumptions from stochastic calculus, i.e. we assume a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and, for $T > 0$, a right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by a standard d-dimensional Brownian motion $W = (W_t)_{t\in[0,T]}$ such that $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ contains all null-sets of $\mathcal{F}$ (cf. [9]). By $||x||$ we denote the Euclidean norm of a vector $x \in \mathbb{R}^d$. A Borel-function $\varphi : B \to \mathbb{R}$ on some set $B \subset \mathbb{R}$ will be extended to $B^d \subset \mathbb{R}^d$ by the notation

$$\varphi(x) := (\varphi(x_1), \varphi(x_2), \dots, \varphi(x_d)),\ x \in B^d.$$

We consider a diffusion

$$X_t^i = x_0^i + \int_0^t b_i(X_u)du + \sum_{j=1}^d \int_0^t \sigma_{ij}(X_u)dW_u^j,\ i = 1, \dots, d,\ a.s. \tag{2.1}$$

where $x_0 \in \mathbb{R}^d$. The process X is obtained through Y given as the unique path-wise continuous solution of (cf. [10, Corollary 2.2.1, p. 101])

$$Y_t^i = y_0^i + \int_0^t \hat{b}_i(Y_u)du + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Y_u)dW_u^j,\ i = 1, \dots, d,\ a.s. \tag{2.2}$$

where

$$\hat{b}_i(x), \hat{\sigma}_{ij}(x) \in C_b^\infty(\mathbb{R}^d)$$

and $\hat{\sigma}\hat{\sigma}^T$, where $(\hat{\sigma}\hat{\sigma}^T)_{ij}(x) = \sum_{k=1}^d \hat{\sigma}_{ik}(x)\hat{\sigma}_{jk}(x)$, is uniformly elliptic i.e.

$$\sum_{i,j=1}^d (\hat{\sigma}\hat{\sigma}^T)_{ij}(x)\xi_i\xi_j \geq \lambda \left|\left|\xi\right|\right|^2,\ \text{for all } x, \xi \in \mathbb{R}^d \text{ and some } \lambda > 0.$$

Under these assumptions the process Y has a transition density Γ with appropriate tail estimates (Theorem 7).

We consider two cases. The first case

($C1$) $x_0 = y_0 \in \mathbb{R}^d$, $\hat{b}_i(x) := b_i(x)$, $\hat{\sigma}_{ij}(x) := \sigma_{ij}(x)$, $X_t = Y_t$,

is related to the Brownian motion and the second case

($C2$) $x_0 = e^{y_0} \in (0,\infty)^d$, $\hat{b}_i(y) := \frac{b_i(e^y)}{e^{y_i}} - \frac{1}{2}\sum_{j=1}^d \hat{\sigma}_{ij}^2(y)$, $\hat{\sigma}_{ij}(y) := \frac{\sigma_{ij}(e^y)}{e^{y_i}}$ and $X_t = e^{Y_t}$,

is close to the geometric Brownian motion. In both cases we have

$$\mathbb{E} \sup_{t\in[0,T]} ||X_t||^p < \infty \tag{2.3}$$

for any $p > 0$ (cf. [10, Corollary 2.2.1, p. 101]).

To summarize the above, we start with the process X by choosing the matrix σ and the vector b such that the matrix $\hat{\sigma}$ and the vector $\hat{b}$ satisfy the required conditions above. In this way we obtain the process Y and deduce properties of the process X from the properties of Y.

To handle both of these cases simultaneously, we define functions $Q_i : \mathbb{R}^d \to \mathbb{R}$ for $i = 1, \dots, d$ by

$$Q_i(x) := \begin{cases} 1, & \text{in case } (C1) \\ x_i, & \text{in case } (C2). \end{cases}$$

In what follows we assume, for some $q \in [2,\infty)$ and $C > 0$, that

$$|f(x)| \le C\left(1 + ||x||^q\right), \ x \in E, \tag{2.4}$$

where the $f : E \to \mathbb{R}$ is a Borel-function and the set E is defined by

$$E := \begin{cases} \mathbb{R}^d, & \text{in case } (C1) \\ (0,\infty)^d, & \text{in case } (C2). \end{cases}$$

Through function the f we define the function g on $\mathbb{R}^d$ by

$$g(y) := \begin{cases} f(y), & \text{in case } (C1) \\ f(e^y), & \text{in case } (C2). \end{cases}$$

Applying Theorem 7 to the stochastic differential equation

$$\begin{cases} Z_t^i = Z_0^i + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Z_u) dW_u^j, & \text{in case } (C1) \\ Z_t^i = Z_0^i - \int_0^t \left(\frac{1}{2}\sum_{j=1}^d \hat{\sigma}_{ij}^2(Z_u)\right) du + \sum_{j=1}^d \int_0^t \hat{\sigma}_{ij}(Z_u) dW_u^j, & \text{in case } (C2) \end{cases}$$

gives a transition density Γ_0 such that we can define the function $G \in C^\infty([0,T)\times\mathbb{R}^d)$ by

$$G(t,y) := \int_{\mathbb{R}^d} \Gamma_0(T-t,y,\xi)g(\xi)d\xi,\ 0 \le t < T$$

so that

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{1}{2}\sum_{k,l=1}^d \left(\hat\sigma\hat\sigma^T(y)\right)_{kl} \frac{\partial^2}{\partial y_k y_l}\right) G(t,y) = 0 & (C1) \\ \left(\frac{\partial}{\partial t} - \sum_{i=1}^d \left(\frac{1}{2}\sum_{j=1}^d \hat\sigma_{ij}^2(y)\right)\frac{\partial}{\partial y_i} + \frac{1}{2}\sum_{k,l=1}^d \left(\hat\sigma\hat\sigma^T(y)\right)_{kl} \frac{\partial^2}{\partial y_k y_l}\right) G(t,y) = 0 & (C2). \end{cases} \tag{2.5}$$

Now we can define the function F on E by

$$F(t,x) := \begin{cases} G(t,x), & \text{in case } (C1) \\ G(t,\log(x)), & \text{in case } (C2). \end{cases}$$

Assumption (2.4) together with Theorem 7 implies that for $0 \le t \le T' < T$

$$|Q_i(x)| \left|\frac{\partial}{\partial x_i}F(t,x)\right| \le C_{d,T'}(1+||x||^q),\ x \in E,\ i=1,\dots,d \tag{2.6}$$

and

$$|Q_i(x)|\,|Q_j(x)| \left|\frac{\partial^2}{\partial x_i x_j}F(t,x)\right| \le C_{d,T'}(1+||x||^q),\quad x \in E,\ i,j=1,\dots,d. \tag{2.7}$$

Let

$$\mathcal{A} := \frac{\partial}{\partial t} + \frac{1}{2}\sum_{k,l=1}^d A_{kl}(x)\frac{\partial^2}{\partial x_k x_l} \tag{2.8}$$

where

$$A_{kl}(x) := \sum_{j=1}^d \sigma_{kj}(x)\sigma_{lj}(x). \tag{2.9}$$

From the definition of F and equation (2.5) it follows that

$$\mathcal{A}F(t,x) = 0 \text{ on } [0,T) \times E. \tag{2.10}$$

Moreover, Itô's formula gives that

$$F(t,X_t) = F(0,X_0) + \sum_{k=1}^d \int_0^t \frac{\partial}{\partial x_k}F(u,X_u)dX_u^k,\ a.s.\ t \in [0,T).$$

Finally, Theorem 7 gives that

$$F(t,X_t) \to f(X_T) \text{ in } L^2 \text{ as } t \nearrow T$$

and

$$f(X_T) = F(0, X_0) + \sum_{k=1}^{d} \int_0^T \frac{\partial}{\partial x_k} F(u, X_u) dX_u^k \ a.s.$$

3. Results

In the rest of the paper we assume the setting from Chapter 2. We start this chapter by stating our main result Theorem 1. It implies that under certain conditions the convergence rate for the supremum of the approximation error is bounded by $n^{-1/2}$, when one optimizes over all deterministic time-nets of cardinality $n+1$. Two examples where Theorem 1 is applied to are presented in Chapter 4.

Theorem 1. *Assume that for all $x \in E$*

$$\left| \frac{\partial^s}{\partial^q_{x_\beta} \partial^r_{x_\alpha}} \sigma_{ij}(x) \right| \leq C_1 \frac{Q_i(x)}{Q^q_\beta(x) Q^r_\alpha(x)}, \ \textit{where } q + r = s, \ q, r, s \in \{0, 1, 2\},$$

$|b_i(x)| \leq C_1 Q_i(x)$ and $A_{ii}(x) \geq \frac{1}{C_1} Q_i^2(x)$ for $i \in \{1, \ldots, d\}$ and some fixed $C_1 > 0$. Moreover, assume that

$$\sup_{\alpha,\beta} \mathbb{E} \left[A_{\alpha\alpha}(X_t) A_{\beta\beta}(X_t) \left| \frac{\partial^2}{\partial x_\alpha x_\beta} F(t, X_t) \right|^2 \right] \leq \frac{C_2}{(T-t)^{2\theta}}, \ \theta \in [0, 1), \ \textit{for some } C_2 > 0. \tag{3.1}$$

Then

$$\left(\mathbb{E} \sup_{t \in [0,T]} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t^\beta_{i-1} \wedge t}^{t^\beta_i \wedge t} \left(\frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t^\beta_{i-1}, X_{t^\beta_{i-1}}) \right) dX^k_u \right|^2 \right)^{\frac{1}{2}} \leq \frac{D_1}{\sqrt{n}},$$

where

$$\tau^\beta_n := \left(T \left(1 - \left(1 - \frac{i}{n} \right)^{\frac{1}{1-\beta}} \right) \right)_{i=0}^{n} \ \textit{and} \ \begin{cases} \beta = 0, & \theta \in [0, \frac{1}{2}) \\ \beta \in (2\theta - 1, 1), & \theta \in [\frac{1}{2}, 1) \end{cases}$$

and $D_1 > 0$ depends at most on β, C_1, C_2, d and T.

In addition, assume that

$$\inf_{u \in (r,s)} H^2(u) = C_H > 0, \tag{3.2}$$

for some $0 \leq r < s < T$, where H is defined by

$$H^2(u) := \mathbb{E} \sum_{\alpha,\beta,i,k=1}^{d} A_{\alpha\beta}(X_u) A_{ik}(X_u) \frac{\partial^2}{\partial x_\alpha x_i} F(u, X_u) \frac{\partial^2}{\partial x_\beta x_k} F(u, X_u), u \in [0, T). \tag{3.3}$$

Then we have the following two cases:

(L1) *In the case that $\theta \in [0, 3/4)$, we have, for any sequence of time-nets $0 = t_0^n \leq t_1^n \leq \ldots \leq t_n^n = T$ with $\sup_{i=1,\ldots,n}(t_i^n - t_{i-1}^n) \leq C_\tau/n$, $C_\tau > 0$, that*

$$\liminf_{n\to\infty} \sqrt{n} \left(\mathbb{E} \sup_{t\in[0,T]} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1}^n \wedge t}^{t_i^n \wedge t} \left(\frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}} \tag{3.4}$$

$$\geq \frac{1}{D_2}.$$

(L2) *If $\theta \in [3/4, 1)$, then we have that*

$$\liminf_{n\to\infty} \sqrt{n} \left(\mathbb{E} \sup_{t\in[0,T]} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1}^\beta \wedge t}^{t_i^\beta \wedge t} \left(\frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^\beta, X_{t_{i-1}^\beta}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}} \tag{3.5}$$

$$\geq \frac{1}{D_2}.$$

The constant $D_2 > 0$ depends at most on C_1, C_2, C_H, d and T.

Remark 2. (1) *In the case that the process $(X_t)_{t\in[0,T]}$ does not have a drift, it follows from Doob's inequality that inequalities (3.4) and (3.5) can be replaced by*

$$\liminf_{n\to\infty} \sqrt{n} \left(\mathbb{E} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1}}^{t_i} \left(\frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}} \geq \frac{1}{D_2}.$$

(2) *In (L2) we have the lower bound only for time-nets τ^β. Compared to (L1) this does not seem natural, since larger θ should correspond to a worse approximation. We need this restriction for technical reason (but believe that it can be removed).*

(3) *Under the setting of the Chapter 2 the assumptions in Theorem 1 which concern the estimates of the matrices A and σ and the vector b by the functions Q_i are always satisfied for some $C_1 > 0$.*

(4) *It follows by a simple calculation that*

$$H^2(u) = \mathbb{E} \sum_{m,n=1}^{d} \left(\sum_{\alpha,\beta=1}^{d} \sigma_{\alpha m}(X_u)\sigma_{\beta n}(X_u) \frac{\partial^2}{\partial x_\alpha x_\beta} F(u, X_u) \right)^2 .$$

Now because of (2.7) we have that $H^2(u) \in [0, \infty)$, *for* $u \in [0, T)$.

(5) *If the matrix* A *defined in (2.9) is a diagonal matrix, then*

$$H^2(u) = \mathbb{E} \sum_{\alpha,\beta=1}^{d} A_{\alpha\alpha}(X_u) A_{\beta\beta}(X_u) \left| \frac{\partial^2}{\partial x_\alpha x_\beta} F(u, X_u) \right|^2$$

and thus it is equivalent to the function

$$\sup_{\alpha,\beta} \mathbb{E} \left[A_{\alpha\alpha}(X_t) A_{\beta\beta}(X_t) \left| \frac{\partial^2}{\partial x_\alpha x_\beta} F(t, X_t) \right|^2 \right]$$

considered for the upper bound in Theorem 1. In the 1-dimensional case our function H *is the same as the function* H *controlling the approximation error in* [5].

Now turn to the proof of Theorem 1. We deal with a multi-step approximation error i.e. the stochastic integral $\sum_{k=1}^{d} \int_0^T \frac{\partial}{\partial x_k} F(u, X_u) dX_u^k$ is approximated by the stochastic integral $\sum_{k=1}^{d} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) dX_u^k$. In order to estimate the multi-step error we need to have information about the one-step error occurring in a time interval $[t_{i-1}, t_i]$. Here Proposition 3 and Proposition 4 below are needed. Proposition 3 gives the upper bound for the one-step error. It is an extension of Temam [12] for the upper estimate and replaces the limit arguments by the inequality (3.6), which can be applied to any fixed time-net to upper bound the approximation error. From Proposition 4 we get the lower bound for the one-step error. In the proof of Proposition 4 we use the same principal decomposition as in Temam [12], but apply it to non-equidistant nets.

Proposition 3. *If for all* $x \in E$

$$\left| \frac{\partial^s}{\partial^q_{x_\beta} \partial^r_{x_\alpha}} \sigma_{ij}(x) \right| \le C_3 \frac{Q_i(x)}{Q^q_\beta(x) Q^r_\alpha(x)}, \ q + r = s, \ q, r, s \in \{0, 1, 2\},$$

$|b_i(x)| \leq C_3 Q_i(x)$ and $A_{ii}(x) \geq \frac{1}{C_3} Q_i^2(x)$ for $i \in \{1,\ldots,d\}$ and for some $C_3 > 0$, then for $0 \leq a \leq u < T$ it holds

$$\begin{aligned} &\sum_{l=1}^{d}\sum_{k=1}^{d} \mathbb{E}\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(a,X_a)\right)^2 \sigma_{kl}(X_u)^2 \qquad (3.6) \\ &\leq D_3 \int_a^u \sup_{\alpha,\beta} \mathbb{E}\left[A_{\alpha\alpha}(X_t)A_{\beta\beta}(X_t)\left|\frac{\partial^2}{\partial x_\alpha x_\beta}F(t,X_t)\right|^2\right] dt, \end{aligned}$$

where $D_3 > 0$ depends at most on C_3, d and T.

Proof. To keep the notation simple, we allow in the following that the constant $C > 0$ may change from line to line.

Set

$$v_a := \left(\frac{\partial}{\partial x_k}F(a,X_a)\right)_{k=1}^{d} \quad \text{and } \phi_{kl}(u,x) := \left(\frac{\partial}{\partial x_k}F(u,x) - v_a^k\right)\sigma_{kl}(x).$$

Using this notation the assertion can be re-written as

$$\sum_{l=1}^{d}\sum_{k=1}^{d} \mathbb{E}\phi_{kl}^2(u,X_u) \leq D\int_a^u \sup_{\alpha,\beta}\mathbb{E}\left[A_{\alpha\alpha}(X_t)A_{\beta\beta}(X_t)\left|\frac{\partial^2}{\partial x_\alpha x_\beta}F(t,X_t)\right|^2\right] dt.$$

By the definition of ϕ_{kl} we have that

$$\begin{aligned} \sum_{l=1}^{d}\sum_{k=1}^{d}\phi_{kl}^2(u,x) &= \sum_{l=1}^{d}\sum_{k=1}^{d}\left(\frac{\partial}{\partial x_k}F(u,x) - v_a^k\right)^2\sigma_{kl}^2(x) \\ &= \sum_{k=1}^{d}\left[\left(\frac{\partial}{\partial x_k}F(u,x) - v_a^k\right)^2\sum_{l=1}^{d}\sigma_{kl}^2(x)\right]. \end{aligned}$$

The assumptions on σ give that

$$\frac{Q_k^2(x)}{C_3} \leq A_{kk}(x) = \sum_{l=1}^{d}\sigma_{kl}^2(x) \leq d\; C_3 Q_k^2(x).$$

This implies the equivalence

$$\begin{aligned} \frac{1}{C_3}\sum_{k=1}^{d}\left[\left(\frac{\partial}{\partial x_k}F(u,x) - v_a^k\right)^2 Q_k^2(x)\right] &\leq \sum_{l=1}^{d}\sum_{k=1}^{d}\phi_{kl}^2(u,x) \qquad (3.7) \\ &\leq d\; C_3\sum_{k=1}^{d}\left[\left(\frac{\partial}{\partial x_k}F(u,x) - v_a^k\right)^2 Q_k^2(x)\right]. \end{aligned}$$

Lemma 10 allows us to use the stopping argument from Lemma 9, which implies that

$$\mathbb{E}\phi_{kl}^2(u, X_u) = \int_a^u \mathbb{E}\left(\mathcal{A}\phi_{kl}^2\right)(v, X_v)dv + \sum_{m=1}^d \int_a^u \mathbb{E}\left(\frac{\partial}{\partial x_m}\phi_{kl}^2(v, X_v)\right) b_m(X_v)dv. \tag{3.8}$$

To prove our theorem we need to compute an upper bound for $\mathcal{A}\phi_{kl}^2(u,x)$ and for $\frac{\partial}{\partial x_m}\phi_{kl}^2(u,x)b_m(x)$. First we consider the term $\mathcal{A}\phi_{kl}^2$:

$$\begin{aligned}
&\left|\mathcal{A}\phi_{kl}^2(u,x)\right| \\
&= \left|2\phi_{kl}(u,x)\left(\mathcal{A}\phi_{kl}\right)(u,x) + \sum_{\alpha,\beta=1}^d \sum_{j=1}^d \left(\sigma_{\alpha j}(x)\frac{\partial}{\partial x_\alpha}\phi_{kl}(u,x)\right)\left(\sigma_{\beta j}(x)\frac{\partial}{\partial x_\beta}\phi_{kl}(u,x)\right)\right| \\
&\leq \phi_{kl}^2(u,x) + \left(\mathcal{A}\phi_{kl}(u,x)\right)^2 + \\
&\quad + \frac{1}{2}\sum_{\alpha,\beta=1}^d \sum_{j=1}^d \left[\left(\sigma_{\alpha j}(x)\frac{\partial}{\partial x_\alpha}\phi_{kl}(u,x)\right)^2 + \left(\sigma_{\beta j}(x)\frac{\partial}{\partial x_\beta}\phi_{kl}(u,x)\right)^2\right] \\
&= \phi_{kl}^2(u,x) + \left(\mathcal{A}\phi_{kl}\right)^2(u,x) + d\sum_{\alpha=1}^d \sum_{j=1}^d \left(\sigma_{\alpha j}(x)\frac{\partial}{\partial x_\alpha}\phi_{kl}(u,x)\right)^2.
\end{aligned} \tag{3.9}$$

Hence equation (3.8) implies that

$$\begin{aligned}
\mathbb{E}\phi_{kl}^2(u, X_u) \leq& \int_a^u \mathbb{E}\phi_{kl}^2(v, X_v)dv + \int_a^u \mathbb{E}\left(\mathcal{A}\phi_{kl}\right)^2(v, X_v)dv \\
&+ d\sum_{\alpha=1}^d \sum_{j=1}^d \int_a^u \mathbb{E}\left(\sigma_{\alpha j}(X_v)\frac{\partial}{\partial x_\alpha}\phi_{kl}(v, X_v)\right)^2 dv \\
&+ \sum_{m=1}^d \int_a^u \mathbb{E}\left|\left(\frac{\partial}{\partial x_m}\phi_{kl}^2(v, X_v)\right) b_m(X_v)\right| dv,
\end{aligned}$$

where the right-hand side is finite because of Lemma 10. From Gronwall's Lemma (see Theorem 8 in Appendix) it follows that

$$\begin{aligned}
\mathbb{E}\phi_{kl}^2(u, X_u) \leq& \left[\int_a^u \mathbb{E}\left(\mathcal{A}\phi_{kl}\right)^2(v, X_v)dv + \right. \\
&+ d\sum_{\alpha=1}^d \sum_{j=1}^d \int_a^u \mathbb{E}\left(\sigma_{\alpha j}(X_v)\frac{\partial}{\partial x_\alpha}\phi_{kl}(v, X_v)\right)^2 dv \\
&\left. + \sum_{m=1}^d \int_a^u \mathbb{E}\left|\frac{\partial}{\partial x_m}\phi_{kl}^2(v, X_v)b_m(X_v)\right| dv\right] e^{(u-a)}.
\end{aligned} \tag{3.10}$$

To continue we need to find an upper bound for the above expression. We start with $\mathcal{A}\phi_{kl}$ and have, by definition, that

$$\begin{aligned}\mathcal{A}\phi_{kl}(u,x) = &\left(\frac{\partial^2}{\partial t x_k}F(u,x)\right)\sigma_{kl}(x)+ \\ &+\frac{1}{2}\sum_{\alpha,\beta=1}^{d} A_{\alpha\beta}(x)\left[\left(\frac{\partial^2}{\partial x_\alpha x_\beta}\sigma_{kl}(x)\right)\left(\frac{\partial}{\partial x_k}F(u,x)-v_a^k\right)\right. \\ &\qquad+\left(\frac{\partial}{\partial x_\beta}\sigma_{kl}(x)\right)\left(\frac{\partial^2}{\partial x_\alpha x_k}F(u,x)\right) \\ &\qquad+\left(\frac{\partial}{\partial x_\alpha}\sigma_{kl}(x)\right)\left(\frac{\partial^2}{\partial x_\beta x_k}F(u,x)\right) \\ &\qquad\left.+\sigma_{kl}(x)\left(\frac{\partial^3}{\partial x_\alpha x_\beta x_k}F(u,x)\right)\right].\end{aligned}$$

Taking the derivative with respect to x_k in the partial differential equation (2.10) we get that

$$\begin{aligned}&\frac{\partial}{\partial x_k}\frac{\partial}{\partial t}F(u,x)+\frac{1}{2}\sum_{\alpha,\beta=1}^{d} A_{\alpha\beta}(x)\frac{\partial^3}{\partial x_k x_\alpha x_\beta}F(u,x) \\ &\quad=-\frac{1}{2}\sum_{\alpha,\beta=1}^{d}\left(\frac{\partial}{\partial x_k}A_{\alpha\beta}(x)\right)\frac{\partial^2}{\partial x_\alpha x_\beta}F(u,x).\end{aligned}$$

Now we can replace the derivative with respect to t and the third order derivatives in the formula for $\mathcal{A}\phi_{kl}(u,x)$ by second order derivatives:

$$\begin{aligned}\mathcal{A}\phi_{kl}(u,x) = \frac{1}{2}\sum_{\alpha,\beta=1}^{d}&\left[A_{\alpha\beta}(x)\left(\frac{\partial^2}{\partial x_\alpha x_\beta}\sigma_{kl}(x)\right)\left(\frac{\partial}{\partial x_k}F(u,x)-v_a^k\right)\right. \\ &+A_{\alpha\beta}(x)\left(\frac{\partial}{\partial x_\beta}\sigma_{kl}(x)\right)\left(\frac{\partial^2}{\partial x_\alpha x_k}F(u,x)\right) \\ &+A_{\alpha\beta}(x)\left(\frac{\partial}{\partial x_\alpha}\sigma_{kl}(x)\right)\left(\frac{\partial^2}{\partial x_\beta x_k}F(u,x)\right) \\ &\left.-\sigma_{kl}(x)\left(\frac{\partial}{\partial x_k}A_{\alpha\beta}(x)\right)\left(\frac{\partial^2}{\partial x_\alpha x_\beta}F(u,x)\right)\right].\end{aligned}$$

It follows from the definition of the matrix A and the assumption on the matrix σ that

$$|A_{\alpha\beta}(x)| \le CQ_\alpha(x)Q_\beta(x)$$

and

$$\left|\frac{\partial}{\partial x_k} A_{\alpha\beta}(x)\right| \le C \frac{Q_\alpha(x) Q_\beta(x)}{Q_k(x)}.$$

Now we can bound the function $\left(\mathcal{A}\phi_{kl}\right)^2(u,x)$ from the above by

$$\begin{aligned}
\left(\mathcal{A}\phi_{kl}\right)^2(u,x) \le C \sum_{\alpha,\beta=1}^{d} \Bigg[& \left(\frac{\partial}{\partial x_k} F(u,x) - v_a^k\right)^2 Q_k^2(x) \\
& + Q_\alpha^2(x) Q_k^2(x) \left(\frac{\partial^2}{\partial x_\alpha x_k} F(u,x)\right)^2 \\
& + Q_\beta^2(x) Q_k^2(x) \left(\frac{\partial^2}{\partial x_\beta x_k} F(u,x)\right)^2 \\
& + Q_\alpha^2(x) Q_\beta^2(x) \left(\frac{\partial^2}{\partial x_\alpha x_\beta} F(u,x)\right)^2 \Bigg].
\end{aligned} \tag{3.11}$$

For $\left(\sigma_{\alpha j}(x) \frac{\partial}{\partial x_\alpha} \phi_{kl}(u,x)\right)^2$ we get that

$$\begin{aligned}
& \left(\sigma_{\alpha j}(x) \frac{\partial}{\partial x_\alpha} \phi_{kl}(u,x)\right)^2 \\
& \le 2\sigma_{\alpha j}^2(x) \left(\frac{\partial}{\partial x_\alpha} \sigma_{kl}(x)\right)^2 \left(\frac{\partial}{\partial x_k} F(u,x) - v_a^k\right)^2 + 2\sigma_{\alpha j}^2(x) \sigma_{kl}^2(x) \left(\frac{\partial^2}{\partial x_\alpha x_k} F(u,x)\right)^2 \\
& \le C \left(Q_k^2(x) \left(\frac{\partial}{\partial x_k} F(u,x) - v_a^k\right)^2 + Q_\alpha^2(x) Q_k^2(x) \left(\frac{\partial^2}{\partial x_\alpha x_k} F(u,x)\right)^2 \right).
\end{aligned} \tag{3.12}$$

The term including b_m can be bounded as follows:

$$\begin{aligned}
& \left| \left(\frac{\partial}{\partial x_m} \phi_{kl}^2(u,x)\right) b_m(x) \right| \\
& \le 2 \left| \frac{\partial}{\partial x_k} F(u,x) - v_a^k \right| |\sigma_{kl}(x)| \, |b_m(x)| \times \\
& \qquad \times \left(\left| \frac{\partial}{\partial x_m} \sigma_{kl}(x) \left(\frac{\partial}{\partial x_k} F(u,x) - v_a^k\right) \right| + |\sigma_{kl}(x)| \left| \frac{\partial^2}{\partial x_m x_k} F(u,x) \right| \right) \\
& \le C \left| \frac{\partial}{\partial x_k} F(u,x) - v_a^k \right| Q_k(x) Q_m(x) \times \\
& \quad \times \left(\frac{Q_k(x)}{Q_m(x)} \left| \frac{\partial}{\partial x_k} F(u,x) - v_a^k \right| + Q_k(x) \left| \frac{\partial^2}{\partial x_m x_k} F(u,x) \right| \right) \\
& \le C \left(\left| \frac{\partial}{\partial x_k} F(u,x) - v_a^k \right|^2 Q_k^2(x) + Q_k^2(x) Q_m^2(x) \left| \frac{\partial^2}{\partial x_m x_k} F(u,x) \right|^2 \right),
\end{aligned}$$

$$(3.13)$$

where we used that

$$\begin{aligned}&\left|\frac{\partial}{\partial x_k}F(u,x)-v_a^k\right|Q_k^2(x)Q_m(x)\left|\frac{\partial^2}{\partial x_m x_k}F(u,x)\right|\\&\le\left|\frac{\partial}{\partial x_k}F(u,x)-v_a^k\right|^2Q_k^2(x)+Q_k^2(x)Q_m^2(x)\left|\frac{\partial^2}{\partial x_m x_k}F(u,x)\right|^2.\end{aligned}$$

Now the expectation of $\phi_{kl}^2(u,X_u)$ can be bounded by

$$\begin{aligned}\mathbb{E}\phi_{kl}^2(u,X_u)\le C\int_a^u\mathbb{E}\left(\frac{\partial}{\partial x_k}F(v,X_v)-v_a^k\right)^2Q_k^2(X_v)dv\\+C\int_a^u\sup_{\alpha,\beta}\mathbb{E}Q_\alpha^2(X_v)Q_\beta^2(X_v)\left(\frac{\partial^2}{\partial x_\alpha x_\beta}F(v,X_v)\right)^2dv,\end{aligned}$$

where we use (3.10), (3.11), (3.12) and (3.13). From the above and (3.7) we get

$$\begin{aligned}&\sum_{k=1}^d\mathbb{E}\left[\left(\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right)^2Q_k^2(X_u)\right]\le C\sum_{l=1}^d\sum_{k=1}^d\mathbb{E}\phi_{kl}^2(u,X_u)\\&\le C\int_a^u\mathbb{E}\sum_{k=1}^d\left(\frac{\partial}{\partial x_k}F(v,X_v)-v_a^k\right)^2Q_k^2(X_v)dv+\\&\quad+C\int_a^u\sup_{\alpha,\beta}\mathbb{E}Q_\alpha^2(X_v)Q_\beta^2(X_v)\left(\frac{\partial^2}{\partial x_\alpha x_\beta}F(v,X_v)\right)^2dv.\end{aligned}$$

Gronwall's lemma (Theorem 8) gives

$$\begin{aligned}&\sum_{k=1}^d\mathbb{E}\left[\left(\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right)^2Q_k^2(X_u)\right]\\&\le e^{C(u-a)}C\int_a^u\sup_{\alpha,\beta}\mathbb{E}Q_\alpha^2(X_v)Q_\beta^2(X_v)\left(\frac{\partial^2}{\partial x_\alpha x_\beta}F(v,X_v)\right)^2dv\end{aligned}\tag{3.14}$$

and the assertion follows from (3.7). □

Proposition 4. *If for all $x\in E$*

$$\left|\frac{\partial^s}{\partial_{x_\beta}^q\partial_{x_\alpha}^r}\sigma_{ij}(x)\right|\le C_4\frac{Q_i(x)}{Q_\beta^q(x)Q_\alpha^r(x)},\ q+r=s,\ q,r,s\in\{0,1,2\},$$

and

$$|b_i(x)|\le C_4Q_i(x)$$

for $i \in \{1, \dots, d\}$ *and for some* $C_4 > 0$*, then for* $0 \le a < t < T$ *it holds that*

$$\sum_{j=1}^{d} \mathbb{E}\left(\sum_{i=1}^{d}\left(\frac{\partial}{\partial x_i}F(t,X_t) - \frac{\partial}{\partial x_i}F(a,X_a)\right)\sigma_{ij}(X_u)\right)^2 \ge \int_a^t H^2(u)du$$

$$- D_4 \int_a^t \left[\left(\int_a^u \sup_{\alpha,\beta} \mathbb{E}Q_\alpha^2(X_v)Q_\beta^2(X_v)\left|\frac{\partial^2}{\partial x_\alpha x_\beta}F(v,X_v)\right|^2 dv\right)^{\frac{1}{2}} \times\right.$$

$$\times \left(\sup_{\alpha,\beta} \mathbb{E}Q_\alpha^2(X_u)Q_\beta^2(X_u)\left|\frac{\partial^2}{\partial x_\alpha x_\beta}F(u,X_u)\right|^2\right)^{\frac{1}{2}} +$$

$$\left. + \int_a^u \sup_{\alpha,\beta} \mathbb{E}Q_\alpha^2(X_v)Q_\beta^2(X_v)\left|\frac{\partial^2}{\partial x_\alpha x_\beta}F(v,X_v)\right|^2 dv\right] du,$$

where the function H *is defined in Theorem 1 and* $D_4 > 0$ *depends at most on* C_4, d *and* T*.*

Proof. To abbreviate the notation we assume again that $C > 0$ may change from line to line. We let

$$\phi_u^{ij} := \left(\frac{\partial}{\partial x_i}F(u,X_u) - v_a^i\right)\sigma_{ij}(X_u),$$

where $u \in [a,t]$ and $v_a^i := \frac{\partial}{\partial x_i}F(a,X_a)$. Itô's formula gives that, a.s.,

$$\begin{aligned}
\phi_t^{ij} = &\int_a^t \frac{\partial^2}{\partial t x_i}F(u,X_u)\sigma_{ij}(X_u)du \\
&+ \sum_{\alpha=1}^{d}\int_a^t b_\alpha(X_u)\left[\frac{\partial^2}{\partial x_\alpha x_i}F(u,X_u)\sigma_{ij}(X_u)\right. \\
&\qquad\left. + \left(\frac{\partial}{\partial x_i}F(u,X_u) - v_a^i\right)\frac{\partial}{\partial x_\alpha}\sigma_{ij}(X_u)\right]du \\
&+ \sum_{\alpha=1}^{d}\sum_{n=1}^{d}\int_a^t \left[\frac{\partial^2}{\partial x_\alpha x_i}F(u,X_u)\sigma_{ij}(X_u)\right. \\
&\qquad\left. + \left(\frac{\partial}{\partial x_i}F(u,X_u) - v_a^i\right)\frac{\partial}{\partial x_\alpha}\sigma_{ij}(X_u)\right]\sigma_{\alpha n}(X_u)dW_u^n \\
&+ \frac{1}{2}\sum_{\alpha,\beta=1}^{d}\int_a^t \left[\frac{\partial^3}{\partial x_\beta x_\alpha x_i}F(u,X_u)\sigma_{ij}(X_u) + \frac{\partial^2}{\partial x_\alpha x_i}F(u,X_u)\frac{\partial}{\partial x_\beta}\sigma_{ij}(X_u) + \right. \\
&\qquad + \frac{\partial^2}{\partial x_\beta x_i}F(u,X_u)\frac{\partial}{\partial x_\alpha}\sigma_{ij}(X_u) + \\
&\qquad\left. + \left(\frac{\partial}{\partial x_i}F(u,X_u) - v_a^i\right)\frac{\partial^2}{\partial x_\beta x_\alpha}\sigma_{ij}(X_u)\right]A_{\alpha\beta}(X_u)du.
\end{aligned}$$

From the above we deduce

$$
\begin{aligned}
&d\langle \phi^{ij}, \phi^{kj}\rangle_u \\
&= \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \Bigg[\left(\frac{\partial^2}{\partial x_\alpha x_i} F(u,X_u)\sigma_{ij}(X_u) + \left(\frac{\partial}{\partial x_i} F(u,X_u) - v_a^i \right) \frac{\partial}{\partial x_\alpha}\sigma_{ij}(X_u) \right) \times \\
&\times \left(\frac{\partial^2}{\partial x_\beta x_k} F(u,X_u)\sigma_{kj}(u,X_u) + \left(\frac{\partial}{\partial x_k} F(u,X_u) - v_a^k \right) \frac{\partial}{\partial x_\beta}\sigma_{kj}(X_u) \right) A_{\alpha\beta}(X_u) \Bigg] du
\end{aligned}
$$

and, using the equality $\mathcal{A}F = 0$ (cf. (2.8) and (2.10)), we get that

$$
\begin{aligned}
&d\phi_u^{ij} \\
&= \sum_{\alpha=1}^{d}\sum_{n=1}^{d} \left(\frac{\partial^2}{\partial x_\alpha x_i} F(u,X_u)\sigma_{ij}(X_u) + \left(\frac{\partial}{\partial x_i} F(u,X_u) - v_a^i \right) \frac{\partial}{\partial x_\alpha}\sigma_{ij}(X_u) \right) \sigma_{\alpha n}(X_u) dW_u^n \\
&+ \frac{1}{2}\sum_{\alpha,\beta=1}^{d} A_{\alpha\beta}(X_u) \Bigg[\frac{\partial^2}{\partial x_\alpha x_i} F(u,X_u) \frac{\partial}{\partial x_\beta}\sigma_{ij}(X_u) + \frac{\partial^2}{\partial x_\beta x_i} F(u,X_u) \frac{\partial}{\partial x_\alpha}\sigma_{ij}(X_u) \\
&\qquad\qquad + \left(\frac{\partial}{\partial x_i} F(u,X_u) - v_a^i \right) \frac{\partial^2}{\partial x_\beta x_\alpha}\sigma_{ij}(X_u) \Bigg] du \\
&- \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \left[\frac{\partial}{\partial x_i} A_{\alpha\beta}(X_u) \frac{\partial^2}{\partial x_\beta x_\alpha} F(u,X_u)\sigma_{ij}(X_u) \right] du \\
&+ \sum_{\alpha=1}^{d} b_\alpha(X_u) \Bigg[\frac{\partial^2}{\partial x_\alpha x_i} F(u,X_u)\sigma_{ij}(X_u) \\
&\qquad\qquad + \left(\frac{\partial}{\partial x_i} F(u,X_u) - v_a^i \right) \frac{\partial}{\partial x_\alpha}\sigma_{ij}(X_u) \Bigg] du.
\end{aligned}
$$

Let

$$
S_N := \inf \left\{ u \geq a : \left|\phi_u^{ij}\right| \geq N \text{ or } ||X_u|| \geq N \text{ for some } i,j \in \{1,\ldots,d\} \right\} \wedge t
$$

for $N = 1, 2, \ldots$. Because of $S_N \nearrow t$ a.s. as $N \to \infty$ and Lemma 10 one has that

$$
\sum_{j=1}^{d} \mathbb{E}\left(\sum_{i=1}^{d} \phi_t^{ij} \right)^2 = \lim_{N\to\infty} \sum_{j=1}^{d} \mathbb{E}\left(\sum_{i=1}^{d} \phi_{S_N}^{ij} \right)^2 .
$$

Using the partial integration formula for semi-martingales, we get

$$
\sum_{j=1}^{d} \mathbb{E}\left(\sum_{i=1}^{d} \phi_{S_N}^{ij} \right)^2 = \sum_{j=1}^{d} 2\mathbb{E}\int_a^{S_N} \sum_{i,k=1}^{d} \phi_u^{kj} d\phi_u^{ij} + \sum_{i,k=1}^{d} \mathbb{E}\int_a^{S_N} d\langle \phi^{ij}, \phi^{kj}\rangle_u .
$$

Because of the choice of the stopping time S_N, the expected value of the "dW_u^n-terms" vanishes. For the rest, we obtain as main term

$$\mathbb{E}\int_a^{S_N}\sum_{\alpha,\beta,i,k=1}^{d} A_{\alpha\beta}(X_u)A_{ik}(X_u)\frac{\partial^2}{\partial x_\alpha x_i}F(u,X_u)\frac{\partial^2}{\partial x_\beta x_k}F(u,X_u)du$$

and (after some computation) terms of the type

$$\mathbb{E}\int_a^{S_N} A_{\alpha\beta}(X_u)\sigma_{kj}(X_u)\frac{\partial}{\partial x_\beta}\sigma_{ij}(X_u)\left(\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right)\frac{\partial^2}{\partial x_\alpha x_i}F(u,X_u)du$$

and

$$\mathbb{E}\int_a^{S_N}\left(\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right)\sigma_{kj}(X_u)b_\alpha(X_u)\left(\frac{\partial}{\partial x_i}F(u,X_u)-v_a^i\right)\frac{\partial}{\partial x_\alpha}\sigma_{ij}(X_u)du.$$

Using the assumptions on the matrix σ and the vector b we can bound these terms by

$$\begin{aligned}
&\left|A_{\alpha\beta}(X_u)\sigma_{kj}(X_u)\frac{\partial}{\partial x_\beta}\sigma_{ij}(X_u)\left(\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right)\frac{\partial^2}{\partial x_\alpha x_i}F(u,X_u)\right| \\
&\le CQ_\alpha(X_u)Q_\beta(X_u)Q_k(X_u)\frac{Q_i(X_u)}{Q_\beta(X_u)}\left|\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right|\left|\frac{\partial^2}{\partial x_\alpha x_i}F(u,X_u)\right| \\
&\le CQ_k(X_u)\left|\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right|Q_\alpha(X_u)Q_i(X_u)\left|\frac{\partial^2}{\partial x_\alpha x_i}F(u,X_u)\right|
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right)\sigma_{kj}(X_u)b_\alpha(X_u)\left(\frac{\partial}{\partial x_i}F(u,X_u)-v_a^i\right)\frac{\partial}{\partial x_\alpha}\sigma_{ij}(X_u)\right| \\
&\le CQ_k(X_u)\left|\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right|Q_i(X_u)\left|\frac{\partial}{\partial x_i}F(u,X_u)-v_a^i\right|.
\end{aligned}$$

Hölder's inequality and (3.14) give

$$\begin{aligned}
&\mathbb{E}Q_k(X_u)\left|\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right|Q_\alpha(X_u)Q_i(X_u)\left|\frac{\partial^2}{\partial x_\alpha x_i}F(u,X_u)\right| \\
&\le\left(\mathbb{E}Q_k^2(X_u)\left|\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right|^2\right)^{\frac{1}{2}}\left(\sup_{\alpha',\beta'}\mathbb{E}Q_{\alpha'}^2(X_u)Q_{\beta'}^2(X_u)\left|\frac{\partial^2}{\partial x_{\alpha'}x_{\beta'}}F(u,X_u)\right|^2\right)^{\frac{1}{2}} \\
&\le C\left(\int_a^u\sup_{\alpha',\beta'}\mathbb{E}Q_{\alpha'}^2(X_v)Q_{\beta'}^2(X_v)\left|\frac{\partial^2}{\partial x_{\alpha'}x_{\beta'}}F(v,X_v)\right|^2dv\right)^{\frac{1}{2}}\times \\
&\quad\times\left(\sup_{\alpha',\beta'}\mathbb{E}Q_{\alpha'}^2(X_u)Q_{\beta'}^2(X_u)\left|\frac{\partial^2}{\partial x_{\alpha'}x_{\beta'}}F(u,X_u)\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Moreover

$$
\begin{aligned}
&\mathbb{E}Q_k(X_u)\left|\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right|Q_i(X_u)\left|\frac{\partial}{\partial x_i}F(u,X_u)-v_a^i\right| \\
&\le\left(\mathbb{E}Q_k^2(X_u)\left|\frac{\partial}{\partial x_k}F(u,X_u)-v_a^k\right|^2\right)^{\frac12}\left(\mathbb{E}Q_i^2(X_u)\left|\frac{\partial}{\partial x_i}F(u,X_u)-v_a^i\right|^2\right)^{\frac12} \\
&\le C\int_a^u\sup_{\alpha,\beta}\mathbb{E}Q_\alpha^2(X_v)Q_\beta^2(X_v)\left|\frac{\partial^2}{\partial x_\alpha x_\beta}F(v,X_v)\right|^2 dv
\end{aligned}
$$

and the assertion follows by $N\to\infty$. $\square$

Proof of Theorem 1.

Also in this proof, we use the same notation for different constants. First we consider the upper bound for the approximation error. Let $\varepsilon\in(0,T)$. Using Doob's inequality together with Hölder's inequality we see that

$$
\begin{aligned}
&\left(\mathbb{E}\sup_{t\in[0,T-\varepsilon]}\left|\sum_{i=1}^n\sum_{k=1}^d\int_{t_{i-1}\wedge t}^{t_i\wedge t}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right)dX_u^k\right|^2\right)^{\frac12} \\
&\le\left(\mathbb{E}\sup_{t\in[0,T-\varepsilon]}\left|\sum_{i=1}^n\sum_{k=1}^d\int_{t_{i-1}\wedge t}^{t_i\wedge t}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right)b_k(X_u)du\right|^2\right)^{\frac12}+ \\
&+\left(\mathbb{E}\sup_{t\in[0,T-\varepsilon]}\left|\sum_{i=1}^n\sum_{l=1}^d\int_{t_{i-1}\wedge t}^{t_i\wedge t}\sum_{k=1}^d\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right)\sigma_{kl}(X_u)dW_u^l\right|^2\right)^{\frac12} \\
&\le\left(\mathbb{E}\left|\sum_{i=1}^n\sum_{k=1}^d\int_{t_{i-1}\wedge T-\varepsilon}^{t_i\wedge T-\varepsilon}\left|\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right||b_k(X_u)|\,du\right|^2\right)^{\frac12}+ \\
&+2\left(\mathbb{E}\left|\sum_{i=1}^n\sum_{l=1}^d\int_{t_{i-1}\wedge T-\varepsilon}^{t_i\wedge T-\varepsilon}\sum_{k=1}^d\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right)\sigma_{kl}(X_u)dW_u^l\right|^2\right)^{\frac12} \\
&\le\sum_{k=1}^d\sqrt{T}\left(\mathbb{E}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\left|\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right|^2|b_k(X_u)|^2\,du\right)^{\frac12}+ \\
&+2\left(\mathbb{E}\left|\sum_{i=1}^n\sum_{l=1}^d\int_{t_{i-1}\wedge T-\varepsilon}^{t_i\wedge T-\varepsilon}\sum_{k=1}^d\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right)\sigma_{kl}(X_u)dW_u^l\right|^2\right)^{\frac12} \\
&:=B_1+B_2.
\end{aligned}
$$

Inequality (3.14) gives that

$$B_1 \le C_1\sqrt{T}\sum_{k=1}^{d}\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\mathbb{E}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right)^2 Q_k^2(X_u)du\right)^{\frac{1}{2}}$$

$$\le \sqrt{T}C\sum_{k=1}^{d}\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{u}\sup_{\alpha,\beta}\mathbb{E}Q_\alpha^2(X_v)Q_\beta^2(X_v)\left(\frac{\partial^2}{\partial x_\alpha x_\beta}F(v,X_v)\right)^2 dvdu\right)^{\frac{1}{2}}.$$

For B_2 the Itô-isometry and the orthogonality of stochastic integrals give

$$B_2^2 = 4\mathbb{E}\left|\sum_{i=1}^{n}\sum_{l=1}^{d}\int_{t_{i-1}\wedge T-\varepsilon}^{t_i\wedge T-\varepsilon}\sum_{k=1}^{d}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right)\sigma_{kl}(X_u)dW_u^l\right|^2$$

$$= 4\sum_{i=1}^{n}\sum_{l=1}^{d}\mathbb{E}\left|\int_{t_{i-1}\wedge T-\varepsilon}^{t_i\wedge T-\varepsilon}\sum_{k=1}^{d}\left[\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right]\sigma_{kl}(X_u)dW_u^l\right|^2$$

$$= 4\sum_{i=1}^{n}\sum_{l=1}^{d}\int_{t_{i-1}\wedge T-\varepsilon}^{t_i\wedge T-\varepsilon}\mathbb{E}\left|\sum_{k=1}^{d}\left[\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right]\sigma_{kl}(X_u)\right|^2 du$$

$$\le 4d\sum_{i=1}^{n}\sum_{l=1}^{d}\sum_{k=1}^{d}\int_{t_{i-1}}^{t_i}\mathbb{E}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right)^2\sigma_{kl}(X_u)^2du.$$

Letting $\varepsilon \searrow 0$ we get by monotone convergence that

$$\left(\mathbb{E}\sup_{t\in[0,T]}\left|\sum_{i=1}^{n}\sum_{k=1}^{d}\int_{t_{i-1}\wedge t}^{t_i\wedge t}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right)dX_u^k\right|^2\right)^{\frac{1}{2}}$$

$$\le \sqrt{T}C\sum_{k=1}^{d}\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{u}\sup_{\alpha,\beta}\mathbb{E}Q_\alpha^2(X_v)Q_\beta^2(X_v)\left(\frac{\partial^2}{\partial x_\alpha x_\beta}F(v,X_v)\right)^2 dv\right)^{\frac{1}{2}}+$$

$$+\left(4d\sum_{i=1}^{n}\sum_{l=1}^{d}\sum_{k=1}^{d}\int_{t_{i-1}}^{t_i}\mathbb{E}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1},X_{t_{i-1}})\right)^2\sigma_{kl}(X_u)^2du\right)^{\frac{1}{2}}.$$

The assertion for the upper bound follows from Proposition 3 and Lemma 11 in Appendix.

Now we continue with the lower bound of the approximation error. Let $[A,B]$ be a subinterval of (r,s) such that

$$0 < \frac{(B-A)C_B}{(T-B)^{2\theta}} \le \frac{C_H}{4}, \tag{3.15}$$

where C_H is taken from (3.2) and the constant $C_B > 0$ satisfies (cf. (3.14))

$$d\sum_{k=1}^{d}\mathbb{E}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(a,X_a)\right)^2 b_k^2(X_u) \le C_B\int_a^u\frac{1}{(T-u)^{2\theta}}dv \tag{3.16}$$

for $A \le a < u \le B$. Let us now consider the approximation error inside the interval $[A, B]$. Denote $I_n := \{i : A \le t_{i-1}^n \le t_i^n \le B\}$ and denote in both cases for the lower estimate (cf. Theorem 1 cases $L1$ and $L2$) the sequence of time-nets by $(t_i^n)_{i=0}^n$. Note that for large n the set I_n is not an empty set because, in both cases, we have that $\sup_{i=1,\dots,n}(t_i^n - t_{i-1}^n) \le C_\tau/n$. Now on $[A, B]$ we get that

$$\left(\mathbb{E}\left|\sum_{i\in I_n}\sum_{k=1}^d \int_{t_{i-1}^n}^{t_i^n}\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}^n, X_{t_{i-1}^n})\right)dX_u^k\right|^2\right)^{\frac{1}{2}}$$
$$\ge \left|\left(\mathbb{E}\left|\sum_{i\in I_n}\sum_{l=1}^d \int_{t_{i-1}^n}^{t_i^n}\sum_{k=1}^d\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}^n, X_{t_{i-1}^n})\right)\sigma_{kl}(X_u)dW_u^l\right|^2\right)^{\frac{1}{2}} -\right.$$
$$\left. - \left(\mathbb{E}\left|\sum_{i\in I_n}\sum_{k=1}^d \int_{t_{i-1}^n}^{t_i^n}\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}^n, X_{t_{i-1}^n})\right)b_k(X_u)du\right|^2\right)^{\frac{1}{2}}\right|.$$

This implies that

$$\mathbb{E}\left|\sum_{i\in I_n}\sum_{k=1}^d \int_{t_{i-1}^n}^{t_i^n}\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}^n, X_{t_{i-1}^n})\right)dX_u^k\right|^2$$
$$\ge \frac{1}{2}\mathbb{E}\left|\sum_{i\in I_n}\sum_{l=1}^d \int_{t_{i-1}^n}^{t_i^n}\sum_{k=1}^d\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}^n, X_{t_{i-1}^n})\right)\sigma_{kl}(X_u)dW_u^l\right|^2 -$$
$$- \mathbb{E}\left|\sum_{i\in I_n}\sum_{k=1}^d \int_{t_{i-1}^n}^{t_i^n}\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}^n, X_{t_{i-1}^n})\right)b_k(X_u)du\right|^2$$
$$\ge \frac{1}{2}\mathbb{E}\left|\sum_{i\in I_n}\sum_{l=1}^d \int_{t_{i-1}^n}^{t_i^n}\sum_{k=1}^d\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}^n, X_{t_{i-1}^n})\right)\sigma_{kl}(X_u)dW_u^l\right|^2$$
$$- (B-A)\sum_{i\in I_n} d\sum_{k=1}^d \mathbb{E}\int_{t_{i-1}^n}^{t_i^n}\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}^n, X_{t_{i-1}^n})\right)^2 b_k^2(X_u)du.$$

Now (3.15) and (3.16) give that

$$(B-A)\sum_{i\in I_n} d\sum_{k=1}^d \mathbb{E}\int_{t_{i-1}^n}^{t_i^n}\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}^n, X_{t_{i-1}^n})\right)^2 b_k^2(X_u)du$$
$$\le (B-A)\sum_{i\in I_n}\int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{u}\frac{C_B}{(T-v)^{2\theta}}dvdu$$
$$\le \sum_{i\in I_n}\int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{u}\frac{H^2(v)}{4}dvdu.$$

Let us now consider the lower bound for

$$\left(\mathbb{E}\left|\sum_{i\in I_n}\sum_{l=1}^{d}\int_{t_{i-1}^n}^{t_i^n}\sum_{k=1}^{d}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1}^n,X_{t_{i-1}^n})\right)\sigma_{kl}(X_u)dW_u^l\right|^2\right)^{\frac{1}{2}}.$$

Using the Itô-isometry, for $0\le a<b<T$, we get that

$$\begin{aligned}B_3 :=&\mathbb{E}\left|\sum_{l=1}^{d}\int_a^b\sum_{k=1}^{d}\left(\frac{\partial}{\partial x_k}F(t,X_t)-\frac{\partial}{\partial x_k}F(a,X_a)\right)\sigma_{kl}(X_t)dW_t^l\right|^2\\ =&\sum_{l=1}^{d}\mathbb{E}\int_a^b\left(\sum_{k=1}^{d}\left(\frac{\partial}{\partial x_k}F(t,X_t)-\frac{\partial}{\partial x_k}F(a,X_a)\right)\sigma_{kl}(X_t)\right)^2 dt.\end{aligned}$$

Assuming $b-a\le 1$, Proposition 4 implies that

$$\begin{aligned}B_3 \ge& \int_a^b\int_a^t H^2(u)dudt - D_4\int_a^b\int_a^t\left(\int_a^u\frac{1}{(T-v)^{2\theta}}dv\right)^{\frac{1}{2}}\left(\frac{1}{(T-u)^{2\theta}}\right)^{\frac{1}{2}}dudt\\ &- D_4\int_a^b\int_a^t\int_a^u\frac{1}{(T-v)^{2\theta}}dvdudt\\ \ge& \int_a^b\int_a^t H^2(u)dudt-2D_4\sqrt{b-a}\int_a^b\int_a^t\frac{1}{(T-u)^{2\theta}}dudt.\end{aligned}$$

Considering the multi-step error for the approximation we get that

$$\begin{aligned}&\mathbb{E}\left|\sum_{i\in I_n}\sum_{k=1}^{d}\int_{t_{i-1}^n}^{t_i^n}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1}^n,X_{t_{i-1}^n})\right)dX_u^k\right|^2\\ &\ge\frac{1}{2}\mathbb{E}\left|\sum_{i\in I_n}\sum_{l=1}^{d}\int_{t_{i-1}^n}^{t_i^n}\sum_{k=1}^{d}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1}^n,X_{t_{i-1}^n})\right)\sigma_{kl}(X_u)dW_u^l\right|^2\\ &-(B-A)\sum_{i\in I_n}d\sum_{k=1}^{d}\mathbb{E}\int_{t_{i-1}^n}^{t_i^n}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1}^n,X_{t_{i-1}^n})\right)^2 b_k^2(X_u)du\\ &\ge\frac{1}{2}\sum_{i\in I_n}\left[\int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{t}H^2(u)dudt - C\int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{t}\frac{\sqrt{t_i^n-t_{i-1}^n}}{(T-u)^{2\theta}}dudt\right]\\ &-\sum_{i\in I_n}\int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{t}\frac{H^2(u)}{4}dudt\\ &\ge\sum_{i\in I_n}\int_{t_{i-1}^n}^{t_i^n}(t_i^n-t)\frac{H^2(t)}{4}dt-\frac{C}{\sqrt{n}}\sum_{i\in I_n}\int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{t}\frac{1}{(T-u)^{2\theta}}dudt.\end{aligned}$$

In the case $L1$ for our lower estimates [5, Remark 6.6] implies that

$$\frac{1}{\sqrt{n}}\sum_{i\in I_n}\int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{t}\frac{1}{(T-u)^{2\theta}}dudt\le\frac{1}{\sqrt{n}}\frac{C}{n^{1/2+\varepsilon}}$$

for some $\varepsilon > 0$. In the case $L2$, Lemma 11 gives

$$\frac{1}{\sqrt{n}}\sum_{i\in I_n}\int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-1}^n}^{t}\frac{1}{(T-u)^{2\theta}}dudt \le \frac{1}{\sqrt{n}}\frac{C}{n}.$$

The term containing H^2 can be bounded from below as

$$\begin{aligned}
\liminf_{n\to\infty} n\sum_{i\in I_n}\int_{t_{i-1}^n}^{t_i^n}(t_i^n-t)\frac{H^2(t)}{4}dt &\ge \liminf_{n\to\infty}\frac{C_H}{4}n\sum_{i\in I_n}\frac{(t_i^n-t_{i-1}^n)^2}{2}\\
&\ge \liminf_{n\to\infty}\frac{C_H}{8}\left(\sum_{i\in I_n}(t_i^n-t_{i-1}^n)\right)^2\\
&= \frac{C_H}{8}(B-A)^2.
\end{aligned}$$

This proves the estimate. □

4. Examples

In this chapter we give two examples as an application of our results. For simplicity, we consider a diffusion

$$X_t^i = x_0^i + \sum_{j=1}^{d}\int_0^t \sigma_{ij}(X_u)dW_u^j,\ i=1,\dots,d$$

in the case $(C2)$. Let $0 = t_0 \le t_1 \le \cdots \le t_n = T$ be any deterministic time-net on $[0,T]$. By $(t_i^\beta)_{i=0}^n$ we denote the time-net

$$(t_i^\beta)_{i=0}^n = \left(T\left(1-\left(1-\frac{i}{n}\right)^{\frac{1}{1-\beta}}\right)\right)_{i=0}^n \quad\text{and}\quad \begin{cases}\beta = 0, & \theta\in[0,\frac{1}{2})\\ \beta\in(2\theta-1,1), & \theta\in[\frac{1}{2},1)\end{cases}$$

where θ is from Theorem 1 equation (3.1).

Example 5. *For a European digital option with strike price $K > 0$,*

$$f(x) := \mathbb{1}_{\sum_{i=1}^d \lambda_i x_i \ge K}(x),\ \textit{where}\ \lambda_1,\dots,\lambda_d > 0,$$

the approximation rate is $n^{-1/4}$ if equidistant time-nets are used [12, Theorem 2.1]. *Theorem 1 gives that this option can be approximated by the rate $n^{-1/2}$, more precisely*

$$\left(\mathbb{E}\sup_{t\in[0,T]}\left|\sum_{i=1}^n\sum_{k=1}^d\int_{t_{i-1}^\beta\wedge t}^{t_i^\beta\wedge t}\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(t_{i-1}^\beta,X_{t_{i-1}^\beta})\right)dX_u^k\right|^2\right)^{\frac{1}{2}} \le \frac{D_1}{\sqrt{n}}$$

and

$$\frac{1}{D_2} \leq \liminf_{n\to\infty} \sqrt{n} \left(\mathbb{E} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1}^\beta}^{t_i^\beta} \left(\frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^\beta, X_{t_{i-1}^\beta}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}}$$

for all $\beta \in (1/2, 1)$. *Assumption (3.1) follows from* [12, Proposition 3.3] *for* $\theta = 3/4$ *and (3.2) is due to* [12, equation (4.2)].

Example 6. *If* $d \geq 2$, $(\sigma_{ij})_{i,j=1}^d$ *is a diagonal matrix, and if* $\sigma_{ij}(x) = \sigma_{ij}(x_i)$, *then the transition density of the process* Y *can be written as the product of the transition densities of the process* Y^i, *i.e.*

$$\Gamma_Y(t, y, \xi) = \prod_{i=1}^d \Gamma_{Y^i}(t, y_i, \xi_i).$$

Assume that $f(x) = \prod_{i=1}^d f_i(x_i)$, *where the functions* f_i *are of at most polynomial growth. The definition of* F *implies that*

$$F(t, x) = \prod_{i=1}^d F_i(t, x_i)$$

with

$$F_i(t, x_i) := \int_{\mathbb{R}} \Gamma_{Y^i}(T - t, \log x_i, \xi_i) f_i(e^{\xi_i}) d\xi_i,$$

and the second order derivatives of the function F *can be written as*

$$\frac{\partial^2}{\partial x_i x_j} F(t, x) = \begin{cases} \frac{\partial}{\partial x_i} F_i(t, x_i) \frac{\partial}{\partial x_j} F_j(t, x_j) \prod_{m=1, m\neq i, m\neq j}^d F_m(t, x_m) & i \neq j, \\ \frac{\partial^2}{\partial x_i x_i} F_i(t, x_i) \prod_{m=1, m\neq i}^d F_m(t, x_m), & i = j. \end{cases}$$

Assume that there exist $C > 0$ *and* $\theta_i \in [0, 1)$ *for all* $i = 1, \ldots, d$ *such that*

$$\mathbb{E} \left[Q_i^2(X_t) \frac{\partial^2}{\partial x_i x_i} F_i(t, X_t^i) \right]^2 \leq \frac{C}{(T - t)^{2\theta_i}}. \tag{4.1}$$

Theorem [3, Theorem 2.3] *gives that*

$$\sup_{t\in[0,T)} (T - t)^\alpha \sqrt{t} \left(\mathbb{E} \left| \sigma_{ii}(X_t) \frac{\partial}{\partial x_i} F_i(t, X_t) \right|^2 \right)^{\frac{1}{2}} < \infty \tag{4.2}$$

if and only if

$$\sup_{t\in[0,T)} (T - t)^\alpha \left(\int_0^t \mathbb{E} \left| \sigma_{ii}^2(X_u) \frac{\partial^2}{\partial x_i x_i} F_i(u, X_u) \right|^2 du \right)^{\frac{1}{2}} < \infty, \tag{4.3}$$

where $\alpha \in [0,1/2)$. *Now we assume, without loss of generality, that* $\theta_i \in (1/2,1)$. *For* $\alpha = \beta$ *we get that*

$$\begin{aligned}
&\mathbb{E}\left[A_{\alpha\alpha}(X_t)A_{\beta\beta}(X_t)\left(\frac{\partial^2}{\partial x_\alpha x_\beta}F(t,X_t)\right)^2\right]\\
&= \mathbb{E}\left[\sigma^2_{\alpha\alpha}(X^\alpha_t)\left|\frac{\partial^2}{\partial x_\alpha x_\alpha}F(t,X^\alpha_t)\right|^2\right]\prod_{m=1,m\neq\alpha}^{d}\mathbb{E}\left|F_m(t,X^m_t)\right|^2\\
&\leq \mathbb{E}\left[\sigma^2_{\alpha\alpha}(X^\alpha_t)\left|\frac{\partial^2}{\partial x_\alpha x_\alpha}F(t,X^\alpha_t)\right|^2\right]\prod_{m=1,m\neq\alpha}^{d}\mathbb{E}\left|f_m(X^m_T)\right|^2
\end{aligned}$$

which is at most of order $(T-t)^{2\theta_\alpha}$. *For* $\alpha \neq \beta$ *we get that*

$$\begin{aligned}
&\mathbb{E}\left[A_{\alpha\alpha}(X_t)A_{\beta\beta}(X_t)\left(\frac{\partial^2}{\partial x_\alpha x_\beta}F(t,X_t)\right)^2\right]\\
&\leq \mathbb{E}\left[\sigma^2_{\alpha\alpha}(X^\alpha_t)\left|\frac{\partial}{\partial x_\alpha}F(t,X^\alpha_t)\right|^2\right]\mathbb{E}\left[\sigma^2_{\beta\beta}(X^\beta_t)\left|\frac{\partial}{\partial x_\beta}F(t,X^\beta_t)\right|^2\right]\times\\
&\quad\times\prod_{m=1,m\neq\alpha,m\neq\beta}^{d}\mathbb{E}\left|f_m(X^m_T)\right|^2.
\end{aligned}$$

The implication (4.3) $\Rightarrow$ *(4.2) implies, for* $\eta_i := (\theta_i - 1/2) \in (0,1/2)$, *that*

$$\sup_{t\in[0,T)}(T-t)^{\eta_\alpha}\sqrt{t}\left(\mathbb{E}\left|\sigma_{ii}(X_t)\frac{\partial}{\partial x_i}F_i(t,X_t)\right|^2\right)^{\frac{1}{2}} < \infty.$$

Using [3, Lemma 5.2] *one can remove the factor* $\sqrt{t}$, *so that*

$$\sup_{t\in[0,T)}(T-t)^{2(\eta_\alpha+\eta_\beta)}\mathbb{E}\left|A_{\alpha\alpha}(X_t)A_{\beta\beta}(X_t)\frac{\partial^2}{\partial x_\alpha x_\beta}F(t,X_t)\right|^2 < \infty.$$

Putting all estimates together, we find a $\theta \in [0,1)$ *such that*

$$\mathbb{E}\left[A_{\alpha\alpha}(X_t)A_{\beta\beta}(X_t)\left(\frac{\partial^2}{\partial x_\alpha x_\beta}F(t,X_t)\right)^2\right] \leq \frac{D}{(T-t)^{2\theta}}.$$

Looking at the above computations, one can take $\theta := \max\{\theta_1,\ldots,\theta_d,1/2\}$ *without the assumption* $\theta_i \in (1/2,1)$.

Let us now consider a simple example of mixing different type of options. Assume, for the dimension $d=3$, *that* σ *is a* 3×3 *matrix defined by*

$$\sigma_{ij}(x) = \begin{cases} 0, & i \neq j,\\ x_i, & i = j,\end{cases}$$

and that $x_0 = (1,1,1)$. *Define the pay-off function* f *by*

$$f(x_1,x_2,x_3) = f_1(x_1)f_2(x_2)f_3(x_3) := (x_1-K_1)_+ \, (x_2-K_2)_+^{\alpha} \, \mathbb{1}_{[K_3,\infty)}(x_3),$$

where $K_i > 0$, $i = 1,2,3$ *and* $\alpha \in (0,\frac{1}{2})$. *For* F_1 *one can compute*

$$\frac{\partial^2}{\partial x_1 x_1} F_1(t,x_1) = \frac{1}{x_1\sqrt{T-t}}\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\left(\frac{\log\left(\frac{x_1}{K_1}\right)+\frac{T-t}{2}}{\sqrt{T-t}}\right)^2}{2}\right]$$

and

$$\mathbb{E}\left[Q_1^2(X_t)\frac{\partial^2}{\partial x_1 x_1}F_1(t,X_t^1)\right]^2 = \frac{K_1}{2\pi\sqrt{T^2-t^2}} \exp\left[-\frac{(T/2+\log(K_1))^2}{T+t}\right].$$

This implies that one can choose $\theta_1 = \frac{1}{4}$. *For* F_2, *we can choose* $\theta_2 = \frac{3-2\alpha}{4}$ *and for* F_3, $\theta_3 = \frac{3}{4}$ *(cf.* [8, Lemma 1 and Lemma 2]*). Now Theorem 1 gives that*

$$\left(\mathbb{E}\sup_{t\in[0,T]}\left|\sum_{i=1}^n\sum_{k=1}^d\int_{t_{i-1}^\beta\wedge t}^{t_i^\beta\wedge t}\left(\frac{\partial}{\partial x_k}F(u,X_u) - \frac{\partial}{\partial x_k}F(t_{i-1}^\beta, X_{t_{i-1}^\beta})\right)dX_u^k\right|^2\right)^{\frac{1}{2}} \le \frac{D_1}{\sqrt{n}}$$

for all $\beta \in (1/2,1)$. *Under the assumptions of this example we have that*

$$\begin{aligned}
H^2(u) &= \mathbb{E}\sum_{\alpha,\beta=1}^3\left(\sigma_{\alpha\alpha}(X_u)\sigma_{\beta\beta}(X_u)\frac{\partial^2}{\partial x_\alpha x_\beta}F(u,X_u)\right)^2\\
&\ge \mathbb{E}\sum_{\alpha=1}^3\left(\sigma_{\alpha\alpha}^2(X_u)\frac{\partial^2}{\partial x_\alpha x_\alpha}F(u,X_u)\right)^2\\
&\ge \frac{1}{C_1}\sum_{\alpha=1}^3\mathbb{E}\,|X_u^\alpha|^4\left|\frac{\partial^2}{\partial x_\alpha x_\alpha}F(u,X_u^\alpha)\right|^2\prod_{m\neq\alpha}\mathbb{E}\,|F_m(u,X_u^m)|^2.
\end{aligned}$$

Since f_1, f_2 *and* f_3 *are not almost surely linear and since*

$$u \mapsto \mathbb{E}\,|X_u^\alpha|^4\left|\frac{\partial^2}{\partial x_\alpha x_\alpha}F_\alpha(u,X_u^\alpha)\right|^2$$

is continuous and increasing [6, proof of Proposition 2.1] *the result from* [5, Theorem 4.6] *implies*

$$\sup_{u\in[0,T)}\mathbb{E}\,|X_u^\alpha|^4\left|\frac{\partial^2}{\partial x_\alpha x_\alpha}F_\alpha(u,X_u^\alpha)\right|^2 > 0.$$

Moreover,

$$\lim_{u\nearrow T}\mathbb{E}\left(F_\alpha(u,X_u^\alpha)\right)^2 = \mathbb{E}f_\alpha^2(X_T^\alpha) > 0.$$

Hence Theorem 1 gives that

$$\frac{1}{D_2} \leq \liminf_{n\to\infty} \sqrt{n} \left(\mathbb{E} \sup_{t\in[0,T]} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1}^{\beta}\wedge t}^{t_i^{\beta}\wedge t} \left(\frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}^{\beta}, X_{t_{i-1}^{\beta}}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}}.$$

If we take $f(x_1,x_2) = f_1(x_1)f_2(x_2)$, *then we can choose* $\theta = \theta_2 < 3/4$ *and get*

$$\frac{1}{D_2} \leq \liminf_{n\to\infty} \sqrt{n} \left(\mathbb{E} \sup_{t\in[0,T]} \left| \sum_{i=1}^{n} \sum_{k=1}^{d} \int_{t_{i-1}^{n}\wedge t}^{t_i^{n}\wedge t} \left(\frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(t_{i-1}, X_{t_{i-1}}) \right) dX_u^k \right|^2 \right)^{\frac{1}{2}},$$

for any sequence of time-nets with $\sup_{i=1,\dots,n}(t_i^n - t_{i-1}^n) \leq C/n$.

Appendix

Theorem 7 (Theorem 8. p. 263 [1], Theorem 5.4. p. 149 [2]). *For* $\hat{b}$, $\hat{\sigma}$ *with* $\hat{\sigma}\hat{\sigma}^T$ *uniformly elliptic, there exists a transition density* $\Gamma : (0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty) \in C^\infty$ *such that* $\mathbb{P}(Y_t \in B) = \int_B \Gamma(t,y,\xi)d\xi$, *for* $t \in (0,T]$ *and* $B \in \mathcal{B}(\mathbb{R}^d)$, *where* $Y = (Y_t)_{t\in[0,T]}$ *is the strong solution of the SDE (2.2) starting from* y*: Moreover, the following are satisfied:*

(i) *For* $(s,y,\xi) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ *one has*

$$\frac{\partial}{\partial s}\Gamma(s,y,\xi) = \frac{1}{2} \sum_{k,l=1}^{d} \sum_{j=1}^{d} \hat{\sigma}_{kj}(y)\hat{\sigma}_{lj}(y) \frac{\partial^2}{\partial y_k y_l}\Gamma(s,y,\xi) + \sum_{i=1}^{d} \hat{b}_i(y) \frac{\partial}{\partial y_i}\Gamma(s,y,\xi).$$

(ii) *For* $a \in \{0,1,2,\dots\}$ *and multi-indices* b *and* c *there exist positive constants* C *and* D, *depending only on* a,b,c *and* d, *such that*

$$\left| \frac{\partial^{a+|b|+|c|}}{\partial^a t\, \partial^b y\, \partial^c \xi} \Gamma(t,y,\xi) \right| \leq \frac{C}{t^{(d+2a+|b|+|c|)/2}} e^{-D\frac{||y-\xi||^2}{t}},$$

where $||\cdot||$ *is the Euclidean norm.*

Theorem 8 (Gronwall's Lemma, [11]). *If, for* $t_0 \leq t \leq t_1$, $\phi(t) \geq 0$ *is a continuous function such that*

$$\phi(t) \leq K + L \int_{t_0}^{t} \phi(s)ds$$

for $t_0 \leq t \leq t_1$ *where* $K, L \geq 0$, *then*

$$\phi(t) \leq K e^{L(t-t_0)}$$

on $t_0 \leq t \leq t_1$.

Lemma 9. *Let* $0 \le a < b < T$ *and define*

$$\phi_{kl}(u,x) := \left(\frac{\partial}{\partial x_k} F(u,x) - v_a^k\right)\sigma_{kl}(x),\ \ u \in [0,T),\ \ x \in E,$$

where v_a^k *is an* $\mathcal{F}_a$*-measurable random variable and assume that*

$$\mathbb{E}\sup_{u\in[a,b]}\left[\phi_{kl}^2(u,X_u) + \left|\left(\mathcal{A}\phi_{kl}^2\right)(u,X_u)\right| + \sum_{m=1}^{d}\left|\frac{\partial}{\partial x_m}\phi_{kl}^2(u,X_u)b_m(X_u)\right|\right] < \infty.$$

Then for $s \in [a,b]$ *one has*

$$\begin{aligned}\mathbb{E}\phi_{kl}^2(s,X_s) =& \mathbb{E}\phi_{kl}^2(a,X_a) + \int_a^s \mathbb{E}\left(\mathcal{A}\phi_{kl}^2\right)(u,X_u)du + \\ &+ \int_a^s \mathbb{E}\sum_{m=1}^{d}\left(\frac{\partial}{\partial x_m}\phi_{kl}^2(u,X_u)\right)b_m(X_u)du.\end{aligned}$$

Proof. By Itô's formula we obtain

$$\begin{aligned}\phi_{kl}^2(s,X_s) = \phi_{kl}^2(a,X_a) &+ \int_a^s \left(\mathcal{A}\phi_{kl}^2\right)(u,X_u)du \\ &+ \sum_{m=1}^{d}\int_a^s \frac{\partial}{\partial x_m}\phi_{kl}^2(u,X_u)dX_u^m.\end{aligned}$$

Define

$$S_n^m := \inf\left\{r \in [a,s] \left| \int_a^r \sum_{j=1}^{d}\left[\frac{\partial}{\partial x_m}\phi_{kl}^2(u,X_u)\right]^2 \sigma_{mj}^2(X_u)du > n\right.\right\} \wedge s$$

and

$$S_n := \min\left\{S_n^m, m \in \{1,\dots,d\}\right\}.$$

This implies that

$$\int_a^{S_n}\sum_{j=1}^{d}\left[\frac{\partial}{\partial x_m}\phi_{kl}^2(u,X_u)\right]^2\sigma_{mj}^2(X_u)du \le n$$

and

$$\mathbb{E}\int_a^{S_n}\frac{\partial}{\partial x_m}\phi_{kl}^2(u,X_u)\sigma_{mj}(X_u)dW_u^j = 0,$$

for $n \in \mathbb{N}$, $m \in \{1,\dots,d\}$ and $j \in \{1,\dots,d\}$. Dominated convergence gives

$$\begin{aligned}\mathbb{E}\phi_{kl}^2(s,X_s) &= \lim_{n\to\infty}\mathbb{E}\phi_{kl}^2(S_n,X_{S_n}) \\ &= \lim_{n\to\infty}\left[\mathbb{E}\phi_{kl}^2(a,X_a) + \mathbb{E}\int_a^{S_n}\left(\mathcal{A}\phi_{kl}^2\right)(u,X_u)du + \right.\end{aligned}$$

$$+ \mathbb{E} \sum_{m=1}^{d} \int_a^{S_n} \frac{\partial}{\partial x_m} \phi_{kl}^2(u, X_u) dX_u^m \Bigg]$$

$$= \mathbb{E}\phi_{kl}^2(a, X_a) + \mathbb{E} \int_a^s \left(\mathcal{A}\phi_{kl}^2\right)(u, X_u) du +$$

$$+ \mathbb{E} \sum_{m=1}^{d} \int_a^{s} \frac{\partial}{\partial x_m} \phi_{kl}^2(u, X_u) b_m(X_u) du.$$

□

Lemma 10. *If for all $x \in E$*

$$\left| \frac{\partial^s}{\partial^q_{x_\beta} \partial^r_{x_\alpha}} \sigma_{ij}(x) \right| \leq C \frac{Q_i(x)}{Q^q_\beta(x) Q^r_\alpha(x)}, \ q + r = s, \ q, r, s \in \{0, 1, 2\},$$

for some $C > 0$, then for all $0 \leq a \leq b < T$ and $k, l \in \{1, \dots, d\}$ we have that

$$\mathbb{E} \sup_{u \in [a,b]} \phi_{kl}^2(u, X_u) < \infty,$$

$$\mathbb{E} \sup_{u \in [a,b]} \left| Q_m(X_u) \frac{\partial}{\partial x_m} \phi_{kl}(u, X_u) \right|^2 < \infty, \ m = 1, \dots, d,$$

$$\mathbb{E} \sup_{u \in [a,b]} \left| Q_m(X_u) \frac{\partial}{\partial x_m} \phi_{kl}^2(u, X_u) \right| < \infty, \ m = 1, \dots, d,$$

and

$$\mathbb{E} \sup_{u \in [a,b]} \left| \left(\mathcal{A}\phi_{kl}^2\right)(u, X_u) \right| < \infty,$$

where

$$\phi_{kl}(u, x) = \left(\frac{\partial}{\partial x_k} F(u, x) - \frac{\partial}{\partial x_k} F(a, X_a) \right) \sigma_{kl}(x), \ u \in [a, b].$$

Proof. This proof uses the same notation for different constants. Equation (2.6) implies that the random variable $\phi_{kl}^2(u, X_u)$ can be bounded by

$$\begin{aligned}
\phi_{kl}^2(u, X_u) &= \left(\frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(a, X_a) \right)^2 \sigma_{kl}^2(X_u) \\
&\leq C \left(\frac{\partial}{\partial x_k} F(u, X_u) - \frac{\partial}{\partial x_k} F(a, X_a) \right)^2 Q_k^2(X_u) \\
&\leq C \left(\left| Q_k(X_u) \frac{\partial}{\partial x_k} F(u, X_u) \right|^2 + \left| Q_k(X_a) \frac{\partial}{\partial x_k} F(a, X_a) \right|^2 \frac{Q_k^2(X_u)}{Q_k^2(X_a)} \right) \\
&\leq C \left[\sup_{u' \in [a,b]} (1 + ||X_{u'}||^q)^2 \right] \left(1 + \frac{Q_k^2(X_u)}{Q_k^2(X_a)} \right).
\end{aligned}$$

Applying Hölder's inequality we get that

$$\mathbb{E}\sup_{u\in[a,b]}\phi^2_{kl}(u,X_u) \le C\mathbb{E}\left[\sup_{u\in[a,b]}(1+||X_u||^q)^2\sup_{u\in[a,b]}\left(1+\frac{Q_k^2(X_u)}{Q_k^2(X_a)}\right)\right]$$
$$\le C\left(\mathbb{E}\sup_{u\in[a,b]}(1+||X_u||^q)^4\right)^{\frac{1}{2}}\left(\mathbb{E}\sup_{u\in[a,b]}\left(1+\frac{Q_k^2(X_u)}{Q_k^2(X_a)}\right)^2\right)^{\frac{1}{2}}.$$

Equation (2.3) gives that $\mathbb{E}\sup_{u\in[a,b]}(1+||X_u||^q)^4$ is finite. In the case $(C1)$ it is trivial that the latter term is finite. Let us now turn to the case $(C2)$. Theorem 7 implies that

$$\begin{aligned}\mathbb{E}(X_u^k)^{-p} &= \mathbb{E}e^{-pY_u^k} \\ &= \int_{\mathbb{R}^d} e^{-py_k}\Gamma_Y(u,y_0,y)dy \\ &\le C\ \exp\left[-py_0^k+\frac{1}{2}\frac{u}{2D}p^2\right]<\infty\end{aligned} \tag{4.4}$$

for all $p\in[0,\infty)$ and some $C>0$ and $D>0$. Hölder's inequality now implies that

$$\mathbb{E}\sup_{u\in[a,b]}\left(1+\frac{Q_k^2(X_u)}{Q_k^2(X_a)}\right)^2 \le 2\left(1+\mathbb{E}\sup_{u\in[a,b]}\frac{Q_k^4(X_u)}{Q_k^4(X_a)}\right)$$
$$\le 2\left(1+\left(\mathbb{E}\sup_{u\in[a,b]}(X_u^k)^8\right)^{\frac{1}{2}}\left(\mathbb{E}\frac{1}{(X_a^k)^8}\right)^{\frac{1}{2}}\right),$$

which is finite by (2.3) and (4.4).

Straightforward calculation gives that

$$\mathbb{E}\sup_{u\in[a,b]}\left|Q_m(X_u)\left(\frac{\partial}{\partial x_m}\phi_{kl}\right)(u,X_u)\right|^2$$
$$=\mathbb{E}\sup_{u\in[a,b]}\left[Q_m^2(X_u)\left|\frac{\partial^2}{\partial x_m x_k}F(u,X_u)\sigma_{kl}(X_u)+\right.\right.$$
$$\left.\left.+\left(\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(a,X_a)\right)\left(\frac{\partial}{\partial x_m}\sigma_{kl}\right)(X_u)\right|^2\right]$$
$$\le C\left(\mathbb{E}\sup_{u\in[a,b]}Q_m^2(X_u)Q_k^2(X_u)\left|\frac{\partial^2}{\partial x_m x_k}F(u,X_u)\right|^2+\right.$$
$$\left.+\mathbb{E}\sup_{u\in[a,b]}Q_k^2(X_u)\left|\frac{\partial}{\partial x_k}F(u,X_u)-\frac{\partial}{\partial x_k}F(a,X_a)\right|^2\right),$$

and this is finite by equations (2.3) and (2.7) and the above argument.

Hölder's inequality together with the above gives that

$$\mathbb{E}\sup_{u\in[a,b]}\left|Q_m(X_u)\left(\frac{\partial}{\partial x_m}\phi_{kl}^2\right)(u,X_u)\right|$$
$$=2\mathbb{E}\sup_{u\in[a,b]}\left|Q_m(X_u)\phi_{kl}(u,X_u)\left(\frac{\partial}{\partial x_m}\phi_{kl}\right)(u,X_u)\right|$$
$$\leq 2\left(\mathbb{E}\sup_{u\in[a,b]}|\phi_{kl}(u,X_u)|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\sup_{u\in[a,b]}Q_m^2(X_u)\left|\left(\frac{\partial}{\partial x_m}\phi_{kl}\right)(u,X_u)\right|^2\right)^{\frac{1}{2}}<\infty.$$

For the last part of the proof, equation (3.9) gives that

$$\mathbb{E}\sup_{u\in[a,b]}\left|\left(\mathcal{A}\phi_{kl}^2\right)(u,X_u)\right|<\infty$$

if

$$\mathbb{E}\sup_{u\in[a,b]}\left(\mathcal{A}\phi_{kl}\right)^2(u,X_u)<\infty.$$

This follows from equation (3.11) and the above arguments. □

Lemma 11. *Assume that a Borel-measurable function $\varphi:[0,T)\to[0,\infty)$ satisfies*

$$\varphi(u)\leq\frac{C}{(T-u)^\theta},\ \ u\in[0,T),$$

for some $C>0$ and some $\theta\in[0,1)$. Then there exists a constant $C'>0$ such that

$$\sum_{t_i\in\mathcal{T}_n^\beta}\int_{t_{i-1}^\beta}^{t_i^\beta}\int_{t_{i-1}^\beta}^{u}\varphi^2(s)dsdu\leq\frac{C'}{n},$$

where

$$\mathcal{T}_n^\beta:=\left(T\left(1-\left(1-\frac{i}{n}\right)^{\frac{1}{1-\beta}}\right)\right)_{i=0}^n\ \ \textit{and}\ \ \begin{cases}\beta=0, & \theta\in[0,\frac{1}{2})\\ \beta\in(2\theta-1,1), & \theta\in[\frac{1}{2},1).\end{cases}$$

Proof. Lemma follows from [4, Lemma 4.14, Proposition 4.16]. □

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